The Hong Kong Polytechnic University
Department of Applied Mathematics

Spectral Hypergraph Theory

Shenglong Hu

A thesis submitted in partial fulfilment of
the requirements for the degree of Doctor of Philosophy

March 2013

Certificate of Originality

I hereby declare that this thesis is my own work and that, to the best of my knowledge and belief, it reproduces no material previously published or written, nor material that has been accepted for the award of any other degree or diploma, except where due acknowledgement has been made in the text.

________________________________(Signed)

Shenglong Hu (Name of student)

Abstract

The main subject of this thesis is the study of a few basic problems in spectral hypergraph theory based on Laplacian-type tensors. These problems are hypergraph analogues of some important problems in spectral graph theory.

As some foundations, we study some new problems of *tensor determinant* and *nonnegative tensor partition*. Then two classes of Laplacian-type tensors for uniform hypergraphs are proposed. One is called *Laplacian*, and the other one *Laplace-Beltrami tensor*. We study the H-spectra of uniform hypergraphs through their Laplacian, and the Z-spectra of even uniform hypergraphs through their Laplace-Beltrami tensors. All the H^+-eigenvalues of the Laplacian can be computed out through the developed partition method. *Spectral component*, an intrinsic notion of a uniform hypergraph, is introduced to characterize the hypergraph spectrum. Many fundamental properties of the spectrum are connected to the underlying hypergraph structures. Basic spectral hypergraph theory based on Laplacian-type tensors are built. With the theory, we study *algebraic connectivity*, *edge connectivity*, *vertex connectivity*, *edge expansion*, and *spectral invariance* of the hypergraph.

This thesis is based on the following papers written by the author during the period of stay at the Department of Applied Mathematics, The Hong Kong Polytechnic University as a graduate student:

1 Hu, S., Qi, L., 2012. Algebraic connectivity of an even uniform hypergraph. Journal of Combinatorial Optimization, 24(4), 564–579.

2 Hu, S., Huang, Z.-H., Ling, C., Qi, L., 2013. On determinant and eigenvalue theory of tensors. Journal of Symbolic Computation, 50, 508–531.

3 Hu, S., Qi, L., 2013. The Laplacian of a uniform hypergraph. Journal of Combinatorial Optimization, in press.

4 Hu, S., Huang, Z.-H., Qi, L., 2013. Strictly nonnegative tensors and nonnegative tensor partition. Science in China Series A: Mathematics, in press.

Acknowledgments

I would like to express my deepest gratitude to my supervisor Professor Liqun Qi for his encouragement, guidance, constant support and numerous help, in the last three years and beyond. Professor Qi taught me many important things, much more than just research. I am greatly indebted to him.

I wish to express my sincere appreciation to my co-advisor Professor Xun Li for his friendship, encouragement and help. My appreciation extends to Professors Zheng-Hai Huang and Chen Ling for the joint work, guidance, encouragement and help for all the time. I also wish to thank Professor Lek-Heng Lim for the guidance, suggestions and help.

I would like to thank Professor An-Min Li and Professor Kung Fu Ng for serving on my reading committee. I would also like to thank the professors, supporting staff members and the fellow postgraduate students at the department for providing me supports, and the university for the finance support.

I wish to thank many other mentors and friends who have supported me in one way or another. The list is most definitely incomplete: Professors Xiaojun Chen, Guoyin Li, Qin Ni, Giorgio Ottaviani, Anthony Man-Cho So, Yiju Wang, Qingzhi Yang, Yinyu Ye, Gaohang Yu, Guofeng Zhang, Liping Zhang; and Yufan Li, Zhibao Li, Luke Oeding, Yisheng Song, Yong Wang, Yi Xu, Qiong Zhang, Xinzhen Zhang, Yanfang Zhang.

At last, I want to thank my family for most supportive and understanding.

Contents

Chapter 1

Introduction

1.1 Overview

Graph theory finds many applications in various fields and itself also becomes an important branch of mathematics since Euler's problem on the seven bridges of Königsberg. Among various branches of graph theory, the spectral graph theory plays a fundamental role since 1950s [7, 11, 12, 19], which is used to better characterize and solve many classical problems in graph theory. Hypergraphs are natural extensions of graphs. As graphs, hypergraphs have many applications in numerous fields [2, 21, 41, 42, 48, 50, 61, 66, 70, 71, 75–77]. However, the theory for hypergraph is not yet as completed as that for graphs. Especially, the analogues of spectral hypergraph theory are still in their infancy, which is mainly due to the lack of spectral theory of tensors in the literature.

In 2005, Lim [49] and Qi [62] introduced, independently, the notion of eigenvalues of tensors. Since then, the spectral theory of tensors has been developed rapidly, see [8, 13, 35, 47, 63, 64] and references therein. Among them, the spectral theory for nonnegative tensors is the most attractive and applicable one. The Perron-Frobenius type theorems and power methods for nonnegative tensors have been studied in deep, see [13, 15–18, 29, 36, 38, 49–51, 57, 60, 67, 78–84, 86] and references therein.

It was first pointed out by Lim [49,50] that the spectral theory of tensors, especially of nonnegative tensors, has intrinsic connection to problems in hypergraph theory. Later on, works based on the adjacency tensor of a uniform hypergraph were carried out by Rota Bulò and Pelillo [70, 71]. The initial work on the Laplacian-type tensor for a uniform hypergraph is [41]. Since that, several approaches have appeared [42,48,66,75, 77].

In this thesis, the *Laplacian* of a uniform hypergraph and the *Laplace-Beltrami tensor* of an even uniform hypergraph are proposed. We study the spectra of uniform hypergraphs through these tensors to establish basic spectral theory of uniform hypergraphs based on the Laplacican-type tensors. The study relies on some new developments on *tensor determinant* and *nonnegative tensor partition*. As applications, we apply the proposed theory to *edge connectivity*, *vertex connectivity*, *algebraic connectivity*, *edge expansion* and *spectral invariance* of a uniform hypergraph.

1.2 Eigenvalues of Tensors

Since eigenvalues of tensors were introduced by Lim [49] and Qi [62] independently, they have been attracted much attention in the literature and found various applications in science and engineering, see [1, 8, 13–15, 37, 39–41, 43, 47, 50, 56, 57, 60, 63–65, 68, 69, 71, 79, 85] and references therein. In the literature, there are several generalizations of eigenvalues, singular values and decompositions from matrices to tensors, see [6, 8, 14, 20, 43, 49, 59, 62, 63] and references therein. In any case, not all the properties of the eigenvalues of matrices are preserved for tensors. In the thesis, we mainly concentrate on the eigenvalues introduced by Qi [62] (see Definition 2.2.1).

Let $\mathbb{C}$ ($\mathbb{R}$) be the field of complex (real) numbers and $\mathbb{C}^n$ ($\mathbb{R}^n$) the n-dimensional complex (real) space. The nonnegative orthant of $\mathbb{R}^n$ is denoted by $\mathbb{R}^n_+$, the interior of $\mathbb{R}^n_+$ is denoted by $\mathbb{R}^n_{++}$. For a real (complex) tensor $\mathcal{T}$ of order k and dimension n with integers $k \geq 3$ and $n \geq 2$, we mean a hypermatrix $(t_{i_1\dots i_k})$ such that $t_{i_1\dots i_k} \in \mathbb{R}(\mathbb{C})$ for all $i_j \in [n] := \{1, \ldots, n\}$ and $j \in [k]$. A tensor is called nonnegative, if all of its entries are real and nonnegative. A tensor $\mathcal{T}$ is called symmetric, if $t_{\tau(i_1)\ldots\tau(i_k)} = t_{i_1\ldots i_k}$ for all permutations τ on $(i_1, \ldots, i_k)$ and all $i_1, \ldots, i_k \in [n]$. Given a vector $\mathbf{x} \in \mathbb{C}^n$, define an n-dimensional vector $\mathcal{T}\mathbf{x}^{k-1}$ with its i-th element being $\sum\limits_{i_2,\ldots,i_k \in [n]} t_{ii_2\ldots i_k} x_{i_2} \cdots x_{i_k}$ for all $i \in [n]$. The notion $\mathcal{T}\mathbf{x}^k$ represents the complex number $\mathbf{x}^T(\mathcal{T}\mathbf{x}^{k-1})$. These are examples of tensor contraction [44]. The order and the dimension of a tensor will be clear from the content. Let $\mathcal{I}$ be the identity tensor of appropriate size, e.g., $i_{i_1\ldots i_k} = 1$ if and only if $i_1 = \cdots = i_k \in [n]$, and zero otherwise when the order is k and the dimension is n. The following definition is introduced by Qi [62, 66].

Definition 1.2.1 *Let $\mathcal{T}$ be a k-th order n-dimensional tensor. For some $\lambda \in \mathbb{C}$, if the polynomial system $(\lambda\mathcal{I} - \mathcal{T})\,\mathbf{x}^{k-1} = 0$ has a solution $\mathbf{x} \in \mathbb{C}^n \setminus \{0\}$, then λ is called an eigenvalue of the tensor $\mathcal{T}$ and $\mathbf{x}$ an eigenvector of $\mathcal{T}$ associated with λ. If $\mathcal{T}$ is real and an eigenvalue λ has an eigenvector $\mathbf{x} \in \mathbb{R}^n$, then λ is called an H-eigenvalue and $\mathbf{x}$ an H-eigenvector. If $\mathbf{x} \in \mathbb{R}^n_+$ ($\mathbb{R}^n_{++}$), then λ is called an H^+-(H^{++}-)eigenvalue.*

It is easy to see that an H-eigenvalue is real. In the sequel, unless stated otherwise, an eigenvector $\mathbf{x}$ would always refer to its normalization $\frac{\mathbf{x}}{\sqrt[k]{\sum_{i\in[n]} |x_i|^k}}$. This convention does not introduce any ambiguities, since the eigenvector defining equations are homogeneous. Hence, when $\mathbf{x} \in \mathbb{R}^n_+$, we always refer to $\mathbf{x}$ satisfying $\sum\limits_{i=1}^{n} x_i^k = 1$.

1.3 Hypergraphs

Hypergraphs are natural extensions of graphs and emerge as a power tool to solve many problems in mathematics and applied science [2, 21, 49, 50, 70, 71]. In this thesis, a hypergraph means an undirected simple k-uniform hypergraph G with vertex set V, which is labeled as $[n] = \{1, \ldots, n\}$, and edge set E. By k-uniformity, we mean that for every edge $e \in E$, the cardinality $|e|$ of e is equal to k. Always, $k \geq 3$ and $n \geq k$. Basic terminologies on graphs and hypergraphs are referred to [2, 7, 11, 12, 19, 28, 53].

For a subset $S \subset [n]$, we denoted by E_S the set of edges $\{e \in E \mid I \cap e \neq \emptyset\}$. For a vertex $i \in V$, we simplify $E_{\{i\}}$ as E_i. It is the set of edges containing the vertex i, i.e., $E_i := \{e \in E \mid i \in e\}$. The cardinality $|E_i|$ of the set E_i is defined as the *degree* of the vertex i, which is denoted by d_i. Then, we have that $k|E| = \sum_{i\in[n]} d_i$. If $d_i = 0$, then we say that the vertex i is *isolated*. Two different vertices i and j are *connected* to each other (or the pair i and j is connected), if there is a sequence of edges $(e_1, \ldots, e_m)$ such that $i \in e_1$, $j \in e_m$ and $e_r \cap e_{r+1} \neq \emptyset$ for all $r \in [m-1]$. A hypergraph is called *connected*, if every pair of different vertices of G is connected. A set $S \subseteq V$ is a *connected component* of G, if every two vertices of S are connected and there is no vertices in $V \setminus S$ that are connected to any vertex in S. For the convenience, an isolated vertex is regarded as a connected component as well. Then, it is easy to see that for every hypergraph G, there is a partition of V as pairwise disjoint subsets $V = V_1 \cup \ldots \cup V_s$ such that every V_i is a connected component of G. Let $S \subseteq V$, the hypergraph with vertex set S and edge set $\{e \in E \mid e \subseteq S\}$ is called the *sub-hypergraph* of G induced by S. We will denoted it by G_S. In the sequel, unless stated otherwise, all the notations introduced above are reserved for the specific meanings.

For a subset $S \subseteq [n]$, S^c denotes the complement of S in $[n]$. Let $G = (V, E)$ be a k-uniform hypergraph. Let $S \subset V$ be a nonempty proper subset. Then, the edge set is partitioned into three pairwise disjoint parts: $E(S) := \{e \in E \mid e \subseteq S\}$, $E(S^c)$ and $E(S, S^c) := \{e \in E \mid e \cap S \neq \emptyset,\ e \cap S^c \neq \emptyset\}$. $E(S, S^c)$ is called the *edge cut* of G with respect to S.

1.4 Outline

This thesis has four other chapters. Here we put an outline of the remaining chapters. It consists of two parts: the theory on tensor determinant and nonnegative tensor partition, and the theory on the spectra of hypergraphs. The aim of this thesis is on the latter theory. While, the former is important and of independent interest as well. Since it is out of the scope of this thesis, only relevant theory is presented. For comprehensive reference, please see [35, 36, 39, 43, 67].

Chapter 2: Section 2.1 gives the definition of tensor determinant and an introduction. In Section 2.2, we present some basic properties of the determinant. Then, in Section 2.3, we show that the solvability of a polynomial system is characterized by the determinant of the leading coefficient tensor of that polynomial system. Block tensors are discussed in Section 2.3 as well. We give an expression of the determinant of a tensor, which has an "upper block triangular" structure, in terms of the determinants of its two diagonal sub-tensors. In Section 2.4.1, we give a trace formula for the determinant. In Section 2.4.2, we analyze various related properties of the characteristic polynomial and the determinant. Especially, a trace formula for the characteristic polynomial is presented, which is helpful for finding the eigenvalues of a hypergraph. We also generalize the eigenvalue representation for the determinant of a matrix to the determinant of a tensor. We show that the r-th order trace of a tensor is equal to the sum of the r-th powers of the eigenvalues of this tensor, and the coefficients of its characteristic polynomial are recursively generated by the higher order traces.

Chapter 3: Section 3.1 gives a literature review. In Section 3.2, we introduce a simple and equivalent definition of weakly irreducible nonnegative tensors. Some properties related to spectral hypergraph theory are reviewed. In Section 3.3, we propose a power method for finding the largest eigenvalue of a weakly irreducible nonnegative tensor, and establish its global R-linear convergence. In Section 3.4, we show that for a nonnegative tensor $\mathcal{T}$, always there exists a partition of the index set $[n]$ such that every tensor induced by the partition is weakly irreducible, and the largest eigenvalue of $\mathcal{T}$ can be obtained from those largest eigenvalues of the induced tensors. This allows us to deal with disconnected hypergraphs smoothly.

The study on the spectra of hypergraphs begins with Chapter 4. Instead of outlining the sections (see Section 4.1 for the outline), we briefly explain the main results.

Chapter 4: The results are summarized in three parts.

I. The first major work is to show some fundamental hypergraph analogues of spectral graph theory. Let $c(n,k) := n(k-1)^{n-1}$. Let $\mathcal{L}$ be the Laplacian of the hypergraph G, see Definition 4.2.3. Let $\sigma(\mathcal{L})$ be the spectrum of $\mathcal{L}$ (the set of eigenvalues, which is also called the spectrum of G). Then, we have the followings.

(i) (Corollary 4.3.6) The smallest H-eigenvalue of $\mathcal{L}$ is zero.

(Proposition 4.3.8) $\sum_{\lambda\in\sigma(\mathcal{L})} m(\lambda)\lambda \leq c(n,k)$ with equality holding if and only if G has no isolated vertices. Here $m(\lambda)$ is the algebraic multiplicity of λ.

(ii) (Theorem 4.3.5) For all $\lambda \in \sigma(\mathcal{L})$, $0 \leq \mathrm{Re}(\lambda)$ with equality holding if and only if $\lambda = 0$; and $\mathrm{Re}(\lambda) \leq 2$ with equality holding if and only if $\lambda = 2$.

(Corollary 4.6.4) When k is odd, we have that $\mathrm{Re}(\lambda) < 2$ for all $\lambda \in \sigma(\mathcal{L})$.

(Theorem 4.6.6/Corollary 4.6.8) When k is even, necessary and sufficient conditions are given for 2 being an eigenvalue/H-eigenvalue of $\mathcal{L}$.

(Corollary 4.6.9) When k is even and G is k-partite, 2 is an eigenvalue of $\mathcal{L}$.

(iii) (Theorem 4.3.5 together with Theorem 2.3.3 and Lemma 4.3.3) Viewed as sets, the spectrum of G is the union of the spectra of its connected components.

Viewed as multisets, an eigenvalue of a connected component with algebraic multiplicity w contributes to G as an eigenvalue with algebraic multiplicity $w(k-1)^{n-s}$. Here s is the number of vertices of the connected component.

(iv) (Corollaries 4.3.6 and 4.4.8) Let all the H$^+$-eigenvalues of $\mathcal{L}$ be ordered in increasing order as $\mu_0 \leq \mu_1 \leq \cdots \leq \mu_{n(G)-1}$. Here $n(G)$ is the number of H$^+$-eigenvalues of $\mathcal{L}$ (with H$^+$-geometric multiplicity), see Definition 4.4.6.

$\mu_0 = 0$, and $\mu_{n(G)-1} \leq 1$ with equality holding if and only if $|E| > 0$.

$\mu_{i-2} = 0$ and $\mu_{i-1} > 0$ if and only if $\log_2 i$ is a positive integer and G has exactly $\log_2 i$ connected components. Thus, $\mu_1 > 0$ if and only if G is connected.

On top of these properties, we also show that the spectral radius of the adjacency tensor of a hypergraph with $|E| > 0$ is equal to one (Lemma 4.3.2). The linear subspace generated by the nonnegative H-eigenvectors of the smallest H-eigenvalue of the Laplacian has dimension exactly the number of the connected components of the hypergraph (Lemma 4.3.7). Equalities that the eigenvalues of the Laplacian should satisfy are given in Proposition 4.3.8. The only two H^+-eigenvalues of the Laplacian of a complete hypergraph are zero and one (Corollary 4.4.11). We give the H^+-geometric multiplicities of the H^+-eigenvalues zero and one of the Laplacian respectively in Lemma 4.4.7 and Proposition 4.4.9. We show that when k is odd and G is connected, the H-eigenvector of $\mathcal{L}$ corresponding to the H-eigenvalue zero is unique (Corollary 4.6.7). The spectrum of the adjacency tensor is invariant under multiplication by any s-th root of unity, here s is the primitive index of the adjacency tensor (Corollary 4.6.5). In particular, the spectrum of the adjacency tensor of a k-partite hypergraph is invariant under multiplication by any k-th root of unity (Corollary 4.6.9).

II. The second major work is that we study the smallest H^+-eigenvalues of the sub-tensors of the Laplacian. We give variational characterizations for these H^+-eigenvalues (Lemma 4.5.1), and show that an H^+-eigenvalue of the Laplacian is the smallest H^+-eigenvalue of a certain sub-tensor of the Laplacian (Theorem 4.4.4 and (4.4.7)). Bounds for these H^+-eigenvalues based on the degrees of the vertices and the second smallest H^+-eigenvalue of the Laplacian are given respectively in Propositions 4.5.4 and 4.5.7. We discuss the relations between these H^+-eigenvalues and the edge connectivity (Proposition 4.5.8) and the edge expansion (Proposition 4.5.11) of the hypergraph.

III. The third major work is that we introduce the concept of spectral components of a hypergraph and investigate their intrinsic roles in the structure of the hypergraph spectrum. We simply interpret the idea of the spectral component first.

Let $G = (V, E)$ be a k-uniform hypergraph and $S \subset V$ be nonempty and proper. Unlike the graph counterpart, the number of intersections $e \cap S^c$ may vary for different $e \in E(S, S^c)$. We say that $E(S, S^c)$ cuts S^c *with depth at least* $r \geq 1$ if $|e \cap S^c| \geq r$ for every $e \in E(S, S^c)$. A subset of V whose edge cut cuts its complement with depth at least two is closely related to an H^+-eigenvalue of the Laplacian. These sets are *spectral* *components* (Definition 4.2.1). With edge cuts of depth at least r, we define *r-th depth edge expansion* which generalizes the edge expansion for graphs (Definition 4.5.9). A *flower heart* of a hypergraph is also introduced (Definition 4.2.2), which is related to the largest H^+-eigenvalue of the Laplacian.

We show that the spectral components characterize completely the H^+-eigenvalues of the Laplacian that are less than one and vice verse, and the flower hearts are in one to one correspondence with the nonnegative eigenvectors of the H^+-eigenvalue one (Theorem 4.4.4). In general, the set of the H^+-eigenvalues of the Laplacian is strictly contained in the set of the smallest H^+-eigenvalues of its sub-tensors (Theorem 4.4.4 and Proposition 4.4.5). We introduce H^+-geometric multiplicity of an H^+-eigenvalue. The second smallest H^+-eigenvalue of the Laplacian is discussed, and a lower bound for it is given in Proposition 4.5.7. Bounds are given for the r-th depth edge expansion based on the second smallest H^+-eigenvalue of $\mathcal{L}$ for a connected hypergraph (Proposition 4.5.10 and Corollary 4.5.12). For a connected hypergraph, necessary and sufficient conditions for the second smallest H^+-eigenvalue of $\mathcal{L}$ being the largest H^+-eigenvalue (i.e., one) are given in Proposition 4.4.10.

Chapter 5: Different from the former chapters, we study the Z-spectrum, another concept of eigenvalues by Lim and Qi [49,62], of an even uniform hypergraph in Chapter 5. As it is a minority in this thesis, the definition of Z-eigenvalues is postponed to Definition 5.3.3. The reason for why only even uniform hypergraphs are considered is given in Section 5.1. We also introduce the notion of the Laplace-Beltrami tensor for an even uniform hypergraph in this section.

In Section 5.2, we show that the Laplace-Beltrami tensor is symmetric, positive semidefinite and has a zero Z-eigenvalue with the normalized vector of all ones as a Z-eigenvector (Proposition 5.3.2). We introduce the *algebraic connectivity* of an even uniform hypergraph as the second smallest Z-eigenvalue of the Laplace-Beltrami tensor (Definition 5.3.9), and show that the algebraic connectivity is larger than zero if and only if the hypergraph is connected (Corollaries 5.3.8 and 5.3.10). We also show that the number of connected components of an even uniform hypergraph is actually the dimension of the set of Z-eigenvectors of the Laplace-Beltrami tensor corresponding to the zero Z-eigenvalue (Theorem 5.3.7). We characterize the algebraic connectivity of an even uniform hypergraph by a generalized Courant-Fischer theorem for the Laplace-Beltrami tensor (Theorem 5.3.11). Hence, computing the algebraic connectivity is transformed into computing the smallest Z-eigenvalue of another tensor resulted by multilinear transformation. Two other technical lemmas concerning algebraic connectivity are established at the end of Section 5.3 (Lemmas 5.3.13 and 5.3.14), while some applications of them that involve the connections of algebraic connectivity with *edge connectivity* (Theorem 5.4.3) and *vertex connectivity* (Theorem 5.4.4) of an even uniform hypergraph are discussed in Section 5.4.

1.5 Notation

The following notation will be frequently used in the sequel. The other unstated notation will be clear from the content.

Usually, scalars are written as lowercase letters $(\lambda, a, \ldots)$; vectors are written as bold lowercase letters $(\mathbf{x} = (x_i), \ldots)$, matrices are written as italic capitals $(A = (a_{ij}), \ldots)$, and tensors are written as calligraphic letters $(\mathcal{T} = (t_{i_1 \ldots i_k}), \ldots)$. The usual symbol $\otimes$ is used to denote the outer product of tensors. For a matrix A, A^T denotes its transpose and $\mathrm{Tr}(A)$ denotes its trace. We denote by $\mathbf{e}$ the vector of all ones, and $\mathbf{e}_i$ the i-th identity vector, i.e., the i-th column vector of the identity matrix I.

Given a ring $\mathbb{K}$ (we always mean a commutative ring with 1 (see [46]), e.g., $\mathbb{C}$), we denote by $\mathbb{K}[E]$ the polynomial ring consisting of polynomials in the set E of indeterminate variables with coefficients in $\mathbb{K}$. Especially, we denote by $\mathbb{K}[\mathcal{T}]$ the polynomial ring consisting of polynomials in indeterminate variables $\{t_{i_1 \ldots i_k}\}$ with coefficients in $\mathbb{K}$, and similarly for $\mathbb{K}[\lambda]$, $\mathbb{K}[A]$, $\mathbb{K}[\lambda, \mathcal{T}]$, etc.

Chapter 2

Tensor Determinant

2.1 Introduction

In this chapter, we introduce the notion of tensor determinant and study some basic properties of it. These properties are closely related to the eigenvalues of tensors and have immediate applications in spectral hypergraph theory, see Chapter 4.

Tensor determinant is a generalization of symmetric hyperdeterminant which is for symmetric tensors. The concept of symmetric hyperdeterminant was introduced by Qi [62] to investigate the eigenvalues of a symmetric tensor. It is based on the resultant of a homogeneous polynomial system, see Definition 2.1.1 or [22, Theorem 3.2.3] and [54, 55, 73].

Definition 2.1.1 *For fixed positive degrees $d_1, \ldots, d_n$, let $f_i := \sum_{|\alpha|=d_i} c_{i,\alpha}\mathbf{x}^\alpha$ be a homogenous polynomial of degree d_i in $\mathbb{C}[\mathbf{x}]$ for $i \in [n]$. Here $\mathbf{x}^\alpha := \prod_{i\in[n]} x_i^{\alpha_i}$. Then the unique polynomial $RES_{d_1,\ldots,d_n} \in \mathbb{Z}[\{u_{i,\alpha}\}]$, which has the following properties, is called the* resultant *of degrees $(d_1, \ldots, d_n)$.*

(i) The system of polynomial equations $f_1 = \cdots = f_n = 0$ has a nontrivial solution

in $\mathbb{C}^n$ *if and only if* $Res_{d_1,\dots,d_n}(f_1,\dots,f_n)=0$.

(ii) $Res_{d_1,\dots,d_n}(x_1^{d_1},\dots,x_n^{d_n})=1$.

(iii) $RES_{d_1,\dots,d_n}$ *is an irreducible polynomial in* $\mathbb{C}[\{u_{i,\alpha}\}]$.

We note that the differences between the capital notion $\mathrm{RES}_{d_1,\dots,d_n}$ and the notion $\mathrm{Res}_{d_1,\dots,d_n}(f_1,\dots,f_n)$ for a specific system $(f_1,\dots,f_n)$ are: the former is understood as a polynomial in the variables $\{u_{i,\alpha} \mid |\alpha|=d_i,\ i\in[n]\}$ and the latter is understood as the evaluation of $\mathrm{RES}_{d_1,\dots,d_n}$ at the point $\{u_{i,\alpha}=c_{i,\alpha}\}$ with $\{c_{i,\alpha}\}$ being given by f_i. Thus, $\mathrm{Res}_{d_1,\dots,d_n}(f_1,\dots,f_n)$ is a number in $\mathbb{C}$. When $d_1=\cdots=d_n=d$, we simplify $\mathrm{RES}_{d,\dots,d}$ (respectively $\mathrm{Res}_{d,\dots,d}$) as RES (respectively Res). The value of d will be clear from the content.

Let $\mathcal{T}=(t_{i_1\dots i_m})$ be an m-th order n-dimensional tensor, $\mathbf{x}=(x_i)\in\mathbb{C}^n$. The symmetric hyperdeterminant for symmetric tensors of order m is defined as the resultant RES of degrees $(m-1,\dots,m-1)$ such that the value of the symmetric hyperdeterminant for a specific symmetric tensor $\mathcal{T}$, which is denoted by $\mathrm{Res}(\mathcal{T}\mathbf{x}^{m-1})$, is the resultant of the polynomial system $\mathcal{T}\mathbf{x}^{m-1}=\mathbf{0}$. The symmetric hyperdeterminant of a symmetric tensor is equal to the product of all of the eigenvalues of that tensor [62].

Li et al. [47] proved that the constant term of the E-characteristic polynomial[1] of tensor $\mathcal{T}$ (not necessarily symmetric) is a power of the resultant $\mathrm{Res}(\mathcal{T}\mathbf{x}^{m-1})$ of the polynomial system $\mathcal{T}\mathbf{x}^{m-1}=\mathbf{0}$. They further found that $\mathrm{Res}(\mathcal{T}\mathbf{x}^{m-1})$ is an invariant of $\mathcal{T}$ under the orthogonal linear transformation group. In the sequel, we generalize the notion of symmetric hyperdeterminant to nonsymmetric tensors and study it systematically. The following is the definition.

Definition 2.1.2 *Let RES be the resultant of degrees* $(m-1,\dots,m-1)$ *which is a polynomial in variables* $\{u_{i,\alpha} \mid |\alpha|=m-1,\ i\in[n]\}$. *Let tensor* $\mathcal{T}=(t_{ii_2\dots i_m})\in\mathbb{T}(\mathbb{C}^n,m)$ *(the space of* m*-th order* n*-dimensional tensors). The determinant DET of* m*-th order*

[1]The characteristic polynomial for another type of eigenvalues, E-eigenvalues, proposed by Qi [62].

n-dimensional tensors is defined as the polynomial with variables $\{v_{ii_2\dots i_m} \mid i, i_2, \dots, i_m \in [n]\}$ through replacing $u_{i,\alpha}$ in the polynomial RES by $\sum_{(i_2,\dots,i_m)\in\mathbb{X}(\alpha)} v_{ii_2\dots i_m}$. Here $\mathbb{X}(\alpha) := \{(i_2, \dots, i_m) \in [n]^{m-1} \mid x_{i_2} \cdots x_{i_m} = \mathbf{x}^{\alpha}\}$. The value of the determinant Det($\mathcal{T}$) of the specific tensor $\mathcal{T}$ is defined as the evaluation of DET at the point $\{v_{ii_2\dots i_m} = t_{ii_2\dots i_m}\}$.

For the convenience of the subsequent analysis, we define $\mathrm{DET}(\mathcal{T})$ as the polynomial with variables $\{t_{ii_2\dots i_m} \mid i, i_2, \dots, i_m \in [n]\}$ through replacing $v_{ii_2\dots i_m}$ in DET by $t_{ii_2\dots i_m}$. There can be some specific relations on the variables $\{t_{ii_2\dots i_m}\}$, such as some being zero. In this case, $\mathcal{T}$ is considered as a tensor of indeterminate variables, while it is considered as a tensor of numbers in $\mathbb{C}$ when we talk about $\mathrm{Det}(\mathcal{T})$.

Given a tensor $\mathcal{T} \in \mathbb{T}(\mathbb{C}^n, m)$, we can associate to it a multilinear function $f : \mathbb{C}^n \times \cdots \times \mathbb{C}^n \to \mathbb{C}$ as $f(\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(m)}) := \sum_{1\le i_1,\dots,i_m\le n} t_{i_1\dots i_m} x^{(1)}_{i_1} \cdots x^{(m)}_{i_m}$. The hyperdeterminant is defined as the unique irreducible polynomial (up to a scalar factor) HDET such that its evaluation $Hdet(\mathcal{T})$ at $\mathcal{T}$ is zero if and only if there are nonzero $\mathbf{x}^{(j)}$ for all $j \in \{1, \dots, m\}$ such that $\frac{\partial f}{\partial x^{(j)}_i} = 0$ for all $i \in [n]$ and $j \in \{1, \dots, m\}$. Then, the tensor determinant is different from the hyperdeterminant investigated in [5, 9, 10, 22, 24, 25, 31, 32, 52, 74].

It is easy to see from Definition 2.1.2 that the tensor determinant generalizes the matrix determinant [33,34,72] and the symmetric hyperdeterminant [62]. Consequently, the notion $\mathrm{Det}(\cdot)$ is meaningful with both a matrix and a tensor as arguments. It should be pointed out that the same thing under the notion resultant[2] has been extensively studied in the monograph by Dolotin and Morozov [26]. In this chapter, we give some new developments of the tensor determinant, and especially investigate some properties related to the eigenvalue theory of tensors proposed by Lim [49] and Qi [62].

The rest of this chapter is organized as follows.

In the next section, we present some basic properties of the determinant. Then, in

[2] As resultant is created for a general polynomial system (Definition 2.1.1), we prefer to Definition 2.1.2 which is unambiguous as well.

Section 2.3, we show that the solvability of a polynomial system is characterized by the determinant of the leading coefficient tensor of that polynomial system. Block tensors are discussed in Section 2.3 as well. We give an expression of the determinant of a tensor, which has an "upper block triangular" structure, in terms of the determinants of its two diagonal sub-tensors.

Based on a result of Morozov and Shakirov [55], in Subsection 2.4.1, we give a trace formula for the determinant. This formula involves some differential operators. The determinant contributes to the characteristic polynomial theory of tensors. In Subsection 2.4.2, we analyze various related properties of the characteristic polynomial and the determinant. Especially, a trace formula for the characteristic polynomial is presented, which is useful for computing eigenvalues of a hypergraph. We also generalize the eigenvalue representation for the determinant of a matrix to the determinant of a tensor. We show that the k-th order trace of a tensor is equal to the sum of the k-th powers of the eigenvalues of this tensor, and the coefficients of its characteristic polynomial are recursively generated by the higher order traces.

2.2 Basic Properties of the Determinant

Let $\mathcal{I}$ be the identity tensor of appropriate order and dimension, e.g., $i_{i_1 \dots i_m} = 1$ if and only if $i_1 = \cdots = i_m \in [n]$, and zero otherwise. The following definitions were introduced by Qi [62].

Definition 2.2.1 *Let $\mathcal{T} \in \mathbb{T}(\mathbb{C}^n, m)$. For some $\lambda \in \mathbb{C}$, if the polynomial system $(\lambda\mathcal{I} - \mathcal{T})\,\mathbf{x}^{m-1} = \mathbf{0}$ has a solution $\mathbf{x} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$, then λ is called an eigenvalue of the tensor $\mathcal{T}$ and $\mathbf{x}$ an eigenvector of $\mathcal{T}$ associated with λ.*

We denote by $\sigma(\mathcal{T})$ the set of all eigenvalues of the tensor $\mathcal{T}$.

Definition 2.2.2 *Let $\mathcal{T}$ be an m-th order n-dimensional tensor of indeterminate variables and λ be an indeterminate variable. The determinant $DET(\lambda\mathcal{I} - \mathcal{T})$ of $\lambda\mathcal{I} - \mathcal{T}$ which is a polynomial in $(\mathbb{C}[\mathcal{T}])[\lambda]$, denoted by $\chi_{\mathcal{T}}(\lambda)$, is called the* characteristic polynomial *of the tensor $\mathcal{T}$.*

For a specific $\mathcal{T} \in \mathbb{T}(\mathbb{C}^n, m)$, $\chi_{\mathcal{T}}(\lambda) \in \mathbb{C}[\lambda]$. When there is no confusion, we simplify $\chi_{\mathcal{T}}(\lambda)$ as $\chi(\lambda)$. Denote by $\mathbb{V}(f)$ the algebraic set associated to the principal ideal $\langle f \rangle$ generated by f [22, 23, 46]. By Definitions 2.1.1, 2.1.2, 2.2.1 and 2.2.2, we have the following result.

Theorem 2.2.3 *Let $\mathcal{T}$ be an m-th order n-dimensional tensor of indeterminate variables. Then $\chi(\lambda) \in \mathbb{C}[\lambda, \mathcal{T}]$ is homogenous of degree $n(m-1)^{n-1}$, and for a specific tensor $\mathcal{T} \in \mathbb{T}(\mathbb{C}^n, m)$,*

$$\mathbb{V}(\chi(\lambda)) = \sigma(\mathcal{T}). \tag{2.2.1}$$

When $\mathcal{T}$ is symmetric, Qi proved (2.2.1), see [62, Theorem 1(a)]. If λ is a root of $\chi(\lambda)$ of multiplicity s, then we call s the *algebraic multiplicity* of eigenvalue λ.

For $f \in \mathbb{K}[\mathbf{x}]$, we denote by $\deg(f)$ the degree of f. If every monomial in f has degree $\deg(f)$, then f is called homogenous of degree $\deg(f)$.

Proposition 2.2.4 *Let $\mathcal{T}$ be an m-th order n-dimensional tensor of indeterminate variables $t_{ii_2\dots i_m}$. Then,*

(i) For every $i \in [n]$, define $\mathbb{K}_i$ as the polynomial ring

$$\mathbb{C}[\{t_{ji_2\dots i_m} \mid j, i_2, \dots, i_m \in [n],\ j \neq i\}].$$

$DET(\mathcal{T}) \in \mathbb{K}_i[\{t_{ii_2\dots i_m} \mid i_2, \dots, i_m \in [n]\}]$ is homogenous of degree $(m-1)^{n-1}$.

(ii) $DET(\mathcal{T}) \in \mathbb{C}[\mathcal{T}]$ is irreducible and homogeneous of degree $n(m-1)^{n-1}$.

(iii) $Det(\mathcal{I}) = 1$.

Proof. Denote by RES the resultant of degrees $(m-1, \dots, m-1)$ which is a polynomial in the variables $\{u_{i,\alpha} \mid |\alpha| = m-1,\ i \in [n]\}$ by Definition 2.1.1. Then, by [32, Proposition 13.1.1] (see also [55, Page 713]), RES is homogeneous of degree $(m-1)^{n-1}$ in the variables $\{u_{i,\alpha} \mid |\alpha| = m-1\}$ for every $i \in [n]$. Consequently, by the replacement in Definition 2.1.2, the determinant DET($\mathcal{T}$) is homogeneous of degree $(m-1)^{n-1}$ in the variables $\{t_{ii_2\dots i_m} \mid i_2, \dots, i_m \in [n]\}$ for every $i \in [n]$. This is exactly statement (i).

By (i), we immediately get that DET($\mathcal{T}$) $\in \mathbb{C}[\mathcal{T}]$ is homogeneous of degree $n(m-1)^{n-1}$. We claim that DET($\mathcal{T}$) $\in \mathbb{C}[\mathcal{T}]$ is irreducible. Suppose on the contrary that DET($\mathcal{T}$) $\in \mathbb{C}[\mathcal{T}]$ can be reduced as the product of two homogenous polynomials as

$$\mathrm{DET}(\mathcal{T}) = f(\mathcal{T})g(\mathcal{T}) \tag{2.2.2}$$

with $\deg(f) \geq 1$ and $\deg(g) \geq 1$. If we replace the indeterminate variable $t_{ii_2\dots i_m}$ with $(i_2, \dots, i_m) \in \mathbb{X}(\alpha)$ in the tensor $\mathcal{T}$ by the variable $\frac{u_{i,\alpha}}{|\mathbb{X}(\alpha)|}$ and denote the resulting tensor by $\mathcal{U}$, then we get that

$$\mathrm{RES} = \mathrm{DET}(\mathcal{U}) = f(\mathcal{U})g(\mathcal{U}).$$

Here the first equality follows from Definition 2.1.2 and the second from (2.2.2). Obviously, $f(\mathcal{U}), g(\mathcal{U}) \in \mathbb{C}[\{u_{i,\alpha}\}]$ are nonzero and of degrees $\deg(f)$ and $\deg(g)$ respectively. Then, RES is reduced as a product of polynomials of positive degrees. This contradicts Definition 2.1.1 (iii). Hence, RES($\mathcal{T}$) $\in \mathbb{C}[\mathcal{T}]$ is irreducible.

(iii) follows from Definitions 2.1.1 (ii) and 2.1.2. $\square$

By Proposition 2.2.4, we have the following corollary.

Corollary 2.2.5 *Let $\mathcal{T} \in \mathbb{T}(\mathbb{C}^n, m)$. If for some i, $t_{ii_2\dots i_m} = 0$ for all $i_2, \dots, i_m \in [n]$, then $Det(\mathcal{T}) = 0$. In particular, the determinant of the zero tensor is zero.*

Proof. Let $\mathcal{V}$ be an m-th order n-dimensional tensor of indeterminate variables $v_{ii_2\dots i_m}$ and $\mathbb{K}_i := \mathbb{C}[\{v_{ji_2\dots i_m} \mid j, i_2, \dots, i_m \in [n],\ j \neq i\}]$. Then by Proposition 2.2.4 (i), $\mathrm{DET}(\mathcal{V})$ is a homogenous polynomial in the variable set $\{v_{ii_2\dots i_m} \mid i_2, \dots, i_m \in [n]\}$ with coefficients in the ring $\mathbb{K}_i$. Moreover, $\mathrm{Det}(\mathcal{T})$ is just the evaluation of $\mathrm{DET}(\mathcal{V})$ at the point $\mathcal{V} = \mathcal{T}$. As $t_{ii_2\dots i_m} = 0$ for all $i_2, \dots, i_m \in [n]$ by the assumption, $\mathrm{Det}(\mathcal{T}) = 0$ as desired. $\square$

By Proposition 2.2.4 (ii), we have another corollary as follows.

Corollary 2.2.6 *Let $\mathcal{T} \in \mathbb{T}(\mathbb{C}^n, m)$ and $\gamma \in \mathbb{C}$. Then*

$$Det(\gamma\mathcal{T}) = \gamma^{n(m-1)^{n-1}} Det(\mathcal{T}).$$

2.3 Block Tensors

Let matrix $A \in \mathbb{T}(\mathbb{C}^n, 2)$, we know that [45, 72]

(i) $\mathrm{Det}(A) = 0$ if and only if $A\mathbf{x} = \mathbf{0}$ has a solution in $\mathbb{C}^n \setminus \{\mathbf{0}\}$, and

(ii) $\mathrm{Det}(A) \neq 0$ if and only if $A\mathbf{x} = \mathbf{b}$ has a unique solution in $\mathbb{C}^n$ for every $\mathbf{b} \in \mathbb{C}^n$.

We generalize such a result to the tensor determinant and the nonlinear polynomial system in this section.

Theorem 2.3.1 *Let $\mathcal{T} \in \mathbb{T}(\mathbb{C}^n, m)$. Then,*

(i) $Det(\mathcal{T}) = 0$ if and only if $\mathcal{T}\mathbf{x}^{m-1} = \mathbf{0}$ has a solution in $\mathbb{C}^n \setminus \{\mathbf{0}\}$.

(ii) If $Det(\mathcal{T}) \neq 0$, then for any $\mathbf{b} \in \mathbb{C}^n$, $A \in \mathbb{T}(\mathbb{C}^n, 2)$, and $\mathcal{B}^j \in \mathbb{T}(\mathbb{C}^n, j)$ for $j \in \{3, \dots, m-1\}$, $\mathcal{T}\mathbf{x}^{m-1} = (\mathcal{B}^{m-1})\,\mathbf{x}^{m-2} + \cdots + (\mathcal{B}^3)\,\mathbf{x}^2 + A\mathbf{x} + \mathbf{b}$ has a solution in $\mathbb{C}^n$.

Proof. (i) It follows from Definitions 2.1.1 and 2.1.2 immediately.

(ii) Suppose that $\mathrm{Det}(\mathcal{T}) \neq 0$. For any $\mathbf{b} \in \mathbb{C}^n$, $A \in \mathbb{T}(\mathbb{C}^n, 2)$, and $\mathcal{B}^j \in \mathbb{T}(\mathbb{C}^n, j)$ for $j \in \{3, \ldots, m-1\}$, we define tensor $\mathcal{U} \in \mathbb{T}(\mathbb{C}^{n+1}, m)$ as follows:

$$u_{i_1 i_2 \ldots i_m} := \begin{cases} t_{i_1 i_2 \ldots i_m} & \forall i_j \in [n],\ j \in [m], \\ -b_{i_1} & \forall i_1 \in [n] \text{ and } i_2 = \cdots = i_m = n+1, \\ -a_{i_1 i_2} & \forall i_1, i_2 \in [n] \text{ and } i_3 = \cdots = i_m = n+1, \\ -b^k_{i_1 \ldots i_k} & \forall i_1, \ldots, i_k \in [n] \text{ and } i_{k+1} = \cdots = i_m = n+1, \\ & \quad \forall k \in \{3, \ldots, m-1\}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.3.3}$$

Actually, the tensor $\mathcal{U}$ is the tensor corresponding to the homogenous polynomial in $n+1$ variables by homogenizing $\mathcal{T}\mathbf{x}^{m-1} = (\mathcal{B}^{m-1})\,\mathbf{x}^{m-2} + \cdots + (\mathcal{B}^3)\,\mathbf{x}^2 + A\mathbf{x} + \mathbf{b}$. By Corollary 2.2.5, we have that $\mathrm{Det}(\mathcal{U}) = 0$ since $u_{i_1 i_2 \ldots i_m} = 0$ whenever $i_1 = n+1$. Hence, by (i), there exists $\mathbf{y} := (\mathbf{x}^T, \alpha)^T \in \mathbb{C}^{n+1} \setminus \{\mathbf{0}\}$ such that $\mathcal{U}\mathbf{y}^{m-1} = \mathbf{0}$. Consequently, by (2.3.3) and the first n equations in $\mathcal{U}\mathbf{y}^{m-1} = \mathbf{0}$, we know that

$$\mathcal{T}\mathbf{x}^{m-1} - \alpha\left(\mathcal{B}^{m-1}\right)\mathbf{x}^{m-2} - \cdots - \alpha^{m-3}\left(\mathcal{B}^3\right)\mathbf{x}^2 - \alpha^{m-2}A\mathbf{x} - \alpha^{m-1}\mathbf{b} = \mathbf{0}. \tag{2.3.4}$$

Furthermore, we claim that $\alpha \neq 0$. Otherwise, from (2.3.4), $\mathcal{T}\mathbf{x}^{m-1} = \mathbf{0}$ which means $\mathrm{Det}(\mathcal{T}) = 0$ by (i). It is a contradiction. Hence, from (2.3.4) we know that $\frac{\mathbf{x}}{\alpha}$ is a solution to

$$\mathcal{T}\mathbf{x}^{m-1} = \left(\mathcal{B}^{m-1}\right)\mathbf{x}^{m-2} + \cdots + \left(\mathcal{B}^3\right)\mathbf{x}^2 + A\mathbf{x} + \mathbf{b}.$$

The proof is complete. $\square$

So, like the matrix determinants of linear equations, the tensor determinants are criterions for the solvability of non-linear polynomial equations. It is interesting to investigate whether $\mathcal{T}\mathbf{x}^{m-1} = (\mathcal{B}^{m-1})\,\mathbf{x}^{m-2} + \cdots + (\mathcal{B}^3)\,\mathbf{x}^2 + A\mathbf{x} + \mathbf{b}$ has only finitely many solutions whenever $\mathrm{Det}(\mathcal{T}) \neq 0$.

In the content of matrices, if a square matrix A can be partitioned as

$$A = \begin{pmatrix} B & D \\ 0 & C \end{pmatrix}$$

with square sub-matrices B and C, and sub-matrix D, then $\mathrm{Det}(A) = \mathrm{Det}(B)\mathrm{Det}(C)$ [45,72]. We now generalize this property to tensors. The following definition is straightforward.

Definition 2.3.2 *Let $\mathcal{T}$ be a k-th order n-dimensional real tensor and $s \in [n]$. The k-th order s-dimensional tensor $\mathcal{U}$ with entries $u_{i_1\dots i_k} = t_{j_{i_1}\dots j_{i_k}}$ for all $i_1, \dots, i_k \in [s]$ is called the* sub-tensor *of $\mathcal{T}$ associated to the subset $S := \{j_1, \dots, j_s\}$. We usually denoted $\mathcal{U}$ as $\mathcal{T}(S)$.*

For a subset $S \subseteq [n]$, we denoted by $|S|$ its cardinality. For $\mathbf{x} \in \mathbb{C}^n$, $\mathbf{x}_S$ is defined as an n vector with its i-th element being x_i if $i \in S$ and zero otherwise, $\mathbf{x}(S)$ is defined as an $|S|$-dimensional sub-vector of $\mathbf{x}$ with its entries being x_i for $i \in S$, and $\sup(\mathbf{x}) := \{i \in [n] \mid x_i \neq 0\}$ is its *support*.

Though Definition 2.3.2 is for a specific tensor, the generalization to tensors of indeterminate variables is straightforward. Given a set $E \subseteq \mathbb{C}^n$, we denote by $\mathbb{I}(E) \subseteq \mathbb{C}[\mathbf{x}]$ the ideal of polynomials in $\mathbb{C}[\mathbf{x}]$ which vanish identically on E. Given a set of polynomials $F := \{f_1, \dots, f_s : f_i \in \mathbb{C}[\mathbf{x}]\}$, we denote by $\mathbb{V}(F) \subseteq \mathbb{C}^n$ the algebraic set associated to F, i.e., the set of the common roots of polynomials in F [22, 46].

Theorem 2.3.3 *Let $\mathcal{T}$ be an m-th order n-dimensional tensor of indeterminate variables such that there exists an integer $k \in \{1, \dots, n-1\}$ satisfying $t_{ii_2\dots i_m} \equiv 0$ for every $i \in \{k+1, \dots, n\}$ and all indices $i_2, \dots, i_m$ such that $\{i_2, \dots, i_m\} \cap [k] \neq \emptyset$. Denote by $\mathcal{U}$ and $\mathcal{V}$ the* sub-tensors *of $\mathcal{T}$ associated to $[k]$ and $\{k+1, \dots, n\}$, respectively. Then, it holds that*

$$DET(\mathcal{T}) = [DET(\mathcal{U})]^{(m-1)^{n-k}} [DET(\mathcal{V})]^{(m-1)^k} . \tag{2.3.5}$$

A word on the notation is necessary before the proof. Though with the same notation, implicitly, $\mathrm{DET}(\mathcal{T})$ is understood as the determinant for m-th order n-dimensional tensors, $\mathrm{DET}(\mathcal{U})$ for m-th order k-dimensional tensors and $\mathrm{DET}(\mathcal{V})$ for m-th order $n-k$-dimensional tensors. The actual meanings are clear from the content. The notation Det is similar.

Proof. We first show that for any specific tensor $\mathcal{T}$ satisfying the hypothesis,

$$\mathrm{Det}(\mathcal{T}) = 0 \quad \Longleftrightarrow \quad \mathrm{Det}(\mathcal{U})\mathrm{Det}(\mathcal{V}) = 0. \tag{2.3.6}$$

Suppose that $\mathrm{Det}(\mathcal{T}) = 0$. Then there exists $\mathbf{x} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ such that $\mathcal{T}\mathbf{x}^{m-1} = \mathbf{0}$ by Theorem 2.3.1 (i). Denote by $\mathbf{u} \in \mathbb{C}^k$ the vector consisting of $x_1, \ldots, x_k$, and $\mathbf{v} \in \mathbb{C}^{n-k}$ the vector consisting of $x_{k+1}, \ldots, x_n$. If $\mathbf{v} \neq \mathbf{0}$, then from $\mathcal{T}\mathbf{x}^{m-1} = \mathbf{0}$ we get that $\mathcal{V}\mathbf{v}^{m-1} = \mathbf{0}$. Consequently, $\mathrm{Det}(\mathcal{V}) = 0$ by Theorem 2.3.1 (i). Otherwise, $\mathbf{u} \neq \mathbf{0}$ and $\mathbf{v} = \mathbf{0}$. This, together with $\mathcal{T}\mathbf{x}^{m-1} = \mathbf{0}$, implies that $\mathcal{U}\mathbf{u}^{m-1} = \mathbf{0}$. Thus, $\mathrm{Det}(\mathcal{U}) = 0$ by Theorem 2.3.1 (i). Hence, we have

$$\mathrm{Det}(\mathcal{T}) = 0 \quad \Longrightarrow \quad \mathrm{Det}(\mathcal{U})\mathrm{Det}(\mathcal{V}) = 0.$$

Conversely, suppose that $\mathrm{Det}(\mathcal{U})\mathrm{Det}(\mathcal{V}) = 0$. If $\mathrm{Det}(\mathcal{U}) = 0$, then there exists $\mathbf{u} \in \mathbb{C}^k \setminus \{\mathbf{0}\}$ such that $\mathcal{U}\mathbf{u}^{m-1} = \mathbf{0}$. Denote $\mathbf{x} := (\mathbf{u}^T, \mathbf{0})^T \in \mathbb{C}^n \setminus \{\mathbf{0}\}$, then $\mathcal{T}\mathbf{x}^{m-1} = \mathbf{0}$, which implies $\mathrm{Det}(\mathcal{T}) = 0$ by Theorem 2.3.1 (i). If $\mathrm{Det}(\mathcal{U}) \neq 0$, then $\mathrm{Det}(\mathcal{V}) = 0$, which implies that there exists $\mathbf{v} \in \mathbb{C}^{n-k} \setminus \{\mathbf{0}\}$ such that $\mathcal{V}\mathbf{v}^{m-1} = \mathbf{0}$. Now, by the vector $\mathbf{v}$ and the tensor $\mathcal{T}$, we construct the vector $\mathbf{b} \in \mathbb{C}^k$ as

$$b_i := \sum_{j_2,\ldots,j_m=k+1}^{n} t_{ij_2\ldots j_m} v_{j_2-k} \cdots v_{j_m-k}, \ \forall i \in [k] \ ; \tag{2.3.7}$$

the matrix $A \in \mathbb{T}(\mathbb{C}^k, 2)$ as

$$a_{ij} := \sum_{(q_2,\ldots,q_m)\in\mathbb{D}(j)} t_{iq_2\ldots q_m} \prod\nolimits_{q_w>k} v_{q_w-k}, \ \forall i, j \in [k] \tag{2.3.8}$$

with $\mathbb{D}(j) := \{(q_2, \ldots, q_m) \mid j = q_p \text{ for some } p \in \{2, \ldots, m\}, \text{ and } q_l \in \{k+1, \ldots, n\}, l \neq p\}$; and, the tensors $\mathcal{B}^s \in \mathbb{T}(\mathbb{C}^k, s)$ for $s \in \{3, \ldots, m-1\}$ as

$$b^s_{ij_2\ldots j_s} := \sum_{(q_2,\ldots,q_m)\in\mathbb{D}^s(j_2,\ldots,j_s)} t_{iq_2\ldots q_m} \prod\nolimits_{q_w>k} v_{q_w-k}, \ \forall i, j_2, \ldots, j_s \in [k] \tag{2.3.9}$$

with

$$\mathbb{D}^s(j_2,\dots,j_s) := \{(q_2,\dots,q_m) \mid \{q_{t_2},\dots,q_{t_s}\} = \{j_2,\dots,j_s\} \text{ for some pairwise different } t_2,\dots,t_s \text{ in } \{2,\dots,m\}, \text{ and } q_l \in \{k+1,\dots,n\}, l \notin \{t_2,\dots,t_s\}\}.$$

Since $\mathrm{Det}(\mathcal{U}) \neq 0$, by Theorem 2.3.1 (ii),

$$\mathcal{U}\mathbf{u}^{m-1} + \left(\mathcal{B}^{m-1}\right)\mathbf{u}^{m-2} + \cdots + \left(\mathcal{B}^3\right)\mathbf{u}^2 + A\mathbf{u} + \mathbf{b} = \mathbf{0}$$

has a solution $\mathbf{u} \in \mathbb{C}^k$. Let $\mathbf{x} := (\mathbf{u}^T, \mathbf{v}^T)^T \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ as $\mathbf{v} \in \mathbb{C}^{n-k} \setminus \{\mathbf{0}\}$. By (2.3.7), (2.3.8) and (2.3.9), we have that

$$\left(\mathcal{T}\mathbf{x}^{m-1}\right)_i = \left(\mathcal{U}\mathbf{u}^{m-1} + \left(\mathcal{B}^{m-1}\right)\mathbf{u}^{m-2} + \cdots + \left(\mathcal{B}^3\right)\mathbf{u}^2 + A\mathbf{u} + \mathbf{b}\right)_i = 0,\ \forall i \in [k].$$

Furthermore,

$$\left(\mathcal{T}\mathbf{x}^{m-1}\right)_i = \left(\mathcal{V}\mathbf{v}^{m-1}\right)_i = 0,\ \forall i \in \{k+1,\dots,n\}.$$

Consequently, $\mathcal{T}\mathbf{x}^{m-1} = \mathbf{0}$ which implies $\mathrm{Det}(\mathcal{T}) = 0$ by Theorem 2.3.1 (i).

Hence, we proved (2.3.6). In the following, we show that (2.3.5) holds. Note that the dimension of $\mathbb{T}(\mathbb{C}^n, m)$ is n^m. The set of tensors satisfying the hypothesis of this theorem forms a vector subspace $\mathbb{S}$ of $\mathbb{T}(\mathbb{C}^n, m)$ with dimension $kn^{m-1} + (n-k)^{m-1}$. Consequently, the number of variables of the polynomial $\mathrm{DET}(\mathcal{T})$ is $kn^{m-1}+(n-k)^{m-1}$. In the following, the ambient space for the algebraic sets is understood as $\mathbb{S}$. As sets of variables, the sets of entries of $\mathcal{U}$ and $\mathcal{V}$ are subsets of the set of entries of $\mathcal{T}$. Hence, we can view $\mathrm{DET}(\mathcal{U}), \mathrm{DET}(\mathcal{V}) \in \mathbb{C}[\mathcal{T}]$. By (2.3.6), we have

$$\mathbb{V}(\mathrm{DET}(\mathcal{U})\mathrm{DET}(\mathcal{V})) = \mathbb{V}(\mathrm{DET}(\mathcal{T})),$$

which implies that

$$\mathbb{I}(\mathbb{V}(\mathrm{DET}(\mathcal{T}))) = \mathbb{I}(\mathbb{V}(\mathrm{DET}(\mathcal{U})\mathrm{DET}(\mathcal{V}))).$$

By Proposition 2.2.4 (ii), both $\mathrm{DET}(\mathcal{U}) \in \mathbb{C}[\mathcal{U}]$ and $\mathrm{DET}(\mathcal{V}) \in \mathbb{C}[\mathcal{V}]$ are irreducible. Consequently,

$$\mathbb{I}(\mathbb{V}(\mathrm{DET}(\mathcal{T}))) = \mathbb{I}(\mathbb{V}(\mathrm{DET}(\mathcal{U})\mathrm{DET}(\mathcal{V}))) = \langle \mathrm{DET}(\mathcal{U})\mathrm{DET}(\mathcal{V}) \rangle.$$

Let $\sqrt{\langle \mathrm{DET}(\mathcal{T})\rangle}$ be the radical ideal of the ideal $\langle \mathrm{DET}(\mathcal{T})\rangle$ [46]. Then, Hilbert's Nullstellensatz (see [23, Theorem 4.2]) implies that

$$\sqrt{\langle \mathrm{DET}(\mathcal{T})\rangle} = \mathbb{I}(\mathbb{V}(\mathrm{DET}(\mathcal{T}))) = \langle \mathrm{DET}(\mathcal{U})\mathrm{DET}(\mathcal{V})\rangle.$$

Since both $\sqrt{\langle \mathrm{DET}(\mathcal{T})\rangle}$ and $\langle \mathrm{DET}(\mathcal{U})\mathrm{DET}(\mathcal{V})\rangle$ are principal ideals and $\mathbb{C}[\mathcal{T}]$ is a unique factorization domain, we have that

$$\mathrm{DET}(\mathcal{T}) = (\mathrm{DET}(\mathcal{U}))^{r_1} (\mathrm{DET}(\mathcal{V}))^{r_2} \tag{2.3.10}$$

for some $r_1, r_2 \in \mathbb{N}_+$.

By Proposition 2.2.4 (i), $\mathrm{DET}(\mathcal{T})$ is homogenous of degree $(m-1)^{n-1}$ in the variables $\{t_{1i_2\dots i_m} \mid i_2,\dots,i_m \in [n]\}$. By the hypothesis, $\mathrm{DET}(\mathcal{V})$ is independent of the variables $\{t_{1i_2\dots i_m} \mid i_2,\dots,i_m \in [n]\}$. By Proposition 2.2.4 (i) again, $\mathrm{DET}(\mathcal{U})$ is homogenous of degree $(m-1)^{k-1}$ in the variables $\{t_{1i_2\dots i_m} \mid i_2,\dots,i_m \in [k]\}$. Thus, $r_1 = (m-1)^{n-k}$ by (2.3.10). Comparing the degrees of the both sides of (2.3.10) with Proposition 2.2.4 (ii), we get $r_2 = (m-1)^k$ and hence (2.3.5). $\square$

2.4 The Characteristic Polynomial

By Definition 2.2.2, for any $\mathcal{T} \in \mathbb{T}(\mathbb{C}^n, m)$, its characteristic polynomial is $\chi(\lambda) = \mathrm{Det}(\lambda\mathcal{I} - \mathcal{T})$. In this section, we discuss some properties of the characteristic polynomial of a tensor related to the determinant. To this end, we give a trace formula for the determinant in Subsection 2.4.1 first. This result, due to Morozov and Shakirov [55], is a corner stone for the subsequent analysis of the characteristic polynomials.

2.4.1 A Trace Formula of the Determinant

Let $\mathcal{T} \in \mathbb{T}(\mathbb{C}^n, m)$. Define the following differential operators:

$$\hat{g}_i := \sum_{i_2=1}^{n} \cdots \sum_{i_m=1}^{n} t_{ii_2\dots i_m} \frac{\partial}{\partial a_{ii_2}} \cdots \frac{\partial}{\partial a_{ii_m}},\ \forall i \in [n], \tag{2.4.11}$$

where A is an auxiliary $n \times n$ matrix consists of indeterminate variables a_{ij}'s. It is clear that for every i, $\hat{g}_i$ is a differential operator which belongs to the operator algebra $\mathbb{C}[\partial A]$, here ∂A is the $n \times n$ matrix with elements $\frac{\partial}{\partial a_{ij}}$'s. The Schur polynomials are defined as:

$$p_0(t_0) = 1, \text{ and } p_k(t_1, \ldots, t_k) := \sum_{i=1}^{k} \sum_{d_j > 0, \sum_{j=1}^{i} d_j = k} \frac{\prod_{j=1}^{i} t_{d_j}}{i!}, \quad \forall k \geq 1, \tag{2.4.12}$$

where $\{t_0, t_1, \ldots\}$ are variables. Motivated by Cooper and Dutle [21] and Morozov and Shakirov [55], we define the *d-th order trace* of the tensor $\mathcal{T}$ as

$$\mathrm{Tr}_d(\mathcal{T}) := (m-1)^{n-1} \left[\sum_{\sum_{i=1}^{n} k_i = d} \prod_{i=1}^{n} \frac{(\hat{g}_i)^{k_i}}{((m-1)k_i)!} \right] \mathrm{Tr}(A^{(m-1)d}). \tag{2.4.13}$$

We show in Proposition 2.4.4 that $\mathrm{Tr}_1(\mathcal{T}) = (m-1)^{n-1} \sum_{i=1}^{n} t_{i \ldots i}$. Hence, it is a generalization of the trace of a matrix. $\mathrm{Tr}_d(\mathcal{T})$'s are called higher order traces for $d > 1$.

We now have the following proposition.

Proposition 2.4.1 *Let $\mathcal{T} \in \mathbb{T}(\mathbb{C}^n, m)$ and the notation be defined as above. Then,*

$$DET(\mathcal{T}) = 1 + \sum_{k=1}^{\infty} p_k \left(-\frac{Tr_1(\mathcal{I}-\mathcal{T})}{1}, \ldots, -\frac{Tr_k(\mathcal{I}-\mathcal{T})}{k} \right). \tag{2.4.14}$$

Proof. This result follows from Proposition II in Morozov and Shakirov [55], the identity $\log(\mathrm{DET}(I-A)) = \mathrm{Tr}(\log(I-A))$ for the matrix A, and the definitions of the Schur polynomials and the higher order traces. As the proof is a restatement of the those from Sections 4–8 in Morozov and Shakirov [55] in the language of tensors, we omit it. $\square$

The following proposition is useful in the sequel, which also helps to give an expression of $\mathrm{DET}(\mathcal{T})$ with only finitely many terms.

Proposition 2.4.2 *Let $\mathcal{T} \in \mathbb{T}(\mathbb{C}^n, m)$ and the notation be defined as above. Then, the following hold:*

(i) for every $d \in \mathbb{N}_+$, $Tr_d(\mathcal{T}) \in \mathbb{C}[\mathcal{T}]$ is homogenous of degree d;

(ii) for every $k \in \mathbb{N}_+$, $p_k\left(-\frac{Tr_1(\mathcal{T})}{1}, \ldots, -\frac{Tr_k(\mathcal{T})}{k}\right) \in \mathbb{C}[\mathcal{T}]$ is homogenous of degree k; and,

(iii) for any integer $k > n(m-1)^{n-1}$, $p_k\left(-\frac{Tr_1(\mathcal{T})}{1}, \ldots, -\frac{Tr_k(\mathcal{T})}{k}\right) \in \mathbb{C}[\mathcal{T}]$ is zero.

Proof. (i) By the formulae of $\hat{g}_i$'s as in (2.4.11), it is easy to see that

$$\sum_{\sum_{i=1}^n k_i = d} \prod_{i=1}^n \frac{(\hat{g}_i)^{k_i}}{((m-1)k_i)!} \in \mathbb{C}[\mathcal{T}, \partial A]$$

is homogeneous, and more explicitly, homogenous of degree d in the variable $\mathcal{T}$ and homogeneous of degree $(m-1)d$ in the variable ∂A. It is also known that

$$\mathrm{Tr}(A^k) = \sum_{i_1=1}^n \cdots \sum_{i_k=1}^n a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{k-1} i_k} a_{i_k i_1} \in \mathbb{C}[A] \tag{2.4.15}$$

is homogeneous of degree k. These, together with (2.4.13), imply that $\mathrm{Tr}_d(\mathcal{T}) \in \mathbb{C}[\mathcal{T}]$ is homogenous of degree d as desired.

(ii) It follows from (i) and the definitions of the Schur polynomials as in (2.4.12) directly.

(iii) From Proposition 2.2.4 (ii), it is clear that $\mathrm{DET}(\mathcal{B})$ is an irreducible polynomial which is homogenous of degree $n(m-1)^{n-1}$ in the variables $\{b_{i_1 \ldots i_m}\}$. In the following, let $\mathcal{B} := \mathcal{I} - \mathcal{T}$. Since the entries of $\mathcal{B}$ consist of 1 and the entries of the tensor $\mathcal{T}$, the highest degree of $\mathrm{DET}(\mathcal{I} - \mathcal{T})$ viewed as a polynomial in $\mathbb{C}[\mathcal{T}]$ is not greater than $n(m-1)^{n-1}$. This, together with (2.4.14) and (ii) which asserts that $p_k(-\frac{\mathrm{Tr}_1(\mathcal{T})}{1}, \ldots, -\frac{\mathrm{Tr}_k(\mathcal{T})}{k}) \in \mathbb{C}[\mathcal{T}]$ is homogenous of degree k, implies the result (iii).

The proof is complete. □

By Proposition 2.4.2 and (2.4.14), we immediately get

$$\mathrm{DET}(\mathcal{T}) = 1 + \sum_{k=1}^{n(m-1)^{n-1}} p_k\left(-\frac{\mathrm{Tr}_1(\mathcal{I}-\mathcal{T})}{1}, \ldots, -\frac{\mathrm{Tr}_k(\mathcal{I}-\mathcal{T})}{k}\right) . \tag{2.4.16}$$

This is a trace formula for the tensor determinant. It provides a way to approach the computation of the tensor determinant. However, it involves the higher order traces of tensors, and hence the differential operators $\hat{g}_i$'s. It is very hard to compute them [26, 55]. For more details on explicit formulae for the second order trace and the determinant of a two dimensional tensor, please refer to [35].

2.4.2 Basic Properties of the Characteristic Polynomial

Some basic properties of the characteristic polynomial are derived in this subsection.

Theorem 2.4.3 *Let $\mathcal{T} \in \mathbb{T}(\mathbb{C}^n, m)$ and the notation be defined as above. Then*

$$\begin{aligned}\chi(\lambda) &= Det(\lambda\mathcal{I} - \mathcal{T}) \\ &= \lambda^{n(m-1)^{n-1}} + \sum_{k=1}^{n(m-1)^{n-1}} \lambda^{n(m-1)^{n-1}-k} p_k \left(-\frac{Tr_1(\mathcal{T})}{1}, \ldots, -\frac{Tr_k(\mathcal{T})}{k}\right) \\ &= \Pi_{\lambda_i \in \sigma(\mathcal{T})} (\lambda - \lambda_i)^{m_i},\end{aligned}$$

where m_i is the algebraic multiplicity of the eigenvalue λ_i.

Proof. The first equality follows from Definition 2.2.2, and the last one from Theorem 2.2.3.

By Proposition 2.4.2 and (2.4.16), we can get that

$$\chi(1) = \mathrm{Det}(\mathcal{I} - \mathcal{T}) = 1 + \textstyle\sum_{k=1}^{n(m-1)^{n-1}} p_k \left(-\frac{\mathrm{Tr}_1(\mathcal{T})}{1}, \ldots, -\frac{\mathrm{Tr}_k(\mathcal{T})}{k}\right). \qquad (2.4.17)$$

Consequently, when $\lambda \neq 0$,

$$\begin{aligned}\chi(\lambda) &= \mathrm{Det}(\lambda\mathcal{I} - \mathcal{T}) \\ &= \lambda^{n(m-1)^{n-1}} \mathrm{Det}(\mathcal{I} - \tfrac{\mathcal{T}}{\lambda}) \\ &= \lambda^{n(m-1)^{n-1}} \left[1 + \textstyle\sum_{k=1}^{n(m-1)^{n-1}} p_k \left(-\frac{\mathrm{Tr}_1(\frac{\mathcal{T}}{\lambda})}{1}, \ldots, -\frac{\mathrm{Tr}_k(\frac{\mathcal{T}}{\lambda})}{k}\right)\right] \\ &= \lambda^{n(m-1)^{n-1}} \left[1 + \textstyle\sum_{k=1}^{n(m-1)^{n-1}} \frac{1}{\lambda^k} p_k \left(-\frac{\mathrm{Tr}_1(\mathcal{T})}{1}, \ldots, -\frac{\mathrm{Tr}_k(\mathcal{T})}{k}\right)\right] \\ &= \lambda^{n(m-1)^{n-1}} + \textstyle\sum_{k=1}^{n(m-1)^{n-1}} \lambda^{n(m-1)^{n-1}-k} p_k \left(-\frac{\mathrm{Tr}_1(\mathcal{T})}{1}, \ldots, -\frac{\mathrm{Tr}_k(\mathcal{T})}{k}\right).\end{aligned}$$

Here the second equality comes from Corollary 2.2.6, the third from (2.4.17), and the fourth from Proposition 2.4.2. Hence, the result follows from the fact that the field $\mathbb{C}$ is of characteristic zero. The proof is complete. $\square$

Theorem 2.4.3 gives a trace formula for the characteristic polynomial of the tensor $\mathcal{T}$ as well as an eigenvalue representation for it.

Here are some properties concerning the coefficients of $\chi(\lambda)$.

Proposition 2.4.4 *Let $\mathcal{T} \in \mathbb{T}(\mathbb{C}^n, m)$ and the notation be defined as above. Then,*

(i) $p_1(-Tr_1(\mathcal{T})) = -Tr_1(\mathcal{T}) = -(m-1)^{n-1}\sum_{i=1}^n t_{ii\dots i}$,

(ii) $p_2(-\frac{Tr_1(\mathcal{T})}{1}, -\frac{Tr_2(\mathcal{T})}{2}) = \frac{1}{2}\left([Tr_1(\mathcal{T})]^2 - Tr_2(\mathcal{T})\right)$, *and*

(iii) $p_{n(m-1)^{n-1}}\left(-\frac{Tr_1(\mathcal{T})}{1}, \dots, -\frac{Tr_{n(m-1)^{n-1}}(\mathcal{T})}{n(m-1)^{n-1}}\right) = (-1)^{n(m-1)^{n-1}} Det(\mathcal{T})$.

Proof. (i) By (2.4.12), we know that $p_1(-\mathrm{Tr}_1(\mathcal{T})) = -\mathrm{Tr}_1(\mathcal{T})$. Furthermore, by (2.4.13), it is easy to see that

$$
\begin{aligned}
\mathrm{Tr}_1(\mathcal{T}) &= (m-1)^{n-1}\sum_{i=1}^n \frac{\hat{g}_i}{(m-1)!}\mathrm{Tr}(A^{m-1}) \\
&= \tfrac{(m-1)^{n-1}}{(m-1)!}\sum_{i=1}^n \left[\sum_{i_2=1}^n \cdots \sum_{i_m=1}^n t_{ii_2\dots i_m}\tfrac{\partial}{\partial a_{ii_2}}\cdots\tfrac{\partial}{\partial a_{ii_m}}\right]\mathrm{Tr}(A^{m-1}) \\
&= \tfrac{(m-1)^{n-1}}{(m-1)!}\sum_{i=1}^n \left[\sum_{i_2=1}^n \cdots \sum_{i_m=1}^n t_{ii_2\dots i_m}\tfrac{\partial}{\partial a_{ii_2}}\cdots\tfrac{\partial}{\partial a_{ii_m}}\right] \\
&\qquad \cdot\left(\sum_{i_1=1}^n \cdots \sum_{i_{m-1}=1}^n a_{i_1i_2}a_{i_2i_3}\cdots a_{i_{m-2}i_{m-1}}a_{i_{m-1}i_1}\right) \\
&= \tfrac{(m-1)^{n-1}}{(m-1)!}\sum_{i=1}^n \left[t_{ii\dots i}\tfrac{\partial}{\partial a_{ii}}\cdots\tfrac{\partial}{\partial a_{ii}}(a_{ii})^{m-1}\right] \\
&= (m-1)^{n-1}\sum_{i=1}^n t_{ii\dots i}.
\end{aligned}
$$

Here, the fourth equality follows from the fact that: (a) the differential operator in the right hand side of the third equality contains only items $\frac{\partial}{\partial a_{i\star}}$'s for $\star \in [n]$ and the total degree is $m-1$, and (b) only terms in $\mathrm{Tr}(A^{m-1})$ that contain the same $\frac{\partial}{\partial a_{i\star}}$'s of total degree $m-1$ can contribute to the result and this case occurs only when every $\star = i$ by (2.4.15). Consequently, the result (i) follows.

(ii) follows from the definition (2.4.12) by direct calculation.

(iii) By Theorem 2.4.3, it is clear that

$$\chi(0) = \mathrm{Det}(-\mathcal{T}) = p_{n(m-1)^{n-1}}\left(-\frac{\mathrm{Tr}_1(\mathcal{T})}{1}, \dots, -\frac{\mathrm{Tr}_{n(m-1)^{n-1}}(\mathcal{T})}{n(m-1)^{n-1}}\right).$$

Moreover, $\mathrm{DET}(-\mathcal{T}) \in \mathbb{C}[\mathcal{T}]$ is homogenous of degree $n(m-1)^{n-1}$ by Proposition 2.2.4 (ii), which implies that $\mathrm{Det}(-\mathcal{T}) = (-1)^{n(m-1)^{n-1}}\mathrm{Det}(\mathcal{T})$. Consequently, the result follows. $\square$

Corollary 2.4.5 *Let $\mathcal{T} \in \mathbb{T}(\mathbb{C}^n, m)$ and the notation be defined as above. Then,*

(i) $\sum_{\lambda_i \in \sigma(\mathcal{T})} m_i\lambda_i = (m-1)^{n-1}\sum_{i=1}^n t_{ii\dots i} = Tr_1(\mathcal{T})$,

(ii) $\sum_{\lambda_i \in \sigma(\mathcal{T})} m_i\lambda_i^2 = Tr_2(\mathcal{T})$, *and*

(iii) $\Pi_{\lambda_i \in \sigma(\mathcal{T})}\lambda_i^{m_i} = Det(\mathcal{T})$.

Here m_i is the algebraic multiplicity of the eigenvalue λ_i.

Proof. The results (i) and (iii) follow from the eigenvalue representation of $\chi(\lambda)$ in Theorem 2.4.3 and the coefficients of $\chi(\lambda)$ in Proposition 2.4.4 immediately. For (ii), by Proposition 2.4.4 (ii) and Newton's identities for the roots and the coefficients of a polynomial, we get that $\sum_{i<j,\lambda_i,\lambda_j\in\sigma(\mathcal{T})} m_im_j\lambda_i\lambda_j = p_2(-\frac{\mathrm{Tr}_1(\mathcal{T})}{1}, -\frac{\mathrm{Tr}_2(\mathcal{T})}{2}) = \frac{1}{2}\left([\mathrm{Tr}_1(\mathcal{T})]^2 - \mathrm{Tr}_2(\mathcal{T})\right)$. Consequently, (ii) follows from (i) and the perfect square formula. $\square$

Remark 2.4.6 In [62], Qi proved the results in Corollary 2.4.5 (i) and (iii) for $\mathcal{T} \in \mathbb{S}(\mathbb{R}^n, m)$ (the space of real symmetric tensors of order m and dimension n). By Theorem 2.3.1 and Corollary 2.4.5, we see that the solvability of a homogeneous polynomial equation is characterized by the zero eigenvalue of the underlying tensor.

In the following, we generalize Corollary 2.4.5 (i) and (ii) to $\mathrm{Tr}_k(\mathcal{T})$ for all $k \in [n(m-1)^{n-1}]$. To this end, we need the following lemmas.

Lemma 2.4.7 *Let $p_k(t_1, \ldots, t_k)$ be the Schur polynomials defined as* (2.4.12)*. Then, for all $k \in \mathbb{N}_+$,*

$$\frac{\partial}{\partial t_i} p_k = p_{k-i}, \ \forall i \in [k]. \tag{2.4.18}$$

Proof. The case for $i = k$ is easy to see, since $p_0 = 1$ and the only monomial in p_k having the variable t_k is t_k by (2.4.12).

In the following, we show the cases $i \in [k-1]$. For each fixed i, we have that $p_{k-i}(t_1, \ldots, t_{k-i}) = \sum_{s=1}^{k-i} \sum_{d_j>0, \sum_{j=1}^{s} d_j = k-i} \frac{\prod_{j=1}^{s} t_{d_j}}{s!}$ by (2.4.12). To prove (2.4.18), it is sufficient to show that there is a one to one correspondence between the monomials in p_{k-i} and these in $p_k(t_1, \ldots, t_k) = \sum_{w=1}^{k} \sum_{d_j>0, \sum_{j=1}^{w} d_j = k} \frac{\prod_{j=1}^{w} t_{d_j}}{w!}$ having variable t_i, and their coefficients satisfying the derivative relation.

First, the one to one correspondence between the monomials in p_{k-i} and these in p_k having variable t_i is obvious: for any monomial $c \prod_{j=1}^{s} t_{d_j}$ with nonzero coefficient c in p_{k-i}, there is the monomial $d t_i \prod_{j=1}^{s} t_{d_j}$ with nonzero coefficient d in p_k, and vice verse.

Second, suppose that $\frac{c}{s!} \prod_{j=1}^{s} t_{d_j}$ with nonzero coefficient c is a monomial in the polynomial p_{k-i} for some $s \in [k-i]$ and $d_1, \ldots, d_s$. Then, by (2.4.12), we see that the number of cases of the ordered s-tuples $(q_1, \ldots, q_s)$ such that $\sum_{j=1}^{s} q_j = k - i$ and $q_j > 0, \ j \in [s]$ resulting in $\prod_{j=1}^{s} t_{d_j}$ is c. For any such ordered $(q_1', \ldots, q_s')$, we get $s+1$ ordered $s+1$-tuples $(i, q_1', \ldots, q_s'), (q_1', i, \ldots, q_s') \ldots, (q_1', \ldots, q_s', i)$ such that every $s+1$-tuple results in $t_i \prod_{j=1}^{s} t_{d_j}$. Note that some of the $s+1$-tuples may be the same. Let r be the degree of the variable t_i in the monomial $t_i \prod_{j=1}^{s} t_{d_j}$. Consequently, the number of cases of the ordered $s+1$-tuples $(q_1, \ldots, q_s, q_{s+1})$ such that $\sum_{j=1}^{s+1} q_j = k$ and $q_j > 0, \ j \in [s+1]$ resulting in $t_i \prod_{j=1}^{s} t_{d_j}$ is $\frac{(s+1)c}{r}$. These, together with (2.4.12), imply that the monomial $\frac{c}{s!}\prod_{j=1}^{s} t_{d_j}$ in p_{k-i} corresponds to the following monomial in p_k:

$$\frac{(s+1)c}{r}\frac{1}{(s+1)!}t_i\prod_{j=1}^{s} t_{d_j}.$$

The derivative of this monomial with respective to t_i is exactly $\frac{c}{s!}\prod_{j=1}^{s} t_{d_j}$. The proof is complete. $\square$

Lemma 2.4.8 *Let* $\mathbf{x} = (x_1, \ldots, x_n)$ *and* $h_1, \ldots, h_n \in \mathbb{C}[\mathbf{x}]$ *be polynomials. If there is some* $f \in \mathbb{C}[\mathbf{x}]$ *satisfying the following system of differential equations*

$$\frac{\partial}{\partial x_i} f = h_i, \ \forall i \in [n]$$

and $f(\mathbf{0}) = 0$*, then* f *is unique.*

Proof. First, $f(\mathbf{0}) = 0$ implies that the constant term of f is zero. Then, as $\mathbb{C}$ is algebraically closed and of characteristic zero, it is sufficient to prove that every monomial of positive degree in f is uniquely determined by the differential equations. This is easy to see: (i) every monomial of f containing the variable x_i is uniquely determined by the i-th differential equation in the hypothesis, and (ii) every monomial of positive degree of f has at least one variable in the set $\{x_i \mid i \in [n]\}$. The proof is complete. $\square$

Lemma 2.4.9 *Let* $p_k(t_1, \ldots, t_k)$ *be the Schur polynomials defined as* (2.4.12)*. Then, for all* $k \in \mathbb{N}_+$*,*

$$kp_k = kt_k + \sum_{i=0}^{k-1} it_i p_{k-i}. \tag{2.4.19}$$

Proof. On the one side, by Lemmas 2.4.7 and 2.4.8, we know that for all $k \in \mathbb{N}_+$, p_k defined as (2.4.12) is the unique polynomial satisfying

$$\frac{\partial}{\partial t_i} p_k = p_{k-i}, \ \forall i \in [k]. \tag{2.4.20}$$

On the other side, we show that polynomials $q_k(t_1, \ldots, t_k)$ defined through the recursive formulae

$$q_0 = 1,\ kq_k = kt_k + \sum_{i=0}^{k-1} it_i q_{k-i},\ \forall k = 1, 2, \ldots \tag{2.4.21}$$

satisfy (2.4.20) through replacing p_k's by q_k's as well. Consequently, $q_k = p_k$ for all $k \in \mathbb{N}$ and (2.4.19) follows.

The proof is by induction, the first step for $k = 1$ is obvious, since $\frac{\partial}{\partial t_1} q_1 = 1 = q_0$. Second, suppose that all of $\{q_1, \ldots, q_k\}$ satisfy (2.4.20) for some $k \geq 1$, we prove that q_{k+1} satisfies (2.4.20) as well. It is easy to see that $\frac{\partial}{\partial t_{k+1}} q_{k+1} = 1 = q_0$ by (2.4.21) and the fact that q_s is independent of t_{k+1} for $s \leq k$. For $s \in [k]$, by (2.4.21)

$$\begin{aligned}
(k+1)\frac{\partial}{\partial t_s} q_{k+1} &= \frac{\partial}{\partial t_s}\left(\sum_{i=0}^{k} it_i q_{k+1-i}\right) \\
&= \sum_{0 \leq i \leq k,\ i \neq s} it_i \frac{\partial}{\partial t_s} q_{k+1-i} + st_s \frac{\partial}{\partial t_s} q_{k+1-s} + sq_{k+1-s} \\
&= \sum_{i=0}^{k} it_i \frac{\partial}{\partial t_s} q_{k+1-i} + sq_{k+1-s} \\
&= \sum_{i=1}^{k+1-s} it_i \frac{\partial}{\partial t_s} q_{k+1-i} + sq_{k+1-s} \\
&= \sum_{i=1}^{k+1-s} it_i q_{k+1-i-s} + sq_{k+1-s} \\
&= \sum_{i=1}^{k-s} it_i q_{k+1-s-i} + (k+1-s)t_{k+1-s} + sq_{k+1-s} \\
&= (k+1-s)q_{k+1-s} + sq_{k+1-s} \\
&= (k+1)q_{k+1-s},
\end{aligned}$$

where the fourth equality follows from the fact that q_w is independent of t_s for $w \leq s-1$, the fifth from the inductive hypothesis, the sixth from the fact that $q_0 = 1$, and the seventh from (2.4.21). Therefore, q_{k+1} satisfies (2.4.20). Then, q_k satisfies (2.4.20) for all $k \in \mathbb{N}_+$ by induction. The proof is complete. $\square$

Now, we are in the position to give the main theorem in this section.

Theorem 2.4.10 *Let $\mathcal{T} \in \mathbb{T}(\mathbb{C}^n, m)$. Denote by p_i the codegree i coefficient of the characteristic polynomial of the tensor $\mathcal{T}$. Then, for all $k \in [n(m-1)^{n-1}]$,*

$$Tr_k(\mathcal{T}) = -kp_k - \sum_{i=1}^{k-1} p_i\, Tr_{k-i}(\mathcal{T}).$$

Moreover, for all $k \in [n(m-1)^{n-1}]$,

$$Tr_k(\mathcal{T}) = \sum_{\lambda_i \in \sigma(\mathcal{T})} m_i \lambda_i^k,$$

where m_i is the algebraic multiplicity of the eigenvalue λ_i.

Proof. The first half of this theorem follows from Theorem 2.4.3 and Lemma 2.4.9 by inserting t_i with $-\frac{\mathrm{Tr}_i(\mathcal{T})}{i}$. The second half follows from the first half and Newton's identities on the roots and the coefficients of a polynomial: for an univariate polynomial equation

$$t^k + a_1 t^{k-1} + \cdots + a_k = 0,$$

let s_i be the sum of the i-th powers of its roots with multiplicity. Then,

$$s_i = -ia_i - \sum_{j=1}^{i-1} s_{i-j} a_j.$$

The proof is complete. □

Remark 2.4.11 Theorem 2.4.10 reveals two facts: (i) the coefficients of the characteristic polynomial of a tensor are recursively generated by the higher order traces of the tensor, and (ii) the higher order traces of a tensor are elementary symmetric functions of powers of the eigenvalues of the tensor. It is a generalization of Newton's identities on the characteristic polynomial for a matrix to a tensor. It also indicates the fundamental roles of the higher order traces in the eigenvalue theory of tensors.

Chapter 3

Nonnegative Tensors

3.1 Introduction

In this chapter, we study the partition of nonnegative tensors and numerical algorithms for finding the spectral radii of nonnegative tensors.

The topic on eigenvalues of nonnegative tensors has attracted much attention [13,15–18,29,40,49–51,57,60,78–84,86]. Researchers studied the Perron-Frobenius theorem for nonnegative tensors and algorithms for finding their largest eigenvalues, i.e., the spectral radii. Most results are based on the notion of irreducibility. In general, there are several classes of nonnegative tensors [13,15,29,36,49,60,84]. In this thesis, we mainly concern on the notion of weak irreducibility, which is related to hypergraphs [42,61,66]. This class of nonnegative tensors was first proposed in [29].

The rest of this chapter contains three sections. In the next section, we introduce a simple and equivalent definition of the weakly irreducible nonnegative tensors. Some properties related to spectral hypergraph theory are reviewed. Then, in Section 3.3, we propose a power method for finding the largest eigenvalue of a weakly irreducible nonnegative tensor, and establish its global R-linear convergence. In Section 3.4, we show that for a nonnegative tensor $\mathcal{T}$, always there exists a partition of the index set $[n]$ such that every sub-tensor induced by the partition is weakly irreducible, and the largest eigenvalue of $\mathcal{T}$ can be obtained from those largest eigenvalues of the induced sub-tensors.

3.2 Weakly Irreducible Nonnegative Tensors

The definition of weak irreducibility for nonnegative tensors by Friedland et al. [29] relies on the strong connectivity of a graph associated to a polynomial map [30]. Alternatively, we use the following simple and equivalent definition [36].

Definition 3.2.1 *Let $\mathcal{T}$ be a nonnegative tensor of order m and dimension n.*

- *We call a nonnegative matrix $R(\mathcal{T})$ the* representation *associated to the nonnegative tensor $\mathcal{T}$, if the (i, j)-th element of $R(\mathcal{T})$ is $\sum_{\{i_2,\ldots,i_m\}\ni j} t_{ii_2\ldots i_m}$.*
- *We call the tensor $\mathcal{T}$* weakly reducible *if its representation $R(\mathcal{T})$ is a reducible matrix, and* weakly primitive *if $R(\mathcal{T})$ is a primitive matrix. If $\mathcal{T}$ is not* weakly reducible*, then it is called* weakly irreducible.

For convenience, a one dimensional tensor (i.e., a scalar) is regarded as weakly irreducible. We summarize the necessary Perron-Frobenius theorem for nonnegative tensors in the next lemma. For comprehensive references on this theory, see [13, 15, 29, 36, 67, 78, 79] and references therein.

Lemma 3.2.2 *Let $\mathcal{T}$ be a nonnegative tensor. Then, we have the followings.*

(i) $\rho(\mathcal{T})$ is an H^+-eigenvalue of $\mathcal{T}$.

(ii) If $\mathcal{T}$ is weakly irreducible, then $\rho(\mathcal{T})$ is the unique H^{++}-eigenvalue of $\mathcal{T}$.

Proof. By Definition 1.2.1, the conclusion (i) follows from [79, Theorem 2.3]. The conclusion (ii) follows from [29, Theorem 4.1]. $\square$

The next lemma is useful.

Lemma 3.2.3 *Let $\mathcal{B}$ and $\mathcal{C}$ be two nonnegative tensors, and $\mathcal{B} \geq \mathcal{C}$ in the sense of componentwise. If $\mathcal{B}$ is weakly irreducible and $\mathcal{B} \neq \mathcal{C}$, then $\rho(\mathcal{B}) > \rho(\mathcal{C})$. Thus, if $\mathbf{x} \in \mathbb{R}^n_+$ is an eigenvector of $\mathcal{B}$ corresponding to $\rho(\mathcal{B})$, then $\mathbf{x} \in \mathbb{R}^n_{++}$ is positive.*

Proof. By [78, Theorem 3.6], $\rho(\mathcal{B}) \geq \rho(\mathcal{C})$ and the equality holding implies that $|\mathcal{C}| = \mathcal{B}$. Since $\mathcal{C}$ is nonnegative and $\mathcal{B} \neq \mathcal{C}$, we must have the strict inequality. The second conclusion follows immediately from the first one and the weak irreducibility of $\mathcal{B}$. For another proof, see [78, Lemma 3.5]. $\square$

Note that the second conclusion of Lemma 3.2.3 is equivalent to that $\rho(\mathcal{S}) < \rho(\mathcal{B})$ for any sub-tensor $\mathcal{S}$ of $\mathcal{B}$ other than the trivial case $\mathcal{S} = \mathcal{B}$. By Theorem 3.4.4, without the weakly irreducible hypothesis, it is easy to construct an example such that the strict inequality in Lemma 3.2.3 fails.

3.3 A Globally R-linearly Convergent Power Method

We present here a modification of the power method proposed in [29].

Algorithm 3.3.1 *(A Higher Order Power Method (HOPM))*

Step 0 *Initialization: choose $\mathbf{x}^{(0)} \in \mathbb{R}^n_{++}$. Let $k := 0$.*

Step 1 *Compute*

$$\bar{\mathbf{x}}^{(k+1)} := \mathcal{T}(\mathbf{x}^{(k)})^{m-1}, \quad \mathbf{x}^{(k+1)} := \frac{\left(\bar{\mathbf{x}}^{(k+1)}\right)^{\left[\frac{1}{m-1}\right]}}{\mathbf{e}^T\left[\left(\bar{\mathbf{x}}^{(k+1)}\right)^{\left[\frac{1}{m-1}\right]}\right]},$$

$$\alpha\left(\mathbf{x}^{(k+1)}\right) := \max_{1\le i\le n} \frac{\left(\mathcal{T}(\mathbf{x}^{(k)})^{m-1}\right)_i}{\left(\mathbf{x}^{(k)}\right)_i^{m-1}} \quad \text{and} \quad \beta\left(\mathbf{x}^{(k+1)}\right) := \min_{1\le i\le n} \frac{\left(\mathcal{T}(\mathbf{x}^{(k)})^{m-1}\right)_i}{\left(\mathbf{x}^{(k)}\right)_i^{m-1}}.$$

Step 2 *If $\alpha\left(\mathbf{x}^{(k+1)}\right) = \beta\left(\mathbf{x}^{(k+1)}\right)$, stop. Otherwise, let $k := k+1$, go to Step 1.*

Algorithm 3.3.1 is well-defined if the underlying tensor $\mathcal{T}$ is weakly irreducible, as in this case, $\mathcal{T}\mathbf{x}^{m-1} > 0$ for any $\mathbf{x} > 0$. The following theorem establishes the convergence of Algorithm 3.3.1 if the underlying tensor $\mathcal{T}$ is weakly primitive, where we need to use the concept of Hilbert's projective metric [58]. We first recall such a concept. For any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n_+ \setminus \{0\}$, if there are $\alpha, \beta > 0$ such that $\alpha\mathbf{x} \le \mathbf{y} \le \beta\mathbf{x}$, then $\mathbf{x}$ and $\mathbf{y}$ are called *comparable*. If $\mathbf{x}$ and $\mathbf{y}$ are comparable, and define

$$m(\mathbf{y}/\mathbf{x}) := \sup\{\alpha > 0 \mid \alpha\mathbf{x} \le \mathbf{y}\} \quad \text{and} \quad M(\mathbf{y}/\mathbf{x}) := \inf\{\beta > 0 \mid \mathbf{y} \le \beta\mathbf{x}\},$$

then, the Hilbert's projective metric d can be defined by

$$d(\mathbf{x}, \mathbf{y}) := \begin{cases} \log\left(\frac{M(\mathbf{y}/\mathbf{x})}{m(\mathbf{y}/\mathbf{x})}\right), & \text{if } \mathbf{x} \text{ and } \mathbf{y} \text{ are comparable,} \\ +\infty, & \text{otherwise} \end{cases}$$

for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n_+ \setminus \{0\}$. Note that if $\mathbf{x}, \mathbf{y} \in \Delta_n := \{\mathbf{z} \in \mathbb{R}^n_{++} \mid \mathbf{e}^T\mathbf{z} = 1\}$, then $d(\mathbf{x}, \mathbf{y}) = 0$ if and only if $\mathbf{x} = \mathbf{y}$. Actually, it is easy to check that d is a metric on Δ_n.

For a nonnegative tensor $\mathcal{T}$, we define a function $F_{\mathcal{T}} : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ as

$$(F_{\mathcal{T}})_i(\mathbf{x}) := \left(\sum_{i_2,\ldots,i_m=1}^{n} t_{ii_2\ldots i_m} x_{i_2} \cdots x_{i_m}\right)^{\frac{1}{m-1}} \tag{3.3.1}$$

for all $i \in [n]$ and $\mathbf{x} \in \mathbb{R}^n_+$, then the eigenvalues of $\mathcal{T}$ is strongly related to the eigenvalue problem for the nonlinear map $F_{\mathcal{T}}$ discussed in [58].

Theorem 3.3.2 *Suppose that $\mathcal{T}$ is a weakly irreducible nonnegative tensor of order m and dimension n. Then, the following results hold.*

(i) $\mathcal{T}$ *has a positive eigenpair* $(\lambda, \mathbf{x})$*, and* $\mathbf{x}$ *is unique up to a multiplicative constant.*

(ii) Let $(\lambda_*, \mathbf{x}^*)$ *be the unique positive eigenpair of* $\mathcal{T}$ *with* $\sum_{i=1}^n x_i^* = 1$*. Then,*

$$\min_{x\in\mathbb{R}^n_{++}} \max_{1\le i\le n} \frac{(\mathcal{T}\mathbf{x}^{m-1})_i}{x_i^{m-1}} = \lambda_* = \max_{x\in\mathbb{R}^n_{++}} \min_{1\le i\le n} \frac{(\mathcal{T}\mathbf{x}^{m-1})_i}{x_i^{m-1}}.$$

(iii) If $(\nu, \mathbf{v})$ *is another eigenpair of* $\mathcal{T}$*, then* $|\nu| \le \lambda_*$*.*

(iv) Suppose that $\mathcal{T}$ *is weakly primitive and the sequence* $\{\mathbf{x}^{(k)}\}$ *is generated by Algorithm 3.3.1. Then,* $\{\mathbf{x}^{(k)}\}$ *converges to the unique vector* $\mathbf{x}^* \in \mathbb{R}^n_{++}$ *satisfying* $\mathcal{T}(\mathbf{x}^*)^{m-1} = \lambda_*(\mathbf{x}^*)^{[m-1]}$ *and* $\sum_{i=1}^n x_i^* = 1$*, and there exist constant* $\theta \in (0,1)$ *and positive integer* M *such that*

$$d(\mathbf{x}^{(k)}, \mathbf{x}^*) \le \theta^{\frac{k}{M}} \frac{d(\mathbf{x}^{(0)}, \mathbf{x}^*)}{\theta} \tag{3.3.2}$$

holds for all $k \ge 1$*.*

Proof. Except the result in (3.3.2), all the other results in this theorem can be easily obtained from [29, Theorem 4.1, Corollaries 4.2, 4.3 and 5.1]. So, we only give the proof of (3.3.2) here. We have the following observations first:

- $\mathbb{R}^n_+$ is a *normal cone* in Banach space $\mathbb{R}^n$ [58], since $\mathbf{y} \ge \mathbf{x} \ge 0$ implies $\|\mathbf{y}\| \ge \|\mathbf{x}\|$;
- $\mathbb{R}^n_+$ has nonempty interior $\mathbb{R}^n_{++}$ which is an open cone, and $F_{\mathcal{T}} : \mathbb{R}^n_{++} \to \mathbb{R}^n_{++}$ is continuous and *order-preserving* by the nonnegativity of tensor $\mathcal{T}$;
- $F_{\mathcal{T}}$ is homogeneous of degree 1 in $\mathbb{R}^n_{++}$;
- the set Δ_n is connected and $\mathcal{T}$ has an eigenvector $\mathbf{x}^*$ in Δ_n by Theorem 3.3.2 (i);
- by (3.3.1), $F_{\mathcal{T}}$ is continuously differentiable in an open neighborhood of $\mathbf{x}^*$, since $\mathbf{x}^* > 0$;
- by Definition 3.2.1, $R(\mathcal{T})$ is primitive, hence there exists an integer N such that $[R(\mathcal{T})]^N > 0$. So, $[R(\mathcal{T})]^N \mathbf{x}$ is comparable with $\mathbf{x}^*$ for any nonzero $\mathbf{x} \in \mathbb{R}^n_+$;

- $R(\mathcal{T}) : \mathbb{R}^n \to \mathbb{R}^n$ is a compact linear map, hence its essential spectrum radius is zero [58, Page 38], while its spectral radius is positive since it is a primitive matrix [3].

Hence, by [58, Corollary 2.5 and Theorem 2.7], we have that there exist a constant $\theta \in (0, 1)$ and a positive integer M such that

$$d(\mathbf{x}^{(Mj)}, \mathbf{x}^*) \leq \theta^j d(\mathbf{x}^{(0)}, \mathbf{x}^*), \tag{3.3.3}$$

where d denotes the Hilbert's projective metric on $\mathbb{R}^n_+ \setminus \{0\}$.

By [58, Proposition 1.5], we also have that

$$d(F_{\mathcal{T}}(\mathbf{x}), F_{\mathcal{T}}(\mathbf{y})) \leq d(\mathbf{x}, \mathbf{y}) \tag{3.3.4}$$

for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n_+$. Since $\lambda_* > 0$, by the property of Hilbert's projective metric d [58, Page 13] we have that

$$\begin{aligned} d(\mathbf{x}^{(k+1)}, \mathbf{x}^*) &= d\left(\frac{F_{\mathcal{T}}(\mathbf{x}^{(k)})}{\mathbf{e}^T F_{\mathcal{T}}(\mathbf{x}^{(k)})}, \mathbf{x}^*\right) = d\left(\frac{F_{\mathcal{T}}(\mathbf{x}^{(k)})}{\mathbf{e}^T F_{\mathcal{T}}(\mathbf{x}^{(k)})}, \frac{1}{(\lambda_*)^{\frac{1}{m-1}}} F_{\mathcal{T}}(\mathbf{x}^*)\right) \\ &= d(F_{\mathcal{T}}(\mathbf{x}^{(k)}), F_{\mathcal{T}}(\mathbf{x}^*)) \leq d(\mathbf{x}^{(k)}, \mathbf{x}^*) \end{aligned}$$

holds for any k. So, for any $k \geq M$, we can find the largest j such that $k \geq Mj$ and $M(j+1) \geq k$. Hence,

$$d(\mathbf{x}^{(k)}, \mathbf{x}^*) \leq d(\mathbf{x}^{(Mj)}, \mathbf{x}^*) \leq \theta^j d(\mathbf{x}^{(0)}, \mathbf{x}^*) \leq \theta^{\frac{k}{M}-1} d(\mathbf{x}^{(0)}, \mathbf{x}^*)$$

which implies (3.3.2) for all $k \geq M$. When $1 \leq k < M$, we have $\theta^{\frac{k}{M}} > \theta$, since $\theta \in (0, 1)$. Therefore, (3.3.2) is true for all $k \geq 1$. □

We denote by $\mathbf{x}^{[p]}$ a vector with its i-th element being x_i^p. By Lemma 3.2.2 and 3.3.2, the following result holds obviously.

Theorem 3.3.3 *Suppose that $\mathcal{T}$ is a weakly irreducible nonnegative tensor of order m and dimension n, and the sequence $\{\mathbf{x}^{(k)}\}$ is generated by Algorithm 3.3.1 with $\mathcal{T}$ being* *replaced by $\mathcal{T}+\mathcal{I}$. Then, $\{\mathbf{x}^{(k)}\}$ converges to the unique vector $\mathbf{x}^* \in \mathbb{R}^n_{++}$ satisfying $\mathcal{T}(\mathbf{x}^*)^{m-1} = \lambda_*(\mathbf{x}^*)^{[m-1]}$ and $\sum_{i=1}^n x_i^* = 1$, and there exist a constant $\theta \in (0,1)$ and a positive integer M such that* (3.3.2) *holds for all $k \geq 1$.*

This method improves the literature very well, please see [36] for the comparisons. Theorem 3.3.3 is one of the corner stones for us to develop a method for finding the spectral radius of a *nonnegative tensor* which is not necessarily weakly irreducible.

3.4 Partition of Nonnegative Tensors

If a nonnegative tensor $\mathcal{T}$ of order m and dimension n is weakly irreducible, then from Theorem 3.3.3, we can find the spectral radius and the corresponding positive eigenvector of $\mathcal{T}$ by using Algorithm 3.3.1. If $\mathcal{T}$ is not weakly irreducible, there is no guarantee for the convergence as that in Theorem 3.3.3.

It is well known for nonnegative matrices that: for a general nonnegative matrix, we can place it into an upper block triangular form with irreducible blocks through simultaneous row/column permutations, and the spectral radius is equal to the largest of the spectral radii of the block sub-matrices. In this section, we show that if a nonnegative tensor $\mathcal{T}$ is not weakly irreducible, then there exists a partition of the index set $[n]$ such that every sub-tensor induced by the set in the partition is weakly irreducible, and the largest eigenvalue of $\mathcal{T}$ can be obtained from these induced sub-tensors. Thus, we can find the spectral radius of a general nonnegative tensor by using Algorithm 3.3.1 for these induced weakly irreducible sub-tensors. The following result is useful.

Theorem 3.4.1 *[79, Theorem 2.3] For a nonnegative tensor $\mathcal{T} \in \mathbb{T}(\mathbb{C}^n, m)$, $\rho(\mathcal{T})$ is an eigenvalue with a nonnegative eigenvector $\mathbf{x} \in \mathbb{R}^n_+$ corresponding to it.*

To develop an algorithm for general nonnegative tensors, we prove the following theorem which is an extension of the corresponding result for nonnegative matrices [3].

Theorem 3.4.2 *Suppose that $\mathcal{T} \in \mathbb{T}(\mathbb{C}^n, m)$ is nonnegative. If $\mathcal{T}$ is weakly reducible, then there is a partition $\{S_1, \ldots, S_k\}$ of $[n]$ such that every tensor in $\{\mathcal{T}(S_j) \mid j \in [k]\}$ is weakly irreducible.*

Proof. Since $\mathcal{T}$ is weakly reducible, by Definition 3.2.1 we can obtain that the matrix $R(\mathcal{T})$ is reducible. Thus, we can find a partition $\{J_1, \ldots, J_l\}$ of $[n]$ such that

- ($\star$) every matrix (a second order tensor) in $\{[R(\mathcal{T})]_{J_i} \mid i \in [l]\}$ is irreducible and $[R(\mathcal{T})]_{st} = 0$ for any $s \in J_p$ and $t \in J_q$ such that $p > q$.

Actually, by the definition of reducibility of a matrix, we can find a partition $\{J_1, J_2\}$ of $[n]$ such that $[R(\mathcal{T})]_{st} = 0$ for any $s \in J_2$ and $t \in J_1$. If both $[R(\mathcal{T})]_{J_1}$ and $[R(\mathcal{T})]_{J_2}$ are irreducible, then we are done. Otherwise, we can repeat the above analysis to any reducible block(s) obtained above. In this way, since $[n]$ is a finite set, we can arrive at the desired result ($\star$).

Now, if every tensor in $\{\mathcal{T}(J_i) \mid i \in [l]\}$ is weakly irreducible, then we are done. Otherwise, we repeat the above procedure to generate a partition of $\mathcal{T}$ to these induced sub-tensors which are not weakly irreducible. Since $[n]$ is finite, this process will stop in finite steps. Hence, the theorem follows. $\square$

By Theorem 3.4.2, we have the following corollary.

Corollary 3.4.3 *Suppose that $\mathcal{T}$ is a nonnegative tensor of order m and dimension n. If $\mathcal{T}$ is weakly irreducible, then $\mathcal{T} + \mathcal{I}$ is weakly primitive; otherwise, there is a partition $\{S_1, \ldots, S_k\}$ of $[n]$ such that every tensor in $\{(\mathcal{T} + \mathcal{I})(S_j) \mid j \in [k]\}$ is weakly primitive.*

Theorem 3.4.4 *Suppose that $\mathcal{T}$ is a weakly reducible nonnegative tensor of order m and dimension n, and $\{S_1, \ldots, S_k\}$ is the partition of $[n]$ determined by Theorem 3.4.2. Then, $\rho(\mathcal{T}) = \rho(\mathcal{T}(S_p))$ for some $p \in [k]$.*

Proof. By the proof of Theorem 3.4.2, for the nonnegative matrix $R(\mathcal{T})$, we can find a partition $\{J_1, \ldots, J_l\}$ of $[n]$ such that

- every matrix in $\{[R(\mathcal{T})]_{J_i} \mid i \in [l]\}$ is irreducible and $[R(\mathcal{T})]_{st} = 0$ for any $s \in J_p$ and $t \in J_q$ such that $p > q$.

First, we have that $\rho(\mathcal{T}_{J_i}) \leq \rho(\mathcal{T})$ for all $i \in [l]$ by Lemma 3.2.3.

Then, denote by $(\rho(\mathcal{T}), \mathbf{x})$ a nonnegative eigenpair of $\mathcal{T}$ which is guaranteed by Theorem 3.4.1. Since $[R(\mathcal{T})]_{ij} = 0$ for all $i \in J_l$ and $j \in \cup_{s=1}^{l-1} J_s$. We must have

$$t_{ii_2\ldots i_m} = 0 \ \ \forall i \in J_l, \ \ \forall \{i_2, \ldots, i_m\} \not\subseteq J_l. \tag{3.4.5}$$

Hence, for all $i \in J_l$, we have

$$\begin{aligned} \rho(\mathcal{T}) x_i^{m-1} &= (\mathcal{T}\mathbf{x}^{m-1})_i = \sum_{i_2, \ldots, i_m = 1}^{n} t_{ii_2\ldots i_m} x_{i_2} \cdots x_{i_m} \\ &= \sum_{\{i_2, \ldots, i_m\} \subseteq J_l}^{n} t_{ii_2\ldots i_m} x_{i_2} \cdots x_{i_m} = \left\{\mathcal{T}(J_l)\left(\mathbf{x}(J_l)\right)^{m-1}\right\}_i, \end{aligned}$$

where the third equality follows from (3.4.5). If $\mathbf{x}(J_l) \neq 0$, then $(\rho(\mathcal{T}), \mathbf{x}(J_l))$ is a nonnegative eigenpair of tensor $\mathcal{T}(J_l)$, and if $\mathbf{x}(J_l) = 0$, then we have

$$\mathcal{T}(\cup_{j=1}^{l-1} J_j)\left(\mathbf{x}(\cup_{j=1}^{l-1} J_j)\right)^{m-1} = \rho(\mathcal{T})\left[\mathbf{x}(\cup_{j=1}^{l-1} J_j)\right]^{[m-1]}.$$

In the later case, repeat the above analysis with $\mathcal{T}$ being replaced by $\mathcal{T}(\cup_{j=1}^{l-1} J_j)$. Since $\mathbf{x} \neq 0$ and l is finite, we must find some $t \in \{1, \ldots, l\}$ such that $\mathbf{x}(J_t) \neq 0$ and $(\rho(\mathcal{T}), \mathbf{x}(J_t))$ is a nonnegative eigenpair of the tensor $\mathcal{T}(J_t)$.

Now, if $\mathcal{T}(J_t)$ is weakly irreducible, we are done since $J_t = S_p$ for some $p \in [k]$ by the proof of Theorem 3.4.2. Otherwise, repeat the above analysis with $\mathcal{T}$ and $\mathbf{x}$ being replaced by $\mathcal{T}(J_t)$ and $\mathbf{x}(J_t)$, respectively. Such a process is finite, since n is finite. Thus, we always obtain a weakly irreducible nonnegative tensor $\mathcal{T}(S_p)$ with $S_p \subseteq [n]$ for some p such that $(\rho(\mathcal{T}), \mathbf{x}(S_p))$ is a nonnegative eigenpair of tensor $\mathcal{T}(S_p)$.

The proof is complete. $\square$

Note that if $\mathcal{T}$ is furthermore symmetric, then we can get a diagonal block representation of $\mathcal{T}$ with diagonal blocks $\mathcal{T}(S_i)$ (after some permutation, if necessary). Now, by Corollary 3.4.3 and Theorems 3.3.3, 3.4.2 and 3.4.4, we can get the following theorem.

Theorem 3.4.5 *Suppose that $\mathcal{T}$ is a nonnegative tensor of order m and dimension n.*

(a) If $\mathcal{T}$ is weakly irreducible, then $\mathcal{T} + \mathcal{I}$ is weakly primitive, and hence, Algorithm 3.3.1 with $\mathcal{T}$ being replaced by $\mathcal{T} + \mathcal{I}$ converges to the unique positive eigenpair $(\rho(\mathcal{T}+\mathcal{I}), \mathbf{x})$ of $\mathcal{T}+\mathcal{I}$. Moreover, $(\rho(\mathcal{T}+\mathcal{I})-1, \mathbf{x})$ is the unique positive eigenpair of $\mathcal{T}$.

(b) If $\mathcal{T}$ is not weakly irreducible, then, we can get a set of weakly irreducible tensors $\{\mathcal{T}(S_j) \mid j \in [k]\}$ with $k > 1$ by Theorem 3.4.2. For each $j \in [k]$, we use item (a) to find the unique positive eigenpair $(\rho(\mathcal{T}(S_j)), \mathbf{x}^j)$ of $\mathcal{T}(S_j)$ which is guaranteed by Corollary 3.4.3 when $|S_j| \geq 2$ or the eigenpair $(\mathcal{T}(S_j), 1)$ when $|S_j| = 1$. Then, $\rho(\mathcal{T}) = \max_{j\in[k]} \rho(\mathcal{T}_{S_j})$ by Theorem 3.4.4. If $\mathcal{T}$ is furthermore symmetric, then, $\mathbf{x}$ with $\mathbf{x}(S_{j^*}) = \mathbf{x}^{j^*}$ is a nonnegative eigenvector of $\mathcal{T}$ where $j^* \in argmax_{j=1,\ldots,k}\rho(\mathcal{T}(S_j))$

Remark 3.4.6 *We can find the partition in the above theorems through the corresponding partition of the nonnegative representation matrix of $\mathcal{T}$ and its induced tensors according to Theorem 3.4.4.*

For symmetric nonnegative tensors, the resulting diagonal block representations for symmetric nonnegative tensors are useful. Recently, Qi proved that for a symmetric nonnegative tensor $\mathcal{T}$, it holds that [67, Theorem 2]

$$\rho(\mathcal{T}) = \max\{\mathcal{T}\mathbf{x}^m \mid \mathbf{x} \in \mathbb{R}^n_+, \sum_{i\in[n]} x_i^m = 1\}. \tag{3.4.6}$$

We summarize the above results in the next theorem with some new observations.

Theorem 3.4.7 *Let $\mathcal{T}$ be a symmetric nonnegative tensor of order m and dimension n. Then, there exists a pairwise disjoint partition $\{S_1, \dots, S_r\}$ of the set $[n]$ such that every tensor $\mathcal{T}(S_j)$ is weakly irreducible. Moreover, we have the followings.*

(i) For any $\mathbf{x} \in \mathbb{C}^n$,

$$\mathcal{T}\mathbf{x}^m = \sum_{j\in[r]} \mathcal{T}(S_j)\mathbf{x}(S_j)^m, \text{ and } \rho(\mathcal{T}) = \max_{j\in[r]} \rho(\mathcal{T}(S_j)).$$

(ii) λ is an eigenvalue of $\mathcal{T}$ with eigenvector $\mathbf{x}$ if and only if λ is an eigenvalue of $\mathcal{T}(S_i)$ with eigenvector $\frac{\mathbf{x}(S_i)}{\sqrt[m]{\sum_{j\in S_i}|x_j|^m}}$ whenever $\mathbf{x}(S_i) \neq 0$.

(iii) $\rho(\mathcal{T}) = \max\{\mathcal{T}\mathbf{x}^m \mid \mathbf{x} \in \mathbb{R}^n_+, \sum_{i\in[n]} x_i^m = 1\}$. Furthermore, $\mathbf{x} \in \mathbb{R}^n_+$ is an eigenvector of $\mathcal{T}$ corresponding to $\rho(\mathcal{T})$ if and only if it is an optimal solution of the maximization problem (3.4.6).

Proof. (i) By Theorem 3.4.2, there exists a pairwise disjoint partition $\{S_1, \dots, S_r\}$ of the set $[n]$ such that every tensor $\mathcal{T}(S_j)$ is weakly irreducible. Moreover, by the proof for Theorem 3.4.2 and the fact that $\mathcal{T}$ is symmetric, $\{\mathcal{T}(S_j),\ j \in [r]\}$ encode all the possible nonzero entries of the tensor $\mathcal{T}$. After a reordering of the index set, if necessary, we get a diagonal block representation of the tensor $\mathcal{T}$. Thus, $\mathcal{T}\mathbf{x}^m = \sum_{j\in[r]} \mathcal{T}(S_j)\mathbf{x}(S_j)^m$ follows for every $\mathbf{x} \in \mathbb{C}^n$. The spectral radii characterization is Theorem 3.4.4.

(ii) follows from the partition immediately.

(iii) Suppose that $\mathbf{x} \in \mathbb{R}^n_+$ is an eigenvector of $\mathcal{T}$ corresponding to $\rho(\mathcal{T})$, then $\rho(\mathcal{T}) = \mathbf{x}^T(\mathcal{T}\mathbf{x}^{m-1})$. Hence, $\mathbf{x}$ is an optimal solution of (3.4.6).

On the other side, suppose that $\mathbf{x}$ is an optimal solution of (3.4.6). Then, by (i), we have

$$\rho(\mathcal{T}) = \mathcal{T}\mathbf{x}^m = \mathcal{T}(S_1)\mathbf{x}(S_1)^m + \cdots + \mathcal{T}(S_r)\mathbf{x}(S_r)^m.$$

Whenever $\mathbf{x}(S_i) \neq 0$, we must have $\rho(\mathcal{T})(\sum_{j\in S_i}(x(S_i))_j^m) = \mathcal{T}(S_i)\mathbf{x}(S_i)^m$, since we have that $\rho(\mathcal{T})(\sum_{j\in S_i}(y(S_i))_j^m) \geq \mathcal{T}(S_i)\mathbf{y}(S_i)^m$ for any $\mathbf{y} \in \mathbb{R}^n_+$ by (3.4.6). Hence, $\rho(\mathcal{T}(S_i)) = \rho(\mathcal{T})$. By Lemma 3.2.3, (3.4.6) and the weak irreducibility of $\mathcal{T}(S_i)$, we must have that $\mathbf{x}(S_i)$ is a positive vector whenever $\mathbf{x}(S_i) \neq 0$. Otherwise, $\rho([\mathcal{T}(S_i)](\sup(\mathbf{x}(S_i)))) = \rho(\mathcal{T}(S_i))$ with $\sup(\mathbf{x}(S_i))$ being the support of $\mathbf{x}(S_i)$. Thus,

$$\max\{\mathcal{T}(S_i)\mathbf{z}^m \mid \mathbf{z} \in \mathbb{R}^{|S_i|}_+, \sum_{i\in S_i} z_i^m = 1\}$$

has an optimal solution $\mathbf{x}(S_i)$ in $\mathbb{R}^{|S_i|}_{++}$. By optimization theory [4], we must have that $(\mathcal{T}(S_i) - \rho(\mathcal{T})\mathcal{I})\mathbf{x}(S_i)^{m-1} = 0$. Then, by (ii), $\mathbf{x}$ is an eigenvector of $\mathcal{T}$. $\square$

Chapter 4

The Laplacian of a Uniform Hypergraph

4.1 Introduction

In this chapter, we establish some basic facts on the spectrum of the normalized Laplacian tensor of a uniform hypergraph. It is an analogue of the spectrum of the normalized Laplacian matrix of a graph [19]. This work is derived by the recently rapid developments on both the spectral hypergraph theory [21, 41, 48–50, 61, 66, 70, 71, 75–77] and the spectral theory of tensors [13, 15, 29, 35, 36, 38, 43, 48–50, 57, 62–64, 67, 78, 79].

In the literature [41, 48, 75, 77], all of these Laplacian tensors are in the spirit of the scheme of sums of powers. We will discuss that in Chapter 5. In formalism, they are not as simple as their matrix counterparts which can be written as $D-A$ or $D+A$ with A the adjacency matrix and D the diagonal matrix of degrees of a graph. Also, this approach only works for even-order hypergraphs. Qi [66] proposed a simple definition $\mathcal{D}-\mathcal{A}$ for the Laplacian tensor and $\mathcal{D}+\mathcal{A}$ for the signless Laplacian tensor. Here $\mathcal{A}=(a_{i_1\dots i_k})$ is the adjacency tensor of a k-uniform hypergraph and $\mathcal{D}=(d_{i_1\dots i_k})$ the diagonal tensor with its diagonal elements being the degrees of the vertices. This is a natural generalization of the definition for $D-A$ and $D+A$ in spectral graph theory [7]. The elements of the adjacency tensor, the Laplacian tensor and the signless Laplacian tensors are rational numbers.

On the other hand, there is another approach in spectral graph theory for the Laplacian of a graph [19]. Suppose that G is a graph without isolated vertices. Let the degree of vertex i be d_i. The Laplacian, or the normalized Laplacian matrix, of G is defined as $L = I - \bar{A}$, where I is the identity matrix, $\bar{A} = (\bar{a}_{ij})$ is the normalized adjacency matrix, $\bar{a}_{ij} = \frac{1}{\sqrt{d_i d_j}}$, if vertices i and j are connected, and $\bar{a}_{ij} = 0$ otherwise. This approach involves irrational numbers in general. However, it is seen that λ is an eigenvalue of the Laplacian L if and only if $1 - \lambda$ is an eigenvalue of the normalized adjacency matrix $\bar{A}$. A comprehensive theory was developed based upon this by Chung [19].

In this chapter, we will investigate the normalized Laplacian tensor approach. A formal definition of the normalized Laplacian tensor and the normalized adjacency tensor will be given in Definition 4.2.3.

In the sequel, the normalized Laplacian tensor is simply called the Laplacian as in [19], and the normalized adjacency tensor simply as the adjacency tensor. In this chapter, hypergraphs refer to k-uniform hypergraphs on n vertices. Recall that, for a positive integer n, we use the convention $[n] := \{1, \dots, n\}$. Let $G = (V, E)$ be a k-uniform hypergraph with vertex set $V = [n]$ and edge set E, and d_i be the degree of the vertex i. If $k = 2$, then G is a graph.

For a graph, let $\lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{n-1}$ be the eigenvalues of L in increasing order. The following results are fundamental in spectral graph theory [19, Section 1.3].

(i) $\lambda_0 = 0$ and $\sum_{i\in[n-1]} \lambda_i \leq n$ with equality holding if and only if G has no isolated vertices.

(ii) $0 \leq \lambda_i \leq 2$ for all $i \in [n-1]$, and $\lambda_{n-1} = 2$ if and only if a connected component of G is bipartite and nontrivial.

(iii) The spectrum of a graph is the union of the spectra of its connected components.

(iv) $\lambda_i = 0$ and $\lambda_{i+1} > 0$ if and only if G has exactly $i+1$ connected components.

Our first major work is to show that the above results can be generalized to the Laplacian $\mathcal{L}$ of a uniform hypergraph. Please check Section 1.4 for the one to one correspondence. Our second major work is that we study the smallest H$^+$-eigenvalues of the sub-tensors of the Laplacian. Our third major work is that we introduce the concept of spectral components of a hypergraph and investigate their intrinsic roles in the structure of the spectrum of the hypergraph.

The rest of this chapter begins with some basic definitions on uniform hypergraphs. The spectral components and the flower hearts of a hypergraph are introduced.

In Section 4.3.1, some basic facts about the spectrum of the adjacency tensor are discussed. Then, some properties on the spectrum of the Laplacian are investigated in Section 4.3.2. We characterize all the H$^+$-eigenvalues of the Laplacian through the spectral components and the flower hearts of the hypergraph in Section 4.4.1. In Section 4.4.2, the H$^+$-geometric multiplicity is introduced, and the second smallest H$^+$-eigenvalue is explored.

The smallest H$^+$-eigenvalues of the sub-tensors of the Laplacian are discussed in Section 4.5. The variational characterizations of these eigenvalues are given in Section 4.5.1. Then their connections to the edge connectivity and the edge expansion are discussed in Section 4.5.2 and Section 4.5.3 respectively.

The eigenvectors of the eigenvalues on the spectral circle of the adjacency tensor are characterized in Section 4.6.1. It gives necessary and sufficient conditions under which the largest H-eigenvalue of the Laplacian being two. In Section 4.6.2, we reformulate the above conditions in the language of linear algebra for modules and give necessary and sufficient conditions under which the eigenvector of an eigenvalue on the spectral circle of the adjacency tensor is unique. Some conclusions are made in the last section.

4.2 Uniform Hypergraphs

In this section, we introduce the notion of Laplacian for uniform hypergraphs.

When G is a usual graph (i.e., $k = 2$), for every edge in an edge cut $E(S, S^c)$ whenever it is nonempty, it contains exactly one vertex from S and the other one from S^c. When G is a k-uniform hypergraph with $k \geq 3$, the situation is much more complicated. We will say that an edge in $E(S, S^c)$ cuts S *with depth at least* r ($1 \leq r < k$) if there are at least r vertices in this edge belonging to S. If every edge in the edge cut $E(S, S^c)$ cuts S with depth at least r, then we say that $E(S, S^c)$ cuts S *with depth at least* r.

Definition 4.2.1 *Let $G = (V, E)$ be a k-uniform hypergraph. A nonempty subset $B \subseteq V$ is called a* **spectral component** *of the hypergraph G if either the edge cut $E(B, B^c)$ is empty or $E(B, B^c)$ cuts B^c with depth at least two.*

It is easy to see that any nonempty subset $B \subset V$ satisfying $|B| \leq k - 2$ is a spectral component. Suppose that G has connected components $\{V_1, \dots, V_r\}$, it is easy to see that $B \subset V$ is a spectral component of G if and only if $B \cap V_i$, whenever nonempty, is a spectral component of G_{V_i}. We will establish the correspondence between the H$^+$-eigenvalues that are less than one with the spectral components of the hypergraph, see Theorem 4.4.4.

Definition 4.2.2 *Let $G = (V, E)$ be a k-uniform hypergraph. A nonempty proper subset $B \subseteq V$ is called a* **flower heart** *if B^c is a spectral component and $E(B^c) = \emptyset$.*

If B is a flower heart of G, then G likes a flower with edges in $E(B, B^c)$ as leafs. It is easy to see that any proper subset $B \subset V$ satisfying $|B| \geq n - k + 2$ is a flower heart. There is a similar characterization between the flower hearts of G and these of its connected components. Theorem 4.4.4 will show that the flower hearts of a hypergraph correspond to its largest H$^+$-eigenvalue.

We give the definition of the normalized Laplacian tensor of a uniform hypergraph.

Definition 4.2.3 *Let G be a k-uniform hypergraph with vertex set $[n] = \{1, \ldots, n\}$ and edge set E. The* **normalized adjacency tensor** $\mathcal{A}$, *which is a k-th order n-dimensional symmetric nonnegative tensor, is defined as*

$$a_{i_1 i_2 \ldots i_k} := \begin{cases} \frac{1}{(k-1)!} \prod\limits_{j \in [k]} \frac{1}{\sqrt[k]{d_{i_j}}} & if\ \{i_1, i_2 \ldots, i_k\} \in E, \\ 0 & otherwise. \end{cases}$$

The **normalized Laplacian tensor** $\mathcal{L}$, *which is a k-th order n-dimensional symmetric tensor, is defined as*

$$\mathcal{L} := \mathcal{J} - \mathcal{A},$$

where $\mathcal{J}$ is a k-th order n-dimensional diagonal tensor with the i-th diagonal element $j_{i \ldots i} = 1$ whenever $d_i > 0$, and zero otherwise.

When G has no isolated points, we have that $\mathcal{L} = \mathcal{I} - \mathcal{A}$. The spectrum of $\mathcal{L}$ is called the spectrum of the hypergraph G, and they are referred interchangeably.

The current definition is motivated by the formalism of the normalized Laplacian matrix of a graph investigated extensively by Chung [19]. We have a similar explanation for the normalized Laplacian tensor to the Laplacian tensor (i.e., $\mathcal{L} = P^k \cdot (\mathcal{D} - \mathcal{B})$ [1]) as that for the normalized Laplacian matrix to the Laplacian matrix [19]. Here P is a diagonal matrix with its i-th diagonal element being $\frac{1}{\sqrt[k]{d_i}}$ when $d_i > 0$ and zero otherwise.

[1] The matrix-tensor product is in the sense of [62, Page 1321]: $\mathcal{L} = (l_{i_1 \ldots i_k}) := P^k \cdot (\mathcal{D} - \mathcal{A})$ is a k-th order n-dimensional tensor with its entries being $l_{i_1 \ldots i_k} := \sum_{j_s \in [n],\ s \in [k]} p_{i_1 j_1} \cdots p_{i_k j_k} (d_{j_1 \ldots j_k} - a_{j_1 \ldots j_k})$.

We have already pointed out one of the advantages of this definition, namely, $\mathcal{L} = \mathcal{I} - \mathcal{A}$ whenever G has no isolated vertices. Such a special structure only happens for regular hypergraphs under the definition in [66]. (A hypergraph is called regular if d_i is a constant for all $i \in [n]$.) By Definition 1.2.1, the eigenvalues of $\mathcal{L}$ are exactly a shift of the eigenvalues of $-\mathcal{A}$. Thus, we can establish many results on the spectra of uniform hypergraphs through the spectral theory of nonnegative tensors without the hypothesis of regularity. We note that, by Definition 1.2.1, $\mathcal{L}$ and $\mathcal{D} - \mathcal{B}$ do not share the same spectrum unless G is regular.

In the sequel, the normalized Laplacian tensor and the normalized adjacency tensor are simply called the Laplacian and the adjacency tensor respectively.

By Definition 3.2.1, the following lemma can be proved similarly to [61, Lemma 3.1].

Lemma 4.2.4 *Let G be a k-uniform hypergraph with vertex set V and edge set E. G is connected if and only if $\mathcal{A}$ is weakly irreducible.*

For a hypergraph $G = (V, E)$, it can be partitioned into connected components $V = V_1 \cup \cdots \cup V_r$ for $r \geq 1$. Reorder the indices, if necessary, $\mathcal{L}$ can be represented by a block diagonal structure according to $V_1, \dots, V_r$. By Definition 1.2.1, the spectrum of $\mathcal{L}$ does not change when reordering the indices. Thus, in the sequel, we assume that $\mathcal{L}$ is in the block diagonal structure with its i-th block tensor being the sub-tensor of $\mathcal{L}$ associated to V_i for $i \in [r]$. By Definition 4.2.3, it is easy to see that $\mathcal{L}(V_i)$ is the Laplacian of the sub-hypergraph G_{V_i} for all $i \in [r]$. Similar convention for the adjacency tensor $\mathcal{A}$ is assumed.

4.3 The Spectrum of a Uniform Hypergraph

Basic properties of the eigenvalues of a uniform hypergraph are established in this section.

4.3.1 The Adjacency Tensor

In this subsection, some basic facts of the eigenvalues of the adjacency tensor are discussed. For a nonempty subset $S \subseteq [n]$ and $\mathbf{x} \in \mathbb{C}^n$, we denoted by $\mathbf{x}^S$ the monomial $\prod_{i \in S} x_i$.

By Definition 1.2.1, H$^+$-eigenvalues of $\mathcal{A}$ should be nonnegative, since $\mathcal{A}$ is nonnegative. For a connected hypergraph G, the following lemma says that the smallest H$^+$-eigenvalue of $\mathcal{A}$ is zero. Moreover, it establishes the relations between the nonnegative eigenvectors of the zero eigenvalue of $\mathcal{A}$ and the flower hearts of G.

Lemma 4.3.1 *Let G be a k-uniform connected hypergraph. Then zero is the smallest H^+-eigenvalue of $\mathcal{A}$. Moreover, a nonzero vector $\mathbf{x} \in \mathbb{R}^n_+$ is an eigenvector of $\mathcal{A}$ corresponding to the eigenvalue zero if and only if $[sup(\mathbf{x})]^c$ is a flower heart of G.*

Proof. Let $\mathbf{x}$ be the vector with its i-th element being one and the other entries being zero. Then, by Definition 4.2.3, it is easy to see

$$\mathcal{A}\mathbf{x}^{k-1} = 0.$$

Thus, zero is an H$^+$-eigenvalue of $\mathcal{A}$ by Definition 1.2.1. The minimality follows from the nonnegativity of H$^+$-eigenvalues.

For the necessity of the second half of this lemma, suppose that $\mathbf{x} \in \mathbb{R}^n_+$ is an eigenvector of $\mathcal{A}$ corresponding to the eigenvalue zero. Since $\mathcal{A}\mathbf{x}^k = 0$ and G is connected with $n \geq k$, we must have that $\text{sup}(\mathbf{x})$ is a proper subset of $[n]$. Thus, $[\text{sup}(\mathbf{x})]^c$ is nonempty. Let $\tilde{\mathbf{d}} \in \mathbb{R}^n$ be the n-vector with its i-th element being $\sqrt[k]{d_i}$ for all $i \in [n]$. Then, by Definition 4.2.3, for all $i \in [\text{sup}(\mathbf{x})]^c$,

$$\begin{aligned} 0 = (\mathcal{A}\mathbf{x}^{k-1})_i &= \sum_{e \in E([\text{sup}(\mathbf{x})]^c),\ i \in e} \frac{\mathbf{x}^{e \setminus \{i\}}}{\tilde{\mathbf{d}}^e} + \sum_{e \in E(\text{sup}(\mathbf{x}), [\text{sup}(\mathbf{x})]^c),\ i \in e} \frac{\mathbf{x}^{e \setminus \{i\}}}{\tilde{\mathbf{d}}^e} \\ &= \sum_{e \in E(\text{sup}(\mathbf{x}), [\text{sup}(\mathbf{x})]^c),\ i \in e} \frac{\mathbf{x}^{e \setminus \{i\}}}{\tilde{\mathbf{d}}^e}. \end{aligned}$$

Thus, we have that $\mathbf{x}^{e\setminus\{i\}}=0$ for all $e\in\{e \mid e\in E(\sup(\mathbf{x}),[\sup(\mathbf{x})]^c),\ i\in e\}$ whenever it is nonempty. Thus, the edge cut $E(\sup(\mathbf{x}),[\sup(\mathbf{x})]^c)$ must satisfy that either it is empty or it cuts $[\sup(\mathbf{x})]^c$ with depth at least two. Then, by Definition 4.2.1, $\sup(\mathbf{x})$ is a spectral component.

For the other $i\in\sup(\mathbf{x})$, we have

$$\begin{aligned} 0=(\mathcal{A}\mathbf{x}^{k-1})_i &= \sum_{e\in E(\sup(\mathbf{x})),\ i\in e}\frac{\mathbf{x}^{e\setminus\{i\}}}{\tilde{\mathbf{d}}^e}+\sum_{e\in E(\sup(\mathbf{x}),[\sup(\mathbf{x})]^c),\ i\in e}\frac{\mathbf{x}^{e\setminus\{i\}}}{\tilde{\mathbf{d}}^e}\\ &= \sum_{e\in E(\sup(\mathbf{x})),\ i\in e}\frac{\mathbf{x}^{e\setminus\{i\}}}{\tilde{\mathbf{d}}^e}. \end{aligned}$$

Hence, $E(\sup(\mathbf{x}))$ must be empty. This, together with the previous result and Definition 4.2.2, implies that $[\sup(\mathbf{x})]^c$ is a flower heart.

For the sufficiency, suppose that there is a nonnegative nonzero vector $\mathbf{x}$ such that $[\sup(\mathbf{x})]^c$ is a flower heart of G. Reversing the above analysis, it is easy to see that $\mathcal{A}\mathbf{x}^{k-1}=0$. Hence, $\mathbf{x}\in\mathbb{R}^n_+$ is an eigenvector of $\mathcal{A}$ corresponding to the eigenvalue zero.

The proof is complete. □

By Lemma 3.2.2, $\rho(\mathcal{A})$ is the largest H$^+$-eigenvalue of $\mathcal{A}$. The next lemma says that $\rho(\mathcal{A})=1$ if and only if $|E|>0$, and $\rho(\mathcal{A})=0$ if and only if $|E|=0$.

Lemma 4.3.2 *Let G be a k-uniform hypergraph. Then $\mathcal{A}$ is a symmetric nonnegative tensor, and $\rho(\mathcal{A})$ is the largest H^+-eigenvalue of $\mathcal{A}$. Moreover, if $E=\emptyset$, then $\mathcal{A}=0$ and hence $\rho(\mathcal{A})=0$; and if G has at least one edge, then $\rho(\mathcal{A})=1$.*

Proof. The first half of the conclusion follows from Lemma 3.2.2 and Definition 4.2.3.

The trivial case $E=\emptyset$ is obvious. In the following, we assume that $E\neq\emptyset$ and prove that $\rho(\mathcal{A})=1$. Let $\mathbf{x}$ be a nonzero nonnegative vector. Then, we have that

$$\mathcal{A}\mathbf{x}^k=\sum_{e\in E}k\prod_{i\in e}\frac{x_i}{\sqrt[k]{d_i}}\leq\sum_{e\in E}k\left(\frac{1}{k}\sum_{i\in e}(\frac{x_i}{\sqrt[k]{d_i}})^k\right)=\sum_{e\in E}\sum_{i\in e}\frac{x_i^k}{d_i}=\sum_{i\in[n],\ d_i>0}\sum_{e\in E_i}\frac{x_i^k}{d_i}=\sum_{i\in[n],\ d_i>0}x_i^k.$$

By Theorem 3.4.7 (iii), we then have that $\rho(\mathcal{A}) \leq 1$.

Let $\tilde{\mathbf{d}} \in \mathbb{R}^n$ be the n-vector with its i-th element being $\sqrt[k]{d_i}$ for all $i \in [n]$. Then, by Definition 4.2.3, we have that

$$\mathcal{A}\tilde{\mathbf{d}}^k = \sum_{e \in E} k\tilde{\mathbf{d}}^e \prod_{i \in e} \frac{1}{\sqrt[k]{d_i}} = \sum_{e \in E} k\tilde{\mathbf{d}}^e \frac{1}{\tilde{\mathbf{d}}^e} = \sum_{e \in E} k = k|E| = \sum_{i \in [n]} d_i > 0.$$

Thus, $\mathcal{A}(\frac{\tilde{\mathbf{d}}}{\sqrt[k]{\sum_{i \in [n]} d_i}})^k = 1$. This, together with $\rho(\mathcal{A}) \leq 1$ and Theorem 3.4.7 (iii), implies that $\rho(\mathcal{A}) = 1$. $\square$

The next lemma is a direct consequence of Theorem 2.3.3, Lemmas 4.3.1 and 4.3.2, and Definition 4.2.3.

Lemma 4.3.3 *Let G be a k-uniform hypergraph. Suppose that G has $s \geq 0$ isolated vertices $\{i_1, \dots, i_s\}$ and $r \geq 0$ connected components $V_1, \dots, V_r$ satisfying $|V_i| > 1$ for $i \in [r]$. Then we have the followings.*

(i) As sets,

$$\sigma(\mathcal{A}) = \sigma(\mathcal{A}_1) \cup \sigma(\mathcal{A}_2) \cup \cdots \cup \sigma(\mathcal{A}_r), \tag{4.3.1}$$

where $\mathcal{A}_i$ is the sub-tensor of $\mathcal{A}$ associated to V_i for $i \in [r]$, and the right hand side of (4.3.1) is understood as $\{0\}$ whenever $r = 0$.

(ii) $\mathcal{A}_i$ defined above is the adjacency tensor of the sub-hypergraph G_i of G induced by V_i for all $i \in [r]$. Thus, $\rho(\mathcal{A}_i) = 1$.

(iii) Let $m_i(\lambda)$ be the algebraic multiplicity of λ as an eigenvalue of $\mathcal{A}_i$. As multisets, we have that zero is an eigenvalue of $\mathcal{A}$ with algebraic multiplicity

$$s(k-1)^{n-1} + \sum_{i \in [r]} m_i(0)(k-1)^{n-|V_i|},$$

and $\lambda \in \sigma(\mathcal{A}_i) \setminus \{0\}$ is an eigenvalue of $\mathcal{A}$ with algebraic multiplicity

$$\sum_{i \in [r]} m_i(\lambda)(k-1)^{n-|V_i|}.$$

The next corollary follows from Lemmas 3.2.2, 4.2.4 and 4.3.3, and Theorem 3.4.7 (ii).

Corollary 4.3.4 *Let G be a k-uniform hypergraph. Then, 1 is the unique H^{++}-eigenvalue of $\mathcal{A}$ if and only if G has no isolated vertices.*

4.3.2 The Laplacian

In this subsection, we discuss some facts on the eigenvalues of the Laplacian of a uniform hypergraph. We start with the following theorem.

Theorem 4.3.5 *Let G be a k-uniform hypergraph. Then, we have the followings.*

(i) If G has at least one edge, then $\lambda \in \sigma(\mathcal{L})$ if and only if $1 - \lambda \in \sigma(\mathcal{A})$. Otherwise, $\sigma(\mathcal{L}) = \sigma(\mathcal{A}) = \{0\}$.

(ii) If $\lambda \in \sigma(\mathcal{L})$, then $Re(\lambda) \geq 0$ with equality holding if and only if $\lambda = 0$, and $2 \geq Re(\lambda)$ with equality holding if and only if $\lambda = 2$.

Proof. Suppose that G has $s \geq 0$ isolated vertices $\{i_1, \ldots, i_s\}$ and $r \geq 0$ connected components $V_1, \ldots, V_r$ satisfying $|V_i| > 1$ for $i \in [r]$. Let $\mathcal{A}_i$ be the adjacency tensor and $\mathcal{L}_i$ the Laplacian of the sub-hypergraph G_{V_i} of G induced by V_i for all $i \in [r]$.

For the conclusion (i), if $s = n$, then $\mathcal{L} = \mathcal{A} = 0$. Thus, $\sigma(\mathcal{L}) = \sigma(\mathcal{A}) = \{0\}$. If $s = 0$, then $\mathcal{L} = \mathcal{I} - \mathcal{A}$ by Definition 4.2.3. We get the conclusion (i) by Definition 1.2.1. In the following, suppose that G has at least one edge and $s \geq 1$. We then have that $r \geq 1$. By Theorems 2.3.3 and 3.4.7 and Definition 4.2.3, $\mathcal{L}$ has a block diagonal structure with diagonal sub-tensors $\{0, \mathcal{L}_1, \ldots, \mathcal{L}_r\}$, and moreover

$$\sigma(\mathcal{L}) = \{0\} \cup \sigma(\mathcal{L}_1) \cup \cdots \cup \sigma(\mathcal{L}_r).$$

Since every G_{V_i} is connected, by the established results, we have that $\lambda \in \sigma(\mathcal{L}_i)$ if and only if $1-\lambda \in \sigma(\mathcal{A}_i)$ for all $i \in [r]$. By Lemmas 4.3.2 and 4.3.3, we have that $\rho(\mathcal{A}_i) = 1$ for all $i \in [r]$. Hence, $\{0\} \subset \sigma(\mathcal{L}_i)$ for all $i \in [r]$. Combining these results, (i) follows.

For the conclusion (ii), if G has no edges, then the results are trivial. In the sequel, suppose that $s < n$. If $\lambda \in \sigma(\mathcal{L})$, then $1 - \lambda \in \sigma(\mathcal{A}_i)$ for some $\mathcal{A}_i$ by (i) and Lemma 4.3.3. Then, by the definition for the spectral radius, it follows that $|1-\lambda| \leq \rho(\mathcal{A}_i) = 1$. Thus, we must have that $0 \leq \mathrm{Re}(\lambda) \leq 2$. By the same reason, we have the necessary and sufficient characterizations, since we must have $\mathrm{Im}(\lambda) = 0$ whenever the equalities are fulfilled. □

In Section 4.6, we will show that $\mathrm{Re}(\lambda) < 2$ if k is odd, i.e., it is impossible that $\lambda = 2$ is an eigenvalue of $\mathcal{L}$ when k is odd. Necessary and sufficient conditions are given for $\lambda = 2$ being an eigenvalue of $\mathcal{L}$ when k is even.

The next corollary says that the H$^+$-eigenvalues of $\mathcal{L}$ have a much more modest behavior than the eigenvalues.

Corollary 4.3.6 *Let G be a k-uniform hypergraph. Then, we have the followings.*

(i) Zero is the unique H^{++}-eigenvalue of $\mathcal{L}$. The smallest H-eigenvalue of $\mathcal{L}$ is zero.

(ii) All the H$^+$-eigenvalues of $\mathcal{L}$ are in the interval $[0,1]$. The largest H$^+$-eigenvalue of $\mathcal{L}$ is one if and only if $|E| > 0$, and it is zero if and only if $|E| = 0$.

(iii) All the H-eigenvalues of $\mathcal{L}$ are nonnegative. If k is even, then $\mathcal{L}$ is positive semidefinite (i.e., $\mathcal{L}\mathbf{x}^k \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$), and $\mathcal{L}\mathbf{x}^k$ can be written as a sum of squares (i.e., $\mathcal{L}\mathbf{x}^k = \sum_{i\in[r]} p_i(\mathbf{x})^2$ for some integer r and polynomials p_i).

Proof. For (i), zero is an H^{++}-eigenvalue follows from Definition 1.2.1, Lemma 3.2.2 and 4.3.2 and Theorem 4.3.5 (i) immediately. The uniqueness follows from Lemma 3.2.2, Corollary 4.3.4 and Theorem 4.3.5 (i), since 1 is the unique H^{++}-eigenvalue of the connected components that have more than one vertices, and the spectra of the isolated vertices are the same set $\{0\}$. Finally, the minimality follows from Theorem 4.3.5 (ii), since all the H-eigenvalues are real.

For the conclusion (ii), first we have that all the H^+-eigenvalues of $\mathcal{L}$ are in the interval $[0, 2]$ by Theorem 4.3.5 (ii). Suppose that $\lambda > 1$ is an H^+-eigenvalue of $\mathcal{L}$. Then, by Definition 1.2.1, Theorems 3.4.7 and 4.3.5 and Lemma 4.3.3, $1 - \lambda < 0$ is an H^+-eigenvalue of some connected component of G. This is a contradiction to Lemma 4.3.1. Thus, $\lambda \in [0, 1]$. The remaining conclusions follow from Theorem 4.3.5 (i), and Lemmas 4.3.1 and 4.3.3 immediately.

By Theorem 4.3.5 (ii), all the H-eigenvalues of $\mathcal{L}$ are nonnegative. When k is even, it is further equivalent to that $\mathcal{L}$ is positive semidefinite by [62, Theorem 5]. Thus, the first two statements of the conclusion (iii) follow. This, together with [27, Corollary 2.8], implies the last statement of the conclusion. $\square$

We remark that, unlike the graph counterpart [19], there would be little hope to write $\mathcal{L}\mathbf{x}^k$ as a sum of powers of linear forms.

The next lemma is on the H-eigenvectors of the smallest H-eigenvalue of $\mathcal{L}$.

Lemma 4.3.7 *Let G be a k-uniform hypergraph. Suppose that G has connected components $V_1, \dots, V_r$. We have the followings.*

(i) Let $L \subseteq \mathbb{R}^n$ be the subspace generated by the H-eigenvectors of $\mathcal{L}$ corresponding to the H-eigenvalue zero. Let $\mathcal{L}_i$ be the Laplacian of G_{V_i}. Let $\tilde{L}_i$ be the subspace generated by the H-eigenvectors of $\mathcal{L}_i$ corresponding to the H-eigenvalue zero, and L_i be the canonical embedding of $\tilde{L}_i$ into $\mathbb{R}^n$ with respect to V_i. Then L has a direct sum decomposition:

$$L = L_1 \oplus \cdots \oplus L_r. \tag{4.3.2}$$

(ii) Let $M \subseteq \mathbb{R}^n$ be the subspace generated by the nonnegative H-eigenvectors of $\mathcal{L}$

corresponding to the H-eigenvalue zero. Let $\tilde{M}_i$ and M_i be defined similarly. Then $M = M_1 \oplus \cdots \oplus M_r$, dim$(M_i) = 1$ for all $i \in [r]$, and hence dim$(M) = r$.

Proof. Let $\mathcal{A}_i$ be the adjacency tensor of the sub-hypergraph G_{V_i} of G induced by V_i for all $i \in [r]$. When V_i is a singleton, then $\mathcal{A}_i$ is the scalar zero. When $|V_i| > 1$, by Lemma 4.2.4, $\mathcal{A}_i$ is a weakly irreducible nonzero tensor.

(i). Suppose $\mathbf{x} \in \mathbb{R}^n$ is an H-eigenvector of $\mathcal{L}$ corresponding to the eigenvalue zero. By Definition 1.2.1 and Theorems 3.4.7 (ii) and 4.3.5 (i), whenever $\mathbf{x}(V_i) \neq 0$, $\mathbf{x}(V_i)$ an H-eigenvector of $\mathcal{L}_i$ corresponding to the eigenvalue zero. Thus, $\mathbf{x}(V_i) \in \tilde{L}_i$ for all $i \in [r]$. The reverse of the statement is true as well: if $0 \neq \mathbf{z} \in \mathbb{R}^{|V_i|}$ is an H-eigenvector of $\mathcal{L}_i$ corresponding to the eigenvalue zero, then its embedding into $\mathbb{R}^n$ is an H-eigenvector of $\mathcal{L}$.

Suppose that $\mathbf{y} \in L$ is nonzero and for some positive integer s, $\mathbf{y} = \sum_{i\in[s]} \mathbf{x}_i$ with $\mathbf{x}_i$ being H-eigenvectors of $\mathcal{L}$ corresponding to the eigenvalue zero. Then, $\mathbf{x}_i(V_j) \in \tilde{L}_j$ for all $j \in [r]$ and $i \in [s]$ by the preceding discussion. Thus,

$$\mathbf{y} = \sum_{i\in[s]} \mathbf{x}_i(V_1) \oplus \cdots \oplus \sum_{i\in[s]} \mathbf{x}_i(V_r) \in L_1 \oplus \cdots \oplus L_r.$$

Here we use the same notation $\mathbf{x}_i(V_j)$ for both $\mathbf{x}_i(V_j) \in \mathbb{R}^{|V_j|}$ and its embedding in $\mathbb{R}^n$.

On the contrary, suppose that $\mathbf{y}_i \in L_i$ is nonzero for $i \in [r]$. Then, we have that $\mathbf{y}_i = \sum_{j\in[s_i]} \mathbf{x}_{i,j}$ for some positive integer s_i and H-eigenvectors $\mathbf{x}_{i,j}(V_i)$ of $\mathcal{L}_i$. Moreover, $\mathbf{x}_{i,j}(V_l) = 0$ whenever $l \neq i$ by the definition of L_i. Thus, $\mathbf{x}_{i,j} \in L$ by the preceding discussion. Hence, $\mathbf{y} = \mathbf{y}_1 \oplus \cdots \oplus \mathbf{y}_r = \sum_{j\in[1_j]} \mathbf{x}_{1,j} \oplus \cdots \oplus \sum_{j\in[r_j]} \mathbf{x}_{r,j} = \sum_{i\in[r]} \sum_{j\in[s_i]} \mathbf{x}_{i,j} \in L$. Combining these results, the direct sum decomposition (4.3.2) follows.

(ii). Note that $\tilde{M}_i$ is the subspace generated by the nonnegative eigenvectors of $\mathcal{L}_i$ corresponding to the eigenvalue zero. If $|V_i| = 1$, then $\tilde{M}_i = \mathbb{R}$. When $|V_i| > 1$, by Lemmas 3.2.2, 3.2.3 and 4.2.4, the nonnegative eigenvectors of $\mathcal{A}_i$ corresponding to $\rho(\mathcal{A}_i) = 1$ is unique and positive. Thus, by Theorem 4.3.5 (i), the nonnegative eigenvector of $\mathcal{L}_i$ corresponding to the eigenvalue zero is unique and positive. Hence, $\dim(\tilde{M_i}) = \dim(M_i) = 1$. A similar proof as that for (i) shows that $M = M_1 \oplus \cdots \oplus M_r$ and hence $\dim(M) = \sum_{i \in [r]} \dim(M_i) = r$. $\square$

Lemma 4.3.7 says that the dimension of the linear subspace generated by the non-negative eigenvectors of the eigenvalue zero of the Laplacian is exactly the number of the connected components of the hypergraph. By Corollary 4.6.7, we will see that if k is odd, then $\dim(L_i) = 1$ for all $i \in [r]$ and hence $\dim(L) = r$.

The next proposition gives equations that the eigenvalues of the Laplacian should satisfy.

Proposition 4.3.8 *Let G be a k-uniform hypergraph. We have the followings.*

(i) Let $m(\lambda)$ be the algebraic multiplicity of $\lambda \in \sigma(\mathcal{L})$ and $c(n,k) = n(k-1)^{n-1}$. Then

$$\sum_{\lambda \in \sigma(\mathcal{L})} m(\lambda)\lambda \le c(n,k)$$

with equality holding if and only if G has no isolated vertices.

(ii) Suppose that G has no isolated vertices. Let $\{\lambda_0, \lambda_1, \ldots, \lambda_h\}$ be the H-eigenvalues of $\mathcal{L}$ in increasing order with algebraic multiplicity; and $\{\alpha_i \pm \sqrt{-1}\beta_i,\ i \in [w]\}$ be the remaining eigenvalues[2] of $\mathcal{L}$ with algebraic multiplicity. Then,

$$\sum_{j \in [h]} \lambda_j + 2 \sum_{j \in [w]} \alpha_j = c(n,k), \tag{4.3.3}$$

and

$$\sum_{j \in [h]} \lambda_j^2 + 2 \sum_{j \in [w]} \alpha_j^2 - 2 \sum_{j \in [w]} \beta_j^2 = c(n,k). \tag{4.3.4}$$

[2]By the discussion on [62, Page 1315], they must appear in conjugate complex pairs. They are called N-eigenvalues in that paper.

If furthermore $k \geq 4$, then we also have

$$\sum_{j\in[h]} \lambda_j^3 + 2\sum_{j\in[w]} \alpha_j^3 - 6\sum_{j\in[w]} \alpha_j\beta_j^2 = c(n,k). \tag{4.3.5}$$

Proof. (i) follows from Definition 4.2.3 and Corollary 2.4.5 (i), which says that the summation of the eigenvalues is equal to $(k-1)^{n-1}$ times the summation of the diagonal elements of $\mathcal{L}$.

(ii). First note that $\lambda_0 = 0$ by Corollary 4.3.6 (i). Second, by Theorem 2.2.3, the degree of $\chi_{\mathcal{T}}(\lambda)$ is $c(n,k)$. Hence, $\sum_{\lambda\in\sigma(\mathcal{L})} m(\lambda) = c(n,k)$. Third, by Definition 4.2.3 and the proof of [21, Corollary 3.14], we see that the h-th order traces of the tensor $\mathcal{A}$ is zero for all $h \in [k-1]$. Theorem 2.4.10 says that the summation of the h-th powers of all the eigenvalues of $\mathcal{A}$ is equal to the h-th trace of $\mathcal{A}$ for all $h \leq c(n,k)$. Thus, by Theorems 4.3.5 (i) and 2.4.10, we have that

$$\sum_{\lambda\in\sigma(\mathcal{L})} m(\lambda)(1-\lambda)^h = 0,\ \forall h \in [k-1].$$

Then, (4.3.3), (4.3.4) and (4.3.5) are just the expansions of the corresponding equalities for $h = 1, 2$ and 3 respectively. □

We can derive more equalities for the other $h \in [k-1]$ similarly.

4.4 H$^+$-Eigenvalues of the Laplacian

In this section, we discuss the H$^+$-eigenvalues of the Laplacian. We denote by $\sigma^+(\mathcal{L})$ the set of all the H$^+$-eigenvalues of $\mathcal{L}$. By Corollary 4.3.6, it is nonempty. We characterize all the H$^+$-eigenvalues and the corresponding nonnegative eigenvectors through the spectral components and the flower hearts of G in Section 4.4.1. Then, in the other subsection, we introduce the H$^+$-geometric multiplicity of an H$^+$-eigenvalue and discuss the second smallest H$^+$-eigenvalue of $\mathcal{L}$.

4.4.1 Characterizations

The next lemma characterizes all the H^+-eigenvalues of $\mathcal{L}$.

Lemma 4.4.1 *Let G be a k-uniform hypergraph. Suppose that G has connected components $V_1, \dots, V_r$ for some positive integer r. Let $\mathcal{L}_i$ be the Laplacian of the sub-hypergraph G_i of G induced by V_i. Then, we have the followings.*

(i) $\lambda = 0$ *is an H^+-eigenvalue of $\mathcal{L}$ with nonnegative eigenvector $\mathbf{x}$ if and only if $\mathbf{x}(V_i)$ is the unique positive eigenvector of $\mathcal{L}_i$ whenever $\mathbf{x}(V_i) \neq 0$.*

(ii) $1 > \lambda > 0$ *is an H^+-eigenvalue of $\mathcal{L}$ with nonnegative eigenvector $\mathbf{x}$ if and only if $\mathbf{x}(V_i) = 0$ whenever $|V_i| = 1$, and $1-\lambda$ is an H^+-eigenvalue of $\mathcal{A}_i$ with eigenvector $\mathbf{x}(V_i)$ whenever $|V_i| > 1$ and $\mathbf{x}(V_i) \neq 0$.*

(iii) $\lambda = 1$ *is an H^+-eigenvalue of $\mathcal{L}$ with nonnegative eigenvector $\mathbf{x}$ if and only if $\mathbf{x}(V_i) = 0$ whenever $|V_i| = 1$, and $[sup(\mathbf{x}(V_i))]^c$ is a flower heart of G_i whenever $|V_i| > 1$ and $\mathbf{x}(V_i) \neq 0$.*

Proof. (i) By Definition 1.2.1 and Theorem 3.4.7, it is easy to see that $\lambda = 0$ is an H^+-Eigenvalue of $\mathcal{L}$ with nonnegative eigenvector $\mathbf{x}$ if and only if $\mathbf{x}(V_i)$ is a nonnegative eigenvector of $\mathcal{L}_i$ whenever $\mathbf{x}(V_i) \neq 0$. In this situation, when $|V_i| = 1$, $\mathbf{x}(V_i) > 0$ is a scalar; and when $|V_i| > 1$, $\mathbf{x}(V_i)$ is a nonnegative eigenvector of the adjacency tensor of the connected sub-hypergraph G_i corresponding to its spectral radius 1. By Lemmas 3.2.3 and 4.2.4, and Theorem 4.3.5, it follows that $\mathbf{x}(V_i)$ is the unique positive eigenvector of $\mathcal{L}_i$. The converse is also true.

(ii) follows from Definitions 1.2.1 and 4.2.3, and Theorem 4.3.5 immediately.

(iii) follows from Definitions 1.2.1 and 4.2.3, Lemma 4.3.1 and Theorem 4.3.5. $\square$

By Lemma 4.4.1 (i) and (iii), the H^+-eigenvalues zero and one of $\mathcal{L}$ and their corresponding nonnegative eigenvectors are clear. In the following, without loss of generality by Lemma 4.4.1 (ii), we consider a connected hypergraph G.

The next lemma, together with Theorem 3.4.7, says that the spectral radius λ of the sub-tensor of $\mathcal{A}$ associated to a spectral component of G contributes to an H$^+$-eigenvalue $1-\lambda$ of $\mathcal{L}$.

Lemma 4.4.2 *Let G be a k-uniform connected hypergraph. Let $S \subseteq [n]$ be a nonempty subset. Suppose that S is a spectral component of G. Let*

$$\lambda = \max\{\mathcal{A}\mathbf{y}^k \mid \mathbf{y} \in \mathbb{R}^n_+,\ \sum_{i\in[n]} y_i^k = 1,\ y_i = 0, \forall i \in S^c\}. \tag{4.4.6}$$

Then, $1 \geq \lambda \geq 0$ and $1-\lambda$ is an H$^+$-eigenvalue of $\mathcal{L}$. Moreover, the optimal solutions of (4.4.6) *and the nonnegative eigenvectors of $\mathcal{L}$ with support being contained in S corresponding to the H$^+$-eigenvalue $1-\lambda$ are in one to one correspondence.*

Proof. If $S = V$, then $\lambda = \rho(\mathcal{A}) = 1$ by Theorem 3.4.7 and Lemma 4.3.2. By Corollary 4.3.6 (i), $1-\lambda = 0$ is an H$^+$-eigenvalue of $\mathcal{L}$. The eigenvector correspondence follows from Theorem 3.4.7 (iii).

In the sequel, suppose that $S \neq V$ is proper. Let $\mathcal{B}$ be the k-th order $|S|$-dimensional sub-tensor of $\mathcal{A}$ corresponding to the set S. Then, we have that

$$\begin{aligned}\lambda &= \max\{\mathcal{A}\mathbf{y}^k \mid \mathbf{y} \in \mathbb{R}^n_+,\ \sum_{i\in[n]} y_i^k = 1,\ y_i = 0, \forall i \in S^c\}\\ &= \max\{\mathcal{B}\mathbf{z}^k \mid \mathbf{z} \in \mathbb{R}^{|S|}_+,\ \sum_{i\in[|S|]} z_i^k = 1\}.\end{aligned}$$

By Theorem 3.4.7, $\lambda = \rho(\mathcal{B})$. Suppose that $\mathbf{y}$ is an optimal solution to (4.4.6) with the optimal value λ. Then, the sub-vector $\mathbf{z}$ of $\mathbf{y}$ corresponding to S is an eigenvector of $\mathcal{B}$ corresponding to λ by Theorem 3.4.7 (iii). Hence, we have

$$\mathcal{B}\mathbf{z}^{k-1} = \lambda \mathbf{z}^{[k-1]},$$

where $\mathbf{z}^{[k-1]}$ is a vector with its i-th entry being z_i^{k-1}. For $i \in S^c$, we have that

$$(\mathcal{A}\mathbf{y}^{k-1})_i = \sum_{\{i,i_2,\dots,i_k\}\in E_i} \frac{1}{\sqrt[k]{d_i}} \prod_{j=2}^{k} \frac{y_{i_j}}{\sqrt[k]{d_{i_j}}} = \sum_{\{i,i_2,\dots,i_k\}\in E_i\cap E(S,S^c)} \frac{1}{\sqrt[k]{d_i}} \prod_{j=2}^{k} \frac{y_{i_j}}{\sqrt[k]{d_{i_j}}} = 0,$$

since S is spectral component which implies that $\{i_2,\ldots,i_k\}\cap S^c \neq \emptyset$ for every $\{i,i_2,\ldots,i_k\}\in E(S,S^c)$. For $i\in S$, we have

$$(\mathcal{A}\mathbf{y}^{k-1})_i = \sum_{\{i,i_2,\ldots,i_k\}\in E_i} \frac{1}{\sqrt[k]{d_i}}\prod_{j=2}^{k}\frac{y_{i_j}}{\sqrt[k]{d_{i_j}}} = \sum_{\{i,i_2,\ldots,i_k\}\in E_i\cap E(S)} \frac{1}{\sqrt[k]{d_i}}\prod_{j=2}^{k}\frac{y_{i_j}}{\sqrt[k]{d_{i_j}}} = (\mathcal{B}\mathbf{z}^{k-1})_i = \lambda y_i^{k-1}.$$

Thus, λ is an H$^+$-eigenvalue of $\mathcal{A}$ with eigenvector $\mathbf{y}$. Then, $1-\lambda$ being an H$^+$-eigenvalue of $\mathcal{L}$ with the eigenvector $\mathbf{y}$ with support being contained in S follows immediately .

The conclusion that a nonnegative eigenvector with support being contained in S of $\mathcal{L}$ corresponding to the eigenvalue $1-\lambda$ is an optimal solution of (4.4.6) follows immediately. $\square$

The next lemma says that the converse of Lemma 4.4.2 is also true.

Lemma 4.4.3 *Let G be a k-uniform connected hypergraph. If $\mathbf{x}\in\mathbb{R}^n_+$ is an eigenvector of $\mathcal{L}$ corresponding to an H^+-eigenvalue λ, then sup$(\mathbf{x})$ is a spectral component of G and $1-\lambda$ is the spectral radius of the sub-tensor of $\mathcal{A}$ corresponding to sup$(\mathbf{x})$.*

Proof. Let $S := \mathrm{sup}(\mathbf{x})$ and S^c be its complement. If $S=V$, then by Lemma 4.4.1 (i), $\lambda=0$ and $1=1-\lambda$ is the spectral radius of $\mathcal{A}$. Obviously, V is a spectral component of G by Definition 4.2.1.

If S is a proper subset of V, then $\{e\in E \mid e\cap S^c\neq\emptyset\}$ is nonempty, since G is connected. By Corollary 4.3.6, we have that $1\geq\lambda\geq 0$. By the hypothesis that $(\mathcal{L}\mathbf{x}^{k-1})_i=0$ for all $i\in S^c$, we have

$$\sum_{\{i,i_2,\ldots,i_k\}\in E_i} \frac{1}{\sqrt[k]{d_i}}\prod_{j=2}^{k}\frac{x_{i_j}}{\sqrt[k]{d_{i_j}}} = \sum_{\{i,i_2,\ldots,i_k\}\in E_i\cap E(S,S^c)} \frac{1}{\sqrt[k]{d_i}}\prod_{j=2}^{k}\frac{x_{i_j}}{\sqrt[k]{d_{i_j}}} = 0.$$

Thus, we must have that $|e\cap S^c|\geq 2$ for every $e\in E(S,S^c)$. Hence, S is a spectral component of G. Let $\mathcal{B}$ be the sub-tensor of $\mathcal{A}$ corresponding to S, and $\mathbf{y}$ be the sub-vector of $\mathbf{x}$ corresponding to S. Then, for all $i \in S$, we have

$$\begin{aligned}(1-\lambda)y_i^{k-1} &= (1-\lambda)x_i^{k-1} = (\mathcal{A}\mathbf{x}^{k-1})_i = \sum_{\{i,i_2,\dots,i_k\}\in E_i} \frac{1}{\sqrt[k]{d_i}} \prod_{j=2}^{k} \frac{x_{i_j}}{\sqrt[k]{d_{i_j}}} \\ &= \sum_{\{i,i_2,\dots,i_k\}\in E_i\cap E(S)} \frac{1}{\sqrt[k]{d_i}} \prod_{j=2}^{k} \frac{x_{i_j}}{\sqrt[k]{d_{i_j}}} = (\mathcal{B}\mathbf{y}^{k-1})_i.\end{aligned}$$

Hence, $\mathbf{y}$ is a positive eigenvector of $\mathcal{B}$. Then, $1-\lambda$ is an H^{++}-eigenvalue of $\mathcal{B}$. By Lemma 3.2.2 (ii), and Theorem 3.4.7 (i) and (ii) (see also [67, Theorem 4]), a symmetric nonnegative tensor has at most one H^{++}-eigenvalue. If it has one, then it should be the spectral radius of this tensor. Hence, we have that $1-\lambda = \rho(\mathcal{B})$. $\square$

By Lemmas 4.4.1, 4.4.2 and 4.4.3, we have the following theorem which characterizes all the nonnegative eigenvectors of $\mathcal{L}$.

Theorem 4.4.4 *Let G be a k-uniform hypergraph. Suppose that G has $r \geq 1$ connected components $V_1, \dots, V_r$. Let $\mathcal{L}_i$ and $\mathcal{A}_i$ be respectively the Laplacian and the adjacency tensor of the sub-hypergraph G_i of G induced by V_i. Then $\mathbf{x} \in \mathbb{R}^n_+$ is an eigenvector of $\mathcal{L}$ corresponding to an H^+-eigenvalue λ if and only if*

(i) when $\lambda = 0$, then $\mathbf{x}(V_i)$ is the unique positive eigenvector of $\mathcal{L}_i$ whenever $\mathbf{x}(V_i) \neq 0$;

(ii) when $1 > \lambda > 0$, then $\mathbf{x}(V_i) = 0$ whenever $|V_i| = 1$, and sup($\mathbf{x}(V_i)$) is a spectral component of G_i and $1-\lambda$ is the spectral radius of the sub-tensor of $\mathcal{A}_i$ corresponding to sup($\mathbf{x}(V_i)$) whenever $\mathbf{x}(V_i) \neq 0$ and $|V_i| > 1$;

(iii) when $\lambda = 1$, then $\mathbf{x}(V_i) = 0$ whenever $|V_i| = 1$, and $[$sup$(\mathbf{x}(V_i))]^c$ is a flower heart of G_i whenever $\mathbf{x}(V_i) \neq 0$ and $|V_i| > 1$.

By Theorem 4.4.4, all the H$^+$-eigenvalues can be computed out, since they correspond to the spectral radii of certain nonnegative tensors. The algorithm proposed in Chapter 3 (Algorithm 3.3.1) can be applied. It is globally R-linearly convergent.

By Theorem 4.4.4, when G has no isolated vertices, if μ is the spectral radius of the sub-tensor of $\mathcal{A}$ corresponding to a spectral component S, then

$$1 - \max\{\mathcal{A}\mathbf{y}^k \mid \mathbf{y} \in \mathbb{R}^n_+,\ \sum_{i\in[n]} y_i^k = 1,\ y_i = 0, \forall i \in S^c\} = 1 - \mu \in \sigma^+(\mathcal{L}).$$

Equivalently, we have

$$\begin{aligned} 1 - \mu &= 1 + \min\{-\mathcal{A}\mathbf{y}^k \mid \mathbf{y} \in \mathbb{R}^n_+,\ \sum_{i\in[n]} y_i^k = 1,\ y_i = 0, \forall i \in S^c\} \\ &= \min\{\mathcal{L}\mathbf{y}^k \mid \mathbf{y} \in \mathbb{R}^n_+,\ \sum_{i\in[n]} y_i^k = 1,\ y_i = 0, \forall i \in S^c\}. \end{aligned} \tag{4.4.7}$$

Define

$$\begin{aligned} \sigma_s(\mathcal{L}) := \{\lambda \mid \lambda = \min\{\mathcal{L}\mathbf{y}^k \mid \sum_{i=1}^n y_i^k = 1,\ \mathbf{y} \in \mathbb{R}^n_+,& \\ y_i = 0,\ \forall i \in A^c\},\ A \in 2^V \setminus \{\emptyset\}\}.& \end{aligned} \tag{4.4.8}$$

Then, Theorem 4.4.4, together with Theorem 4.3.5, says that $\sigma^+(\mathcal{L}) \subseteq \sigma_s(\mathcal{L})$. Here a natural question arises. Are the two sets equal to each other? The next proposition gives a negative answer, it says that the hypothesis in Lemma 4.4.2 is necessary, i.e., if S is not a spectral component, then the optimal value of (4.4.7) may not be an H$^+$-eigenvalue of $\mathcal{L}$. More properties on the set $\sigma_s(\mathcal{L})$ are discussed in Section 4.5.

A hypergraph $G = (V, E)$ is complete if E contains all the possible edges.

Proposition 4.4.5 *Let G be a k-uniform complete hypergraph with $n > k$. Then,*

$$\sigma^+(\mathcal{L}) \neq \sigma_s(\mathcal{L}). \tag{4.4.9}$$

Proof. Since G is complete, it is easy to see that the sub-tensors of $\mathcal{A}$ corresponding to the sets with the same cardinality are the same. Thus, there are at most n values in $\sigma_s(\mathcal{L})$. By Lemmas 3.2.2 and 3.2.3 and the fact that G is complete, the values corresponding to sets with different cardinalities larger than $k-1$ are strictly smaller than one and different. The values corresponding to sets with cardinalities not larger than $k-1$ are one, since the sub-tensors are all the identity tensors with appropriate dimensions. Hence, there are exactly $n-k+2$ values in $\sigma_s(\mathcal{L})$.

Since G is complete, every set A satisfying $|A| \geq k-1$ cannot be a spectral component. Hence, the value corresponding to $\{i\}^c$ for every i cannot be in $\sigma^+(\mathcal{L})$ by Theorem 4.4.4. Since otherwise, this value can be expressed by some spectral component. It should be one by the preceding discussion. This would contradict the fact that $\rho(\mathcal{A}(\{i\}^c)) > 0$ (which implies $1-\rho(\mathcal{A}(\{i\}^c)) < 1$) by Lemma 3.2.3 and Theorem 3.4.7, since $\mathcal{A}(\{i\}^c)$ is nonzero.

Hence, the result (4.4.9) follows. The proof is complete. $\square$

Actually, by Proposition 4.4.10 and Corollary 4.4.11 in Section 4.4.2, $\sigma^+(\mathcal{L}) = \{0, 1\}$ for a complete hypergraph. While, by the proof of Lemma 4.3.2, it can be calculated that $\sigma_s(\mathcal{L}) = \{1 - \frac{d(s)}{d(n)},\ \ s \in \{k-1, \dots, n\}\}$ with $d(s) := \binom{s-1}{k-1}$.

4.4.2 H$^+$-Geometric Multiplicity

In this subsection, we discuss the second smallest H$^+$-eigenvalue of the Laplacian. To this end, we need to order the H$^+$-eigenvalues first. Since the eigenvectors of an eigenvalue of a tensor do not form a linear subspace of $\mathbb{C}^n$ like its matrix counterpart in general, it is subtle to define geometric multiplicity of an eigenvalue of a tensor. However, by Theorem 4.4.4 and the fact that the number of the spectral components of a hypergraph is always finite, we can define the *H$^+$-geometric multiplicity* of an H$^+$-eigenvalue of $\mathcal{L}$ in the following way.

Definition 4.4.6 *Let G be a k-uniform hypergraph. Let μ be an H$^+$-eigenvalue of $\mathcal{L}$. The* **H$^+$-geometric multiplicity** *of the H$^+$-eigenvalue μ is defined to be the number of nonnegative eigenvectors (up to multiscalar multiplication) corresponding to μ.*

For a hypergraph G, we denote by $n(G)$ the number of the H^+-eigenvalues of $\mathcal{L}$ (with H^+-geometric multiplicity). By Corollary 4.3.6 (i), $\mathcal{L}$ always has the H^+-eigenvalue zero and hence $n(G) \geq 1$. When $|E| > 0$, by Lemma 4.3.1 and Theorem 4.3.5, 1 is also an H^+-eigenvalue of $\mathcal{L}$. Then, in this case $n(G) \geq 2$. By Definition 4.4.6 and Corollary 4.3.6 (ii), we can order all the H^+-eigenvalues (with H^+-geometric multiplicity) of $\mathcal{L}$ in increasing order as:

$$0 = \mu_0 \leq \mu_1 \leq \ldots \leq \mu_{n(G)-1} \leq 1. \tag{4.4.10}$$

The next lemma establishes the relation between the number of the connected components of a hypergraph G and the H^+-geometric multiplicity of the eigenvalue zero.

Lemma 4.4.7 *Let G be a k-uniform hypergraph. Suppose that G has r connected components. Then, the H^+-geometric multiplicity $c(G, 0)$ of the H^+-eigenvalue zero of $\mathcal{L}$ is $c(G, 0) = 2^r - 1$.*

Proof. Suppose that $\{V_1, \ldots, V_r\}$ are the connected components of G. For all $i \in [r]$, let $\mathcal{L}_i$ be the Laplacian of the sub-hypergraph G_i of G induced by V_i. For any choice of s ($1 \leq s \leq r$) connected components $\{V_{i_1}, \ldots, V_{i_s}\}$ of G, let $\mathbf{x}(V_{i_j})$ be the unique positive eigenvector of $\mathcal{L}_{i_j}$ by Theorem 4.4.4 (i). Let $\mathbf{x}(V_i) = 0$ for the other V_i. By Theorem 4.4.4 (i), the vector $\mathbf{x}$ formed by the components $\mathbf{x}(V_i)$ is a nonnegative eigenvector of $\mathcal{L}$ corresponding to the eigenvalue zero. By Theorem 4.4.4 (i) again, the correspondence between the choices of the connected components of G and the nonnegative eigenvectors of $\mathcal{L}$ corresponding to the eigenvalue zero in the above sense is one to one. Thus, by Definition 4.4.6, the H^+-geometric multiplicity $c(G, 0)$ of the H^+-eigenvalue zero of $\mathcal{L}$ is $\sum_{s\in[r]} \binom{r}{s} = 2^r - 1$. $\square$

The next corollary is a direct consequence of Lemma 4.4.7.

Corollary 4.4.8 *Let G be a k-uniform hypergraph. Then, $\mu_{i-2} = 0$ and $\mu_{i-1} > 0$ if and only if $\log_2 i$ is a positive integer and G has exactly $\log_2 i$ connected components. In particular, $\mu_1 > 0$ if and only if G is connected.*

The next proposition gives the H$^+$-geometric multiplicity of the H$^+$-eigenvalue one of $\mathcal{L}$.

Proposition 4.4.9 *Let G be a k-uniform hypergraph and $|E| > 0$. Suppose that G has $r \geq 0$ connected components $\{V_1, \dots, V_r\}$ with $|V_i| > 1$. Let G_i be the sub-hypergraph of G induced by V_i. Suppose that G_i has $t_i \geq 0$ flower hearts for all $i \in [r]$. Then the H$^+$-geometric multiplicity $c(G, 1)$ of the H$^+$-eigenvalue one of $\mathcal{L}$ is*

$$c(G, 1) = \sum_{i \in [r]} s_i(t_1, \dots, t_r),$$

where $s_i(t_1, \dots, t_r)$ is the elementary symmetric polynomial on the variables $\{t_1, \dots, t_r\}$ of degree i, and the vacuous summation is understood as zero.

Proof. Note that $s_i(t_1, \dots, t_r) = \sum_{1 \leq j_1 < \ldots < j_i \leq r} t_{j_1} \cdots t_{j_i}$. By Theorem 4.4.4 (iii), the result follows from a similar proof to that for Lemma 4.4.7. $\square$

Proposition 4.4.9 says that $c(G, 1)$ is independent of the number of isolated vertices of the hypergraph G. For the other H$^+$-eigenvalues, by Theorem 4.4.4, their H$^+$-geometric multiplicities are determined by the number of the connected components, and the spectral components of every connected component. Similarly, these H$^+$-geometric multiplicities are independent of the number of isolated vertices of the hypergraph.

The next proposition gives necessary and sufficient conditions for $\mu_1 = \mu_{n(G)-1} = 1$. By Corollary 4.4.8, the underlying hypergraph should be connected.

Proposition 4.4.10 *Let G be a k-uniform connected hypergraph. Then, $\mu_1 = 1$ if and only if the complements of all the proper spectral components are the flower hearts. In this situation, we have $\sigma^+(\mathcal{L}) = \{0, 1\}$.*

Proof. The first half follows from Theorem 4.4.4 immediately. The result $\sigma^+(\mathcal{L}) = \{0, 1\}$ in this situation follows from the fact that $\mu_{n(G)-1} = 1$ when $|E| > 0$. $\square$

The next corollary completes Proposition 4.4.5.

Corollary 4.4.11 *Let G be a k-uniform complete hypergraph. Then, $\sigma^+(\mathcal{L}) = \{0, 1\}$.*

Proof. Suppose that $A \neq V$ is a nonempty subset of $\{1, \ldots, n\}$. Since G is complete, A is a spectral component if and only if $|A| \leq k-2$. On the other side, we also have that $E(A) = \emptyset$ whenever $|A| \leq k-2$. Hence, A^c is a flower heart by Definition 4.2.2. Then, the result follows from Proposition 4.4.10. $\square$

We will give lower bounds for μ_1 in Section 4.5.1.

4.5 The Smallest H$^+$-Eigenvalues of the Sub-Tensors of the Laplacian

Suppose that G is a k-uniform hypergraph without isolated vertices. By Theorem 4.4.4, if λ is an H$^+$-eigenvalue of the Laplacian $\mathcal{L}$, there exists a spectral component of G such that λ has the characterization (4.4.7). However, Proposition 4.4.5 says that $\sigma^+(\mathcal{L}) \neq \sigma_s(\mathcal{L})$ in general. In this section, we show that every $\lambda \in \sigma_s(\mathcal{L})$ is the smallest H$^+$-eigenvalue of some sub-tensor of $\mathcal{L}$. This is the mean of the subscript "s" of $\sigma_s(\mathcal{L})$. Then, we discuss the relations between these H$^+$-eigenvalues and μ_1, the edge connectivity and the edge expansion.

4.5.1 Characterization

We establish the equivalence between the smallest H$^+$-eigenvalues of the sub-tensors of $\mathcal{L}$ and $\sigma_s(\mathcal{L})$. Let $S \subseteq [n]$ be nonempty and $\kappa(S)$ the smallest H$^+$-eigenvalue of $\mathcal{L}(S)$.

The next lemma says that $\{\kappa(S) \mid S \in 2^V \setminus \{\emptyset\}\} = \sigma_s(\mathcal{L})$.

Lemma 4.5.1 *Let G be a k-uniform hypergraph without isolated vertices and $S \subseteq [n]$ be nonempty. We have that $\kappa(S) = 1 - \rho(\mathcal{A}(S)) \in [0,1]$, and*

$$\kappa(S) = \min\{\mathcal{L}\mathbf{y}^k \mid \mathbf{y} \in \mathbb{R}^n_+,\ \sum_{i\in[n]} y_i^k = 1,\ y_i = 0, \forall i \in S^c\}. \tag{4.5.11}$$

Proof. Note that $\mathcal{L}(S) = \mathcal{I} - \mathcal{A}(S)$. Hence, λ is an H$^+$-eigenvalue of $\mathcal{L}(S)$ if and only if $1-\lambda$ is an H$^+$-eigenvalue of $\mathcal{A}(S)$. Thus, by Lemmas 3.2.2, 3.2.3 and 4.3.2, we have that $\kappa(S) = 1 - \rho(\mathcal{A}(S)) \in [0,1]$. This, together with Theorem 3.4.7, further implies that

$$\begin{aligned}
\kappa(S) &= 1 - \max\{\mathcal{A}(S)\mathbf{y}^k \mid \mathbf{y} \in \mathbb{R}^{|S|}_+,\ \sum_{i\in[|S|]} y_i^k = 1\} \\
&= 1 - \max\{\mathcal{A}\mathbf{y}^k \mid \mathbf{y} \in \mathbb{R}^n_+,\ \sum_{i\in[n]} y_i^k = 1,\ y_i = 0, \forall i \in S^c\} \\
&= \min\{1 - \mathcal{A}\mathbf{y}^k \mid \mathbf{y} \in \mathbb{R}^n_+,\ \sum_{i\in[n]} y_i^k = 1,\ y_i = 0, \forall i \in S^c\} \\
&= \min\{\mathcal{L}\mathbf{y}^k \mid \mathbf{y} \in \mathbb{R}^n_+,\ \sum_{i\in[n]} y_i^k = 1,\ y_i = 0, \forall i \in S^c\}.
\end{aligned}$$

Thus, (4.5.11) follows. The proof is complete. □

The next corollary is a direct consequence of Lemma 4.5.1.

Corollary 4.5.2 *Let G be a k-uniform hypergraph without isolated vertices and $S, T \subseteq [n]$ be nonempty such that $S \subset T$. Then, $\kappa(T) \leq \kappa(S)$.*

Corollary 4.5.3 *Let G be a k-uniform hypergraph without isolated vertices. Then, $\mu_1 = \min\{\kappa(S) \mid S$ is a proper spectral component$\}$.*

Proof. We first prove that there is a proper spectral component S of G such that $\mu_1 = \kappa(S)$. Then, the minimality follows immediately from Theorem 4.4.4 and (4.4.10).

By Theorem 4.4.4 and Lemma 4.5.1, there is a spectral component of G such that $\kappa(S) = \mu_1$. If G is connected, then $\mu_1 > 0$ by Corollary 4.4.8. While, $\kappa(V) = 0$ by

Lemmas 4.3.2 and 4.5.1. Thus, $S \neq V$. If G has at least two connected components V_1 and V_2, then V_1 is a proper spectral component and $\kappa(V_1) = 0$ by Lemmas 4.5.1 and 4.3.2. Since $\mu_1 = 0$ by Corollary 4.4.8, V_1 can be chosen as S. $\square$

Recall that d_i is the degree of the vertex i. In the following, we define $d_{\min} := \min_{i\in[n]} d_i$ and $d_{\max} := \max_{i\in[n]} d_i$. For a nonempty subset $S \subset V$, define $\mathrm{vol}(S) := \sum_{i\in S} d_i$ as the *volume* of S. The volume $\mathrm{vol}([n])$ of the hypergraph is simply denoted as d_{vol}.

Proposition 4.5.4 *Let G be a k-uniform hypergraph without isolated vertices. For any nonempty subset $S \subseteq V$, we have*

$$\kappa(S) \leq \frac{(k-1)vol(S^c)}{vol(S)} \tag{4.5.12}$$

with the convention that $vol(\emptyset) = 0$. In particular, for any $i \in [n]$, we have that

$$\kappa(\{i\}^c) \leq \frac{(k-1)d_{\max}}{d_{vol} - d_{\max}}. \tag{4.5.13}$$

Proof. When $S = V$, then $\kappa(S) = \kappa(V) = 0$ by (4.5.11) and Lemma 4.3.2. Thus, the result follows. In the following, we assume that $S \neq V$. Let $\tilde{\mathbf{d}} \in \mathbb{R}^n$ be the n-vector with its i-th element being $\sqrt[k]{d_i}$ for all $i \in [n]$. Let $\mathbf{y}$ be the vector with its j-th element being $\frac{\tilde{d}_j}{\sqrt[k]{\mathrm{vol}(S)}}$ for $j \in S$ and $y_j = 0$ for $j \in S^c$. Then, by Lemma 4.5.1, we have that

$$\begin{aligned}\kappa(S) &\leq \mathcal{L}\mathbf{y}^k = 1 - k\sum_{e\in E\setminus E_{S^c}} \frac{\mathbf{y}^e}{\tilde{\mathbf{d}}^e} = 1 - k\sum_{e\in E\setminus E_{S^c}} \frac{1}{\mathrm{vol}(S)} = 1 - \frac{k|E| - k|E_{S^c}|}{\mathrm{vol}(S)}\\ &= \frac{\mathrm{vol}(S) + k|E_{S^c}| - d_{vol}}{\mathrm{vol}(S)} = \frac{k|E_{S^c}| - \mathrm{vol}(S^c)}{\mathrm{vol}(S)} \leq \frac{(k-1)\mathrm{vol}(S^c)}{\mathrm{vol}(S)}.\end{aligned}$$

Here, the fourth equality follows from the fact that $k|E| = d_{vol}$, and the last inequality from the fact that: for every $e \in E_{S^c}$, e contributes to $\mathrm{vol}(S^c)$ at least one. Thus, the number of edges in E_{S^c} is at most $\mathrm{vol}(S^c)$. Thus, $k|E_{S^c}| \leq k\mathrm{vol}(S^c)$.

(4.5.13) follows from the fact that $d_i \leq d_{\max}$ for any $i \in [n]$. $\square$

Note that (4.5.12) is nontrivial only if $\mathrm{vol}(S) > (k-1)\mathrm{vol}(S^c)$; and the bound is tight.

A hypergraph is d-regular, if $d_i = d \geq 0$ for all $i \in [n]$. The following corollary is a direct consequence of Proposition 4.5.4.

Corollary 4.5.5 *Let G be a k-uniform hypergraph and d-regular for some $d > 0$. For any $i \in [n]$, we have that*

$$\kappa(\{i\}^c) \leq \frac{k-1}{n-1}.$$

By the proof of Proposition 4.5.4, if $d_i = d_{\min}$, then $\kappa(\{i\}^c) \leq \frac{k-1}{n-1}$, since $d_{vol} - d_{min} \geq (n-1)d_{min}$. Hence, the next corollary follows.

Corollary 4.5.6 *Let G be a k-uniform hypergraph without isolated vertices. Then*

$$\min_{i\in[n]} \kappa(\{i\}^c) \leq \frac{k-1}{n-1}.$$

The next proposition gives lower bounds on μ_1 in terms of $\kappa(\{i\}^c)$.

Proposition 4.5.7 *Let G be a k-uniform hypergraph without isolated vertices. Then, for any proper spectral component S of G such that $\mu_1 = \kappa(S)$,*

$$\mu_1 \quad \geq \quad \max_{i\in S^c} \kappa(\{i\}^c) \geq \min_{i\in[n]} \kappa(\{i\}^c). \tag{4.5.14}$$

Proof. The result follows from Theorem 4.4.4, Lemma 4.5.1 and Corollaries 4.5.2 and 4.5.3. □

In Section 4.5.2, $\min_{i\in[n]} \kappa(\{i\}^c)$ is related to the edge connectivity of the hypergraph. This value is similar to the algebraic connectivity defined in [66, Section 8].

4.5.2 Edge Connectivity

In this short subsection, we discuss the relation between the smallest H$^+$-eigenvalues of the sub-tensors of $\mathcal{L}$ and the edge connectivity. Recall that the minimum of the cardinalities of the edge cuts corresponding to nonempty proper subsets is called the **edge connectivity** of G. We denote it by $e(G)$. Note that G is disconnected if and only if $e(G) = 0$. It is also easy to see that $e(G) \le d_{\min}$.

Proposition 4.5.8 *Let G be a k-uniform hypergraph without isolated vertices. We have that*

$$\min_{i\in[n]} \kappa(\{i\}^c) \le \frac{k}{d_{vol}} e(G). \tag{4.5.15}$$

Proof. Let $\tilde{\mathbf{d}} \in \mathbb{R}^n$ be the n-vector with its i-th element being $\sqrt[k]{d_i}$ for all $i \in [n]$. Let S be a nonempty proper subset of $[n]$. Let $\mathbf{y}$ be the vector with its j-th element being $\frac{\tilde{d}_j}{\sqrt[k]{\sum_{i\in S} d_i}}$ for $j \in S$ and $y_j = 0$ for $j \in S^c$. Then, by Lemma 4.5.1,

$$\kappa(S) \le \mathcal{L}\mathbf{y}^k \;=\; 1 - k \sum_{e\in E(S)} \frac{\mathbf{y}^e}{\tilde{\mathbf{d}}^e} = 1 - k \sum_{e\in E(S)} \frac{1}{\mathrm{vol}(S)} = 1 - \frac{k|E(S)|}{\mathrm{vol}(S)}.$$

Similarly, we have that $\kappa(S^c) \le 1 - \frac{k|E(S^c)|}{\mathrm{vol}(S^c)}$. Thus,

$$\begin{aligned}
\mathrm{vol}(S)\kappa(S) + \mathrm{vol}(S^c)\kappa(S^c) &\le \mathrm{vol}(S) + \mathrm{vol}(S^c) - k(|E(S)| + |E(S^c)|) \\
&= d_{vol} - k(|E| - |E(S, S^c)|) \\
&= k|E(S, S^c)|.
\end{aligned}$$

Since both S and S^c are nonempty, we have that $S \subseteq \{r\}^c$ and $S^c \subseteq \{s\}^c$ for some r and s respectively. By Corollary 4.5.2, we have that

$$d_{vol} \min_{i\in[n]} \kappa(\{i\}^c) \le d_{vol} \min\{\kappa(\{r\}^c), \kappa(\{s\}^c)\} \le \sum_{i\in S} d_i\kappa(S) + \sum_{i\in S^c} d_i\kappa(S^c) \;\le\; k|E(S, S^c)|.$$

Thus, (4.5.15) follows. □

4.5.3 Edge Expansion

In this subsection, we define and discuss the edge expansion of a hypergraph.

The next definition is a generalization of the edge expansion of a graph.

Definition 4.5.9 *Let G be a k-uniform hypergraph without isolated vertices and $r \in [k-1]$. The* r**-th depth edge expansion**, *denoted by $h_r(G)$, of G is defined as*

$$h_r(G) = \min_{S \subset V,\ vol(S) \leq \lceil \frac{d_{vol}}{2} \rceil} \frac{|E(S, S^c)|}{vol(S)}, \tag{4.5.16}$$

where the minimum takes additionally over all nonempty subsets S such that either $E(S, S^c)$ is empty or it cuts S^c with depth at least r.

When $r = 1$ and G reduces to a usual graph, this definition is the same as that in [19, Section 2.2]. Moreover, in this situation, it is easy to see that

$$h_1(G) = \min_{S \subset V} \frac{|E(S, S^c)|}{\min\{\mathrm{vol}(S), \mathrm{vol}(S^c)\}},$$

since in this case, $E(S, S^c)$, whenever nonempty, cuts both S and S^c with depth exactly one. For a hypergraph, the situation is more complicated. Thus, we need the generalized definition (4.5.16).

Definition 4.5.9 is well defined for all $r \in [k-1]$, since $E(\{i\}, \{i\}^c)$, whenever nonempty, always cuts $\{i\}^c$ with depth $k - 1 \geq r$ for all $i \in [n]$. The next proposition gives bound on $h_2(G)$ in terms of μ_1.

Proposition 4.5.10 *Let G be a k-uniform hypergraph without isolated vertices and $r \in [k-1]$. We have that $\kappa(S) \leq \frac{(k-2)|E(S,S^c)|}{vol(S)}$ for any spectral component S. Thus,*

$$\mu_1 \leq (k-2) h_2(G).$$

Proof. Let $\tilde{\mathbf{d}} \in \mathbb{R}^n$ be the n-vector with its i-th element being $\sqrt[k]{d_i}$ for all $i \in [n]$. Let S be a spectral component, then either $E(S, S^c)$ is empty or it cuts S^c with depth at least two. The empty case is trivial, in the following, we assume that $E(S, S^c)$ is nonempty. Let $\mathbf{y}$ be the vector with its j-th entry being $\frac{\tilde{d}_j}{\sqrt[k]{\mathrm{vol}(S)}}$ for $j \in S$ and $y_j = 0$ for $j \in S^c$. By Lemma 4.5.1, we have

$$\begin{aligned} \kappa(S) &= \min\{\mathcal{L}\mathbf{z}^k \mid \mathbf{z} \in \mathbb{R}^n_+,\ \sum_{i \in [n]} z_i^k = 1,\ z_i = 0, \forall i \in S^c\} \leq \mathcal{L}\mathbf{y}^k \\ &= 1 - k \sum_{e \in E(S)} \frac{\mathbf{y}^e}{\tilde{\mathbf{d}}^e} = 1 - k \sum_{e \in E(S)} \frac{1}{\mathrm{vol}(S)} = 1 - \frac{k|E(S)|}{\mathrm{vol}(S)} \leq \frac{(k-2)|E(S, S^c)|}{\mathrm{vol}(S)}. \end{aligned}$$

The last inequality follows from the fact that $k|E(S)| \geq \text{vol}(S) - (k-2)|E(S,S^c)|$, since $E(S,S^c)$ cuts S^c with depth at least two. Then, the first conclusion follows. This, together with Definition 4.5.9 and Corollary 4.5.3, implies that

$$\mu_1 \leq (k-2)h_2(G),$$

since $d_{vol} > \lceil \frac{d_{vol}}{2} \rceil$ which implies that the minimum (4.5.16) involves only proper subsets.

$\square$

By Proposition 4.5.7 and a similar proof of Proposition 4.5.10 and the fact that $\min_{i\in[n]} \kappa(\{i\}^c) \leq \kappa(S)$ for any nonempty proper subset S, we have the following proposition.

Proposition 4.5.11 *Let G be a k-uniform hypergraph without isolated vertices. We have that for all $r \in [k-1]$, $\kappa(S) \leq \frac{(k-r)|E(S,S^c)|}{vol(S)}$ for any nonempty subset S such that either $E(S,S^c)$ is empty or it cuts S^c with depth at least r. In particular,*

$$\min_{i\in[n]} \kappa(\{i\}^c) \leq (k-r)h_r(G). \tag{4.5.17}$$

Note that any nonempty subset S such that either $E(S,S^c)$ is empty or it cuts S^c with depth at least r with $r \geq 2$ is a spectral component by Definition 4.2.1. This, together with Corollary 4.5.3 and Proposition 4.5.11, implies the following corollary immediately.

Corollary 4.5.12 *Let G be a k-uniform hypergraph without isolated vertices. For all r such that $2 \leq r \leq k-1$, we have that $\mu_1 \leq (k-r)h_r(G)$.*

4.6 The Largest H-Eigenvalue of the Laplacian

By Theorem 4.3.5, if λ is an H-eigenvalue of $\mathcal{L}$, then $\lambda \leq 2$. Does $\mathcal{L}$ has an eigenvalue 2? If it does, is it an H-eigenvalue of $\mathcal{L}$? In this section, we discuss these questions. By

Lemma 4.3.3, it is sufficient to consider connected hypergraphs. Note that when $\lambda = 2$, we have that -1 is an eigenvalue of the adjacency tensor $\mathcal{A}$.

4.6.1 Eigenvectors of the Largest H-Eigenvalue

As $\rho(\mathcal{A}) = 1$, the set of complex numbers with module one is called the *spectral circle* of the adjacency tensor $\mathcal{A}$. By [78, Theorem 3.9], if there are $r \geq 1$ eigenvalues of $\mathcal{A}$ with module one, then they are uniformly distributed on the spectral circle, i.e., they appear in the form $\exp(\frac{2s\pi\sqrt{-1}}{r})$ for $s \in [r]$. In this subsection, we establish necessary and sufficient conditions for a nonzero vector being an eigenvector of an eigenvalue on the spectral circle.

The next technical lemma is useful.

Lemma 4.6.1 *Let G be a k-uniform connected hypergraph. If $\mathbf{x} \in \mathbb{C}^n$ is an eigenvector of $\mathcal{A}$ with eigenvalue $\exp(\sqrt{-1}\theta)$, then there exist $\theta_i \in \mathbb{R}$ such that*

$$x_i = \exp(\sqrt{-1}\theta_i)\frac{\sqrt[k]{d_i}}{\sqrt[k]{d_{vol}}},\ \forall i \in [n], \tag{4.6.18}$$

and for all $i \in [n]$, there exists $\gamma_i \in \mathbb{C}$ such that

$$\exp(\sqrt{-1}\theta_{i_2})\cdots\exp(\sqrt{-1}\theta_{i_k}) = \gamma_i,\ \forall e = \{i, i_2, \ldots, i_k\} \in E_i. \tag{4.6.19}$$

Proof. Let $\tilde{\mathbf{d}} \in \mathbb{R}^n$ be the n-vector with its i-th element being $\sqrt[k]{d_i}$ for all $i \in [n]$. Then we have the following

$$\begin{aligned}\sum_{i\in[n]}|x_i||(\mathcal{A}\mathbf{x}^{k-1})_i| &= \sum_{i\in[n]}|x_i|\left|\sum_{e\in E_i}\frac{1}{\sqrt[k]{d_i}}\prod_{j\in e\setminus\{i\}}\frac{x_j}{\sqrt[k]{d_j}}\right| \leq \sum_{e\in E}k\prod_{i\in e}\frac{|x_i|}{\sqrt[k]{d_i}}\\ &\leq \sum_{e\in E}k\left(\frac{1}{k}\sum_{i\in e}(\frac{|x_i|}{\sqrt[k]{d_i}})^k\right) = \sum_{e\in E}\sum_{i\in e}\frac{|x_i|^k}{d_i} = \sum_{i\in[n]}\sum_{e\in E_i}\frac{|x_i|^k}{d_i} = \sum_{i\in[n]}|x_i|^k.\end{aligned}$$

This, together with the hypothesis $\sum_{i\in[n]}|x_i||(\mathcal{A}\mathbf{x}^{k-1})_i| = \sum_{i\in[n]}|x_i|^k$, implies that all of the inequalities should be equalities.

By the fact that the second (the arithmetic-geometric mean) inequality is an equality, we have that $|x_i| = \alpha|\sqrt[k]{d_i}|$ for some $\alpha > 0$ for all $i \in [n]$, since G is connected. When normalizing the vectors $\mathbf{x}$ and $\tilde{\mathbf{d}}$, we get (4.6.18).

By the fact that the first inequality is an equality, we have (4.6.19). $\square$

Let $\mathbb{Z}$ be the ring of integers. For a positive integer k, let $\langle k \rangle$ be the ideal in $\mathbb{Z}$ generated by k. Let $\mathbb{K} := \{\bar{0}, \bar{1}, \dots, \overline{k-1}\}$ be the quotient ring $\mathbb{Z}/\langle k \rangle$. The image of $\alpha \in \mathbb{Z}$ under the natural homomorphism $\mathbb{Z} \to \mathbb{K}$ is denoted by $\bar{\alpha}$. For basic definitions, see [22, 23, 46].

The next theorem gives necessary and sufficient conditions for $\exp(\sqrt{-1}\theta)$ being an eigenvalue of $\mathcal{A}$.

Theorem 4.6.2 *Let G be a k-uniform connected hypergraph. Then, $\exp(\sqrt{-1}\theta)$ is an eigenvalue of $\mathcal{A}$ if and only if $\theta = \frac{2\alpha\pi}{k}$ for some integer α, and there exist integers α_i for $i \in [n]$ such that*

$$\sum_{j \in e} \bar{\alpha}_j = \bar{\alpha}, \ \forall e \in E. \tag{4.6.20}$$

Proof. Suppose that $\exp(\sqrt{-1}\theta)$ is an eigenvalue of $\mathcal{A}$ with an eigenvector $\mathbf{x}$. By Lemma 4.6.1, for all $i \in [n]$, there exist $\theta_i \in \mathbb{R}$ such that $x_i = \exp(\sqrt{-1}\theta_i)\frac{\sqrt[k]{d_i}}{\sqrt[k]{d_{vol}}}$, and

$$\exp(\sqrt{-1}\theta_{i_2}) \cdots \exp(\sqrt{-1}\theta_{i_k}) = \gamma_i, \ \forall e = \{i, i_2, \dots, i_k\} \in E_i$$

for some $\gamma_i \in \mathbb{C}$. Let $\tilde{\mathbf{d}} \in \mathbb{R}^n$ be the n-vector with its i-th element being $\sqrt[k]{d_i}$ for all $i \in [n]$. Since

$$\exp(\sqrt{-1}\theta)x_i^{k-1} = (\mathcal{A}\mathbf{x}^{k-1})_i = \gamma_i \frac{(\mathcal{A}\tilde{\mathbf{d}}^{k-1})_i}{(\sqrt[k]{d_{vol}})^{k-1}} = \gamma_i \left(\frac{\sqrt[k]{d_i}}{\sqrt[k]{d_{vol}}} \right)^{k-1},$$

we have that for all $i \in [n]$,

$$\exp(\sqrt{-1}\theta_{i_2}) \cdots \exp(\sqrt{-1}\theta_{i_k}) = \exp(\sqrt{-1}\theta)[\exp(\sqrt{-1}\theta_i)]^{k-1}, \ \forall e = \{i, i_2, \dots, i_k\} \in E_i.$$

Thus, likewise, we must have

$$\sum_{j \in e} \theta_j = \theta + k\theta_i + 2\alpha_{i,e}\pi, \ \forall i \in e, \ \forall e \in E, \tag{4.6.21}$$

for some integer $\alpha_{i,e}$. Since the eigenvalue equations are homogeneous, we can scale $\mathbf{x}$ such that $\theta_1 = 0$ without loss of generality. Consequently,

$$\theta + k\theta_i + 2\alpha_{i,e}\pi = \sum_{j \in e} \theta_j = \theta + 2\alpha_{1,e}\pi, \ \forall i \in e, \ \forall e \in E_1.$$

Hence, $\theta_i = \alpha_i \frac{2\pi}{k}$ for some integer α_i for all $i \in V(1)$ (the set of vertices that share an edge with the vertex 1). Since G is connected, by a similar proof, we can show that $\theta_i = \alpha_i \frac{2\pi}{k}$ for some integer α_i for all $i \in [n]$. Then, we have

$$\theta + 2\alpha_{1,e}\pi = \sum_{j \in e} \theta_j = (\sum_{j \in e} \alpha_j)\frac{2\pi}{k}, \ \forall e \in E_1.$$

Hence, $\theta = \alpha\frac{2\pi}{k}$ for some integer α. With these, we have that (4.6.21) becomes

$$(\sum_{j \in e} \alpha_j)\frac{2\pi}{k} = \alpha\frac{2\pi}{k} + 2(\alpha_i + \alpha_{i,e})\pi, \ \forall i \in e, \ \forall e \in E.$$

Equivalently,

$$\sum_{j \in e} \bar{\alpha}_j = \bar{\alpha}, \ \forall e \in E.$$

Reversing the above proof, we see that the converse statement is true as well. $\square$

As we remarked at the beginning of this subsection, $\theta = \frac{2\alpha\pi}{k}$ with some integer α is not by accident. The eigenvalues of $\mathcal{A}$ with module $\rho(\mathcal{A}) = 1$ distribute uniformly on the spectral circle $\{\lambda \mid |\lambda| = 1\}$ [78]. The number of the eigenvalues on this circle is called the **primitive index** of the tensor $\mathcal{A}$.

If $\exp(\frac{2\alpha\pi}{k}\sqrt{-1})$ is an eigenvalue of $\mathcal{A}$, then there exist integers α_i for $i \in [n]$ such that for all $e \in E$, $\sum_{j \in e} \bar{\alpha}_j = \bar{\alpha}$. It is then easy to see that for all $s \in [k]$, $\sum_{j \in e} \overline{s\alpha_j} = \overline{s\alpha}$ for all $e \in E$. Thus, $\exp(\frac{2s\alpha\pi}{k}\sqrt{-1})$ is an eigenvalue of $\mathcal{A}$ for all $s \in [k]$ by Theorem 4.6.2. So, the primitive index must be a factor of k. We include it in the next corollary. Recall that (r, s) denotes the greatest common divisor of the integers r and s.

Corollary 4.6.3 *Let G be a k-uniform connected hypergraph. If $\alpha \in [k]$ is the smallest positive integer such that $\exp(\frac{2\alpha\pi}{k}\sqrt{-1})$ is an eigenvalue of $\mathcal{A}$, then the primitive index of $\mathcal{A}$ is $\frac{k}{(k,\alpha)}$. Thus, the primitive index of $\mathcal{A}$ is a factor of k; when k is a prime number, either 1 is the unique eigenvalue of $\mathcal{A}$ on the spectral circle or $\exp(j\frac{2\pi}{k}\sqrt{-1})$ is an eigenvalue of $\mathcal{A}$ for all $j \in [k]$.*

The next corollary, together with Theorem 4.3.5, says that when k is odd, $\mathcal{L}$ does not have an eigenvalue being 2.

Corollary 4.6.4 *Let G be a k-uniform connected hypergraph and k be odd. Then for any $\lambda \in \sigma(\mathcal{L})$, we have $Re(\lambda) < 2$.*

Proof. Note that by Theorem 4.3.5 (ii), $\mathrm{Re}(\lambda) \leq 2$ with equality holding if and only if $\lambda = 2$. In this case -1 is an eigenvalue of $\mathcal{A}$. While, Theorem 4.6.2 says that -1 cannot be an eigenvalue of $\mathcal{A}$, since k is odd. Thus, the result follows. $\square$

The next corollary says that the spectrum of the adjacency tensor is invariant under the multiplication by $\exp(\frac{2j\pi}{s}\sqrt{-1})$ for all $j \in [s]$ with $s \geq 1$ being the primitive index of $\mathcal{A}$.

Corollary 4.6.5 *Let G be a k-uniform connected hypergraph and the primitive index of $\mathcal{A}$ be $s \geq 1$. Let $(\alpha_1, \dots, \alpha_n)$ be a set of integers satisfying the equations (4.6.20) for $\alpha = \frac{k}{s}$. If λ is an eigenvalue of $\mathcal{A}$ with an eigenvector $\mathbf{x}$, then $\exp(\frac{2\pi}{s}\sqrt{-1})\lambda$ is also an eigenvalue of $\mathcal{A}$ with an eigenvector $\mathbf{z}$ with $z_i := \exp(\frac{2\alpha_i\pi}{k}\sqrt{-1})x_i$ for all $i \in [n]$.*

Proof. Suppose that λ is an eigenvalue of $\mathcal{A}$ with an eigenvector $\mathbf{x} \in \mathbb{C}^n$. Then, we have that $\mathcal{A}\mathbf{x}^{k-1} = \lambda\mathbf{x}^{[k-1]}$. By the hypothesis, we have that

$$\sum_{j\in e} \bar{\alpha}_j = \overline{(\frac{k}{s})},\ \forall e \in E. \tag{4.6.22}$$

Let $\mathbf{z} \in \mathbb{C}^n$ be an n-vector with $z_i := \exp(\frac{2\alpha_i\pi}{k}\sqrt{-1})x_i$ for all $i \in [n]$. Then, for all $i \in [n]$,

$$\begin{aligned}
(\mathcal{A}\mathbf{z}^{k-1})_i &= \sum_{e\in E_i}\frac{1}{\sqrt[k]{d_i}}\prod_{j\in e\setminus\{i\}}\frac{z_j}{\sqrt[k]{d_j}} = \sum_{e\in E_i}\frac{\exp(\frac{2\pi}{s}\sqrt{-1})}{\exp(\frac{2\alpha_i\pi}{k}\sqrt{-1})}\frac{1}{\sqrt[k]{d_i}}\prod_{j\in e\setminus\{i\}}\frac{x_j}{\sqrt[k]{d_j}} \\
&= \frac{\exp(\frac{2\pi}{s}\sqrt{-1})}{\exp(\frac{2\alpha_i\pi}{k}\sqrt{-1})}\lambda x_i^{k-1} = \frac{\exp(\frac{2\pi}{s}\sqrt{-1})}{\exp(\left[\frac{2\alpha_i\pi}{k}+(k-1)\frac{2\alpha_i\pi}{k}\right]\sqrt{-1})}\lambda z_i^{k-1} \\
&= \exp(\frac{2\pi}{s}\sqrt{-1})\lambda z_i^{k-1},
\end{aligned}$$

where the second equality follows from (4.6.22). Thus, $\exp(\frac{2\pi}{s}\sqrt{-1})\lambda$ is an eigenvalue of $\mathcal{A}$ with the eigenvector $\mathbf{z}$. The proof is complete. □

The invariant of the eigenvalues under the multiplication by $\exp(\frac{2j\pi}{s}\sqrt{-1})$ in Corollary 4.6.5 follows from [78, Theorem 3.17] as well. While, our proof is more constructive, and it reveals the relations between the eigenvectors of the eigenvalues on the same orbit.

By the proof of Theorem 4.6.2, we actually get the next theorem, which characterizes all the eigenvectors of $\mathcal{A}$ corresponding to the eigenvalues on the spectral circle.

Theorem 4.6.6 *Let G be a k-uniform connected hypergraph and $\alpha \in [k]$. Then, a nonzero vector $\mathbf{x}$ is an eigenvector of $\mathcal{A}$ corresponding to the eigenvalue $\exp(\frac{2\alpha\pi}{k}\sqrt{-1})$ if and only if there exist θ and integers α_i such that $x_i = \exp(\sqrt{-1}\theta)\exp(\frac{2\alpha_i\pi}{k}\sqrt{-1})\frac{\sqrt[k]{d_i}}{\sqrt[k]{d_{vol}}}$ for $i \in [n]$, and*

$$\sum_{j\in e}\bar{\alpha}_j = \bar{\alpha},\ \forall e \in E.$$

The next corollary says that when k is odd, the H-eigenvector of $\mathcal{A}$ corresponding to the spectral radius is unique up to scalar multiplication.

Corollary 4.6.7 *Let G be a k-uniform connected hypergraph and k be odd. If $\mathbf{x} \in \mathbb{R}^n$ is an eigenvector of $\mathcal{A}$ corresponding to the eigenvalue one, then $\mathbf{x}$ or $-\mathbf{x}$ is the unique positive eigenvector.*

Proof. By Theorem 4.6.6, the real vector $\mathbf{x}$ that is an eigenvector of $\mathcal{A}$ corresponding to the eigenvalue one should satisfy

$$\exp(\sqrt{-1}\theta)\exp(\frac{2\alpha_i\pi}{k}\sqrt{-1}) = \pm 1,\ \forall i \in [n].$$

These constraints say that $\theta + \frac{2\alpha_i\pi}{k} = \beta_i\pi$ for some integers β_i for all $i \in [n]$. Hence, $\frac{2(\alpha_i-\alpha_j)\pi}{k} = (\beta_i - \beta_j)\pi$ for all $i, j \in [n]$. Since k is odd, we must have that $\beta_i - \beta_j \in \langle 2\rangle \subset \mathbb{Z}$. Thus, $\exp(\sqrt{-1}\theta)\exp(\frac{2\alpha_i\pi}{k}\sqrt{-1}) = 1$ for all $i \in [n]$ or $\exp(\sqrt{-1}\theta)\exp(\frac{2\alpha_i\pi}{k}\sqrt{-1}) = -1$ for all $i \in [n]$, since $\exp(\beta_i\pi) = \exp(\beta_j\pi)$ for all $i, j \in [n]$. The result follows. $\square$

The next corollary gives necessary and sufficient conditions for 2 being an H-eigenvalue of $\mathcal{L}$.

Corollary 4.6.8 *Let G be a k-uniform connected hypergraph and k be even. Then, 2 is an H-eigenvalue of $\mathcal{L}$ if and only if there exists a pairwise disjoint partition of the vertex set $V = V_1 \cup V_2$ with $V_1 \neq \emptyset$ such that for every edge $e \in E$, $|e \cap V_1|$ is an odd number.*

Proof. The sufficiency is obvious: let $\theta = 0$ and $\alpha_i := \frac{k}{2}$ whenever $i \in V_1$ and $\alpha_i = 0$ whenever $i \in V_2$. Since then $\sum_{j\in e}\bar{\alpha}_j = \frac{\bar{k}}{2}$ for all $e \in E$, by Theorem 4.6.6, we see that -1 is an H-eigenvalue of $\mathcal{A}$. Hence, 2 is an H-eigenvalue of $\mathcal{L}$.

For the necessity, suppose that -1 is an H-eigenvalue of $\mathcal{A}$ with an H-eigenvector $\mathbf{x}$. By Theorem 4.6.6, we have that there exist θ and integers α_i satisfying

$$\exp(\sqrt{-1}\theta)\exp(\frac{2\alpha_i\pi}{k}\sqrt{-1}) = \pm 1,\ \forall i \in [n],\ \text{and} \sum_{j\in e}\bar{\alpha}_j = \frac{\bar{k}}{2},\ \forall e \in E.$$

The former constraints say that $\theta + \frac{2\alpha_i\pi}{k} = \beta_i\pi = \frac{\beta_i k\pi}{k} = \frac{2(\frac{k}{2}\beta_i)\pi}{k}$ for some integers β_i for all $i \in [n]$. Thus, $\theta = \frac{2\beta\pi}{k}$ for some integer β. Since $\overline{k\beta} = \bar{0}$, we can absorb θ into α_i for all $i \in [n]$. Without loss of generality, we denote the absorbed integers still by α_i. Then, we have

$$\exp(\frac{2\alpha_i\pi}{k}\sqrt{-1}) = \pm 1,\ \forall i \in [n] \Longleftrightarrow \bar{\alpha}_i = \bar{0},\ \text{or}\ \bar{\alpha}_i = \frac{\bar{k}}{2},\ \forall i \in [n],$$

and still

$$\sum_{j\in e}\bar{\alpha}_j = \frac{\bar{k}}{2},\ \forall e\in E.$$

Since $\overline{2\alpha_i} = \bar{0}$ for all $i\in[n]$, the latter constraints imply that there exists a pairwise disjoint partition of the vertex set $V = V_1\cup V_2$ with $V_1\neq\emptyset$ such that for every edge $e\in E$, $|e\cap V_1|$ is an odd number. Actually, V_1 can be chosen as $\{i\in[n]\mid \bar{\alpha}_i = \frac{\bar{k}}{2}\}$. $\square$

A hypergraph is called k-partite, if there is a pairwise disjoint partition of $V = V_1\cup\cdots\cup V_k$ such that every edge $e\in E$ intersects V_i nontrivially (i.e., $e\cap V_i\neq\emptyset$) for all $i\in[k]$.

Corollary 4.6.9 *Let G be a k-uniform connected hypergraph. If G is k-partite, then the primitive index of $\mathcal{A}$ is k.*

Proof. Since G is k-partite, let $V_1,\dots,V_k$ be one of its k-partition. For any $j\in[k]$, let $\theta=0$, $\bar{\alpha}_i=\bar{j}$ for $i\in V_1$, and $\bar{\alpha}_i=\bar{0}$ for all $i\notin V_1$. Thus, we fulfill $\sum_{i\in e}\bar{\alpha}_i=\bar{j}$ for all $e\in E$. Hence, for all $j\in[k]$, $\exp(j\frac{2\pi}{k}\sqrt{-1})$ is an eigenvalue of $\mathcal{A}$ by Theorem 4.6.6. $\square$

Corollaries 4.6.9 and 4.6.5 imply that the spectrum of the adjacency tensor of a k-partite hypergraph is invariant under multiplication by any k-th root of unity. Thus, we recover [21, Theorem 4.2] for the spectrum of the normalized adjacency tensor of a k-partite hypergraph.

4.6.2 Algebraic Reformulation

In this short subsection, we reformulate Theorems 4.6.2 and 4.6.6 into the language of linear algebra over modules.

For a positive integer k, let the ring $\mathbb{K}$ be defined as above. Let $\mathbb{A} := \mathbb{K}^n$ be the free $\mathbb{K}$-module of rank n. For every hypergraph $G=(V,E)$, we can associated it a submodule $\mathbb{G} \subseteq \mathbb{A}$ with $\mathbb{G}$ being generated by $\tilde{G} := \{\mathbf{z}(e) \in \mathbb{A} \mid z(e)_i = \bar{1},\ i \in e, z(e)_i = \bar{0},\ i \notin e,\ \forall e \in E\}$. Let $\hat{G}$ be the $|E| \times n$ representation matrix of $\mathbb{K}^{|E|} \to \mathbb{G}$ with its rows being given by $\mathbf{z}^T$ with $\mathbf{z} \in \tilde{G}$. Denote by $\mathbf{1} \in \mathbb{A}$ the vector consisting of all ones. Then we have the following result.

Theorem 4.6.10 *Let G be a k-uniform connected hypergraph. Let $\theta = \frac{2\alpha\pi}{k}$ with some nonnegative integer α. Then, $\exp(\sqrt{-1}\theta)$ is an eigenvalue of $\mathcal{A}$ if and only if there is a vector $\mathbf{y} \in \mathbb{A}$ such that $\mathbf{z}^T\mathbf{y} = \bar{\alpha}$ for all $\mathbf{z} \in \tilde{G}$. Moreover, when $\exp(\sqrt{-1}\theta)$ is an eigenvalue of $\mathcal{A}$, it has a unique eigenvector (up to scalar multiplication) if and only if the kernel of $\hat{G}$ is $\langle \mathbf{1} \rangle$.*

The merit of Theorem 4.6.10 is that it states the nonlinear eigenvalue problem of tensors as a linear algebra problem. In the classic linear algebra over fields, for a matrix $A \in \mathbb{R}^{m\times n}$, we have $\dim(\ker(A)) + \dim(\mathrm{im}(A^T)) = n$ [33]. Here $\ker(A)$ and $\mathrm{im}(A)$ mean respectively the kernel and the image of the linear map $A : \mathbb{R}^n \to \mathbb{R}^m$. While, in above case for $\hat{G}$, the situation is more complicated, since the set of the first syzygies of $\mathbb{G}$ could be nontrivial (i.e., other than $\{0\}$) [22, 23].

4.7 Conclusions

In this chapter, the Laplacian of a uniform hypergraph is introduced and investigated. Various basic facts about the spectrum of the Laplacian are explored. These basic facts are related to the structures of the hypergraph. Among them, the sets of the H$^+$-eigenvalues and the nonnegative eigenvectors of the Laplacian are characterized through the spectral components and the flower hearts of the hypergraph. It is shown that all the H$^+$-eigenvalues of the Laplacian can be computed out efficiently. Thus, they are applicable. We also characterized the eigenvectors of the eigenvalues on the spectral circle of the adjacency tensor. It is formulated in the language of linear algebra over modules. That would be our next topic to investigate.

Chapter 5

The Laplace-Beltrami Tensor of an Even Uniform Hypergraph

5.1 Introduction

In this chapter, we introduce Laplace-Beltrami tensors for even uniform hypergraphs. The reason why we restrict the study on even uniform hypergraphs is that positive semidefiniteness is an intrinsic property for the Laplace-Beltrami tensors, while there is no nontrivial odd order tensor which is positive semidefiniteness. For simplicity, the results are presented only for 4-uniform hypergraphs. We note that all the results can be extended to the content of r-uniform hypergraphs with even $r \geq 6$ routinely.

This chapter is summarized as follows. In Section 5.1, we introduce the notion of the Laplace-Beltrami tensor of an even uniform hypergraph. In Section 5.2, we show that this tensor is symmetric, positive semidefinite and has a zero Z-eigenvalue with the normalized vector of all ones as a Z-eigenvector. We introduce the *algebraic connectivity* of an even uniform hypergraph as the second smallest Z-eigenvalue of the Laplace-Beltrami tensor like that for graphs [19, 28], and show that the algebraic connectivity is larger than zero if and only if the hypergraph is connected. We also show that the number of connected components of an even uniform hypergraph is actually the dimension of the set of Z-eigenvectors of the Laplace-Beltrami tensor corresponding to the zero Z-eigenvalue. We characterize the algebraic connectivity of an even uniform hypergraph by a generalized Courant-Fischer theorem [33] for the Laplace-Beltrami tensor. Hence, computing the algebraic connectivity of an even uniform hypergraph is transformed into computing the smallest Z-eigenvalue of another tensor resulted by multilinear transformation [50]. Two other technical lemmas concerned algebraic connectivity are established at the end of Section 5.3, while some applications of them that involve the connections of algebraic connectivity with *edge connectivity* and *vertex connectivity* of an even uniform hypergraph are discussed in Section 5.4. Some conclusions are given in the last section.

5.2 The Laplace-Beltrami Tensor

In this section, we introduce the notion of Laplacian-Beltrami tensors for even uniform hypergraphs. The hypergraph is denoted as $G = (V, E)$.

Let L be the Laplacian matrix of a graph $G = (V, E)$, then for any $\mathbf{x} \in \mathbb{R}^n$ [53]

$$\mathbf{x}^T L\mathbf{x} = \sum_{\{i,j\}\in E} (x_i - x_j)^2. \tag{5.2.1}$$

So, L is positive semidefinite with $\mathbf{e}$ being its eigenvector corresponding to zero eigenvalue. A natural generalization of (5.2.1) to fourth order is as follows: for a 4-uniform hypergraph $G = (V, E)$, its Laplace-Beltrami tensor $\mathcal{T}$ corresponds to the following quartic form:

$$\mathcal{T}\mathbf{x}^4 := \sum_{e_p\in E} \mathcal{C}(e_p)\mathbf{x}^4,\ \forall \mathbf{x} \in \mathbb{R}^4 \tag{5.2.2}$$

with

$$\begin{aligned}\mathcal{C}(e_p)\mathbf{x}^4 \quad = \quad & \frac{1}{84}\left[(x_i + x_j + x_k - 3x_l)^4 + (x_i + x_j + x_l - 3x_k)^4\right.\\ & \left.+(x_i + x_k + x_l - 3x_j)^4 + (x_j + x_k + x_l - 3x_i)^4\right],\end{aligned} \tag{5.2.3}$$

here $\mathcal{C}(e_p)$ is a tensor associated to edge e_p. It is easy to see that $t_{iiii} = d_i$ for all $i \in V$ as those for 2-uniform graphs [53]. This is one of reasons why $\frac{1}{84}$ appears in (5.2.3).

Now, we collect the above idea into the following formal definitions.

Definition 5.2.1 *Given any nonempty subset $S \subseteq V$, we associate it an n dimensional tensor $\mathcal{C}(S)$, called the* core tensor with respect to S, *as:*

$$[\mathcal{C}(S)]_{ijkl} := \begin{cases} 1 & i = j = k = l \in S; \\ -\frac{1}{3} & \{i,j,k,l\} \subseteq S, \text{ three of them equal, but not all}; \\ \frac{5}{21} & \{i,j,k,l\} \subseteq S, \text{ two different pairs of them equal}; \\ \frac{1}{21} & \{i,j,k,l\} \subseteq S, \text{ one pair equal, three of them different}; \\ -\frac{1}{7} & \{i,j,k,l\} \subseteq S, \text{pairwise different}; \\ 0 & \text{otherwise.} \end{cases} \tag{5.2.4}$$

We call $\mathcal{C}(V)$ the core tensor *of hypergraph $G = (V, E)$, denoted by $\mathcal{C}$.*

Definition 5.2.2 *Given a hypergraph $G = (V, E)$, we associate it an n dimensional nonnegative integer tensor $\mathcal{K}$, called the* degree tensor *of G, as k_{ijst} being the cardinality of the set $D := \{e_p \in E \mid \{i,j,s,t\} \subseteq e_p\}$. It is easy to see that $k_{iiii} = d_i$ for all $i \in [n]$.*

Definition 5.2.3 *Given a hypergraph $G = (V, E)$, let $\mathcal{K}$ be the degree tensor of G, and $\mathcal{C}$ be its core tensor. The* Laplace-Beltrami tensor *$\mathcal{T}$ of G is defined as tensor $\mathcal{K} * \mathcal{C}$. Here $*$ represents the Hadamard product of tensors, i.e., the componentwise product.*

It is easy to see that the Laplace-Beltrami tensor $\mathcal{T}$ indeed satisfies (5.2.2).

Definition 5.2.4 *The* symmetric rank *r of a symmetric tensor $\mathcal{T}$ is the minimum nonnegative integer k such that $\mathcal{T}$ has the following representation:*

$$\mathcal{T} = \sum_{j=1}^{k} \alpha_j \mathbf{u}^j \otimes \mathbf{u}^j \otimes \mathbf{u}^j \otimes \mathbf{u}^j,$$

here $\alpha_j \in \mathbb{C}$ and $\mathbf{u}^j \in \mathbb{C}^n$ for all $j \in [k]$.

5.3 Algebraic Connectivity

In this section, we introduce *algebraic connectivity* of a hypergraph and discuss its properties.

Lemma 5.3.1 *For any $e_p = \{i, j, k, l\}$ with $1 \le i, j, k, l \le n$, let $\mathcal{C}(e_p)$ be the core tensor with respect to e_p and $\mathbf{x} \in \mathbb{R}^n$. We have*

$$\mathcal{C}(e_p) = \frac{1}{84}\sum_{s=1}^{4} \mathbf{u}_{e_p}^s \otimes \mathbf{u}_{e_p}^s \otimes \mathbf{u}_{e_p}^s \otimes \mathbf{u}_{e_p}^s \tag{5.3.5}$$

with $\mathbf{u}_{e_p}^1 := \mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_k - 3\mathbf{e}_l$, $\mathbf{u}_{e_p}^2 := \mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_l - 3\mathbf{e}_k$, $\mathbf{u}_{e_p}^3 := \mathbf{e}_i + \mathbf{e}_k + \mathbf{e}_l - 3\mathbf{e}_j$ and $\mathbf{u}_{e_p}^4 := \mathbf{e}_j + \mathbf{e}_k + \mathbf{e}_l - 3\mathbf{e}_i$. So, $\mathcal{C}(e_p)$ is positive semidefinite.

Proof. It is easy to see that (5.3.5) follows from (5.2.3) and Definition 5.2.1 directly. The positive semidefiniteness of $\mathcal{C}(e_p)$ follows directly from (5.2.3). $\Box$

Proposition 5.3.2 *For any hypergraph $G = (V, E)$, its associated Laplace-Beltrami tensor $\mathcal{T}$ is symmetric, positive semidefinite with symmetric rank at most $4|E|$.*

Proof. By Definitions 5.2.1 and 5.2.2, the core tensor $\mathcal{C}$ and the degree tensor $\mathcal{K}$ of a hypergraph are both symmetric, then their Hadamard product $\mathcal{T}$ is symmetric as well. Actually, by Definitions 5.2.1 and 5.2.2,

$$\mathcal{T} = \mathcal{K} * \mathcal{C} = \sum_{e_p \in E} \mathcal{C}(e_p). \tag{5.3.6}$$

For any $\mathbf{x} \in \mathbb{R}^n$

$$\mathcal{T}\mathbf{x}^4 \;=\; \sum_{e_p \in E} \mathcal{C}(e_p)\mathbf{x}^4 = \frac{1}{84}\sum_{e_p \in E}\sum_{s=1}^{4} (\mathbf{u}_{e_p}^s \bullet \mathbf{x})^4 \ge 0$$

with $\mathbf{u}_{e_p}^s$'s are defined in (5.3.5) and $\bullet$ the usual inner product in $\mathbb{R}^n$. Hence, $\mathcal{T}$ is positive semidefinite. The rank estimation follows from (5.3.6) and (5.3.5) directly. $\Box$

The concept of Z-eigenvalues is important in the sequel, which is defined as follows.

Definition 5.3.3 *For a fourth order tensor $\mathcal{T}$, a pair $(\lambda, \mathbf{x})$ is a Z-eigenpair of $\mathcal{T}$ if the follows hold:*

$$\begin{cases} \mathcal{T}\mathbf{x}^3 = \lambda \mathbf{x}, \\ \lambda \in \mathbb{R},\ \mathbf{x} \in \mathbb{R}^n,\ \mathbf{x}^T\mathbf{x} = 1. \end{cases} \tag{5.3.7}$$

λ is called a Z-eigenvalue and $\mathbf{x}$ is the associated Z-eigenvector [62, 63].

From Definition 5.3.3 and the fact that the gradient of $\mathcal{T}\mathbf{x}^4$ with respect to $\mathbf{x}$ is $4\mathcal{T}\mathbf{x}^3$ when $\mathcal{T}$ is symmetric, the following theorem is easy to get. See also the proofs for [62, Theorems 3 and 5].

Theorem 5.3.4 *The Z-eigenvectors of a symmetric tensor $\mathcal{T}$ and the critical points of the following minimization problem have a one to one correspondence:*

$$\begin{aligned} \min \quad & \mathcal{T}\mathbf{x}^4 \\ s.t. \quad & \|\mathbf{x}\|_2 = 1,\ \mathbf{x} \in \mathbb{R}^n. \end{aligned} \tag{5.3.8}$$

Here $\|\cdot\|_2$ represents 2-norm in $\mathbb{R}^n$. Furthermore, if $\mathbf{x}$ is a Z-eigenvector of $\mathcal{T}$, then the corresponding Z-eigenvalue is $\mathcal{T}\mathbf{x}^4$.

Since the minimization problem (5.3.8) is minimizing a continuous function on a compact set, it must have at least one critical point. Hence, there is at least one Z-eigenpair for a symmetric tensor.

Theorem 5.3.5 *For any hypergraph $G = (V, E)$, let $\mathcal{T}$ be its Laplace-Beltrami tensor. Then, $\frac{\mathbf{e}}{\|\mathbf{e}\|_2}$ is a Z-eigenvector of $\mathcal{T}$ with the corresponding Z-eigenvalue zero.*

Proof. For any $\{i, j, k, l\} = e_p \in E$, we have $\mathcal{C}(e_p)\mathbf{e}^4 = \mathcal{C}(e_p)(\mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_k + \mathbf{e}_l)^4 = 0$ by Lemma 5.3.1. So,

$$\mathcal{T}\mathbf{e}^4 = \sum_{e_p \in E} \mathcal{C}(e_p)\mathbf{e}^4 = \sum_{\{i,j,k,l\} = e_p \in E} \mathcal{C}(e_p)(\mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_k + \mathbf{e}_l)^4 = 0.$$

This, together with Proposition 5.3.2, implies that $\frac{\mathbf{e}}{\|\mathbf{e}\|_2}$ is a global minimizer of problem (5.3.8). By Theorem 5.3.4, $\frac{\mathbf{e}}{\|\mathbf{e}\|_2}$ is a Z-eigenvector of $\mathcal{T}$ with Z-eigenvalue zero. $\square$

Lemma 5.3.6 *Let $\{i, j, k, l\} = e_p$. $\mathcal{C}(e_p)\mathbf{x}^4 = 0$ if and only if $x_i = x_j = x_k = x_l$.*

Proof. By Lemma 5.3.1, we have that $\mathcal{C}(e_p)\mathbf{x}^4 = 0$ if and only if

$x_i + x_j + x_k = 3x_l,\ x_i + x_j + x_l = 3x_k,\ x_i + x_k + x_l = 3x_j,$ and $x_k + x_j + x_l = 3x_i.$

It is easy to see that the latter is equivalent to $x_i = x_j = x_k = x_l$. $\square$

Theorem 5.3.7 *Given a hypergraph $G = (V, E)$, let $\mathcal{T}$ be its Laplace-Beltrami tensor. Let*

$$S_0 := \left\{ \mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\} \;\middle|\; \frac{\mathbf{x}}{\|\mathbf{x}\|_2} \textit{ is a Z-eigenvector of } \mathcal{T} \textit{ with Z-eigenvalue } 0 \right\} \cup \{\mathbf{0}\}.$$

Then, S_0 is a linear subspace of $\mathbb{R}^n$, and hypergraph G has exactly Dim(S_0) connected components.

Proof. Suppose that $\{S_1, \ldots, S_q\}$ are the connected components of hypergraph G.

For every $\mathbf{y} := \mathbf{e}_{S_t}$, we have that $\mathcal{T}\mathbf{y}^4 = 0$ by Lemma 5.3.1 and (5.3.6). Hence, by Theorem 5.3.4 and Proposition 5.3.2, $\frac{\mathbf{e}_{S_t}}{\|\mathbf{e}_{S_t}\|_2}$ is a Z-eigenvector of $\mathcal{T}$ with Z-eigenvalue zero. So, $\mathbf{e}_{S_t} \in S_0$ for every $t \in \{1, \ldots, q\}$. Obviously, the set of vectors $\{\mathbf{e}_{S_1}, \ldots, \mathbf{e}_{S_q}\}$ is linearly independent. By Theorem 5.3.4 and Lemma 5.3.6, every nonzero linear combination of $\{\mathbf{e}_{S_1}, \ldots, \mathbf{e}_{S_q}\}$ is in $S_0 \setminus \{\mathbf{0}\}$.

Now, for any $\mathbf{x} \in S_0 \setminus \{\mathbf{0}\}$, by Theorem 5.3.4, we have

$$0 = \mathcal{T}\mathbf{x}^4 = \sum_{e_p \in E} \mathcal{C}(e_p)\mathbf{x}^4.$$

By Lemma 5.3.1, every $\mathcal{C}(e_p)$ is positive semidefinite. Hence, $\mathcal{C}(e_p)\mathbf{x}^4 = 0$ for every $e_p \in E$. Thus, by Lemma 5.3.6, x_i's are a constant for all $i \in S_t$ for every $t \in \{1, \ldots, q\}$. This, together with the fact that $\mathbf{x} \neq \mathbf{0}$, implies that $\mathbf{x} = \alpha_1 \mathbf{e}_{S_1} + \cdots + \alpha_q \mathbf{e}_{S_q}$ for some $\alpha \in \mathbb{R}^q$ satisfying $\sum_{i=1}^q \alpha_i^2 > 0$.

So, S_0 is a linear space of dimension q, i.e., $\text{Dim}(S_0) = q$, which is the exact number of connected components of hypergraph G. $\square$

As in linear algebra [33], $\text{Dim}(S_0)$ is called the *geometrical multiplicity* of the zero Z-eigenvalue of $\mathcal{T}$. By Theorems 5.3.5 and 5.3.7, we get the following result.

Corollary 5.3.8 *Hypergraph $G = (V, E)$ is connected if and only if its geometrical multiplicity of the zero Z-eigenvalue of its Laplace-Beltrami tensor is one.*

By Proposition 5.3.2, the Laplace-Beltrami tensor $\mathcal{T}$ of a hypergraph G is positive semidefinite. By [62, Theorem 5], $\mathcal{T}$ is positive semidefinite if and only if all its Z-eigenvalues are nonnegative. Thus, using these and Theorem 5.3.5, we can order all the Z-eigenvalues of $\mathcal{T}$ with multiplicity as:

$$0 = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_b.$$

It is easy to see that

$$\begin{array}{rcll} \lambda_b & = & \max & \mathcal{T}\mathbf{x}^4 \\ & & \text{s.t.} & \|\mathbf{x}\|_2 = 1 \end{array} \tag{5.3.9}$$

by Theorem 5.3.4 since (5.3.8) and (5.3.9) have the same critical points. By [8, 56], we know that

$$1 \leq b \leq \frac{3^n - 1}{2}.$$

So it is not vacuous to talk about λ_1. As in the literature [19, 28], we introduce the following concept.

Definition 5.3.9 *We call λ_1 the* algebraic connectivity *of hypergraph $G = (V, E)$, denoted as $\alpha(G)$.*

Corollary 5.3.10 *For a hypergraph $G = (V, E)$, $\alpha(G) > 0$ if and only if Dim$(S_0) = 1$.*

Here we give a variational characterization of $\alpha(G)$.

Theorem 5.3.11 *For any hypergraph $G = (V, E)$, we have that*

$$\begin{array}{rcll}\alpha(G) & = & \lambda_1 = & \min \quad \mathcal{T}\mathbf{x}^4 \\ & & & s.t. \quad \|\mathbf{x}\|_2 = 1, \mathbf{e}^T\mathbf{x} = 0.\end{array} \tag{5.3.10}$$

Proof. The result is true when G is disconnected. Since then $\alpha(G) = 0$ by Corollary 5.3.10. Let $X \subset V$ be one of the connected components of G, and $\mathbf{y} := \sum_{i\in X} \mathbf{e}_i$. We have an orthogonal decomposition of $\mathbf{y}$ as $\mathbf{y} = \beta\mathbf{e}+\mathbf{x}$ such that $\mathbf{e}^T\mathbf{x} = 0$. Actually, $\beta = \frac{|X|}{n}$ and $\mathbf{x} = \left(\sum_{i\in X} \frac{n-|X|}{n}\mathbf{e}_i - \sum_{i\notin X} \frac{|X|}{n}\mathbf{e}_i\right)$. Note that $\mathcal{T}\mathbf{x}^4 = \mathcal{T}\mathbf{y}^4 = 0$ by Lemmas 5.3.1 and 5.3.6. This, together with the positive semidefiniteness of $\mathcal{T}$ by Proposition 5.3.2, implies that the optimal value of minimization problem (5.3.10) is actually $\alpha(G) = 0$.

In the following, we assume that G is connected.

We first show that a global minimizer $\mathbf{x}$ of the minimization problem (5.3.10) is indeed a Z-eigenvector of $\mathcal{T}$. By the first order necessary optimality condition [4], a minimizer $\mathbf{x}$ of (5.3.10) satisfies $\|\mathbf{x}\|_2 = 1$ and

$$\mathcal{T}\mathbf{x}^3 = \kappa\mathbf{x} + \nu\mathbf{e}$$

with some $\kappa \in \mathbb{R}$ and $\nu \in \mathbb{R}$. Taking inner products of the both sides with $\mathbf{e}$, we get

$$\begin{aligned} n\nu &= \kappa\mathbf{x} \bullet \mathbf{e} + \nu\mathbf{e} \bullet \mathbf{e} = \mathbf{e} \bullet \mathcal{T}\mathbf{x}^3 = \mathbf{e} \bullet \left[\sum_{e_p\in E} \mathcal{C}(e_p)\mathbf{x}^3\right] \\ &= \mathbf{e} \bullet \left[\frac{1}{84}\sum_{e_p\in E}\sum_{s=1}^{4}(\mathbf{u}^s_{e_p} \bullet \mathbf{x})^3\mathbf{u}^s_{e_p}\right] = \frac{1}{84}\sum_{e_p\in E}\sum_{s=1}^{4}(\mathbf{u}^s_{e_p} \bullet \mathbf{x})^3(\mathbf{u}^s_{e_p} \bullet \mathbf{e}) \\ &= 0. \end{aligned}$$

Here the first equality follows from the fact that $\mathbf{x} \bullet \mathbf{e} = 0$, the fourth from Lemma 5.3.1, and the last from the fact that $\mathbf{u}^s_{e_p} \bullet \mathbf{e} = 0$ by the definition of $\mathbf{u}^s_{e_p}$ in Lemma 5.3.1. Hence, $\nu = 0$, and then $\mathcal{T}\mathbf{x}^3 = \kappa\mathbf{x}$. So, $\mathbf{x}$ is a Z-eigenvector of $\mathcal{T}$ with Z-eigenvalue $\kappa = p^*$. Here we denote by p^* the optimal value of the minimization problem (5.3.10). Furthermore, by the hypothesis of that G is connected, Theorem 5.3.5 and Corollary 5.3.8, we get that $p^* > 0$.

Then, we prove that if $\mathbf{y} \in \mathbb{R}^n$ with $\mathbf{y}^T\mathbf{y} = 1$ is a Z-eigenvector of $\mathcal{T}$ with Z-eigenvalue $\lambda > 0$, then $\lambda \geq p^*$. Hence, by the definition of algebraic connectivity of hypergraph G, $\alpha(G) = \lambda_1 = p^*$.

To this end, suppose that $\mathbf{y} \in \mathbb{R}^n$ with $\mathbf{y}^T\mathbf{y} = 1$ is a Z-eigenvector of $\mathcal{T}$ with Z-eigenvalue $\lambda > 0$. We have an orthogonal decomposition of $\mathbf{y}$ as $\mathbf{y} = \beta\mathbf{e} + \mathbf{x}$ for some $\beta \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$ with $\mathbf{x}^T\mathbf{e} = 0$ and $\mathbf{x} \neq 0$ by Corollary 5.3.8 and the assumption $\lambda > 0$. Moreover, we have

$$\begin{aligned}
\mathcal{T}\mathbf{y}^3 &= \sum_{e_p \in E} \mathcal{C}(e_p)\mathbf{y}^3 = \sum_{e_p \in E} \sum_{s=1}^{4} \left(\mathbf{u}_{e_p}^s \otimes \mathbf{u}_{e_p}^s \otimes \mathbf{u}_{e_p}^s \otimes \mathbf{u}_{e_p}^s\right) \mathbf{y}^3 \\
&= \sum_{e_p \in E} \sum_{s=1}^{4} \left(\mathbf{u}_{e_p}^s \bullet \mathbf{y}\right)^3 \mathbf{u}_{e_p}^s \\
&= \sum_{e_p \in E} \sum_{s=1}^{4} \left(\mathbf{u}_{e_p}^s \bullet \mathbf{x}\right)^3 \mathbf{u}_{e_p}^s,
\end{aligned}$$

and $\mathcal{T}\mathbf{y}^3 = \lambda(\beta\mathbf{e}+\mathbf{x})$. Taking inner products of the both sides with $\mathbf{e}$, we get $0 = \lambda\beta n+0$ since $\mathbf{u}_{e_p}^s \bullet \mathbf{e} = 0$ by the definition of $\mathbf{u}_{e_p}^s$ in Lemma 5.3.1. So, $\beta = 0$ as $\lambda > 0$. Hence, $\mathbf{y} = \mathbf{x}$ and $\mathbf{x}^T\mathbf{e} = 0$. That is to say $\mathbf{y}$ is feasible for the minimization problem (5.3.10). By the fact that $\lambda = \mathcal{T}\mathbf{y}^4$, we conclude that $\lambda \geq p^*$. $\square$

Remark 5.3.12 Here are several remarks.

- Similar results for Theorem 5.3.11 are true for Laplacian matrices, namely the Courant-Fischer theorem [33]. Nevertheless, Theorem 5.3.11 is not true for general tensors, even for general positive semidefinite tensors. One reason why Theorem 5.3.11 is true is that the Z-eigenvalue problem (5.3.7) has the property of orthogonally transformational invariance [62, 63].

- For usual graphs, similar results of Theorem 5.3.11 [19] imply

$$\alpha(G) = \inf_{\mathbf{x} \perp \mathbf{e}, \mathbf{x} \neq 0} \frac{M\mathbf{x}^2}{\|\mathbf{x}\|_2^2} = \inf_{\mathbf{x} \perp \mathbf{e}, \mathbf{x} \neq 0} \frac{\sum_{\{i,j\}=e_p \in E}(x_i - x_j)^2}{\sum_{i=1}^n x_i^2} \tag{5.3.11}$$

with $M = D(G) - A(G)$ as the Laplacian matrix [53] and $M\mathbf{x}^2 := \mathbf{x}^T M\mathbf{x}$, which corresponds to the eigenvalue problem of the Laplace-Beltrami operator in Riemannian manifolds of the following form:

$$\lambda_M := \inf \frac{\int_M |\nabla h|^2}{\int_M |h|^2}, \tag{5.3.12}$$

where h ranges over functions satisfying $\int_M h = 0$. Here the measure on edges $e_p \in E$ and vertices $i \in V$ is 1. In an equivalent form,

$$\lambda_M := \inf \int_M |\nabla h|^2$$

where h ranges over functions satisfying $\int_M h = 0$ and $\int_M |h|^2 = 1$. One of the generalizations to fourth order is:

$$\lambda_{\mathcal{T}} := \inf \int_{\mathcal{T}} |\nabla h|^4$$

where h ranges over functions satisfying $\int_{\mathcal{T}} h = 0$ and $\int_{\mathcal{T}} |h|^2 = 1$. When it is discreted, the resulting problem is actually (5.3.10). This is one of our motivations to define the core tensors in Definition 5.2.1 and the terminology "Laplace-Beltrami tensor".

Lemma 5.3.13 *Let $\mathcal{T}$ be the Laplace-Beltrami tensor of hypergraph $G = (V, E)$ and $\alpha(G)$ be the algebraic connectivity of G. We have*

$$\alpha(G) \le \frac{2n^2}{n^2 - 2} \min_{1 \le i \le n} d_i. \tag{5.3.13}$$

Proof. Denote by the feasible solution set of (5.3.10) as $F := \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\|_2 = 1, \mathbf{e}^T\mathbf{x} = 0\}$. For any $\mathbf{y} \in \mathcal{S}^{n-1} := \{\mathbf{y} \in \mathbb{R}^n \mid \|\mathbf{y}\|_2 = 1\}$, we can get a decomposition of $\mathbf{y}$ as $\mathbf{y} = c_1\mathbf{e} + c_2\mathbf{x}$ for some $c_1, c_2 \in \mathbb{R}$ and $\mathbf{x} \in F$. So,

$$\begin{aligned}
& \left[2\mathcal{T} - \alpha(G)(I \otimes I - \frac{2}{n^2}\mathbf{e} \otimes \mathbf{e} \otimes \mathbf{e} \otimes \mathbf{e})\right] \mathbf{y}^4 \\
= \;& 2\mathcal{T}(c_2\mathbf{x})^4 - \alpha(G)\left[(nc_1^2 + c_2^2\|\mathbf{x}\|_2^2)^2 - 2c_1^4 n^2\right] \\
\ge \;& 2\mathcal{T}(c_2\mathbf{x})^4 - \alpha(G)\left[2(n^2c_1^4 + c_2^4\|\mathbf{x}\|_2^4) - 2c_1^4 n^2\right] \\
= \;& 2\mathcal{T}(c_2\mathbf{x})^4 - 2c_2^4\alpha(G) \\
\ge \;& 0
\end{aligned}$$

for any $y \in \mathcal{S}^{n-1}$. Here the first inequality follows from the facts that $\alpha(G) \geq 0$ and $(a+b)^2 \leq 2(a^2+b^2)$ for $a, b \in \mathbb{R}$, and the second from the fact that $\mathbf{x} \in F$ and Theorem 5.3.11. Hence, tensor $\mathcal{W} := 2\mathcal{T} - \alpha(G)(I \otimes I - \frac{2}{n^2}\mathbf{e}\otimes\mathbf{e}\otimes\mathbf{e}\otimes\mathbf{e})$ is positive semidefinite. Especially, the diagonal elements of the tensor $\mathcal{W}$ are nonnegative. So,

$$\min_{1\leq i\leq n} w_{iiii} = 2\min_{1\leq i\leq n} t_{iiii} - \alpha(G)(1 - \frac{2}{n^2}) \geq 0$$

which, together with the definition of degrees, implies (5.3.13) directly. $\square$

Lemma 5.3.14 [1] *Let $G = (V, E)$ be a hypergraph, and G' be a hypergraph by removing a vertex from G and all the adjacent edges. Then,*

$$\alpha(G') \geq \alpha(G) - \frac{11(|V|-1)^2}{18}. \tag{5.3.14}$$

Proof. Let G_1 be the hypergraph by adding a vertex to G' and all the possible adjacent edges. Then G is a subhypergraph of G_1. Denote by $F(p) := \{\mathbf{x} \in \mathbb{R}^p \mid \|\mathbf{x}\|_2 = 1, \mathbf{e}^T\mathbf{x} = 0\}$ for $p \geq 4$. Let $\mathcal{W}$ be the Laplace-Beltrami tensor of the hypergraph G_1, and $\mathcal{A}$ be the Laplace-Beltrami tensor of the hypergraph G. Let $\bar{E}$ be the set of edges of the hypergraph G_1. Then, $E \subseteq \bar{E}$ by the construction of G_1. Now, by Theorem 5.3.11, $\alpha(G) = \min\{\mathcal{A}\mathbf{x}^4 \mid \mathbf{x} \in F(|V(G)|)\}$, and $\alpha(G_1) = \min\{\mathcal{W}\mathbf{x}^4 \mid \mathbf{x} \in F(|V(G)|)\}$. While,

$$\mathcal{A}\mathbf{x}^4 = \sum_{e_p\in E}\mathcal{C}(e_p)\mathbf{x}^4, \text{ and } \mathcal{W}\mathbf{x}^4 = \sum_{e_p\in\bar{E}}\mathcal{C}(e_p)\mathbf{x}^4 = \sum_{e_p\in E}\mathcal{C}(e_p)\mathbf{x}^4 + \sum_{e_p\in\bar{E}\setminus E}\mathcal{C}(e_p)\mathbf{x}^4.$$

So, $\mathcal{A}\mathbf{x}^4 \leq \mathcal{W}\mathbf{x}^4$ for any $\mathbf{x} \in F(|V(G)|)$ since every $\mathcal{C}(e_p)$ is positive semidefinite by Lemma 5.3.1. Hence,

$$\alpha(G_1) \geq \alpha(G). \tag{5.3.15}$$

Let $\mathcal{T}$ be the Laplace-Beltrami tensor of the hypergraph G'. Then, by Theorem 5.3.11, $\alpha(G') = \mathcal{T}\mathbf{x}^4$ for some $\mathbf{x} \in F(|V(G)|-1)$. Let $l \in V(G)$ be the removed vertex and $\mathbf{y} \in \mathbb{R}^{|V(G)|}$ with $\mathbf{y}_{V(G)-\{l\}} = \mathbf{x}$ and $y_l = 0$, we have

$$\mathcal{W}\mathbf{y}^4 = \mathcal{T}\mathbf{x}^4 + \sum_{e_p\in M}\mathcal{C}(e_p)\mathbf{y}^4,$$

[1] Special thanks are devoted to Professor Chang An and Mr. Xie Jinshan for pointing out an error to us in the journal version.

where $M = \{e_p \mid e_p = \{i,j,k,l\},\ i,j,k \in V(G')\}$. For any $\{i,j,k,l\} = e_p \in M$, we have that

$$\begin{aligned}
\mathcal{C}(e_p)\mathbf{y}^4 &= \frac{1}{84}\big[(x_i+x_j+x_k)^4 \\
&\quad +(x_i+x_j-3x_k)^4+(x_i+x_k-3x_j)^4+(x_j+x_k-3x_i)^4\big] \\
&= \frac{1}{84}\big[84(x_i^4+x_j^4+x_k^4)-112(x_i^3x_j+x_i^3x_k+x_j^3x_i+x_j^3x_k+x_k^3x_i+x_k^3x_j) \\
&\quad +120(x_i^2x_j^2+x_j^2x_k^2+x_k^2x_i^2)+48(x_i^2x_jx_k+x_j^2x_ix_k+x_k^2x_ix_j)\big].
\end{aligned}$$

Denote by $q := |V(G')| = |V(G)|-1 \geq 3$ as assumed in Introduction, we have that $|M| = \binom{q}{3}$, and

$$\begin{aligned}
\sum_{e_p\in M}\mathcal{C}(e_p)\mathbf{y}^4 &= 3\binom{q}{3}\frac{1}{q}\sum_{i=1}^q x_i^4 - \binom{q-2}{1}\frac{112}{84}\sum_{i=1}^q x_i^3(\sum_{j\neq i}x_j) \\
&\quad +\frac{1}{2}\binom{q-2}{1}\frac{120}{84}\sum_{i=1}^q x_i^2(\sum_{j\neq i}x_j^2)+\frac{1}{2}\frac{48}{84}\sum_{i\neq j}x_ix_j\sum_{k\neq i,\,k\neq j}x_k^2 \\
&= 3\binom{q}{3}\frac{1}{q}\sum_{i=1}^q x_i^4 + \binom{q-2}{1}\frac{112}{84}\sum_{i=1}^q x_i^3x_i \\
&\quad +\binom{q-2}{1}\frac{60}{84}\sum_{i=1}^q x_i^2(1-x_i^2)+\frac{24}{84}\sum_{i\neq j}x_ix_j(1-x_i^2-x_j^2) \\
&= 3\binom{q}{3}\frac{1}{q}\sum_{i=1}^q x_i^4 + (q-2)\frac{112}{84}\sum_{i=1}^q x_i^3x_i \\
&\quad -(q-2)\frac{60}{84}\sum_{i=1}^q x_i^4+(q-2)\frac{60}{84}+\frac{24}{84}\sum_{i\neq j}(x_ix_j-x_i^3x_j-x_ix_j^3) \\
&= 3\binom{q}{3}\frac{1}{q}\sum_{i=1}^q x_i^4 + (q-2)\frac{112}{84}\sum_{i=1}^q x_i^3x_i \\
&\quad -(q-2)\frac{60}{84}\sum_{i=1}^q x_i^4+(q-2)\frac{60}{84}-\frac{24}{84}\sum_{i=1}^q x_i^2+2\frac{24}{84}\sum_{i=1}^q x_i^4 \\
&= \left(\frac{3(q-1)(q-2)}{6}+\frac{112(q-2)-60(q-2)+48}{84}\right)\sum_{i=1}^q x_i^4 \\
&\quad +\frac{60(q-2)-24}{84} \\
&\leq \frac{3(q-1)(q-2)}{6}+\frac{112(q-2)-60(q-2)+48+60(q-2)-24}{84} \\
&= \frac{3(q-1)(q-2)}{6}+\frac{112(q-2)+24}{84} = \frac{q^2}{2}-\frac{q}{6}+\frac{29}{21} \leq \frac{11q^2}{18},
\end{aligned}$$

where the second and the fourth equalities follow from the fact that $\|\mathbf{x}\|_2 = 1$ and $\mathbf{x}^T\mathbf{e} = 0$, the first inequality from the fact that $\|\mathbf{x}\|_4 \leq \|\mathbf{x}\|_2$ and $\|\mathbf{x}\|_2 = 1$, and the last inequality from the fact that $q \geq 3$.

So, by the fact that $\|\mathbf{y}\|_2 = \|\mathbf{x}\|_2$, $\sum_{i\in V(G)} y_i = \mathbf{e}^T\mathbf{x} = 0$ and Theorem 5.3.11,

$$\alpha(G_1) \leq \mathcal{W}\mathbf{y}^4 \leq \alpha(G') + \frac{11q^2}{18}. \tag{5.3.16}$$

Hence, (5.3.16), together with (5.3.15), implies (5.3.14). □

The following corollary is a direct consequence of Lemma 5.3.14.

Corollary 5.3.15 *Let $G = (V, E)$ be a hypergraph, and G' be a hypergraph by removing $k \leq n := |V(G)|$ vertices from G and all the adjacent edges. Then,*

$$\alpha(G') \geq \alpha(G) - \frac{11k}{18}(n-1)^2.$$

5.4 Applications

In this section, we discuss some issues of hypergraphs that relate to its algebraic connectivity. Let $G = (V, E)$ be a hypergraph. The edge cut means: given any nonempty proper subset $X \subset V$, the edge cut of X is the set of edges

$$E_X := \{e_p \in E \mid \exists i \in X,\ \exists j \notin X,\ \text{s.t. } \{i, j\} \subset e_p\}.$$

The edge connectivity of G, denoted by $e(G)$, is defined as the minimum cardinality of E_X over all nonempty proper subsets X of V such that the resulting hypergraph is disconnected.

Lemma 5.4.1 *Let $G = (V, E)$ be a hypergraph, $\mathcal{T}$ be its Laplace-Beltrami tensor, $\alpha(G)$ be its algebraic connectivity and λ_b be the largest Z-eigenvalue of $\mathcal{T}$. Then, for all $X \subset V$*

$$\frac{|X|^2(n-|X|)^2}{n^2}\alpha(G) \leq |E_X| \leq \frac{21|X|^2(n-|X|)^2}{16n^2}\lambda_b. \tag{5.4.17}$$

Proof. Let X be a nonempty proper subset of V and E_X its associated edge cut. Let $\mathbf{x} := \sum_{i\in X} \mathbf{e}_i$, we have an orthogonal decomposition of $\mathbf{x}$ as $\mathbf{x} = \beta\mathbf{e} + \mathbf{g}$ such that $\mathbf{e}^T\mathbf{g} = 0$. Actually, $\beta = \frac{|X|}{n}$ and $\mathbf{g} = \left(\sum_{i\in X} \frac{n-|X|}{n}\mathbf{e}_i - \sum_{i\notin X} \frac{|X|}{n}\mathbf{e}_i\right)$. So,

$$\begin{aligned}\mathcal{T}\mathbf{g}^4 = \mathcal{T}\mathbf{x}^4 &= \sum_{\{i,j,k,l\}=e_p\in E_X} \frac{1}{84}\left[(x_i + x_j + x_k - 3x_l)^4 + (x_i + x_j + x_l - 3x_k)^4\right. \\ &\qquad \left.+(x_i + x_k + x_l - 3x_j)^4 + (x_j + x_k + x_l - 3x_i)^4\right].\end{aligned}$$

For every $\{i, j, k, l\} = e_p \in E_X$, there are three situations:

- Three of $\{x_i, x_j, x_k, x_l\}$ are zero and one of them is 1.
- Two of $\{x_i, x_j, x_k, x_l\}$ are zero and two of them are 1.
- One of $\{x_i, x_j, x_k, x_l\}$ is zero and three of them are 1.

So, we have $\frac{16}{21} \leq \mathcal{C}(e_p)\mathbf{x}^4 \leq 1$ by a direct computation for the three cases. Thus, $\frac{16|E_X|}{21} \leq \mathcal{T}\mathbf{g}^4 \leq |E_X|$. Hence, by (5.3.9), Theorem 5.3.11 and the fact that $\|\mathbf{g}\|_2^2 = \frac{|X|(n-|X|)}{n}$, we get that

$$\frac{16|E_X|}{21} \leq \frac{|X|^2(n-|X|)^2}{n^2}\lambda_b, \text{ and } \frac{|X|^2(n-|X|)^2}{n^2}\alpha(G) \leq |E_X|. \tag{5.4.18}$$

Then, (5.4.18) implies (5.4.17) directly. □

Here we give an intuitive example for the lower bound in Lemma 5.4.1.

Example 5.4.2 *Consider hypergraph $G = (V, E)$ with vertices set $V = \{1, 2, 3, 4, 5\}$ and edges set $E = \{\{1, 2, 3, 4\}, \{1, 2, 4, 5\}, \{1, 3, 4, 5\}, \{1, 2, 3, 5\}, \{2, 3, 4, 5\}\}$. Since $|E_X|$ is easy to compute when $|X| = 1$ for any hypergraphs. We consider the more nontrivial cases. The lower bound for $|E_X|$ when $|X| = 2$ provided by Lemma 5.4.1 is $\frac{36}{25}\alpha(G)$. It is easy to see that $|E_X| = 5$ when $|X| = 2$. It is difficulty to solve minimization problem* (5.3.10)*, so we randomly generate* 100000 *points in the feasible set of* (5.3.10) *to get an approximation $\alpha(G) = 2.98$. Then, the lower bound computed is* 4.29*. Since $|E_X|$ is an integer, we see that the computed lower bound is tight.*

The following result is a direct corollary from Lemma 5.4.1.

Theorem 5.4.3 *Let $G = (V, E)$ be a hypergraph, $\mathcal{T}$ be its Laplace-Beltrami tensor, $\alpha(G)$ be the algebraic connectivity of G, λ_b be the largest Z-eigenvalue of $\mathcal{T}$, and $e(G)$ be the edge connectivity of G. Then,*

$$\frac{(n-1)^2}{n^2}\alpha(G) \le e(G) \le \begin{cases} \frac{21n^2}{256}\lambda_b & \text{if } n \text{ is even,} \\ \frac{21(n^2-1)^2}{256n^2}\lambda_b & \text{if } n \text{ is odd.} \end{cases}$$

The vertex connectivity of G, denoted by $v(G)$, is defined as the minimum cardinality of $X \subset V$ such that the resulting hypergraph by removing vertices in X and their associated edges is disconnected.

Theorem 5.4.4 *Let $G = (V, E)$ be a hypergraph, $\alpha(G)$ be the algebraic connectivity of G and $v(G)$ be the vertex connectivity of G. We have*

$$\alpha(G) \le v(G)\frac{11(n-1)^2}{18}.$$

Proof. Let X be a subset of vertices such that X is the vertex cut to disconnect the hypergraph G. Then $|X| = v(G)$, and the resulting hypergraph is disconnected. Hence, its algebraic connectivity is zero by Corollaries 5.3.8 and 5.3.10. Thus, the result follows from Corollary 5.3.15 directly. $\square$

5.5 Conclusions

We introduced in this chapter the Laplace-Beltrami tensor for an even uniform hypergraph, and the algebraic connectivity through the concept of Z-eigenvalues of tensors. We established several properties of the algebraic connectivity for an even uniform hypergraph and its connections with the edge connectivity and the vertex connectivity.

Chapter 6

Conclusions

In this thesis, the notions of the Laplacian of a uniform hypergraph and the Laplace-Beltrami tensor of an even uniform hypergraph, and the notion of the tensor determinant are introduced. The theory of the tensor determinant has applications in the spectral hypergraph theory. Besides this, the theory on nonnegative tensor partition contributes to the spectral hypergraph theory as well. Especially, based on these results, we studied the spectra of uniform hypergraphs through these Laplacican-type tensors and established the basic spectral theory of uniform hypergraphs. All the H^+-eigenvalues of the Laplacian of a uniform hypergraph are characterized by the newly introduced concepts of spectral components and flowers hearts. They are closely related to the hypergraph structures. The eigenvectors of the eigenvalues on the spectral circle of the normalized adjacency tensor are completely characterized. It sheds lights on further research about the symmetry of the spectra of uniform hypergraphs as well as the structures of hypergraphs. The algebraic connectivity of an even uniform hypergraph is characterized variationally, which gives the philosophy to compute it out.

We also applied the theory to edge connectivity, vertex connectivity, edge expansion and spectral invariance of a uniform hypergraph, which indicates the feasibility of the established spectral hypergraph theory.

Bibliography

[1] Balan, V., 2012. Spectra of symmetric tensors and m-root Finsler models. Linear Algebra Appl. 436, 152–162.

[2] Berge, C., 1973. Hypergraphs, Combinatorics of finite sets, third edition. North-Holland, Amsterdam.

[3] Berman, A., Plemmom, R., 1979. Nonnegative Matrices in the Mathematical Sciences, Acadamic Press.

[4] Bertsekas, D.P., 1999. Nonlinear Programming, Second edition. Athena Scientific.

[5] Bézout, E., 1779. Théorie générale des equations algébriques, De l'Imprimerie de Ph.-D. Pierres, Paris.

[6] Brachat, J., Comon, P., Mourrain, B., Tsigaridas, E., 2010. Symmetric tensor decomposition. Linear Algebra Appl. 433, 1851–1872.

[7] Brouwer, A.E., Haemers, W.H., 2011. Spectra of Graphs, Springer. New York.

[8] Cartwright, D., Sturmfels, B., 2013. The number of eigenvalues of a tensor. Linear Algebra Appl. 438, 942–952.

[9] Cayley, A., 1845. On the theory of linear transformations. Cambridge Math. J. 4, 193–209.

[10] Cayley, A., 1843. On the theory of determinants. Trans. Cambridge Phil Soc. VIII, 1–16.

[11] Cvetković, D.M., Doob, M., Gutman, I., Torgasev, A., 1988. Recent Results in the Theory of Graph Spectra, North Holland, Amsterdam.

[12] Cvetković, D.M., Doob, M., Sachs, H., 1980. Spectra of Graphs, Theory and Application, Academic Press.

[13] Chang, K.C., Pearson, K., Zhang, T., 2008. Perron-Frobenius theorem for nonnegative tensors. Commun. Math. Sci. 6, 507–520.

[14] Chang, K.C., Pearson, K., Zhang, T., 2009. On eigenvalue problems of real symmetric tensors. J. Math. Anal. Appl. 350, 416–422.

[15] Chang, K.C., Pearson, K., Zhang, T., 2011. Primitivity, the convergence of the NZQ method, and the largest eigenvalue for nonnegative tensors. SIAM J. Matrix Anal. Appl. 32, 806–819.

[16] Chang, K.C., Pearson, K., Zhang, T., 2013. Some variational principles of the Z-eigenvalues for nonnegative tensors. Linear Algebra Appl. 438, 4166–4182.

[17] Chang, K.C., Qi, L., Zhang, T., 2012. A survey on the spectral theory of nonnegative tensors. Peking University.

[18] Chang, K.C., Zhang, T., 2013. On the uniqueness and non-uniqueness of the positive Z-eigenvector for transition probability tensors. J. Math. Anal. Appl. in press.

[19] Chung, F.R.K., 1997. Spectral Graph Theory, Am. Math. Soc.

[20] Comon, P., Golub, G., Lim, L.-H., Mourrain, B., 2008. Symmetric tensors and symmetric tensor rank. SIAM J. Matrix Anal. Appl. 30, 1254–1279.

[21] Cooper, J., Dutle, A., 2012. Spectra of uniform hypergraphs. Linear Algebra Appl. 436, 3268–3292.

[22] Cox, D., Little, J., O´Shea, D., 1998. Using Algebraic Geometry, Springer-Verlag, New York.

[23] Cox, D., Little, J., O´Shea, D., 2006. Ideals, Varieties, and Algorithms, Springer-Verlag, New York.

[24] de Silva, V., Lim, L.-H., 2008. Tensor rank and the ill-posedness of the best low-rank approximation problem. SIAM J. Matrix Anal. Appl. 30, 1084–1127.

[25] Dixon, A.L., 1906. On a form of the eliminant of two quantics. P. Lond. Math. Soc. 6, 468–478.

[26] Dolotin, V., Morozov, A., 2007. Introduction to Non-Linear Algebra, World Scientific.

[27] Fidalgo, C., Kovacec, A., 2011. Positive semidefinite diagonal minus tail forms are sums of squares. Math. Z. 269, 629–645.

[28] Fiedler, M., 1973. Algebraic connectivity of graphs. Czech. Math. J. 23, 298–305.

[29] Friedland, S., Gaubert, S., Han, L., 2013. Perron-Frobenius theorem for nonnegative multilinear forms and extensions. Linear Algebra Appl. 438, 738–749.

[30] Gaubert, S., Gunawardena, J., 2004. The Perron-Frobenius theorem for homogeneous, monotone function. Trans. Amer. Math. Soc. 356, 4931–4950.

[31] Gelfand, I.M., Kapranov, M.M., Zelevinsky, A.V., 1992. Hyperdeterminants. Adv. Math. 96, 226–263.

[32] Gelfand, I.M., Kapranov, M.M., Zelevinsky, A.V., 1994. Discriminants, Resultants and Multidimensional Determinants, Birkhäuser, Boston.

[33] Horn, R.A., Johnson, C.R., 1985. Matrix Analysis, Cambridge University Press, New York.

[34] Horn, R.A., Johnson, C.R., 1994. Topics in Matrix Analysis, Cambridge University Press, New York.

[35] Hu, S., Huang, Z.H., Ling, C., Qi, L., 2013. On determinants and eigenvalue theory of tensors. J. Symb. Comput. 50, 508–531.

[36] Hu, S., Huang, Z.H., Qi, L., 2013. Strictly nonnegative tensors and nonnegative tensor partition. Sci. China Ser. A. Mathematics. in press.

[37] Hu, S., Huang, Z.H., Ni, H.Y., Qi, L., 2012. Positive definiteness of diffusion kurtosis imaging. Inverse Probl. Imag. 6, 57–75.

[38] Hu, S., Li, G., Qi, L., Song, Y., 2013. Finding the maximum eigenvalue of essentially nonnegative symmetric tensors via sum of squares programming. J. Optim. Theory Appl. in press.

[39] Hu, S., Qi, L., 2013. E-characteristic polynomial of a tensor of dimension two. Appl. Math. Lett. 26, 225–231.

[40] Hu, S., Qi, L., 2011. Convergence of a second order Markov chain. The Hong Kong Polytechnic University.

[41] Hu, S., Qi, L., 2012. Algebraic connectivity of an even uniform hypergraph. J. Comb. Optim. 24, 564–579.

[42] Hu, S., Qi, L., 2013. The Laplacian of a uniform hypergraph. J. Comb. Optim. in press.

[43] Hu, S., Qi, L., 2013. The E-eigenvectors of tensors. Linear Multilinear A. in press.

[44] Landsberg, J.M., 2011. Tensors: Geometry and Applications, Graduate Studies in Mathematics, vol. 128, AMS, Providence.

[45] Lang, S., 1987. Linear Algebra, Revised Third Edition, Springer-Verlag, New York.

[46] Lang, S., 2002. Algebra, Revised Third Edition, Springer-Verlag, New York.

[47] Li, A.-M., Qi, L., Zhang, B., 2013. E-characteristic polynomials of tensors. Commun. Math. Sci. 11, 33–53.

[48] Li, G., Qi, L., Yu, G., 2013. The Z-eigenvalues of a symmetric tensor and its application to spectral hypergraph theory. Numer. Linear Algebra Appl. in press.

[49] Lim, L.-H., 2005. Singular values and eigenvalues of tensors: a variational approach. Proceedings of the IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing, CAMSAP ’05, 1, 129–132.

[50] Lim, L.-H., 2007. Foundations of Numerical Multilinear Algebra: decomposition and approximation of tensors, PhD thesis, Standford University, USA.

[51] Liu, Y., Zhou, G.-L., Ibrahim, N.F., 2010. An always convergent algorithm for the largest eigenvalue of an irreducible nonnegative tensor. J. Comput. Appl. Math. 235, 286–292.

[52] Macaulay, F.S., 1902. On some formulas in elimination. P. Lond. Math. Soc. 3, 3–27.

[53] Merris, R., 1994. Laplacian matrics of graphs: A survey. Linear Algebra and its Applications. 198, 143–176.

[54] Morozov, A., Shakirov, Sh., 2010. New and old results in resultant theory. Theoret. Math. Phys. 163, 587–617.

[55] Morozov, A., Shakirov, Sh., 2011. Analogue of the identity Log Det=Trace Log for resultants. J. Geom. Phys. 61, 708–726.

[56] Ni, G., Qi, L., Wang, F., Wang, Y., 2007. The degree of the E-characteristic polynomial of an even order tensor. J. Math. Anal. Appl. 329, 1218–1229.

[57] Ng, M., Qi, L., Zhou, G., 2009. Finding the largest eigenvalue of a non-negative tensor. SIAM J. Matrix Anal. Appl. 31, 1090–1099.

[58] Nussbaum, R.D., 1988. Hilbert's projective metric and iterated nonlinear maps, Mem. AMS.

[59] Oeding, L., Ottaviani, G., 2013. Eigenvectors of tensors and algorithms for Waring decomposition. J. Symbolic Comput. in press.

[60] Pearson, K.J., 2010. Essentially positive tensors. International Journal of Algebra. 4, 421–427.

[61] Pearson, K.J., Zhang, T., 2012. On spectral hypergraph theory of the adjacency tensor. arXiv:1209.5614.

[62] Qi, L., 2005. Eigenvalues of a real supersymmetric tensor. J. Symb. Comput. 40, 1302–1324.

[63] Qi, L., 2006. Rank and eigenvalues of a supersymmetric tensor, a multivariate homogeneous polynomial and an algebraic surface defined by them. J. Symb. Comput. 41, 1309–1327.

[64] Qi, L., 2007. Eigenvalues and invariants of tensors. J. Math. Anal. Appl. 325, 1363–1377.

[65] Qi, L., 2011. The best rank-one approximation ratio of a tensor space. SIAM J. Matrix Anal. Appl. 32, 430–442.

[66] Qi, L., 2013. H^+-eigenvalues of Laplcian and signless Laplacian tensors. Commun. Math. Sci. in press.

[67] Qi, L., 2013. Symmetric nonnegative tensors and copositive tensors. Linear Algebra Appl. 439, 228–238.

[68] Qi, L., Yu, G., Wu, E.X., 2010. Higher order positive semi-definite diffusion tensor imaging. SIAM J. Imag. Sci. 3, 416–433.

[69] Ragnarsson, S., Van Loan, C.F., 2013. Block tensors and symmetric embeddings. Linear Algebra Appl. 438, 853–874.

[70] Rota Bulò, S., 2009. A game-theoretic framework for similarity-based data clustering, PhD thesis, Università Ca' Foscari di Venezia, Italy.

[71] Rota Bulò, S., Pelillo, M., 2009. A generalization of the Motzkin-Straus theorem to hypergraphs. Optim. Lett. 3, 187–295.

[72] Strang, G., 1993. Introduction to Linear Algebra, Wellesley-Cambridge Press.

[73] Sturmfels, B., 2002. Solving Systems of Polynomial Equations, AMS, Providence.

[74] Sylvester, J.J., 1840. A method of determining by mere inspection the derivatives from two equations of any degree. Philos. Magazine. 16, 132–135.

[75] Xie, J., Chang, A., 2012. On the Z-eigenvalues of the signless Laplacian tensor for an even uniform hypergraph. Manuscript, Fuzhou University.

[76] Xie, J., Chang, A., 2013. On the Z-eigenvalues of the adjacency tensors for uniform hypergraphs. Linear Algebra Appl. in press.

[77] Xie, J., Chang, A., 2013. H-eigenvalues of the signless Laplacian tensor for an even uniform hypergraph. Front. Math. China 8, 107–127.

[78] Yang, Q., Yang, Y., 2011. Further results for Perron-Frobenius Theorem for nonnegative tensors II. SIAM J. Matrix Anal. Appl. 32, 1236–1250.

[79] Yang, Y., Yang, Q., 2010. Further results for Perron-Frobenius Theorem for nonnegative tensors. SIAM J. Matrix Anal. Appl. 31, 2517–2530.

[80] Yang, Y., Yang, Q., 2013. Singular values of nonnegative rectangular tensors. Front. Math. China. 6, 363–378.

[81] Yang, Y., Yang, Q., 2011. Geometric simplicity of spectral radius of nonnegative irreducible tensors. Front. Math. China 8, 129–140.

[82] Zhang, L., Qi, L., 2012. Linear convergence of an algorithm for computing the largest eigenvalue of a nonnegative tensor. Numer. Linear Algebra Appl. 19, 830–841.

[83] Zhang, L., Qi, L., Luo, Z., Xu, Y., 2013. The dominant eigenvalue of an essentially nonnegative tensor. Numer. Linear Algebra Appl. in press.

[84] Zhang, L., Qi, L., Xu, Y., 2012. Linear convergence of the LZI algorithm for weakly positive tensors, J. Comput. Math. 30, 24–33.

[85] Zhang, X., Qi, L., Ye, Y., 2012. The cubic spherical optimization problems. Math. Comput. 81, 1513–1525.

[86] Zhou, G., Caccetta, L., Qi, L., 2013. Convergence of an algorithm for the largest singular value of a nonnegative rectangular tensor. Linear Algebra Appl. 438, 959–968.