The Hong Kong Polytechnic University
Department of Applied Mathematics

Some Nonlinear Spectral Properties of Higher Order Tensors

SONG Yisheng

A thesis submitted in partial fulfilment of
the requirements for the degree of Doctor of Philosophy

April 2013

Certificate of Originality

I hereby declare that this thesis is my own work and that, to the best of my knowledge and belief, it reproduces no material previously published or written, nor material that has been accepted for the award of any other degree or diploma, except where due acknowledgement has been made in the text.

_______________(Signed)

SONG Yisheng (Name of student)

To my family

Abstract

The main purposes of this thesis focus on the nonlinear spectral properties of higher order tensor with the help of the spectral theory and fixed point theory of nonlinear positively homogeneous operator as well as the constrained minimization theory of homogeneous polynomial. The main contributions of this thesis are as follows.

We obtain the Fredholm alternative theorems of the eigenvalue (included E-eigenvalue, H-eigenvalue, Z-eigenvalue) of a higher order tensor $\mathcal{A}$. Some relationship between the Gelfand formula and the spectral radius are discussed for the spectra induced by such several classes of eigenvalues of a higher order tensor. This content is mainly based on the paper 5 in Underlying Papers.

We show that the eigenvalue problem of a nonnegative tensor $\mathcal{A}$ can be viewed as the fixed point problem of the Edelstein Contraction with respect to Hilbert's projective metric. Then by means of the Edelstein Contraction Theorem, we deal with the existence and uniqueness of the positive eigenvalue-eigenvector of such a tensor, and give an iteration sequence for finding positive eigenvalue of such a tensor, i.e., a nonlinear version of the famous Krein-Rutman Theorem. This content is mainly based on the paper 2 in Underlying Papers.

We introduce the concept of eigenvalue to the additively homogeneous mapping pairs (f, g), and establish existence and uniqueness of such a eigenvalue under the boundedness of some orbits of f, g in the Hilbert semi-norm. In particular, the nonlinear

Perron-Frobenius property for nonnegative tensor pairs $(\mathcal{A}, \mathcal{B})$ is given without involving the calculation of the tensor inversion. Moreover, we also present the iteration methods for finding generalized $H-$eigenvalue of nonnegative tensor pairs $(\mathcal{A}, \mathcal{B})$. This content is mainly based on the paper 1 in Underlying Papers.

We introduce the concepts of Pareto H-eigenvalue and Pareto Z-eigenvalue of higher order tensor for studying constrained minimization problem and show the necessary and sufficient conditions of such eigenvalues. We obtain that a symmetric tensor has at least one Pareto H-eigenvalue (Pareto Z-eigenvalue). What is more, the minimum Pareto H-eigenvalue (or Pareto Z-eigenvalue) of a symmetric tensor is exactly equal to the minimum value of constrained minimization problem of homogeneous polynomial deduced by such a tensor, which gives an alternative methods for solving the minimum value of constrained minimization problem. In particular, a symmetric tensor $\mathcal{A}$ is copositive if and only if every Pareto H-eigenvalue ($Z-$eigenvalue) of $\mathcal{A}$ is non-negative. This content is mainly based on the papers 3 and 4 in Underlying Papers.

Underlying papers

This thesis is based on the following papers written by the author during the period of stay at the Department of Applied Mathematics, The Hong Kong Polytechnic University as a graduate student:

1. Yisheng Song, Liqun Qi, The existence and uniqueness of eigenvalue for monotone homogeneous mapping pairs, Nonlinear Analysis Series A: Theory, Methods & Applications, Volume 75, Issue 13, (September 2012) Pages 5283-5293.

2. Yisheng Song, Liqun Qi, Positive eigenvalue-eigenvector of nonlinear positive mappings, to appear in: Frontiers of Mathematics in China.DOI 10.1007/s11464-013-0258-1 in Press

3. Yisheng Song, Liqun Qi, Eigenvalue analysis of constrained minimization problem for homogeneous polynomial, Department of Applied Mathematics, The Hong Kong Polytechnic University, arXiv:1302.6085 [math.OC], 2013 Submitted.

4. Yisheng Song, Liqun Qi, The necessary and sufficient conditions of copositive tensors, Department of Applied Mathematics, The Hong Kong Polytechnic University, arXiv:1302.6084 [math.OC], 2013 Submitted.

5. Yisheng Song, Liqun Qi, Spetral properties of positively homogeneous operators induced by higher order tensors, to appear in: SIAM Journal on Matrix Analysis and Applications in Press

In addition, the following is a list of other papers written by the author during the period of his Ph.D study.

- Shenglong Hu, Guoyin Li, Liqun Qi, Yisheng Song, Finding the Maximum Eigenvalue of Essentially Nonnegative Symmetric Tensors via Sum of Squares Programming, Journal of Optimization Theory and Applications, September 2013, Volume 158, Issue 3, pp 717-738.
- Yisheng Song, Qingnian Zhang, Proximal algorithms for a class of mixed equilibrium problems, Fixed Point Theory and Applications, 2012, 2012:166 (2 October 2012).
- Chang'An Tian, Yisheng Song, Strong convergence of A Regularization Method for Rockafellar's Proximal Point Algorithm, Journal of Global Optimization, Volume 55, Issue 4 (April 2013) Pages 831-837.
- Qingnian Zhang, Yisheng Song, Halpern type proximal point algorithm of accretive operators, Nonlinear Analysis: Theory, Methods & Applications, Volume 75, Issue 4 (March 2012) Pages 1859-1868.

Acknowledgments

This thesis would not have been possible without the support, encouragement, input and ideas of many people. First and foremost, my sincere appreciation goes to my supervisor Professor Li Qun Qi for his valuable advice, motherly patient guidance, and generous support which made this study possible. During the past three years, Professor Qi, as a brilliant mathematician, has never ceased to inspire; as a kind teacher, has never lacked encouraging words; as a dedicated scholar, has never yielded to challenges. He has been an exemplary role model for my work and study in future. It has been a precious experience for me to study for my PhD degree under Professor Qi's guidance in the Department of Applied Mathematics of The Hong Kong Polytechnic University.

I wish to express my sincere appreciation to many other mentors and friends for his friendship, encouragement, guidance, suggestions and help. The following is an incomplete lists: Professor Bingsheng He (Nanjing University), Professor NG Kung Fu (The Chinese University of Hong Kong), Professor Xiaojun Chen (The Hong Kong Polytechnic University), Professor Chen Ling (Hangzhou Dianzi University), Professor Rudong Chen (Tianjin Polytechnic University), Professor Hong-Kun Xu (National Sun Yat-sen University), Professor Zheng-Hai Huang (Tianjin University), Professor Wenyu Sun (Nanjing Normal University), Professor Qin Ni (Nanjing University of Aeronautics and Astronautics), Professor Philippe G. Citarlet (City University of Hong Kong), Professor Anthony Man-Cho So (The Chinese University of Hong Kong), Professor Yiju Wang (Qufu Normal University), Professor Qingzhi Yang (Nankai University), Professor Yinyu Ye (Stanford University), Professor Shisheng Zhang (Sichuan University), Professor Guoyin Li (University of New South Wales), Professor Gaohang Yu (Gannan Normal University), Professor Guofeng Zhang (The Hong Kong Polytechnic University), Professor Liping Zhang (Tsinghua University), and Dr. Shenglong Hu, Dr. Yi Xu, Dr. Xinzhen Zhang, Dr. Zhibao Li, Dr. Yanfang Zhang.

I gratefully acknowledge The Hong Kong Polytechnic University for the financial support during the entire period of my candidature. I also express my gratitude to the supporting staffs in Department of Applied Mathematics and Research Office for their kindly help. I shall always treasure the stimulating research environment and the excellent research facilities of The Hong Kong Polytechnic University.

And finally, my deepest gratitude goes to my parents, my wife, Yuchun Zheng, and my son, Shuo Song, for their sacrifices and encouragement. Without their love and support, this thesis would not be completed on time.

Contents

Lists of Notations

Symbol		**Page**
$\alpha \vee \beta$	$\max\{\alpha, \beta\}$ for two real numbers α, β	51
$\alpha \wedge \beta$	$\min\{\alpha, \beta\}$ for two real numbers α, β	51
$\|\alpha\|$	the absolute value of a number α	1
$\\|\cdot\\|$	the norm of a vector space	11
$\\|x\\|_k$	$(\|x_1\|^k + \|x_2\|^k + \cdots + \|x_n\|^k)^{\frac{1}{k}}$ for $x = (x_1, \cdots, x_n)^T$	$22, 80$
θ	the zero element of a vector space	$1, 9$
Θ	the zero operator	$1, 11$
$\sigma(T)$	the spectrum of an operator T	15
$\mathcal{A}x^m$	$\sum\limits_{i_1, i_2, \cdots, i_m = 1}^{n} a_{i_1 i_2 \cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m}$ for a tensor $\mathcal{A}$	$1, 69$
$\mathcal{A}x^{m-1}$	a vector in $\mathbb{C}^n$ with $(\mathcal{A}x^{m-1})_i = \sum\limits_{i_2, \cdots, i_m = 1}^{n} a_{i i_2 \cdots i_m} x_{i_2} \cdots x_{i_m}$	$3, 69$
$\mathcal{A}^{\mathcal{N}}$	the principal sub-tensor of a tensor $\mathcal{A}$	76
$b(x)$	$x_1 \wedge x_2 \wedge \cdots \wedge x_n$ for a vector $x = (x_1, \cdots, x_n)^T$	51
$\mathcal{CH}(X)$	the set of all compact, continuous and positively homogeneous operators from X to itself	11
$\mathbb{C}$	the set of all complex number	1
e	$(1, 1, \cdots, 1)^T$	55
$\mathcal{E}$	a tensor with all entries being 1	24
$e^{(i)}$	$(e_1^{(i)}, e_2^{(i)}, \cdots, e_n^{(i)})^T$, the ith unit vector in $\mathbb{R}^n$	80
$e_j^{(i)}$	1 if $i = j$, 0 if $i \neq j$	81
$\exp(x)$	$(\exp(x_1), \exp(x_2), \cdots, \exp(x_n))^T$ for $x = (x_1, \cdots, x_n)^T$	$4, 49$
$F_{\mathcal{A}}(x)$	$(\mathcal{A}x^{m-1})^{[\frac{1}{m-1}]}$ for $x = (x_1, \cdots, x_n)^T$, a tensor $\mathcal{A}$	$5, 43$
$f_{\mathcal{A}}(x)$	$\frac{1}{m-1} \log(\mathcal{A}(\exp(x))^{m-1})$ for $x = (x_1, \cdots, x_n)^T$, a tensor $\mathcal{A}$	$3, 49$
$G(\mathcal{A})$	the directed graph of $\mathcal{A}$	64
$\mathcal{I}$	the unit tensor	$4, 46$

$\mathcal{L}(X)$	the set of all bounded linear operators from X to itself	14
$\log(x)$	$(\log(x_1), \log(x_2), \cdots, \log(x_n))^T$ for $x = (x_1, \cdots, x_n)^T$	4, 49
$\mathcal{N}$	a subset of the index set $\{1, 2, \cdots, n\}$	76
$\|\mathcal{N}\|$	the cardinality of a set $\mathcal{N}$	76
$\mathring{P}$	the interior of a cone P	29
P^+	$\{x \in P; x \neq \theta\}$ for a cone P	29
$\mathbb{R}$	the set of all real number	1
$\mathbb{R}^n_+$	$\{x \in \mathbb{R}^n; x \geq \theta\}$	27, 49
$\mathbb{R}^n_-$	$\{x \in \mathbb{R}^n; x \leq \theta\}$	27
$\mathbb{R}^n_{++}$	$\{x \in \mathbb{R}^n; x > \theta\}$	27, 49
$r(T)$	$\lim\limits_{k\to\infty} \|T^k\|^{\frac{1}{k}}$	14
$r_\sigma(T)$	the spectral radius of an operator T	15
$S^\lambda(f, g)$	$\{x \in \mathbb{R}^n;\ f(x) \leq \lambda + g(x)\}$	56
$S_\lambda(f, g)$	$\{x \in \mathbb{R}^n;\ f(x) \geq \lambda + g(x)\}$	56
$S^\lambda_\mu(f, g)$	$\{x \in \mathbb{R}^n;\ \mu + g(x) \leq f(x) \leq \lambda + g(x)\}$	57
$\Lambda(f, g)$	$\{\lambda \in \mathbb{R}; S^\lambda(f, g) \neq \emptyset\}$	57
$t(x)$	$x_1 \vee x_2 \vee \cdots \vee x_n$ for a vector $x = (x_1, \cdots, x_n)^T$	51
$T_{\mathcal{A}}x$	$(x^T x)^{-\frac{m-2}{2}} \mathcal{A}x^{m-1}$ if $x \neq \theta, \theta$ if $x = \theta$	9, 19
x^T	the transposition of a vector x	1
$(x)_i$	x_i for a vector $x = (x_1, x_2, \cdots, x_n)^T$	3
$x^{[m]}$	$(x_1^m, x_2^m, \cdots, x_n^m)^T$	18
X	a Banach space	11
$\underline{\chi}(f, g)$	$\lim\limits_{k\to\infty} b(\frac{f^k(y)-g^k(y)}{k})$	55
$\overline{\chi}(f, g)$	$\lim\limits_{k\to\infty} t(\frac{f^k(y)-g^k(y)}{k})$	55
$\chi(f, g)$	$\lim_{k\to\infty} \frac{f^k(y)-g^k(y)}{k}$	55
$\overline{\chi}(f)$	$\lim\limits_{k\to\infty} t(\frac{f^k(y)}{k})$	55

Chapter 1

Overview

1.1 Introduction

As a natural extension of the concept of matrices, an m-order n-dimensional tensor $\mathcal{A}$ consists of n^m elements in the real field $\mathbb{R}$:

$$\mathcal{A} = (a_{i_1 \cdots i_m}), \qquad a_{i_1 \cdots i_m} \in \mathbb{R}, \quad i_1, i_2, \cdots, i_m = 1, 2, \cdots, n.$$

Throughout this thesis, the superscript 'T' always indicates transposition and $|\alpha|$ always stands for the absolute value of a number α. For an element $x = (x_1, x_2, \cdots, x_n)^T \in \mathbb{R}^n$ or $\mathbb{C}^n$ (here $\mathbb{C}$ is the set of all complex number), $\mathcal{A}x^m$ is defined by

$$\mathcal{A}x^m = \sum_{i_1, i_2, \cdots, i_m = 1}^{n} a_{i_1 i_2 \cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m} \tag{1.1}$$

An $m-$order n-dimensional tensor $\mathcal{A}$ is called *nonnegative* (or respectively *positive*), denoted $\mathcal{A} \geq \Theta$ (or respectively $\mathcal{A} > \Theta$), if $a_{i_1 i_2 \cdots i_m} \geq 0$ (or respectively $a_{i_1 i_2 \cdots i_m} > 0$), where Θ is the zero operator ($\Theta x = \theta$ for all x, here $\theta = (0, 0, \cdots, 0)^T$). An m-order n-dimensional tensor $\mathcal{A}$ is said to be *symmetric* if its entries $a_{i_1 \cdots i_m}$ are invariant for any permutation of the indices.

In 2005, Qi [59] and Lim [40] independently defined eigenvalues and eigenvectors of a real symmetric tensor, and explored their practical application in determining positive definiteness of an even degree multivariate form. Qi [60, 61] defined E-eigenvalues and the E-characteristic polynomial of a tensor, and obtained an E-eigenvalue of a tensor is a root of the E-characteristic polynomial if $\mathcal{A}$ is regular. These works extended the classical concept of eigenvalues of square matrices, forms an important part of numerical multi-linear algebra, and has found applications or links with automatic control, statistical data analysis, optimization, magnetic resonance imaging, solid mechanics, quantum physics, higher order Markov chains, spectral hypergraph theory, Finsler geometry, etc, and attracted attention of mathematicians from different disciplines. For more details, see Chang [10], Chang, Pearson and Zhang [11], Chang, Pearson and Zhang [12], Friedland, Gauber and Han [19], Hu, Huang and Qi [26], Hu and Qi [29], Hu and Qi [27], Hu, Huang, Ling and Qi [28], Li, Qi and Zhang [39], Ni, Qi, Wang and Wang [54], Yang and Yang [69, 70], Zhang [74] and the references cited therein.

Pareto eigenvalue

It is obvious that each m-order n-dimensional symmetric tensor $\mathcal{A}$ defines a homogeneous polynomial $\mathcal{A}x^m$ of degree m with n variables and vice versa. When $m = 2$, the corresponding homogeneous polynomial is a homogeneous polynomial of degree 2 with n variables, i.e. a quadratic form induced by symmetric matrix. Seeger [67] first introduced and used the concept of Pareto eigenvalue for studying the equilibrium processes defined by linear complementarity conditions.

A real number μ is called *Pareto eigenvalue* of the symmetric matrix A if there exists a non-zero element $x \in \mathbb{R}^n$ such that

$$\begin{cases} Ax^2 = \mu x^T x \\ Ax - \mu x \geq \theta \\ x \geq \theta, \end{cases} \tag{1.2}$$

where $x^T y = \sum_{i=1}^{n} x_i y_i$, the usual inner product in the Euclidean space $\mathbb{R}^n$. Seeger [67] gave the necessary and sufficient conditions of Pareto eigenvalue of a symmetric matrix A. Hiriart-Urruty and Seeger [25] showed that a symmetric matrix A is copositive if and only if its Pareto eigenvalues are all nonnegative.

A real symmetric matrix A is said to be *copositive* if $x \geq \theta$ implies $x^T A x \geq 0$. The concept of copositive matrix was introduced by Motzkin [47] in 1952, which is an important concept in applied mathematics and graph theory. Recently, Qi [63] extended this concept to the higher order symmetric tensor and obtained its some nice properties as ones of copositive matrices. Let $\mathcal{A}$ be a real symmetric tensor of order m and dimension n. $\mathcal{A}$ is said to be *copositive* if $\mathcal{A}x^m \geq 0$ for all $x \in \mathbb{R}^n_+$.

In this thesis, we will introduce the notion of Pareto eigenvalue to a higher order and symmetric tensor, and seek some methods to generalize and develop the Pareto eigenvalue properties from symmetric matrices to symmetric tensors. Moreover, we will give the relationship between the Pareto eigenvalue problem of symmetric tensor and the constrained minimization problem of homogeneous polynomial deduced by such a tensor, and will study the copositivity of a symmetric tensor by means of the Pareto eigenvalue of such a tensor.

Additively homogeneous mappings

For $x \in \mathbb{R}^n$ and a nonnegative tensor $\mathcal{A}$ of order m and dimension n, let

$$f_{\mathcal{A}}(x) = \frac{1}{m-1} \log(\mathcal{A}(\exp(x))^{m-1}), \tag{1.3}$$

where $\mathcal{A}x^{m-1}$ is a vector in $\mathbb{C}^n$ with its ith component defined by

$$(\mathcal{A}x^{m-1})_i = \sum_{i_2,\cdots,i_m=1}^{n} a_{ii_2\cdots i_m} x_{i_2} \cdots x_{i_m} \text{ for } i = 1, 2, \ldots, n$$

and $\log(x) = (\log(x_1), \cdots, \log(x_n))^T$ and $\exp(x) = (\exp(x_1), \cdots, \exp(x_n))^T$. Clearly, for $x \in \mathbb{R}^n$ and $\mu \in \mathbb{R}$, $\gamma = \exp(\mu) > 0$ is a positive real number, and so

$$\begin{aligned} f_{\mathcal{A}}(\mu + x) =& \frac{1}{m-1} \log(\mathcal{A}(\exp(\mu + x))^{m-1}) \\ =& \frac{1}{m-1} \log(\mathcal{A}(\exp(\mu)\exp(x))^{m-1}) \\ =& \frac{1}{m-1} \log(\mathcal{A}(\gamma \exp(x))^{m-1}) \\ =& \frac{1}{m-1} \log(\gamma^{m-1}\mathcal{A}(\exp(x))^{m-1}) \\ =& \frac{1}{m-1} \log(\gamma)^{m-1}) + \frac{1}{m-1} \log(\mathcal{A}(\exp(x))^{m-1}) \\ =& \log(\gamma) + f_{\mathcal{A}}(x) = \log(\exp(\mu)) + f_{\mathcal{A}}(x) \\ =& \mu + f_{\mathcal{A}}(x), \end{aligned}$$

where $\mu + x = (\mu + x_1, \mu + x_2, \cdots, \mu + x_n)^T$. The mapping $f_{\mathcal{A}}$ in the above equation is called *additively homogeneous* (Gunawardena and Keane [21]). Such a class of mappings appears classically in a remarkable variety of mathematical disciplines such as the matrices over the max-plus semiring, Markov decision theory, the theory of stochastic games, the optimal control problems, the discrete event systems models (see [2, 4, 22, 23, 42, 43, 68] for different applications). Batap [4] established the max version of the Perron-Frobenius theorem. Recently, Gaubert and Gunawardena [22] proved the nonlinear version of Perron-Frobenius theorem about the additively homogeneous mappings.

Recently, Chang, Pearson Zhang [11] firstly used and studied the notion of eigenvalues of higher order tensors pairs (or tensor pencils). For two m-order n-dimensional real tensors $\mathcal{A}$ and $\mathcal{B}$, a number μ is called a $\mathcal{B}$-*eigenvalue* of $\mathcal{A}$ if both $\mathcal{A}x^{m-1}$ and $\mathcal{B}x^{m-1}$ are not identical to zero and there exists $x \in \mathbb{C}^n \setminus \{\theta\}$ such that

$$\mathcal{A}x^{m-1} = \mu\mathcal{B}x^{m-1},$$

and call x a $\mathcal{B}$-*eigenvector* of $\mathcal{A}$. If $\mathcal{B} = \mathcal{I}$, the unit tensor, then the $\mathcal{B}$-eigenvalue is said to be *eigenvalue*, and the real $\mathcal{B}$-eigenvalue with real eigenvector is called H-*eigenvalue*.

In this thesis, we will introduce the concept of generalized eigenvalue of the additively homogeneous mapping pairs (f,g), and study the properties of such eigenvalues, and then discuss the nonlinear Perron-Frobenius properties of such mappings pairs. Furthermore, we will establish the Perron-Frobenius properties of the generalized $H-$eigenvalue problem for strictly nonnegative tensor pairs $(\mathcal{A},\mathcal{B})$.

Positively homogeneous mappings

For an m-order n-dimensional tensor $\mathcal{A}$, when $(\mathcal{A}x^{m-1})^{[\frac{1}{m-1}]}$ can be well defined, let

$$F_{\mathcal{A}}(x) = (\mathcal{A}x^{m-1})^{[\frac{1}{m-1}]}. \tag{1.4}$$

It is easy to see that the operators $F_{\mathcal{A}}$ is continuous and positively homogeneous, i.e.,

$$F_{\mathcal{A}}(tx) = tF_{\mathcal{A}}(x) \text{ for all } t > 0.$$

Many mathematical workers studied the spectral properties of nonlinear positively homogeneous operators, and gave different definition of the spectrum about such a class of operators. For more details, see Appell, Pascale and Vignoli [1], Deimling [15], Fucik, Necas, Soucek, Soucek [18], Feng and Webb [17], Meghea [49] and Nussbaum [50–52].

For $m = 2$, Birkhoff [6] proved that the famous Perron Theorem can be considered a special case of the Banach Contraction Principle. That is, a nonnegative primitive matrix A is a Banach contraction with respect to Hilbert's projective metric (the detail definition of such a metric see Chapter 3, Page 30). So, A has a unique positive eigenvalue λ with a positive eigenvector x^*. Moreover, for all $x \geq \theta$ and $x \neq \theta$, we have the convergence of the following iterative sequence

$$\lim_{k\to\infty} \frac{A^k x}{\|A^k x\|} = x^* > 0 \text{ and } \lim_{k\to\infty} \frac{\|A^{k+1}x\|}{\|A^k x\|} = \|Ax^*\| = \lambda > 0. \tag{1.5}$$

Still by means of the properties of Hilbert's projective metric, Bushell [7,8] proved that a linear and strictly positive mapping T ($T(\mathring{P}) \subset \mathring{P}$ for a cone P) with finite projective diameter $\triangle(T) = \sup\{d(Tx,Ty); x,y \in P^+\}$ is a Banach contraction. Namely,

$$d(Tx,Ty) \leq \beta d(x,y),$$

where $\beta = \tanh(\frac{\triangle(T)}{4}) = \frac{\exp(\frac{\triangle(T)}{2})-1}{\exp(\frac{\triangle(T)}{2})+1} < 1$. So, by Banach Contraction Principle, the same conclusions still hold. Kohlberg and Pratt [34] showed that the Hilbert's projective metric is the only metric that can turn strictly positive linear mappings into Banach contractions. When a mapping is positively homogeneous of degree p ($T(\lambda x) = \lambda^p Tx,\ \lambda > 0$, $0 < p < 1$), Bushell [8] showed for Hilbert's projective metric d,

$$d(Tx, Ty) \leq pd(x, y).$$

That is, T was a Banach contraction. So, by Banach Contraction Principle, the conclusions (1.5) remain true also. On $\mathbb{R}^n$, Kohlberg [35] successfully turned a continuous, positively homogeneous and strongly increasing ($Tx < Ty$ for $x \leq y$ with $x \neq y$) mapping into an Edelstein contraction, i.e.

$$d(Tx, Ty) < d(x, y) \text{ with } x \neq y. \tag{1.6}$$

By Edelstein Contraction Theorem (Edelstein [16]), the conclusions (1.5) were easily obtained. Krause [36] proved a mapping T with the following condition (i) is an Edelstein contraction with respect to Hilbert's projective metric d,

(i) For any $x, y \in U$ and $0 \leq \lambda \leq 1$: If $\lambda x \leq y$, then $\lambda Tx \leq Ty$ and $\lambda < 1$, then $\lambda Tx < Ty$.

In this thesis, we will explore the spectral properties (for example, the Fredholm alternative type results and the Gelfand formula) and fixed point theorems of such a class of operator, and then employing them, to discuss the corresponding properties of eigenvalue (E-eigenvalues) of higher order tensor.

1.2 Outline of the Thesis

The main objection to this thesis will centre around the nonlinear spectral properties of higher order tensor. Our studied techniques is based on the spectral theory and fixed point theory of nonlinear positively homogeneous operator as well as the constrained minimization problem of homogeneous polynomial.

In chapter 2, we will study the nonlinear Fredholm alternative type results and the Gelfand formula for such a class of positively homogeneous operators induced by higher order tensors. Moreover, we will investigate the Fredholm alternative theorems of the eigenvalue (included E-eigenvalue, H-eigenvalue, Z-eigenvalue) of a higher order tensor $\mathcal{A}$. For the spectra induced by such several classes of eigenvalues of a higher order tensor, some relationship between the Gelfand formula and the corresponding spectral radius will be discussed.

In chapter 3, we will discuss the existence and uniqueness of the positive eigenvalue-eigenvector for such a class of nonlinear mappings in a Banach space by means of the Edelstein Contraction Theorem, and will deal with an iteration sequence for finding positive eigenvalue of such a tensor. As an application, we will consider that the eigenvalue problem of a nonnegative tensor $\mathcal{A}$ can be viewed as the fixed point problem of the Edelstein Contraction with respect to Hilbert's projective metric. Furthermore, we will explore the nonlinear Perron-Frobenius property to a nonnegative tensor $\mathcal{A}$.

In chapter 4, we will introduce the concept of eigenvalue to the increasing and additively homogeneous mapping pairs (f, g), and will discuss existence and uniqueness of such a eigenvalue. Also, the Collatz-Wielandt min-max type property will be studied for such a class of mapping pairs. As an application, we will investigate the nonlinear Perron-Frobenius property for nonnegative tensor pairs $(\mathcal{A}, \mathcal{B})$ without involving the calculation of the tensor inversion and the iteration methods for finding generalized $H-$eigenvalue of nonnegative tensor pairs $(\mathcal{A}, \mathcal{B})$.

In chapter 5, the concepts of Pareto H-eigenvalue and Pareto Z-eigenvalue of higher order tensor will be introduced for studying constrained minimization problem. We will study the existence of Pareto H-eigenvalue (Pareto Z-eigenvalue) of a symmetric tensor and will establish the relationship between the minimum Pareto H-eigenvalue (or Pareto Z-eigenvalue) of a symmetric tensor and the minimum value of constrained minimization problem of homogeneous polynomial deduced by such a tensor. We also investigative the correspondence of the copositivity of a symmetric tensor $\mathcal{A}$ to the Pareto H-eigenvalue ($Z-$eigenvalue) of $\mathcal{A}$.

In chapter 6, we will sum up this thesis and give some suggestions for future studies.

Chapter 2

Fredholm alternative type results and Gelfand formula

2.1 Introduction

For an element $x = (x_1, x_2, \cdots, x_n)^T \in \mathbb{R}^n$ or $\mathbb{C}^n$ and an m-order n-dimensional tensor $\mathcal{A}$, $\mathcal{A}x^{m-1}$ is a vector in $\mathbb{R}^n$ (or $\mathbb{C}^n$) with its ith component defined by

$$(\mathcal{A}x^{m-1})_i = \sum_{i_2,\cdots,i_m=1}^{n} a_{ii_2\cdots i_m} x_{i_2} \cdots x_{i_m} \text{ for } i = 1, 2, \ldots, n.$$

Let

$$F_{\mathcal{A}}(x) = (\mathcal{A}x^{m-1})^{[\frac{1}{m-1}]} \text{ and } T_{\mathcal{A}}x = \begin{cases} (x^T x)^{-\frac{m-2}{2}} \mathcal{A}x^{m-1}, & x \neq \theta \\ \theta, & x = \theta \end{cases} \tag{2.1}$$

when $(\mathcal{A}x^{m-1})^{[\frac{1}{m-1}]}$ is well defined, where $x^{[k]} = (x_1^k, x_2^k, \cdots, x_n^k)^T$ and $\theta = (0, 0, \cdots, 0)^T$, the zero element of a vector space $\mathbb{C}^n$. It is easy to see that both operators $F_{\mathcal{A}}$ and $T_{\mathcal{A}}$ defined by higher order tensor $\mathcal{A}$ are continuous and positively homogeneous. So, our main interests are to study the spectral properties of this class of operators, and employing them, to discuss the properties of higher order tensor.

Many mathematical workers gave different definition of the spectrum for nonlinear operators, for example, the Rhodius spectra and Neuberger spectra for Fréchet differentiable operators, the Kachurovskij spectra and Dorfner spectra for Lipschitz continuous operators and so on. For more details, see Appell, Pascale and Vignoli [1]. The spectra defined by the measure of noncompactness also can be found in Deimling [15], Nussbaum [50–52].

The well-known Fredholm alternative theorem says that for a linear bounded and compact operator T on a Banach space X, $\lambda \in \mathbb{C} \setminus \{0\}$ either is an eigenvalue of T or belongs to its resolvent set $\rho(T)$.

The Fredholm alternative theorem was extended to nonlinear operator. Fucik, Necas, Soucek, Soucek [18] gave a nonlinear Fredholm alternative type theorem of an odd compact operator. Feng and Webb [17] obtained the surjectivity results of Fredholm alternative type for nonlinear operator equations. Recently, Meghea [49] showed a result of Fredholm alternative type of $a-$homogeneous odd compact operator in a normed space.

Motivated by the above results, we will study the Fredholm alternative type results and the spectral radius for a class of positively homogeneous operators in this chapter. In order to reaching this purpose, we first study the completeness of the space $\mathcal{CH}(X)$, the set of all compact, continuous and positively homogeneous operators defined on a Banach space X. On the basis the completeness of $\mathcal{CH}(X)$, the nonlinear Fredholm alternative type results and the Gelfand formula of the spectral radius are obtained respectively. As an application, we will give the Fredholm alternative theorems for the eigenvalue (E-eigenvalue, H-eigenvalue, Z-eigenvalue) of a higher order tensor $\mathcal{A}$. We will show the Gelfand formula of the spectral radius for the spectra induced by such some classes of eigenvalue of a higher order tensor.

2.2 The norm of positively homogeneous operators

Let X be a Banach space over the field $\mathbb{K} = \mathbb{R}$ (the set of all real number) or $\mathbb{K} = \mathbb{C}$ (the set of all complex number) with the norm $\|\cdot\|$, and $T : X \to X$ be an operator. T is called

- *homogeneous* if $T(tx) = tTx$ for each $t \in \mathbb{K}$ and all $x \in X$;
- *positively homogeneous* if $T(tx) = tTx$ for each $t > 0$ and all $x \in X$;
- *compact* if it takes bounded subsets of X into relatively compact subsets of X.

A real (or complex) number λ is said to be an *eigenvalue* of the operator T if there exists a non-zero element $x \in X$ such that $Tx = \lambda x$, and x is called an *eigenvector* corresponding to λ.

By $\mathcal{CH}(X)$ we denote the set of all compact, continuous and positively homogeneous operators $T : X \to X$. $\forall T \in \mathcal{CH}(X)$, define

$$\|T\| = \sup\{\|Tx\| : \|x\| = 1\}. \tag{2.2}$$

As usual, we denote by θ the zero element of a vector space X, and by Θ the zero operator $\Theta x = \theta$ for all $x \in X$. Then $\|\cdot\|$, defined by (2.2) is a norm on the operator space $\mathcal{CH}(X)$ with the algebraic operations defined in the usual way:

$$(\alpha T_1 + \beta T_2)(x) = \alpha T_1 x + \beta T_2 x, \quad \text{for } \alpha, \beta \in \mathbb{K} \text{ and } x \in X.$$

In fact, for $T, F \in \mathcal{CH}(X)$, the following results are easy to be proved.

(1) $\|T\| = \sup_{\|x\| \le 1} \|Tx\| = \sup_{x \ne \theta} \frac{\|Tx\|}{\|x\|}$.

(2) $\|T\| < +\infty$ since T is compact.

(3) $\|T\| = 0 \Leftrightarrow T = \Theta$.

(4) $\|T+F\| \leq \sup_{\|x\|=1} \|Tx\| + \sup_{\|x\|=1} \|Fx\| = \|T\| + \|F\|$.

(5) $\|\lambda T\| = \sup_{\|x\|=1} \|\lambda Tx\| = |\lambda| \sup_{\|x\|=1} \|Tx\| = |\lambda|\|T\|$, for all $\lambda \in \mathbb{K}$.

So the operator space $\mathcal{CH}(X)$ is a norm space. Furthermore, we also have the following results resembling the properties of the bounded linear operators.

Lemma 2.2.1 *Let X be a Banach space and $\|\cdot\|$ be defined by (2.2). Then*
(i) for all $x \in X$ and $T \in \mathcal{CH}(X)$, $\|Tx\| \leq \|T\|\|x\|$.
(ii) $\|TF\| \leq \|T\|\|F\|$ for all $T, F \in \mathcal{CH}(X)$.
(iii) $(\mathcal{CH}(X), \|\cdot\|)$ is a Banach space.
(iv) $\mathcal{CH}(\mathbb{C}^n)$ is a set of all continuous and positively homogeneous operators.

Proof. (i) If $x = \theta$, then $T\theta = \theta$, and so the conclusion is obvious. If $x \neq \theta$, then $\|x\| > 0$. By the definition (2.2) of the operator norm, we have

$$\frac{\|Tx\|}{\|x\|} = \|T(\frac{x}{\|x\|})\| \leq \sup_{\|x\|=1} \|Tx\| = \|T\|.$$

(ii) If $F = \Theta$, then the conclusion is obvious. If $F \neq \Theta$, we may assume $Fx \neq \theta$ for $x \in X$ with $\|x\| = 1$, then by the positively homogeneous property of T, F and the definition (2.2) of the operator norm, we have

$$\|T(Fx)\| = \|T(\frac{Fx}{\|Fx\|}\|Fx\|) = \|Fx\|\|T(\frac{Fx}{\|Fx\|})\| \leq \|T\|\|Fx\| \leq \|T\|\|F\|.$$

So $\|TF\| = \sup_{\|x\|=1} \|TFx\| \leq \|T\|\|F\|$.
(iii) Let $\{T_k\}$ be a Cauchy sequence of $\mathcal{CH}(X)$. Then as $k, i \to \infty$,

$$\|T_k - T_i\| = \sup_{\|x\|=1} \|T_k x - T_i x\| \to 0. \tag{2.3}$$

So, for all $x \in S = \{x \in X; \|x\| = 1\}$, $\|T_k x - T_i x\| \to 0$ (uniformly for x). Thus for $x \in X$ and $x \neq \theta$, we have

$$\|T_k(\frac{x}{\|x\|}) - T_i(\frac{x}{\|x\|})\| = \frac{\|T_k x - T_i x\|}{\|x\|} \to 0,$$

and so for each $x \in X$,

$$\|T_k x - T_i x\| \to 0 \ (k, i \to \infty). \tag{2.4}$$

That is, $\{T_k x\} \subset X$ is a Cauchy sequence of X. Since X is complete, for each $x \in X$, there is $x^* \in X$ such that $T_k x \to x^*$. Let $Tx = x^* = \lim_{k\to\infty} T_k x$. Then T is an operator from X to itself. Since $\|T_k x - Tx\| = \lim_{i\to\infty} \|T_k x - T_i x\|$, we have

$$\frac{\|T_k x - Tx\|}{\|x\|} = \frac{\|(T_k - T)x\|}{\|x\|} = \lim_{i\to\infty} \frac{\|(T_k - T_i)x\|}{\|x\|} \le \lim_{i\to\infty} \|T_k - T_i\|$$

which means

$$\|T_k - T\| = \sup_{x\neq\theta} \frac{\|(T_k - T)x\|}{\|x\|} \le \lim_{i\to\infty} \|T_k - T_i\|.$$

From (2.3), we obtain that $\lim_{k\to\infty} \|T_k - T\| = 0$.

Now we show that $T \in \mathcal{CH}(X)$. Obviously, T is positively homogeneous. Next we show T is continuous. Indeed, take a sequence $\{x_k\} \subset X$ such that $x_k \to x \in X$. Using (i), we have

$$\begin{aligned}
\|Tx_k - Tx\| &\le \|Tx_k - T_i x_k\| + \|T_i x_k - T_j x_k\| + \|T_j x_k - T_j x\| + \|T_j x - Tx\| \\
&\le \|T - T_i\|\|x_k\| + \|T_i - T_j\|\|x_k\| + \|T_j x_k - T_j x\| + \|T_j - T\|\|x\|.
\end{aligned}$$

Since T_j is continuous for each j, $\lim_{k\to\infty} \|T_j x_k - T_j x\| = 0$. So we have

$$\limsup_{k\to\infty} \|Tx_k - Tx\| \le \|T - T_i\|\|x\| + \|T_i - T_j\|\|x\| + \|T_j - T\|\|x\|.$$

Let $i, j \to \infty$. Then we have $\lim_{k\to\infty} \|Tx_k - Tx\| = 0$, which implies T is continuous.

Finally, we show T is compact. Let M be a bounded subset of X. Then we only need prove that $T(M)$ is a relatively compact subsets of X, i.e. $\forall \varepsilon > 0$, there exists a finite $\varepsilon-$net $K_\varepsilon \subset X$ for $T(M)$. In fact, for all $\varepsilon > 0$, since $\lim_{k\to\infty} \|T_k - T\| = 0$, there is an $l \in \mathbb{N}$ such that

$$\|T - T_l\| < \frac{\varepsilon}{3c},$$

where c is a positive real number with $\sup\{\|x\|; x \in M\} \le c$. Using the fact that $T_l(M)$ is relatively compact, choose a finite subset $M_\varepsilon = \{x_1, x_2, \ldots, x_N\}$ such that

$$\forall y \in T_l(M), \exists\ x_i \in M_\varepsilon \text{ with } \|y - T_l x_i\| < \frac{\varepsilon}{3}.$$

That is, $T_l(M_\varepsilon)$ is a finite $\frac{\varepsilon}{3}-$net for $T_l(M)$. For any $z \in T(M)$, there is $x_0 \in M$ such that $z = Tx_0$. Then

$$T_l x_0 \in T_l(M), \exists\, x_j \in M_\varepsilon \text{ with } \|T_l x_0 - T_l x_j\| < \frac{\varepsilon}{3}.$$

Let $K_\varepsilon = T(M_\varepsilon) = \{Tx_1, Tx_2, \ldots, Tx_N\}$. Without loss of generality, we may assume $\sup\{\|x\|; x \in M_\varepsilon\} \leq c$. Then we have

$$\begin{aligned}\|z - Tx_j\| &\leq \|Tx_0 - T_l x_0\| + \|T_l x_0 - T_l x_j\| + \|T_l x_j - Tx_j\| \\ &\leq \|T - T_l\|\|x_0\| + \frac{\varepsilon}{3} + \|T_l - T\|\|x_j\| \\ &\leq 2c\|T - T_l\|\|| + \frac{\varepsilon}{3} < \varepsilon,\end{aligned}$$

which shows that for $\varepsilon > 0$, K_ε is a finite $\varepsilon-$net for $T(M)$. Since ε is arbitrary, $T(M)$ is totally bounded, and hence relatively compact.

(iv) It follows from (i) that $T(K)$ is a bounded subset for any bounded subset K of $\mathbb{C}^n$, and hence $T(K)$ is relatively compact. So T is compact. ∎

2.3 Some auxiliary results of positively homogeneous operators

Let X be a Banach space. When $T \in \mathcal{L}(X)$ ($\mathcal{L}(X)$ is the set of all bounded linear operators), the spectral radius of T may be calculated in case of a complex Banach space by the Gelfand formula

$$r_\sigma(T) = \lim_{k\to\infty} \|T^k\|^{\frac{1}{k}} = \inf_{k\in\mathbb{N}} \|T^k\|^{\frac{1}{k}}. \tag{2.5}$$

Then for $T \in \mathcal{CH}(X)$, whether or not its Gelfand formula holds also. In this section, we will study some properties of the spectral of T when $T \in \mathcal{CH}(X)$.

Lemma 2.3.1 *Let X be a Banach space and $T \in \mathcal{CH}(X)$. Then there exists $r(T) \in \mathbb{R}$ such that $\lim_{k\to\infty} \|T^k\|^{\frac{1}{k}} = r(T)$ and $r(T) \leq \|T\|$. More generally, $r(T) \leq \|T^k\|^{\frac{1}{k}}$ for all positive integer k.*

Proof. It follows from Lemma 2.2.1 (ii) that

$$\|T^k\| \le \|T^{k-1}\|\|T\| \le \cdots \le \|T\|^k, \text{ and so } \|T^k\|^{\frac{1}{k}} \le \|T\|.$$

Then the sequence $\{\|T^k\|^{\frac{1}{k}}\}$ is bounded. So both $\limsup_{k\to\infty} \|T^k\|^{\frac{1}{k}}$ and $\liminf_{k\to\infty} \|T^k\|^{\frac{1}{k}}$ exist. If there exists a sufficiently large k such that $\|T^k\| = 0$, then the conclusions obviously hold.

If for arbitrary positive integer k, $\|T^k\| > 0$, then we only need prove that

$$\limsup_{k\to\infty} \|T^k\|^{\frac{1}{k}} \le \liminf_{k\to\infty} \|T^k\|^{\frac{1}{k}}.$$

Fixed a positive integer k. Let $\mu = \max\{\|T^i\|;\ i = 1, 2, \ldots, k-1\} > 0$. Then for any positive integer l, we have

$$l = i_l k + r_l,\ 0 \le r_l < k.$$

So $\lim_{l\to\infty} \frac{i_l k}{l} = 1$ since $1 = \frac{i_l k}{l} + \frac{r_l}{l}$ and $0 \le \frac{r_l}{l} < \frac{k}{l}$. By Lemma 2.2.1 (ii), we have

$$\|T^l\| = \|T^{i_l k} T^{r_l}\| = \|(T^k)^{i_l} T^{r_l}\| \le \|(T^k)^{i_l}\|\|T^{r_l}\| \le \|T^k\|^{i_l}\|T^{r_l}\|.$$

Thus,

$$\|T^l\|^{\frac{1}{l}} \le (\|T^k\|^{\frac{1}{k}})^{\frac{i_l k}{l}} \mu^{\frac{1}{l}}.$$

Taking the superior limit on the two sides as $l \to \infty$,

$$\limsup_{l\to\infty} \|T^l\|^{\frac{1}{l}} \le \|T^k\|^{\frac{1}{k}}$$

since $\lim_{l\to\infty} \frac{i_l k}{l} = 1$ and $\lim_{l\to\infty} \mu^{\frac{1}{l}} = 1$ $(\mu > 0)$. Taking the inferior limit on the two sides as $k \to \infty$, we have $\limsup_{k\to\infty} \|T^k\|^{\frac{1}{k}} \le \liminf_{k\to\infty} \|T^k\|^{\frac{1}{k}}$. The desired conclusion follows. ∎

Let $T \in \mathcal{CH}(\mathbb{C}^n)$. Following the definition of spectrum of the matrices, we define the *spectrum* of T, say $\sigma(T)$, to be the set of all eigenvalues of T. Assume that $\sigma(T) \ne \emptyset$, then we call the number

$$r_\sigma(T) = \sup\{|\lambda|; \lambda \in \sigma(T)\} \tag{2.6}$$

the *spectral radius* of T, where $|\lambda|$ stands for the absolute value of a number λ. Let $\mathcal{CH}_0(\mathbb{C}^n)$ be a set of all continuous and homogeneous operators. Clearly, $\mathcal{CH}_0(\mathbb{C}^n)$ is a closed subspace of $\mathcal{CH}(\mathbb{C}^n)$.

Theorem 2.3.2 *Let $T \in \mathcal{CH}(\mathbb{C}^n)$ and $r(T) = \lim_{k\to\infty} \|T^k\|^{\frac{1}{k}}$. Then*

(i) $r_\sigma(T) \le \|T\|$;

(ii) $\sigma(T)$ is a compact subset in $\mathbb{C}$;

(iii) λ is not an eigenvalue of T if $\lambda > r(T)$;

(iv) $r_\sigma(T) \le r(T)$ for all $T \in \mathcal{CH}_0(\mathbb{C}^n)$.

Proof. (i) Let $\lambda \in \sigma(T)$. It follows from the definition of the eigenvalue that there exists a non-zero vector $x \in \mathbb{C}^n$ such that $Tx = \lambda x$, and so,

$$\|Tx\| = \|\lambda x\| = |\lambda|\|x\|.$$

Therefore, we have

$$|\lambda| = \frac{\|Tx\|}{\|x\|} = \left\| T\left(\frac{x}{\|x\|}\right)\right\| \le \|T\|.$$

Since λ is arbitrary in $\sigma(T)$, then $r_\sigma(T) \le \|T\|$.

(ii) It follows from (i) that $\sigma(T)$ is bounded. Now we show that $\sigma(T)$ is closed. Choose a sequence $\{\lambda_k\} \subset \sigma(T)$ and $\lambda \in \mathbb{C}$ such that $\lim_{k\to\infty} \lambda_k = \lambda$. We only need to show that $\lambda \in \sigma(T)$. In fact, for each k, there exists $x_k \ne \theta$ such that $Tx_k = \lambda_k x_k$. Since $\|x_k\| > 0$ for all k and $T \in \mathcal{CH}(\mathbb{C}^n)$, then we have

$$\lambda_k\left(\frac{x_k}{\|x_k\|}\right) = \frac{Tx_k}{\|x_k\|} = T\left(\frac{x_k}{\|x_k\|}\right) \text{ for all } k.$$

Let $y_k = \frac{x_k}{\|x_k\|}$. Then the sequence $\{y_k\}$ is bounded and $\|y_k\| = 1$, and hence there is a subsequence $\{y_{k_i}\}$ of $\{y_k\}$ and some $y \in \mathbb{C}^n$ such that $\lim_{i\to\infty} y_{k_i} = y$. By the continuity, we obtain that

$$\lim_{i\to\infty} Ty_{k_i} = Ty \text{ and } \lim_{i\to\infty} \lambda_{k_i} y_{k_i} = \lambda y,$$

and so, $Ty = \lambda y$ and $\|y\| = 1$, i.e., $\lambda \in \sigma(T)$.

(iii) Suppose λ is an eigenvalue of T. It follows from the definition of the eigenvalue that $\exists x \ne \theta$ such that $Tx = \lambda x$. Then by $\lambda > r(T) \ge 0$ and the positive homogeneity of T, we have

$$T^k x = T^{k-1}(Tx) = \lambda T^{k-1}x = \cdots = \lambda^k x,$$

and hence

$$\|T^k x\| = \lambda^k \|x\|, \text{ i.e. } \lambda^k = \frac{\|T^k x\|}{\|x\|} = \|T^k(\frac{x}{\|x\|})\|.$$

Thus

$$\lambda^k = \left\| T^k \left(\frac{x}{\|x\|} \right) \right\| \le \sup_{\|x\|=1} \|T^k x\| = \|T^k\|,$$

and so $\lambda \le \|T^k\|^{\frac{1}{k}}$. Let $k \to \infty$, we have $\lambda \le r(T)$, a contradiction.

(iv) Take $\lambda \in \sigma(T)$. Then $\exists x \neq \theta$ such that $Tx = \lambda x$. It follows from the homogeneity of T that

$$T^k x = T^{k-1}(Tx) = \lambda T^{k-1} x = \cdots = \lambda^k x,$$

and hence $|\lambda^k| \|x\| = \|T^k x\| \le \|T^k\| \|x\|$. Thus $|\lambda|^k \le \|T^k\|$, i.e., $|\lambda| \le \|T^k\|^{\frac{1}{k}}$. So we have that $|\lambda| \le \lim_{k\to\infty} \|T^k\|^{\frac{1}{k}}$, and so $r_\sigma(T) \le r(T) = \lim_{k\to\infty} \|T^k\|^{\frac{1}{k}}$. ■

Following Theorem 2.3.2 (ii), for $T \in \mathcal{CH}(\mathbb{C}^n)$, we can modify the definition of the spectral radius $r_\sigma(T)$ to be

$$r_\sigma(T) = \max\{|\lambda|; \lambda \in \sigma(T)\}.$$

Theorem 2.3.3 *Let X be a Banach space and $T \in \mathcal{CH}(X)$. If $\lambda \in \mathbb{K}$ with $\lambda \neq 0$ is not an eigenvalue of T, then there exists $\alpha > 0$ such that*

$$\|\lambda x - Tx\| \ge \alpha \|x\| \text{ for all } x \in X. \tag{2.7}$$

Proof. $x = \theta$, then the conclusion is obvious. Suppose for any $\varepsilon > 0$, there is $x_\varepsilon \neq \theta$ such that $\|\lambda x_\varepsilon - Tx_\varepsilon\| < \varepsilon \|x_\varepsilon\|$. Then

$$\inf_{\|x\|=1} \|\lambda x - Tx\| \le \|\lambda(\frac{x_\varepsilon}{\|x_\varepsilon\|}) - T(\frac{x_\varepsilon}{\|x_\varepsilon\|})\| = \frac{\|\lambda x_\varepsilon - Tx_\varepsilon\|}{\|x_\varepsilon\|} < \varepsilon.$$

Since ε is arbitrary,

$$\inf_{\|x\|=1} \|\lambda x - Tx\| = 0.$$

Then there exists x_k with $\|x_k\| = 1$ such that

$$\lim_{k\to\infty} \|\lambda x_k - Tx_k\| = 0.$$

Since T is compact, there is x_{k_i} such that $Tx_{k_i} \to y \in X$ as $i \to \infty$, and hence

$$\|\lambda x_{k_i} - y\| \le \|\lambda x_{k_i} - Tx_{k_i}\| + \|Tx_{k_i} - y\| \to 0.$$

Then $x_{k_i} \to \frac{y}{\lambda} = x$ as $i \to \infty$, and so $Tx_{k_i} \to Tx = y (i \to \infty)$. Therefore, $Tx = y = \lambda x$. Since $\|x_k\| = 1$ for all k, $\|\frac{y}{\lambda}\| = \|x\| = 1$. So λ is an eigenvalue of T, a contradiction. ■

Obviously, if the inquality (2.7) holds, then λ is not an eigenvalue of T. So it is easy to obtain the nonlinear Fredholm Alternative type results.

Corollary 2.3.4 *Let X be a Banach space and $T \in \mathcal{CH}(X)$. If $\lambda \in \mathbb{K}$ with $\lambda \neq 0$, then either λ is an eigenvalue of T or there exists $\alpha > 0$ satisfying (2.7).*

For an m-order n-dimensional tensor $\mathcal{A}$, $\lambda \in \mathbb{C}$ is called an *eigenvalue of* $\mathcal{A}$, if there exists a vector $x \in \mathbb{C}^n \setminus \{\theta\}$ such that

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]}, \tag{2.8}$$

where $x^{[m-1]} = (x_1^{m-1}, \cdots, x_n^{m-1})^T$, and call x an *eigenvector* of $\mathcal{A}$ associated with the eigenvalue λ. We call such an eigenvalue *H-eigenvalue* if it is real and has a real eigenvector x, and call such a real eigenvector x an *H-eigenvector*. These concepts were first introduced by Qi [59] for the higher order symmetric tensors. Lim [40] independently introduced this notion but restricted x to be a real vector and λ to be a real number. Qi [59, 60] extended some nice properties of matrices to the higher order symmetric tensors. The Perron-Frobenius theorem for nonnegative matrix had been generalized to the higher order nonnegative tensors with various conditions by Chang, Pearson and Zhang [11, 14] and others.

For an m-order n-dimensional tensor $\mathcal{A}$, when $(\mathcal{A}x^{m-1})^{[\frac{1}{m-1}]}$ is well defined, let

$$F_{\mathcal{A}}(x) = (\mathcal{A}x^{m-1})^{[\frac{1}{m-1}]}.$$

Then $F_{\mathcal{A}} \in \mathcal{CH}(\mathbb{C}^n)$ and λ^{m-1} is an eigenvalue of $\mathcal{A}$ if and only if λ is an eigenvalue of $F_{\mathcal{A}}$.

For an m-order n-dimensional tensor $\mathcal{A}$, a number $\mu \in \mathbb{C}$ is called *E-eigenvalue of* $\mathcal{A}$, if there exists a vector $x \in \mathbb{C}^n \setminus \{\theta\}$ such that

$$\mathcal{A}x^{m-1} = \mu x (x^T x)^{\frac{m-2}{2}}, \tag{2.9}$$

and call a vector x an *E-eigenvector* of $\mathcal{A}$ associated with the E-eigenvalue μ. If x is real, then μ is also real. In this case, μ and x are called *Z-eigenvalue* of $\mathcal{A}$ and *Z-eigenvector* of $\mathcal{A}$ associated with μ, respectively. These concepts were first introduced by Qi [59, 60] for a higher order tensor. Qi [61] defined the E-characteristic polynomial of a tensor, and showed that if $\mathcal{A}$ is regular, then a complex number is an E-eigenvalue if and only if it is a root of the E-characteristic polynomial. Let

$$T_{\mathcal{A}}x = \begin{cases} \|x\|_2 \mathcal{A}\left(\frac{x}{\|x\|_2}\right)^{m-1}, & x \neq \theta \\ \theta, & x = \theta, \end{cases} \tag{2.10}$$

where $\|x\|_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$. Clearly, $T_{\mathcal{A}} \in \mathcal{CH}(\mathbb{C}^n)$ and μ is an $E-$eigenvalue of $\mathcal{A}$ if and only if μ is an eigenvalue of $T_{\mathcal{A}}$.

For $x \in \mathbb{C}^n$ and $p \geq 1$, it is known well that

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \text{ and } \|x\|_\infty = \max_{1 \geq i \geq n} |x_i|$$

are the norms defined on $\mathbb{C}^n$. Then for $T \in \mathcal{CH}(\mathbb{C}^n)$, it is obvious that

$$\|T\|_p = \max_{\|x\|_p = 1} \|Tx\|_p \text{ and } \|T\|_\infty = \max_{\|x\|_\infty = 1} \|Tx\|_\infty$$

are the norms defined on $\mathcal{CH}(\mathbb{C}^n)$. Now we give the norm of the positively homogeneous operators $F_{\mathcal{A}}$ and $T_{\mathcal{A}}$.

Theorem 2.3.5 *Let $\mathcal{A}$ be an m-order n-dimensional tensor. Then*

(i) $\|F_{\mathcal{A}}\|_\infty \leq \max_{1 \leq i \leq n} \left(\sum_{i_2, \cdots, i_m = 1}^{n} |a_{ii_2 \cdots i_m}|\right)^{\frac{1}{m-1}}$;

(ii) $\|T_{\mathcal{A}}\|_\infty \leq \max_{1 \leq i \leq n} \left(\sum_{i_2, \cdots, i_m = 1}^{n} |a_{ii_2 \cdots i_m}|\right)$.

Proof. (i) It follows from the definition of the norm that

$$\begin{aligned}
\|F_{\mathcal{A}}\|_\infty &= \max_{\|x\|_\infty=1} \|F_{\mathcal{A}}x\|_\infty \\
&= \max_{\|x\|_\infty=1} \max_{1\le i\le n} \left| \left(\sum_{i_2,\cdots,i_m=1}^{n} a_{ii_2\cdots i_m} x_{i_2} x_{i_3} \cdots x_{i_m} \right)^{\frac{1}{m-1}} \right| \\
&\le \max_{\|x\|_\infty=1} \max_{1\le i\le n} \left(\sum_{i_2,\cdots,i_m=1}^{n} |a_{ii_2\cdots i_m}||x_{i_2}||x_{i_3}| \cdots |x_{i_m}| \right)^{\frac{1}{m-1}} \\
&\le \max_{\|x\|_\infty=1} \max_{1\le i\le n} \left(\sum_{i_2,\cdots,i_m=1}^{n} |a_{ii_2\cdots i_m}| \|x\|_\infty^{m-1} \right)^{\frac{1}{m-1}} \\
&= \max_{\|x\|_\infty=1} \max_{1\le i\le n} \left(\sum_{i_2,\cdots,i_m=1}^{n} |a_{ii_2\cdots i_m}| \right)^{\frac{1}{m-1}} \|x\|_\infty \\
&= \max_{1\le i\le n} \left(\sum_{i_2,\cdots,i_m=1}^{n} |a_{ii_2\cdots i_m}| \right)^{\frac{1}{m-1}}.
\end{aligned}$$

(ii) It follows from the definition of the norm that $\|x\|_2 \ge \|x\|_\infty$ and

$$\begin{aligned}
\|T_{\mathcal{A}}\|_\infty &= \max_{\|x\|_\infty=1} \|T_{\mathcal{A}}x\|_\infty \\
&= \max_{\|x\|_\infty=1} \max_{1\le i\le n} \left| \|x\|_2^{-(m-2)} \sum_{i_2,\cdots,i_m=1}^{n} a_{ii_2\cdots i_m} x_{i_2} x_{i_3} \cdots x_{i_m} \right| \\
&\le \max_{\|x\|_\infty=1} \|x\|_\infty^{-(m-2)} \max_{1\le i\le n} \left(\sum_{i_2,\cdots,i_m=1}^{n} |a_{ii_2\cdots i_m}||x_{i_2}||x_{i_3}| \cdots |x_{i_m}| \right) \\
&\le \max_{\|x\|_\infty=1} \|x\|_\infty^{-(m-2)} \|x\|_\infty^{m-1} \max_{1\le i\le n} \left(\sum_{i_2,\cdots,i_m=1}^{n} |a_{ii_2\cdots i_m}| \right) \\
&= \max_{1\le i\le n} \left(\sum_{i_2,\cdots,i_m=1}^{n} |a_{ii_2\cdots i_m}| \right).
\end{aligned}$$

This completes the proof. ■

Let $x^{(0)} = (1, 1, \cdots, 1)^T$. Then $\|x^{(0)}\|_\infty = 1$, and so a simple calculation of $\|F_{\mathcal{A}}(x^{(0)})\|_\infty$ (or $\|T_{\mathcal{A}}(x^{(0)})\|_\infty$) yields to the following conclusions.

Theorem 2.3.6 *Let $\mathcal{A}$ be an m-order n-dimensional nonnegative tensor. Then* *(i)* $\|F_{\mathcal{A}}\|_\infty = \max_{1\le i\le n}\left(\sum_{i_2,\cdots,i_m=1}^{n} a_{ii_2\cdots i_m}\right)^{\frac{1}{m-1}}$;

(ii) $\max_{1\le i\le n}\left(\sum_{i_2,\cdots,i_m=1}^{n} a_{ii_2\cdots i_m}\right) \ge \|T_{\mathcal{A}}\|_\infty \ge n^{-\frac{m-2}{2}} \max_{1\le i\le n}\left(\sum_{i_2,\cdots,i_m=1}^{n} a_{ii_2\cdots i_m}\right)$.

Combining Theorems 2.3.2 (i) and 2.3.5, the following conclusions are obtained.

Corollary 2.3.7 *Let $\mathcal{A}$ be an m-order n-dimensional tensor. Then*
(i) $|\lambda| \le (r_\sigma(F_{\mathcal{A}}))^{m-1} \le \max_{1\le i\le n}\left(\sum_{i_2,\cdots,i_m=1}^{n} |a_{ii_2\cdots i_m}|\right)$ *for all eigenvalues λ of $\mathcal{A}$;*
(ii) $|\mu| \le r_\sigma(T_{\mathcal{A}}) \le \max_{1\le i\le n}\left(\sum_{i_2,\cdots,i_m=1}^{n} |a_{ii_2\cdots i_m}|\right)$ *for all E-eigenvalues μ of $\mathcal{A}$.*

2.4 Fredholm alternative type results and Gelfand formula of higher order tensors

Theorem 2.4.1 *Let $\mathcal{A}$ be an m-order n-dimensional tensor. If $\lambda \in \mathbb{C}\setminus\{0\}$, then either λ^{m-1} is an eigenvalue (H-eigenvalue) of $\mathcal{A}$ or*

$$\inf_{\|x\|=1} \|\lambda x - F_{\mathcal{A}}x\| > 0. \tag{2.11}$$

Proof. It is obvious that $F_{\mathcal{A}} \in \mathcal{CH}(\mathbb{C}^n)$. Then it follows from the Corollary 2.3.4 that either λ is an eigenvalue (H-eigenvalue) of $F_{\mathcal{A}}$ or there exists $\alpha > 0$ satisfying

$$\|\lambda x - F_{\mathcal{A}}x\| \ge \alpha\|x\| \text{ for all } x \in \mathbb{C}^n. \tag{2.12}$$

If λ is an eigenvalue of $F_{\mathcal{A}}$, then a simple conversion between $F_{\mathcal{A}}$ and $\mathcal{A}$, it is easy to show that λ^{m-1} is an eigenvalue of $\mathcal{A}$. Suppose λ satisfies (2.15). Then for all $x \in \mathbb{C}^n \setminus \{\theta\}$,

$$\|\lambda(\frac{x}{\|x\|}) - F_{\mathcal{A}}(\frac{x}{\|x\|})\| = \frac{\|\lambda x - F_{\mathcal{A}}x\|}{\|x\|} \ge \alpha > 0,$$

and hence the desired conclusion follows. ■

Theorem 2.4.2 *Let $\mathcal{A}$ be an m-order n-dimensional tensor. If $\mu \in \mathbb{C}\setminus\{0\}$, then either μ is an E-eigenvalue of $\mathcal{A}$ or*

$$\inf_{\|x\|_2=1} \|\mu x - \mathcal{A}x^{m-1}\|_2 > 0. \tag{2.13}$$

Proof. Let $T_{\mathcal{A}}$ be defined by (2.10). Then $T_{\mathcal{A}} \in \mathcal{CH}(\mathbb{C}^n)$. From the Corollary 2.3.4, it follows that either μ is an eigenvalue of $T_{\mathcal{A}}$ or there exists $\alpha > 0$ satisfying

$$\|\mu x - T_{\mathcal{A}}x\|_2 \geq \alpha\|x\|_2 \text{ for all } x \in \mathbb{C}^n. \tag{2.14}$$

If μ is an eigenvalue of $T_{\mathcal{A}}$, then μ is an E-eigenvalue of $\mathcal{A}$. Suppose μ satisfies (2.14). Then for all $x \in \mathbb{C}^n \setminus \{\theta\}$, we have

$$\|\mu(\frac{x}{\|x\|_2}) - T_{\mathcal{A}}(\frac{x}{\|x\|_2})\|_2 = \|\mu(\frac{x}{\|x\|_2}) - \mathcal{A}(\frac{x}{\|x\|_2})^{m-1}\|_2 \geq \alpha > 0,$$

and hence the desired conclusion follows. ■

If $\mathcal{A}$ is an m-order n-dimensional tensor and m is even, then the operators $F_{\mathcal{A}}$, $T_{\mathcal{A}} \in \mathcal{CH}_0(\mathbb{C}^n)$. Following Theorems 2.3.2 (iv), the following conclusions are obvious.

Theorem 2.4.3 *Let $\mathcal{A}$ be an m-order n-dimensional tensor where m is even. Then*
(i) $|\lambda|^{\frac{1}{m-1}} \leq r_\sigma(F_{\mathcal{A}}) \leq \lim_{k\to\infty} \|F_{\mathcal{A}}^k\|^{\frac{1}{k}}$ for all eigenvalues (H-eigenvalues) λ of $\mathcal{A}$;
(ii) $|\mu| \leq r_\sigma(T_{\mathcal{A}}) \leq \lim_{k\to\infty} \|T_{\mathcal{A}}^k\|^{\frac{1}{k}}$ for all E-eigenvalues (Z-eigenvalues) μ of $\mathcal{A}$.

Recalled that a tensor $\mathcal{A}$ is called *reducible* if there exists a nonempty proper index subset $\mathcal{N} \subset \{1, 2, \cdots, n\}$ such that

$$a_{i_1 i_2 \cdots i_m} = 0 \text{ for all } i_1 \in \mathcal{N}, \text{ for all } i_2, i_3, \cdots, i_m \notin \mathcal{N}.$$

If $\mathcal{A}$ is not reducible, then we call it *irreducible*. The notion of irreducible tensor is first introduced by Lim [40]. Chang, Pearson, Zhang [11] adopted this notion in their subsequent work. Let $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n; x_i \geq 0 \text{ for all } i\}$ and $\mathbb{R}^n_{++} = \{x \in \mathbb{R}^n; x_i > 0 \text{ for all } i\}$. If $x \in \mathbb{R}^n_+$ ($x \in \mathbb{R}^n_{++}$), then it is said that a vector x is nonnegative (positive).

A tensor $\mathcal{A}$ is called *weakly symmetric* if the associated homogeneous polynomial $\mathcal{A}x^m$ satisfies

$$\nabla \mathcal{A}x^m = m\mathcal{A}x^{m-1}.$$

This concept was first introduced and used by Chang, Pearson, and Zhang [14] for studying the properities of $Z-$eigenvalue of nonnegative tensors. In the sequel, Θ denotes zero tensor, and the symbol $\mathcal{A} \leq \mathcal{B}$ ($\mathcal{A} < \mathcal{B}$) means that $a_{i_1i_2\cdots i_m} \leq b_{i_1i_2\cdots i_m}$ ($a_{i_1i_2\cdots i_m} < b_{i_1i_2\cdots i_m}$) for all $i_1, i_2, \cdots, i_m$, and the operation $\mathcal{A}+\mathcal{B}$ ($k\mathcal{A}$) implies that its elements are defined by $a_{i_1i_2\cdots i_m} + b_{i_1i_2\cdots i_m}$ ($ka_{i_1i_2\cdots i_m}$) for all $i_1, i_2, \cdots, i_m$.

Lemma 2.4.4 *Suppose that both m-order n-dimensional tensors $\mathcal{B}$ and $\mathcal{A}$ are weakly symmetric and nonnegative.*
(i) If $\mathcal{B}$ is irreducible, then $r_\sigma(T_\mathcal{B})$ is a positive $Z-$eigenvalue with a nonnegative $Z-$eigenvector;
(ii) If $\Theta \leq \mathcal{A} < \mathcal{B}$, then $r_\sigma(T_\mathcal{A}) \leq r_\sigma(T_\mathcal{B})$.

Proof. (i) It follows from Theorems 2.5 and 2.6 of Chang, Pearson and Zhang [14] that there exists Z-eigenvalue $\lambda_0 > 0$ with a positive Z-eigenvector x such that $\|x\|_2 = 1$, and hence, $T_\mathcal{B}x = \mathcal{B}x^{m-1} = \lambda_0 x$. Let

$\lambda^* = \max\{\lambda'; \lambda'$ is nonnegative Z-eigenvalue of $\mathcal{B}$ with a nonnegative $Z-$eigenvector$\}$.

Then $\lambda^* > 0$ and from Theorem 4.7 of Chang, Pearson and Zhang [14], it follows that

$$\lambda^* = \max_{\|x\|_2=1, x\in\mathbb{R}^n_+} \min_{x_i>0} \frac{(\mathcal{B}x^{m-1})_i}{x_i}.$$

We claim that

$$r_\sigma(T_\mathcal{B}) = \lambda^*, \text{ i.e., } \lambda^* \geq |\lambda| \text{ for all } \lambda \in \sigma(T_\mathcal{B}).$$

Indeed, let $\lambda \in \sigma(T_\mathcal{B})$ and y be a E-eigenvector corresponding to λ satisfying $\|y\|_2 = 1$. Then

$$|\lambda||y| = |\lambda y| = |T_\mathcal{B}y| = \left|\mathcal{B}y^{m-1}\right| \leq \mathcal{B}|y|^{m-1},$$

where $|y| = (|y_1|, |y_2|, \cdots, |y_n|)^T$ for $y \in \mathbb{C}^n$. This implies that

$$|\lambda||y_i| = \left|\left(\mathcal{B}y^{m-1}\right)_i\right| \leq \left(\mathcal{B}|y|^{m-1}\right)_i, \ \forall i,$$

and hence

$$|\lambda| \leq \min_{|y_i|>0} \frac{(\mathcal{B}|y|^{m-1})_i}{|y_i|} \leq \max_{\|x\|_2=1, x\in\mathbb{R}^n_+} \min_{x_i>0} \frac{(\mathcal{B}x^{m-1})_i}{x_i} = \lambda^*. \tag{2.15}$$

(ii) Let $\mu \in \sigma(T_\mathcal{A})$ and z be a E-eigenvector corresponding to μ satisfying $\|z\|_2 = 1$. Then $|\mu||z| = |\mathcal{A}z^{m-1}| \leq \mathcal{A}|z|^{m-1} \leq \mathcal{B}|z|^{m-1}$. Since $\mathcal{B}$ is irreducible, $r_\sigma(T_\mathcal{B})$ is the largest $Z-$eigenvalue of $\mathcal{B}$ and

$$r_\sigma(T_\mathcal{B}) = \max_{\|x\|_2=1, x\in\mathbb{R}^n_+} \min_{x_i>0} \frac{(\mathcal{B}x^{m-1})_i}{x_i}$$

by (i) and Theorem 4.7 of Chang, Pearson and Zhang [14]. By the same argumentation of (2.15), we obtain that $|\mu| \leq r_\sigma(T_\mathcal{B})$, and so $r_\sigma(T_\mathcal{A}) \leq r_\sigma(T_\mathcal{B})$. ■

Lemma 2.4.5 *Let an m-order n-dimensional tensor $\mathcal{A}$ be weakly symmetric and nonnegative. Then*
(i) $(r_\sigma(F_\mathcal{A}))^{m-1}$ is an $H-$eigenvalue of $\mathcal{A}$ with a nonnegative $H-$eigenvector;
(ii) $r_\sigma(T_\mathcal{A})$ is an $Z-$eigenvalue of $\mathcal{A}$ with a nonnegative $Z-$eigenvector.

Proof. (i) It follows from Theorem 2.3 of Yang and Yang [69] that $(r_\sigma(F_\mathcal{A}))^{m-1}$ is an $H-$eigenvalue of $\mathcal{A}$ with a nonnegative $H-$eigenvector.
(ii) Let $\mathcal{A}_k = \mathcal{A} + \frac{1}{k}\mathcal{E}$, where $\mathcal{E}$ is a tensor with all entries being 1. Then $\Theta \leq \mathcal{A} < \mathcal{A}_{k+1} < \mathcal{A}_k$, and hence, $r_\sigma(T_\mathcal{A}) \leq r_\sigma(T_{\mathcal{A}_{k+1}}) \leq r_\sigma(T_{\mathcal{A}_k})$ for all positive integer k by Lemma 2.4.4 (ii). Thus, there exists a real number μ_0 such that

$$\lim_{k\to\infty} r_\sigma(T_{\mathcal{A}_k}) = \mu_0 \geq r_\sigma(T_\mathcal{A}).$$

The remainders of the proof are the same as ones of Theorem 2.3 of Yang and Yang [69] other than a few small changes, we omit it. ■

Theorem 2.4.6 *Let $\mathcal{A}$ be an m-order n-dimensional weakly symmetric and nonnegative tensor. Then*
(i) $|\lambda|^{\frac{1}{m-1}} \leq r_\sigma(F_\mathcal{A}) \leq \lim\limits_{k\to\infty} \|F_\mathcal{A}^k\|^{\frac{1}{k}}$ for all eigenvalues (H-eigenvalues) λ of $\mathcal{A}$;
(ii) $|\mu| \leq r_\sigma(T_\mathcal{A}) \leq \lim\limits_{k\to\infty} \|T_\mathcal{A}^k\|^{\frac{1}{k}}$ for all E-eigenvalues (Z-eigenvalues) μ of $\mathcal{A}$.

Chapter 3

Eigenvalue problem and fixed point theory

3.1 Introduction

The well-known Banach Contraction Principle says that if T is a *strict contraction* from a complete metric space (X, d) to itself, i.e.,

$$\text{for some } \beta \text{ with } 0 \leq \beta < 1, d(Tx, Ty) \leq \beta d(x, y) \text{ for all } x, y \in X, \tag{3.1}$$

then T has unique fixed point $x^* \in X$ $(x^* = Tx^*)$ and $\lim\limits_{k \to \infty} T^k x = x^*$ for all $x \in X$.

Edelstein [16] relaxed the strict contraction condition (3.1) by permitting $\beta = 1$ and obtained the following result which is referred to as the *Edelstein Contraction Theorem.*

Theorem 3.1.1 (The Edelstein Contraction Theorem) *Let (X, d) be a metric space and $T : X \to X$ be a contraction (which is called* Edelstein Contraction *), that is*

$$d(Tx, Ty) < d(x, y) \text{ for all } x, y \in X \text{ with } x \neq y. \tag{3.2}$$

If T also satisfies the condition

$$\exists x, x^* \in X \text{ such that } \exists \{T^{k_i}x\} \subset \{T^k x\} \text{ with } \lim_{i\to\infty} d(T^{k_i}x, x^*) = 0, \tag{3.3}$$

then T has a unique fixed point $x^* \in X$ and $\lim_{k\to\infty} d(T^k x, x^*) = 0$.

Clearly, the condition (3.3) is replaced by the fact that metric space X is compact or the range $R(T) = T(X)$ is relatively compact, then the conclusions still hold.

For solving the eigenvalue-eigenvector problem of linear mapping, many proof methods have been used. Among them, the proof technique applying the Banach Contraction Principle is very straightforward and simple, which has been widely employed by many mathematicians. Birkhoff [6] showed that the famous Perron Theorem can be viewed as a special case of the Banach Contraction Principle. That is, if the range $R(T)$ of a bounded positive linear transformation T has finite diameter $\triangle(T)$ with respect to Hilbert's projective metric d, then T can be viewed as a strict contraction with contraction coefficient $\beta = \tanh(\frac{\triangle(T)}{4}) < 1$. So by the Banach Contraction Principle, the Perron-Frobenius theorem can be easily obtained. Furthermore, for a nonnegative primitive square matrix A, there exists $x^* > 0$ such that

$$\frac{A^k x}{\|A^k x\|} \to x^* \ (k \to \infty) \text{ for all } x \geq 0 \text{ and } x \neq 0. \tag{3.4}$$

Still by means of the properties of Hilbert's projective metric d, Bushell [7, 8] proved that a positive linear mapping T can be properly turned into a strict contraction with contraction coefficient $\beta \leq \tanh(\frac{\triangle(T)}{4})$. It is natural to try to define another metric, a little simpler than Hilbert's projective metric, which would also turn positive linear mappings into strict contractions. Kohlberg and Pratt [34] showed that the above experiment was essentially impossible. Also see Nussbaum [51, 52] for the results of contraction ratio of linear mappings.

The above results dealt with linear mappings. Spontaneously, we have the following question:

Problem 1 *Does there exist a class of nonlinear positive mappings such that they can be turned into (strict) contraction?*

When a mapping is positively homogeneous of degree p ($T(\lambda x) = \lambda^p Tx,\ \lambda > 0$), Bushell [8] showed for Hilbert's projective metric d, $d(Tx, Ty) \leq pd(x, y)$. Under the condition $0 < p < 1$, Bushell [8,9] proved that a monotone increasing mapping T which is positively homogeneous of degree p is a Banach strict contraction with contraction coefficient $\beta = p$ with respect to Hilbert's projective metric, and hence there exists unique x^* such that $x^* = Tx^*$. Potter [57] proved that a continuous, monotone increasing and $\gamma-$concave mapping T $(0 < \gamma < 1)$ is a Banach strict contraction with contraction coefficient $\beta = \gamma$ with respect to Hilbert's projective metric, where a $\gamma-$concave mapping $T : K \to K$ says that $T(\lambda x) \geq \lambda^\gamma Tx$ for all $x \in K$ and $0 < \lambda \leq 1$. Nussbaum [51] developed Potter's work to study the eigenvalue-eigenvector problem of the sum mapping of $1-$concave mapping and $\gamma-$concave mapping. Recently, with the help of Banach Contraction Principle, Huang, Huang and Tsai [31] extend Bushell's results to $\gamma-$concave mappings. Applying the techniques of Nussbaum [51], Huang, Huang and Tsai [32] continued their early work to extend Bushell's results to the eigenvalue-eigenvector problem of the sum mapping of $1-$concave mapping and $\gamma-$concave mapping.

Clearly, with the aid of the nice properties of Hilbert's projective metric, both positively homogeneous mapping of degree p with $0 < p < 1$ and $\gamma-$concave mapping with $0 < \gamma < 1$ boil down to the Banach strict contraction. On $m-$dimensional Euclidean space $\mathbb{R}^m$, a class of positively homogeneous mappings of degree 1 (for short, positively homogeneous mappings), a class of more general mappings, has also beeen viewed as a special case of the Edelstein contraction with respect to Hilbert's projective metric under adding some conditions by Kohlberg [35] and Krause [36].

Let $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n ; x \geq \theta\}$, $\mathbb{R}^n_- = -\mathbb{R}^n_+$ and $\mathbb{R}^n_{++} = \{x \in \mathbb{R}^n ; x > \theta\}$, where $\theta = (0, 0, \cdots, 0)^T$. On $\mathbb{R}^n_+$, without the additivity ($T(x + y) = Tx + Ty$), Kohlberg [35] successfully turned a continuous, positively homogeneous and primitive mapping into an Edelstein contraction, defined by (3.2) with respect to Hilbert's projective metric d, and so obtained the similar results as the Perron-Frobenius theorem. Also in $\mathbb{R}^n_+$, instead of the additivity, homogeneity and primitivity of linear mapping, Krause [36] used the following conditions:

(i) There exist numbers $a > 0, b > 0$ and a vector $v > \theta$ such that $av \le Tx \le bv$ for all $x \in U = \{x \in \mathbb{R}^m_+; p(x) = 1\}$, where $p : \mathbb{R}^m_+ \to \mathbb{R}_+$ is a continuous mapping which is not identically 0 and positively homogeneous and increasing;

(ii) For any $x, y \in U$ and $0 \le \lambda \le 1$: If $\lambda x \le y$, then $\lambda Tx \le Ty$ and if $\lambda < 1$, then $\lambda Tx < Ty$.

Such a nonlinear mapping T successfully becomes an Edelstein contraction with respect to Hilbert's projective metric d by Krause [36]. The condition (i) guarantees the compactness of a subset with respect to Hilbert's projective metric, and hence by the Edelstein Contraction Theorem, the conclusion (3.4) is reached obviously. Moreover, the limit x^* is a positive eigenvector with positive eigenvalue as required.

In this chapter, we will turn a class of nonlinear mappings defined on a Banach space into the Edelstein contractions with respect to Hilbert's projective metric, and furthermore, applying the Edelstein Contraction Theorem to achieve our purposes. It is our final goal to study the existence and uniqueness of the positive eigenvalue-eigenvector for a (eventually) strongly increasing, positively homogeneous, (eventually) strongly positive and compact mapping in a Banach space, and to give an iteration sequence for finding a positive eigenvector with positive eigenvalue. That is, a nonlinear version of the famous Krein-Rutman Theorem is presented.

The Krein-Rutman Theorem deals with the existence of the positive eigenvalue-eigenvector for a linear, positive and compact mapping (Deimling [15, Theorem 19.2]). Ogiwara [55, 56] studied the nonlinear Perron-Frobenius problem for order-preserving mappings on a positive cone of an ordered Banach space. Recently, the nonlinear

Krein-Rutman Theorem has been studied by Chang in [10] and by Mahadeva in [41], respectively. Mahadeva [41] considered a strongly positive and strongly increasing, positively homogeneous, compact continuous mapping T on a Banach space together with the condition:

$$\exists u \in P, u \neq \theta \text{ such that } MTu \geq u \text{ for some constant } M > 0. \tag{3.5}$$

Chang [10] extended Mahadeva's result by removing the condition (3.5) and provided a unified proof for linear and nonlinear, also for finite dimension and infinite dimension.

Our proof techniques are different from those of Chang [10] and Mahadeva [41]. We turn a (eventually) strongly increasing and positively homogeneous mapping into an Edelstein contraction with respect to Hilbert's projective metric, and then verify such a mapping satisfies the condition (3.3) under adding some proper conditions. So by the Edelstein Contraction Theorem, the desired aims are reached.

As an application, we will show that the eigenvalue problem of a nonnegative tensor $\mathcal{A}$ can be viewed as the fixed point problem of the Edelstein Contraction with respect to Hilbert's projective metric. As a result, the nonlinear Perron-Frobenius property of a nonnegative tensor $\mathcal{A}$ is reached easily.

3.2 Preliminaries and basic results

Let P be a closed cone of a real Banach space X and θ be the zero element of X. A *partial ordering* $\leq$ with respect to P is defined in X by saying $x, y \in X$,

$$x \leq y (x < y) \text{ if and on if } \; y - x \in P (y - x \in \mathring{P}),$$

where $\mathring{P}$ is the interior of P. Let $P^+ = \{x \in P; x \neq \theta\}$. If $x, y \in P^+$, we define

$$M(x,y) = \begin{cases} \inf\{\mu; x \leq \mu y\}, \\ \infty, \{\mu; x \leq \mu y\} = \emptyset \end{cases} \quad \text{and } m(x,y) = \sup\{\lambda; x \geq \lambda y\}. \tag{3.6}$$

The following basic properties of M, m are proved by Bushell in [8]; see also Nussbaum [51, 52] for more details.

Lemma 3.2.1 (Bushell [8, Lemma 2.1, Corollary, Theorem 2.3]) *For* $x, y \in P^+$,

(i) $0 \le m(x, y) \le M(x, y) \le \infty$;

(ii) $m(x, y) = \frac{1}{M(x,y)}$;

(iii) $m(x, y)y \le x \le M(x, y)y$ *whenever* $M(x, y) < \infty$;

(iv) $0 < m(x, y) \le M(x, y) < +\infty$ *whenever* $x, y \in \mathring{P}$.

In P^+, Hilbert's projective metric $d(\cdot, \cdot)$ is defined by

$$d(x, y) = \log \left(\frac{M(x, y)}{m(x, y)} \right).$$

This concept was introduced by David Hilbert in 1895 in a paper [24] on the foundations of geometry and has been widely used in proving the existence and uniqueness of fixed point for a nonlinear positive mapping and others. Furthermore, the existence and uniqueness of the positive eigenvalue can be obtained easily for such a mapping. The following basic properties of d are proved by Bushell in [8, 9] see also Nussbaum [51, 52] for more details.

Lemma 3.2.2 (Bushell [8, Theorem 2.1, Lemma 2.2]) (1) *For all* $x, y \in \mathring{P}$,

$$d(\lambda x, \mu y) = d(x, y) \text{ for all } \lambda > 0, \mu > 0.$$

(2) $(\mathring{P}, d)$ *is a pseudo-metric space, i.e. the metric* d *satisfies the conditions:* $\forall x, y, z \in \mathring{P}$

(i) $0 \le d(x, y) < +\infty, \ d(x, x) = 0$;

(ii) $d(x, y) = d(y, x)$;

(iii) $d(x,y) \le d(x,z) + d(z,y)$;

(iv) $d(x,y) = 0$ *if and only if* $x = \lambda y$*, for some* $\lambda > 0$.

(3) $(\mathring{P} \bigcap S_1, d)$ *is a metric space, where* $S_r = \{x \in X; \|x\| = r\}$. *Obviously, for all* $r > 0$*, so is* $(\mathring{P} \bigcap S_r, d)$ *also.*

Using similar techniques of Nussbaum [51, Eq.(1.21)] and Bushell [9, Theorem 2.2], the following lemmas are obtained.

Lemma 3.2.3 *Let* $\{x_k\} \subset \mathring{P}$ *and* $x \in \mathring{P}$. *If* $\lim_{k\to\infty} \|x_k - x\| = 0$*, then*

$$\lim_{k\to\infty} d(x_k, x) = 0.$$

Proof. Since $x \in \mathring{P}$, then there exists $\delta > 0$ such that

$$B_\delta(x) = \{y \in X; \|x - y\| < \delta\} \subset P.$$

Without loss of generality, we may assume $\|x_k - x\| > 0$. Since $\lim_{k\to\infty} \|x_k - x\| = 0$, there exists a positive integral N such that

$$0 < \|x_k - x\| < \frac{\delta}{2}, \quad \text{for all } k > N,$$

and hence $1 - \frac{2\|x_k - x\|}{\delta} > 0$. So $x \pm \frac{\delta(x_k - x)}{2\|x_k - x\|} \in B_\delta(x) \subset P$, and hence

$$x \ge \pm \frac{\delta(x_k - x)}{2\|x_k - x\|} = \pm \frac{\delta}{2}\left(\frac{x_k}{\|x_k - x\|} - \frac{x}{\|x_k - x\|}\right).$$

Therefore, we have

$$\left(1 + \frac{\delta}{2\|x_k - x\|}\right)x \ge \frac{\delta x_k}{2\|x_k - x\|} \ge \left(\frac{\delta}{2\|x_k - x\|} - 1\right)x,$$

and so, multiplying $\frac{2\|x_k - x\|}{\delta}$ to the two sides of the above inequality, we have

$$\left(1 + \frac{2\|x_k - x\|}{\delta}\right)x \ge x_k \ge \left(1 - \frac{2\|x_k - x\|}{\delta}\right)x.$$

By the definition of $M(x_k, x)$ and $m(x_k, x)$, we obtain for all $k > N$,

$$1 + \frac{2\|x_k - x\|}{\delta} \ge M(x_k, x) \ge m(x_k, x) \ge 1 - \frac{2\|x_k - x\|}{\delta} > 0.$$

Thus, we have

$$\frac{M(x_k, x)}{m(x_k, x)} \leq \frac{\delta + 2\|x_k - x\|}{\delta - 2\|x_k - x\|},$$

and hence

$$d(x_k, x) = \log\left(\frac{M(x_k, x)}{m(x_k, x)}\right) \leq \log\left(\frac{\delta + 2\|x_k - x\|}{\delta - 2\|x_k - x\|}\right).$$

Consequently, $\lim_{k\to\infty} d(x_k, x) = 0$. ■

Lemma 3.2.4 *Let the norm $\|\cdot\|$ in X be monotonic, i.e. $\theta \leq x \leq y$ means $\|x\| \leq \|y\|$. Then for all $x, y \in E_r = \mathring{P} \bigcap S_r$ $(r > 0)$, we have*

(i) $0 < m(x, y) \leq 1 \leq M(x, y) < \infty$;

(ii) $\|x - y\| \leq r(\exp(d(x, y)) - 1)$.

If $\{x_k\} \subset E_r$, $x \in E_r$ $(r > 0)$ and $\lim_{k\to\infty} d(x_k, x) = 0$, then

$$\lim_{k\to\infty} \|x_k - x\| = 0.$$

Proof. It follows from Lemma 3.2.1 that $0 < m(x, y) \leq M(x, y) < \infty$ and $m(x, y)y \leq x \leq M(x, y)y$ for all $x, y \in E_r$. Then we have

$$m(x, y)r = m(x, y)\|y\| \leq \|x\| = r \leq M(x, y)\|y\| = M(x, y)r,$$

and hence

$$0 < m(x, y) \leq 1 \leq M(x, y) < \infty.$$

So,

$$m(x, y)y \leq y \text{ and } x \leq M(x, y)y.$$

Then, we have

$$x - y \leq M(x, y)y - m(x, y)y = m(x, y)(\frac{M(x, y)}{m(x, y)} - 1)y,$$

and so,

$$\|x - y\| \leq m(x, y)(\frac{M(x, y)}{m(x, y)} - 1)\|y\| \leq r(\exp(d(x, y)) - 1).$$

In particular,

$$\|x_k - x\| \le r(\exp(d(x_k, x)) - 1).$$

As a result, the limits $\lim\limits_{k\to\infty} d(x_k, x) = 0$ implies

$$\lim_{k\to\infty} \|x_k - x\| = 0.$$

This completes the proof. ■

Let X be a real Banach space with the usual partial ordering induced by a closed cone P, and $T : X \to X$ be a mapping. We also need the following definitions and facts for T. T is called

- *positively homogeneous* if $T(tx) = tTx$ for each $t > 0$ and all $x \in X$,
- *increasing* if $Tx \le Ty$ for $x \le y$,
- *strongly increasing* if $Tx < Ty$ for $x \le y$ with $x \ne y$ in case $\mathring{P} \ne \emptyset$,
- *eventually strongly increasing* if $T^k x < T^k y$ for some positive integer k and all $x \le y$ with $x \ne y$ in case $\mathring{P} \ne \emptyset$,
- *strictly increasing* if $Tx < Ty$ for $x < y$ in case $\mathring{P} \ne \emptyset$,
- *positive* if $T(P) \subset P$,
- *strongly positive* if $T(P^+) \subset \mathring{P}$ in case $\mathring{P} \ne \emptyset$,
- *eventually strongly positive* $T^k(P^+) \subset \mathring{P}$ for some positive integer k,
- *strictly positive* if $T(\mathring{P}) \subset \mathring{P}$ in case $\mathring{P} \ne \emptyset$.

Clearly, for a linear mapping T, its (strong, eventually strong, strict) increasing coincides with its (respectively strong, eventually strong, strict) positiveness.

We say that a mapping T is *compact* if it takes bounded subsets of X into relatively compact subsets of X. A real number λ is said to be an *eigenvalue* of the mapping T if there exists a non-zero $x \in X$ such that $Tx = \lambda x$.

Lemma 3.2.5 *Let $T : X \to X$ be a mapping. If T is positively homogeneous and strongly (strictly) increasing, then T is strongly (strictly) positive.*

Proof. It follows from the positive homogeneousness of T that $T\theta = \theta$ because $T\theta = T(t\theta) = tT\theta$ for each $t > 0$. Since $x \geq \theta$ ($x > \theta$) for all $x \in P^+$ ($x \in \mathring{P}$) and T is strongly (strictly) increasing, $Tx > T\theta = \theta$ ($Tx > T\theta = \theta$), and so $T(P^+) \subset \mathring{P}$ ($T(\mathring{P}) \subset \mathring{P}$). ∎

3.3 Existence and uniqueness of the positive eigenvalue

In this section, let X be a real Banach space with the usual partial ordering induced by a closed cone P, and $\mathring{P} \neq \emptyset$. Now we show that a strongly increasing and positively homogeneous nonlinear mapping can be an Edelstein contraction on the metric space $(\mathring{P} \bigcap S_r, d)$ for each $r > 0$.

Theorem 3.3.1 *Let $T : X \to X$ be a strongly increasing and positively homogeneous mapping. If the norm $\|\cdot\|$ in X is monotonic, then T is an Edelstein contraction with respect to Hilbert's projective metric d on $E_r = \mathring{P} \bigcap S_r$ $(r > 0)$, i.e.*

$$d(Tx, Ty) < d(x, y) \text{ for all } x, y \in E_r \text{ with } x \neq y. \tag{3.7}$$

Proof. Let $x, y \in E_r$ with $x \neq y$. Then following Lemmas 3.2.4 and 3.2.1, we have for all $x, y \in E_r \subset \mathring{P}$,

$$0 < m(x, y) \leq 1 \leq M(x, y) < \infty \text{ and } m(x, y)y \leq x \leq M(x, y)y.$$

Since $x \neq y$, then $M(x, y)$ and $m(x, y)$ can not be both equal to 1, and so $M(x, y)y$ and $m(x, y)y$ can not be both equal to x. Without loss of generality, we may assume

$M(x,y)y \neq x$. By the strong monotonicity and positive homogeneousness of T, we obtain

$$m(x,y)Ty = T(m(x,y)y) \leq Tx < T(M(x,y)y) = M(x,y)Ty.$$

Therefore we have

$$0 < m(x,y) \leq m(Tx,Ty) \text{ and } M(Tx,Ty) < M(x,y),$$

and hence

$$\frac{M(Tx,Ty)}{m(Tx,Ty)} < \frac{M(x,y)}{m(x,y)}.$$

It follows from the properties of the function log that

$$\log\left(\frac{M(Tx,Ty)}{m(Tx,Ty)}\right) < \log\left(\frac{M(x,y)}{m(x,y)}\right).$$

Consequently, $d(Tx,Ty) < d(x,y)$ as required. ■

Now we prove a property of a strongly positive, positively homogeneous and compact continuous nonlinear mapping under Hilbert's projective metric d. Namely, the condition (3.3) of Edelstein Contraction Theorem (Theorem 3.1.1) can be satisfied by such a mapping.

Theorem 3.3.2 *Let $T : X \to X$ be a strongly positive, positively homogeneous and continuous mapping. If $F_r y = \frac{rTy}{\|Ty\|}$ for $r > 0$ is compact, then for all $y \in P^+$, the sequence $\{F_r^{k+1}y\}$ contains a subsequence which converges to a point of E_r with respect to Hilbert's projective metric d, that is, $x = \frac{rTy}{\|Ty\|}$,*

$$\exists x^* \in E_r \text{ such that } \exists \{F_r^{k_i}x\} \subset \{F_r^k x\} \text{ with } \lim_{i\to\infty} d(F_r^{k_i}x, x^*) = 0. \tag{3.8}$$

Proof. Take $y \in P^+$. From the strong positiveness of T, it follows that $Ty \in \mathring{P}$ and $\|Ty\| > 0$. Let $x = \frac{rTy}{\|Ty\|}$. Clearly, $x \in E_r = \mathring{P} \bigcap S_r$. By positive homogeneousness of T, we have

$$F_r(F_r x) = F_r\left(\frac{rTx}{\|Tx\|}\right) = \frac{rT(\frac{rTx}{\|Tx\|})}{\|T(\frac{rTx}{\|Tx\|})\|} = \frac{rT^2x}{\|T^2x\|} = \frac{rT^3y}{\|T^3y\|}.$$

Similarly, we also have

$$F_r^k x = \frac{rT^k x}{\|T^k x\|} = \frac{rT^{k+1} y}{\|T^{k+1} y\|} = F_r^{k+1} y \in E_r \text{ for all } k > 0.$$

From compact continuity of F_r, it follows that there exists $\{F^{k_i} x\} \subset \{F^k x\}$ such that for some $x' \in X$,

$$\lim_{i\to\infty} \|F_r^{k_i} x - x'\| = 0,$$

and hence

$$\lim_{i\to\infty} \|F_r^{k_i+1} x - F_r x'\| = 0.$$

Then by the closedness of P and $\{F_r^k x\} \subset E_r$, we have $x' \in P$ and $\|x'\| = r$. Thus we obtain $Tx' \in \mathring{P}$. Let $x^* = F_r x' = \frac{rTx'}{\|Tx'\|}$. Then $x^* \in E_r \subset \mathring{P}$.

As a result, we have shown that there exists $\{F_r^{k_i+1} x\} \subset \{F_r^k x\} \subset \mathring{P}$ such that

$$\lim_{i\to\infty} \|F_r^{k_i+2} y - x^*\| = \lim_{i\to\infty} \|F_r^{k_i+1} x - x^*\| = 0 \text{ and } x^* \in \mathring{P}.$$

By Lemma 3.2.3, the desired result follows. ■

Combining Theorems 3.3.1 and 3.3.2, a nonlinear version of the famous Krein-Rutman Theorem can be obtained.

Theorem 3.3.3 *Let $T : X \to X$ be a strongly increasing, positively homogeneous and continuous mapping. If the norm $\|\cdot\|$ in X is monotonic and $F_r y = \frac{rTy}{\|Ty\|}$ is compact for $r > 0$, then*

- *T has a unique positive eigenvalue $\lambda > 0$ with a positive eigenvector $y^* \in \mathring{P}$;*
- *the equation $Tx = \lambda x$ for $x \in P^+$ implies that $x = \gamma y^*$ for some real number $\gamma > 0$.*

Furthermore, for each $y \in P^+$, we have

$$\lim_{k\to\infty} \frac{T^k y}{\|T^k y\|} = y^* > \theta \textit{ and } \lim_{k\to\infty} \frac{\|T^{k+1} y\|}{\|T^k y\|} = \lambda > 0.$$

Proof. It follows from Theorem 3.3.1 that T is an Edelstein contraction with respect to Hilbert's projective metric d on $E_r = \mathring{P} \bigcap S_r$ $(r > 0)$, that is

$$d(Tx, Ty) < d(x, y) \text{ for all } x, y \in E_r \text{ with } x \neq y.$$

Let $F_r z = \frac{rTz}{\|Tz\|}$ for $r > 0$ and all $z \in P^+$. Then for all $x, y \in E_r$ with $x \neq y$, we have $Tx, Ty \in \mathring{P}$ and so $\|Tx\| > 0$ and $\|Ty\| > 0$. By Lemma 3.2.2(1), we obtain

$$d(F_r x, F_r y) = d(\frac{r}{\|Tx\|}Tx, \frac{r}{\|Ty\|}Ty) = d(Tx, Ty) < d(x, y).$$

From Lemma 3.2.2(3) and 3.2.5, it follows that (E_r, d) is a metric space and T is strongly positive. So the Theorem 3.3.2 assures that the condition (3.3) of Edelstein Contraction Theorem (Theorem 3.1.1) holds. For all $z \in E_r$, clearly, $F_r z = \frac{rTz}{\|Tz\|} \in E_r$, and so $F_r(E_r) \subset E_r$.

Consequently, following the Edelstein Contraction Theorem, for each $r > 0$, there exists unique $x_r \in E_r$ such that

$$F_r x_r = \frac{rTx_r}{\|Tx_r\|} = x_r \text{ and } \lim_{k\to\infty} d(F_r^{k+1} y, x_r) = 0 \text{ for all } y \in P^+.$$

Let $x = \frac{rTy}{\|Ty\|}$. Then $x \in E_r$. From Lemma 3.2.4, we obtain

$$\lim_{k\to\infty} \|\frac{rT^k x}{\|T^k x\|} - x_r\| = \lim_{k\to\infty} \|F_r^k x - x_r\| = 0 \text{ for all } y \in P^+.$$

In particular $(r = 1)$, there exists a unique $x_1 \in E_1$ such that

$$F_1 x_1 = \frac{Tx_1}{\|Tx_1\|} = x_1 \text{ and } \lim_{k\to\infty} \|\frac{T^k x}{\|T^k x\|} - x_1\| = 0.$$

Let $y^* = x_1$, $\lambda = \|Ty^*\|$ and $\lambda_r = \frac{\|Tx_r\|}{r}$. Clearly, $x_r, y^* \in \mathring{P}$, and $\lambda_r > 0, \lambda > 0$ with

$$\text{for each } r > 0,\ \|x_r\| = r,\ Tx_r = \lambda_r x_r,\ Ty^* = \lambda y^*,\ \|y^*\| = 1.$$

Now we show the uniqueness of λ. For any given $r > 0$, from $F_r x_r = \frac{rTx_r}{\|Tx_r\|} = x_r$, it follows that $\frac{Tx_r}{\|Tx_r\|} = \frac{x_r}{r}$, and so

$$\frac{x_r}{r} \in E_1 \text{ and } F_1(\frac{x_r}{r}) = \frac{T(\frac{x_r}{r})}{\|T(\frac{x_r}{r})\|} = \frac{Tx_r}{\|Tx_r\|} = \frac{x_r}{r}.$$

By the uniqueness of $y^* = x_1$ in E_1, we obtain

$$\frac{x_r}{r} = y^*, \ i.e., \ x_r = ry^*.$$

Thus,

$$\lambda_r = \frac{\|Tx_r\|}{r} = \frac{r\|Ty^*\|}{r} = \|Ty^*\| = \lambda \text{ for each } r > 0,$$

and so $Tx_r = \lambda x_r$ for each $r > 0$.

Next we construct an iteration with initial value $y \in P^+$. Take $y \in P^+$. Then $Ty \in \mathring{P}$. Let $x = \frac{Ty}{\|Ty\|}$. Then $x \in E_1$. Therefore, by the positive homogeneousness of T, we have

$$F_1^k(x) = \frac{T^k x}{\|T^k x\|} = \frac{T^k(\frac{Ty}{\|Ty\|})}{\|T^k(\frac{Ty}{\|Ty\|})\|} = \frac{T^{k+1}y}{\|T^{k+1}y\|},$$

and hence,

$$\lim_{k\to\infty} \|\frac{T^{k+1}y}{\|T^{k+1}y\|} - y^*\| = \lim_{k\to\infty} \|F_1^k(x) - y^*\| = 0 \text{ for all } y \in P^+.$$

The desired conclusion follows. ■

Corollary 3.3.4 *Let $T : X \to X$ be a strongly increasing, positively homogeneous and continuous mapping. If the norm $\|\cdot\|$ in X is monotonic and $F_r y = \frac{rTy}{\|Ty\|}$ is compact for $r > 0$, then*

- *T has a unique positive eigenvalue $\lambda > 0$ with a unique positive eigenvector $y^* \in S_1 \cap \mathring{P}$;*
- *for all $y \in P^+$, we have*

$$\lim_{k\to\infty} \|T^k y\|^{\frac{1}{k}} = \lambda > 0.$$

Proof. It follows from Theorem 3.3.3 that T has a unique positive eigenvalue $\lambda > 0$ with a unique positive eigenvector $y^* \in S_1 \cap \mathring{P}$ and

$$\lim_{k\to\infty} \frac{\|T^{k+1}y\|}{\|T^k y\|} = \lambda > 0 \text{ for all } y \in P^+.$$

Then for all $\varepsilon > 0$, there exists a positive integer N such that for $k \geq N$,

$$\lambda - \varepsilon < \frac{\|T^{k+1}y\|}{\|T^k y\|} < \lambda + \varepsilon,$$

and so

$$(\lambda - \varepsilon)\|T^k y\| < \|T^{k+1}y\| < (\lambda + \varepsilon)\|T^k y\|.$$

Therefore, we have

$$\begin{aligned}(\lambda - \varepsilon)^{k-N+1}\|T^N y\| <& (\lambda - \varepsilon)^{k-N}\|T^{N+1}y\| < \cdots < (\lambda - \varepsilon)^2\|T^{k-1}y\| \\ < (\lambda - \varepsilon)\|T^k y\| <& \|T^{k+1}y\| < (\lambda + \varepsilon)\|T^k y\| \\ &< (\lambda + \varepsilon)^2\|T^{k-1}y\| < \cdots < (\lambda + \varepsilon)^{k-N}\|T^{N+1}y\| \\ &< (\lambda + \varepsilon)^{k-N+1}\|T^N y\|,\end{aligned}$$

and hence

$$(\lambda - \varepsilon)^{\frac{k-N+1}{k+1}}\|T^N y\|^{\frac{1}{k+1}} < \|T^{k+1}y\|^{\frac{1}{k+1}} < (\lambda + \varepsilon)^{\frac{k-N+1}{k+1}}\|T^N y\|^{\frac{1}{k+1}}.$$

So, we obtain

$$\lambda - \varepsilon \leq \lim_{k\to\infty}\|T^k y\|^{\frac{1}{k}} \leq \lambda + \varepsilon.$$

Since ε is arbitrary, the desired conclusion follows. ■

Theorem 3.3.5 *Let $T : X \to X$ be eventually strongly increasing, positively homogeneous and continuous. Suppose that the norm $\|\cdot\|$ of X is monotonic and $F_r y = \frac{rTy}{\|Ty\|}$ is compact for $r > 0$. Then*

- *T has a unique positive eigenvalue $\lambda > 0$ with a positive eigenvector $y^* \in \mathring{P}$;*
- *the equation $Tx = \lambda x$ for $x \in P^+$ implies that $x = \gamma y^*$ for some real number $\gamma > 0$;*
- *for each $y \in P^+$, we have*

$$\lim_{k\to\infty}\frac{T^k y}{\|T^k y\|} = y^* > \theta \text{ and } \lim_{k\to\infty}\frac{\|T^{k+1}y\|}{\|T^k y\|} = \lim_{k\to\infty}\|T^k y\|^{\frac{1}{k}} = \lambda.$$

Proof. It follows from the definition of eventually strongly increasing mapping that $S = T^t$ is strongly increasing for some positive integer t. Then $Fz = \frac{Sz}{\|Sz\|}$ is an Edelstein contraction with respect to Hilbert's projective metric d on $E_1 = \mathring{P} \bigcap S_1$. Consequently, following the Edelstein Contraction Theorem, there exists unique $y^* \in E_1$ such that

$$Fy^* = \frac{Sy^*}{\|Sy^*\|} = y^* \text{ and } \lim_{k\to\infty} d(F^{k+1}y, y^*) = 0 \text{ for all } y \in P^+.$$

Then $Sy^* = \|Sy^*\|y^*$, and so

$$F(\frac{Ty^*}{\|Ty^*\|}) = \frac{S(\frac{Ty^*}{\|Ty^*\|})}{\|S(\frac{Ty^*}{\|Ty^*\|})\|} = \frac{T(Sy^*)}{\|T(Sy^*)\|} = \frac{T(y^*\|Sy^*\|)}{\|T(y^*\|Sy^*\|)\|} = \frac{Ty^*}{\|Ty^*\|},$$

and hence, $\frac{Ty^*}{\|Ty^*\|}$ is a fixed point of F. So by the uniqueness, we have

$$\frac{Ty^*}{\|Ty^*\|} = y^*, \text{ i.e., } Ty^* = \lambda y^* \text{ and } \lambda = \|Ty^*\| > 0.$$

From Theorem 3.3.3, we easily obtain that λ is unique and for all $y \in P^+$,

$$\lim_{k\to\infty} \frac{T^k y}{\|T^k y\|} = \lim_{k\to\infty} \frac{T^{k+t} y}{\|T^{k+t} y\|} = \lim_{k\to\infty} \frac{S^k y}{\|S^k y\|} = y^*.$$

The desired conclusion follows. ■

Remark 3.3.1 *Our proof techniques are totally different from the ones of Chang [10] and Mahadeva [41], which may be regarded as a unified proof for linear and nonlinear mapping in Banach space. We directly reach our main goals by means of Edelstein Contraction Theorem.*

A cone P is called *solid* if $\mathring{P} \neq \emptyset$, *normal* if there exists a constant M such that $\|x\| \leq M\|y\|$ for all $x, y \in P$ with $x \leq y$. Clearly, the cone $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n; x \geq 0\}$ of the finite dimensional space $\mathbb{R}^n$ is normal and solid. It is well known that if P is a normal cone in a Banach space with norm $\|\cdot\|$, then there exists an equivalent norm $\|\cdot\|_1$ such that

$$\|x\|_1 \leq \|y\|_1 \text{ for all } x, y \in P \text{ with } x \leq y.$$

That is, the norm $\|\cdot\|_1$ in X is monotonic (Nussbaum [51]). So the following results are obtained easily, which are the improvement, development and complement of main results in Krause [36] and Kohlberg [35].

Corollary 3.3.6 *Let X be a real Banach space with the usual partial ordering induced by a normal solid cone P, $T : X \to X$ be a (eventually) strongly increasing, positively homogeneous and continuous mapping. Suppose that $F_r y = \frac{rTy}{\|Ty\|}$ is compact for $r > 0$. Then*

- *T has a unique positive eigenvalue λ with a positive eigenvector $y^* \in \mathring{P}$;*
- *the equation $Tx = \lambda x$ for $x \in P^+$ implies that $x = \gamma y^*$ for some real number $\gamma > 0$;*
- *for all $y \in P^+$, we have*
$$\lim_{k\to\infty} \frac{T^k y}{\|T^k y\|} = y^* > \theta \text{ and } \lim_{k\to\infty} \frac{\|T^{k+1}y\|}{\|T^k y\|} = \lambda = \lim_{k\to\infty} \|T^k y\|^{\frac{1}{k}}.$$

On a finite dimensional Euclidean space $\mathbb{R}^n$, the compactness of T can be removed.

Theorem 3.3.7 *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a strongly increasing, positively homogeneous and continuous mapping. Then*

- *T has a unique positive eigenvalue $\lambda > 0$ with a positive eigenvector $y^* \in \mathbb{R}^n_{++}$;*
- *the equation $Tx = \lambda x$ for $x \in \mathbb{R}^n_+ \setminus \{\theta\}$ implies that $x = ry^*$ for some real number $r > 0$.*

Furthermore, for all $y \in \mathbb{R}^n_+ \setminus \{\theta\}$, we have
$$\lim_{k\to\infty} \frac{T^k y}{\|T^k y\|} = y^* > \theta \text{ and } \lim_{k\to\infty} \frac{\|T^{k+1}y\|}{\|T^k y\|} = \lambda = \lim_{k\to\infty} \|T^k y\|^{\frac{1}{k}}.$$

Proof. Take $y \in \mathbb{R}^n_+ \setminus \{\theta\}$. From the strong positiveness of T (Lemma 3.2.5), it follows that $Ty \in \mathbb{R}^n_{++}$ and $\|Ty\| > 0$. Let $F_r z = \frac{rTz}{\|Tz\|}$ and $x = \frac{rTy}{\|Ty\|}$. Clearly, $x \in E_r = \{z \in \mathbb{R}^n_{++}; \|z\| = r > 0\}$. By positive homogeneousness of T, we have

$$F_r^k x = \frac{rT^k x}{\|T^k x\|} = \frac{rT^{k+1} y}{\|T^{k+1} y\|} \in E_r \text{ for all } k > 0.$$

Since $\|F_r^k x\| = r$ for all $k > 0$, then there exists $\{F^{k_i} x\} \subset \{F^k x\}$ such that for some $x' \in X$, $F_r^{k_i} x \xrightarrow{\|\cdot\|} x'$, and hence $F_r^{k_i+1} x \xrightarrow{\|\cdot\|} F_r x'$ and $\|x'\| = r$. Thus $Tx' \in \mathbb{R}^n_{++}$. Let $y^* = F_r x' = \frac{rTx'}{\|Tx'\|}$. Clearly, $y^* \in E_r$.

Therefore, we have $F_r^{k_i+1} x \xrightarrow{\|\cdot\|} y^* \in E_r$. By Lemma 3.2.3, we obtain

$$F_r^{k_i+1} x \xrightarrow{d(\cdot,\cdot)} y^* \in E_r.$$

That is, the conclusion of Theorem 3.3.2 follows. Obviously, the norm $\|\cdot\|$ in $\mathbb{R}^n_+$ is monotonic. By the same proof techniques of Theorem 3.3.3, the desired result follows. ■

Theorem 3.3.8 *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a eventually strongly increasing, positively homogeneous and continuous mapping. Then*

- *T has a unique positive eigenvalue $\lambda > 0$ with a positive eigenvector $y^* \in \mathbb{R}^n_{++}$;*
- *the equation $Tx = \lambda x$ for $x \in \mathbb{R}^n_+ \setminus \{\theta\}$ implies that $x = \gamma y^*$ for some real number $\gamma > 0$.*

Furthermore, for all $y \in \mathbb{R}^n_+ \setminus \{\theta\}$, we have

$$\lim_{k\to\infty} \frac{T^k y}{\|T^k y\|} = y^* > \theta \text{ and } \lim_{k\to\infty} \frac{\|T^{k+1} y\|}{\|T^k y\|} = \lambda = \lim_{k\to\infty} \|T^k y\|^{\frac{1}{k}}.$$

For a nonnegative square matrix A, let $Tx = Ax$. Obviously, T is positively homogeneous and compact continuous, and moreover, its (eventually) strong increasing coincides with (eventually) strong positiveness. The Perron-Frobenius theorem is derived easily. So our proof may be viewed as a unified proof for linear and nonlinear mapping in Banach space.

Corollary 3.3.9 *Let A be a nonnegative, primitive square matrix. Then A has a unique positive eigenvalue $\lambda > 0$ with a positive eigenvector $y^* \in \mathbb{R}^n_{++}$, and the equation $Tx = \lambda x$ for $x \in \mathbb{R}^n_+ \setminus \{\theta\}$ implies that $x = ry^*$ for some real number $r > 0$.*

Furthermore, for all $y \in \mathbb{R}^n_+ \setminus \{\theta\}$, we have

$$\lim_{k\to\infty} \frac{A^k y}{\|A^k y\|} = y^* > \theta \text{ and } \lim_{k\to\infty} \frac{\|A^{k+1} y\|}{\|A^k y\|} = \lambda = \lim_{k\to\infty} \|A^k y\|^{\frac{1}{k}}.$$

We also have the following problems awaiting further research and thought.

Problem 2 *In Theorem 3.3.1–3.3.3, whether or not the strong positiveness (strong increase) of T is replaced by the strict positiveness (strict increase, respectively) of T or the other weaker conditions.*

Problem 3 *Whether is there really necessary for the norm $\|\cdot\|$ in Theorem 3.3.3 and 3.3.5 to be monotonic?*

3.4 Positive eigenvalue of nonnegative tensors

For an m-order n-dimensional tensor $\mathcal{A}$, when $(\mathcal{A}x^{m-1})^{[\frac{1}{m-1}]}$ can be well defined, let

$$F_{\mathcal{A}}(x) = (\mathcal{A}x^{m-1})^{[\frac{1}{m-1}]}.$$

An $m-$order n-dimensional tensor $\mathcal{A}$ is called *nonnegative* (or respectively *positive*), denoted $\mathcal{A} \geq \Theta$ (or respectively $\mathcal{A} > \Theta$), if $a_{i_1 i_2 \cdots i_m} \geq 0$ (or respectively $a_{i_1 i_2 \cdots i_m} > 0$).

Chang, Pearson, and Zhang [12] showed that the strong increase of the mapping $F_{\mathcal{A}}$ with a nonnegative tensor $\mathcal{A}$ coincides with its strong positiveness (Also see Hu, Huang and Qi [26]).

Theorem 3.4.1 *Let $F_{\mathcal{A}}$ with a nonnegative tensor $\mathcal{A}$ be strongly positive. Then $F_{\mathcal{A}}$ is an Edelstein contraction with respect to Hilbert's projective metric d on $E_r = \{x \in \mathbb{R}^n_{++}; \|x\| = r > 0\}$, i.e.*

$$d(F_{\mathcal{A}}x, F_{\mathcal{A}}y) < d(x, y) \text{ for all } x, y \in E_r \text{ with } x \neq y. \tag{3.9}$$

Proof. It follows from the definitions of $F_{\mathcal{A}}$ and $\mathcal{A}x^{m-1}$ that $F_{\mathcal{A}}$ is positively homogeneous. Clearly, the norm $\|\cdot\|$ in $\mathbb{R}^n$ is monotonic. Since the (eventually) strong positiveness of $F_{\mathcal{A}}$ is equivalent to its (eventually) strong increasing (see Chang, Pearson, and Zhang [12], Hu, Huang and Qi [26]), then following Theorem 3.3.1, the desired result is reached. ■

Corollary 3.4.2 *Let $F_{\mathcal{A}}$ with a nonnegative tensor $\mathcal{A}$ be strongly positive. Then*

- *the tensor $\mathcal{A}$ has a unique positive eigenvalue λ^{m-1} with a positive eigenvector $y^* \in \mathbb{R}^n_{++}$;*
- *if $x \geq \theta (x \neq \theta)$ is a eigenvector corresponding to λ^{m-1}, then $x = \gamma y^*$ for some real number $\gamma > 0$.*

Furthermore, for all $y \in \mathbb{R}^n_+ \setminus \{\theta\}$, we have

$$\lim_{k\to\infty} \frac{F^k_{\mathcal{A}}y}{\|F^k_{\mathcal{A}}y\|} = y^* > \theta \text{ and } \lim_{k\to\infty} \|F^k_{\mathcal{A}}y\|^{\frac{1}{k}} = \lambda. \tag{3.10}$$

Proof. Similarly to Theorem 3.4.1, we also have $F_{\mathcal{A}}$ is strongly increasing, positively homogeneous. It follows from the definitions of $F_{\mathcal{A}}$ and $\mathcal{A}x^{m-1}$ that $F_{\mathcal{A}}$ is compact and continuous. Following Theorem 3.3.7 or 3.3.3, there exists an unique $\lambda > 0$ and $y^* \in \mathbb{R}^m_{++}$ such that

$$F_{\mathcal{A}} y^* = (\mathcal{A}(y^*)^{m-1})^{[\frac{1}{m-1}]} = \lambda y^*.$$

Furthermore, for all $y \in \mathbb{R}^n_+ \setminus \{0\}$, we have

$$\lim_{k\to\infty} \frac{F^k_{\mathcal{A}} y}{\|F^k_{\mathcal{A}} y\|} = y^* > 0 \text{ and } \lim_{k\to\infty} \frac{\|F^{k+1}_{\mathcal{A}} y\|}{\|F^k_{\mathcal{A}} y\|} = \lambda = \lim_{k\to\infty} \|F^k_{\mathcal{A}} y\|^{\frac{1}{k}}.$$

Let $\mu = \lambda^{m-1}$. Then $\mu > 0$ and

$$\mathcal{A}(y^*)^{m-1} = \lambda^{m-1} (y^*)^{[m-1]} = \mu (y^*)^{[m-1]}.$$

This yields the desired conclusion. ∎

A higher order tensor $\mathcal{A}$ is called *primitive* if $F_{\mathcal{A}}$ is eventually strongly positive (Chang, Pearson, and Zhang [12]). So, by Theorem 3.3.8 or 3.3.5, the following result is obtained easily.

Corollary 3.4.3 *Assume that $F_{\mathcal{A}}$ is eventually strongly positive (or equivalently, $\mathcal{A}$ is primitive). Then*

- *the tensor $\mathcal{A}$ has a unique positive eigenvalue λ^{m-1} with a positive eigenvector $y^* \in \mathbb{R}^n_{++}$;*
- *if $x \geq \theta (x \neq \theta)$ is a eigenvector corresponding to λ^{m-1}, then $x = \gamma y^*$ for some real number $\gamma > 0$.*

Furthermore, for all $y \in \mathbb{R}^n_+ \setminus \{\theta\}$, we have

$$\lim_{k\to\infty} \frac{F^k_{\mathcal{A}} y}{\|F^k_{\mathcal{A}} y\|} = y^* > \theta \text{ and } \lim_{k\to\infty} \|F^k_{\mathcal{A}} y\|^{\frac{1}{k}} = \lambda. \tag{3.11}$$

A tensor $\mathcal{A}$ is called *reducible*, if there exists a nonempty proper index subset $\mathcal{N} \in \{1, 2, \cdots, n\}$ such that

$$a_{i_1 i_2 \cdots i_m} = 0 \text{ for all } i_1 \in \mathcal{N}, \text{ for all } i_2, i_3, \cdots, i_m \notin \mathcal{N}.$$

If $\mathcal{A}$ is not reducible, then we call it *irreducible.*

The notion of irreducible tensor is first introduced by Lim [40]. Chang, Pearson, Zhang [11] adopted this notion in their subsequent work. Chang, Pearson, and Zhang [12] showed that $\mathcal{A}+\mathcal{I}$ is primitive whenever $\mathcal{A}$ is irreducible, where $\mathcal{I}$ is unit tensor (its entries are $\delta_{i_1 i_2\cdots i_m}$ with $\delta_{i_1 i_2\cdots i_m}=1$ if and only if $i_1=i_2=\cdots=i_m$ and the others are zero). So, the following result is obtained easily.

Corollary 3.4.4 *Assume that $\mathcal{A}$ is irreducible. Then*

- *$\mathcal{A}$ has a unique positive eigenvalue $(\lambda^{m-1}-1)$ with a positive eigenvector $y^*\in\mathbb{R}^n_{++}$;*
- *if $x\geq\theta(x\neq\theta)$ is a eigenvector corresponding to $(\lambda^{m-1}-1)$, then $x=\gamma y^*$ for some real number $\gamma>0$.*

Furthermore, for all $y\in\mathbb{R}^n_+\setminus\{\theta\}$, we have

$$\lim_{k\to\infty}\frac{F^k_{\mathcal{A}+\mathcal{I}}y}{\|F^k_{\mathcal{A}+\mathcal{I}}y\|}=y^*>\theta \text{ and } \lim_{k\to\infty}\|F^k_{\mathcal{A}+\mathcal{I}}y\|^{\frac{1}{k}}=\lambda>1. \tag{3.12}$$

Proof. It follows from Theorem 3.3.8 (or 3.3.5 or Corollary 3.4.4) that $\lambda y^*=F_{\mathcal{A}+\mathcal{I}}y^*$ and

$$\lambda=\|F_{\mathcal{A}+\mathcal{I}}y^*\|=\|(\mathcal{A}(y^*)^{m-1}+(y^*)^{[m-1]})^{[\frac{1}{m-1}]}\|>\|y^*\|=1.$$

Since

$$\lambda y^*=F_{\mathcal{A}+\mathcal{I}}y^*=(\mathcal{A}(y^*)^{m-1}+(y^*)^{[m-1]})^{[\frac{1}{m-1}]},$$

we have

$$\lambda^{m-1}(y^*)^{[m-1]}=\mathcal{A}(y^*)^{m-1}+(y^*)^{[m-1]},$$

and hence

$$(\lambda^{m-1}-1)(y^*)^{[m-1]}=\mathcal{A}(y^*)^{m-1}.$$

The desired conclusion follows. ■

Chapter 4

Generalized eigenvalue problem

4.1 Introduction

Recently, Chang, Pearson Zhang [11,13] generalized the notion of eigenvalues of higher order tensors to tensor pairs (or tensor pencils). For two m-order n-dimensional real tensor $\mathcal{A}$, $\mathcal{B}$, $(\mu, x) \in \mathbb{C} \times (\mathbb{C}^n \setminus \{\theta\})$ is called an *eigenvalue-eigenvector pairs of* $\mathcal{A}$ *relative to* $\mathcal{B}$, if both $\mathcal{A}x^{m-1}$ and $\mathcal{B}x^{m-1}$ are not identical to zero and the n-system of equations:

$$(\mathcal{A} - \mu\mathcal{B})x^{m-1} = \theta,$$

i.e.,

$$\sum_{i_2,\cdots,i_m=1}^{n} (a_{ii_2\cdots i_m} - \mu b_{ii_2\cdots i_m})x_{i_2}\cdots x_{i_m} = 0, \quad i = 1, 2, \cdots, n, \tag{4.1}$$

possesses a solution, where $\theta = (0, 0, \cdots, 0)^T$. μ is called a $\mathcal{B}$-*eigenvalue* of $\mathcal{A}$, and x is called a $\mathcal{B}$-*eigenvector* of $\mathcal{A}$ associated with μ. If $\mathcal{B} = \mathcal{I}$, the unit tensor, then the $\mathcal{B}$-eigenvalues are the eigenvalues, and the real $\mathcal{B}$-eigenvalues with real eigenvectors are the H-eigenvalues. Chang, Pearson and Zhang [11] also gave the Perron-Frobenius theorem to the tensor pairs $(\mathcal{A}, \mathcal{B})$ involving the calculation of the inversion of $\mathcal{B}$. Very recently, Zhang [74] established the existence of real eigenvalue of higher order real tensor pairs with the help of the Brouwer degree.

In this chapter, with the aid of the property of topical mapping, we will present a new iterative technique to extend the nonlinear Perron-Frobenius property to the tensor pairs $(\mathcal{A}, \mathcal{B})$ without the requirement of the tensor inversion, where a condition sufficiently guarantees that $\mathcal{A}$ has a unique positive $\mathcal{B}$-eigenvalue with a corresponding positive $\mathcal{B}$-eigenvector. Also the well-known Collatz-Wielandt min-max theorem is extended to the tensor pairs $(\mathcal{A}, \mathcal{B})$.

A mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ is called *additively homogeneous* if

$$\text{for all } \lambda \in \mathbb{R}, x \in \mathbb{R}^n,\ f(x+\lambda) = f(x) + \lambda, \tag{4.2}$$

where $x+\lambda = (x_1+\lambda, x_2+\lambda, \cdots, x_n+\lambda)^T$. In the sequel, we will omit the terminology "additively", when the additive characteristic is clear from the context. f is said to be *increasing* if

$$\text{for all } x, y \in \mathbb{R}^n, x \leq y \implies f(x) \leq f(y), \tag{4.3}$$

where $\leq$ is the usual partial order on $\mathbb{R}^n (x \leq y \iff x_i \leq y_i$, for all $i = 1, 2, \ldots, n)$. Following Gunawardena and Keane [21], a mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ which satisfies (4.2) and (4.3) is known as *topical mapping*. It is easy to obtain that a topical mapping must necessarily be nonexpansive in the l^∞ norm,

$$\text{for all } x, y \in \mathbb{R}^n,\ \|f(x) - f(y)\| \leq \|x - y\|, \tag{4.4}$$

where $\|x\| = \max\limits_{1 \leq i \leq n} |x_i|$.

The theory of the topical mappings is very interesting because it appears classically in a remarkable variety of mathematical disciplines such as the matrices over the max-plus semiring, Markov decision theory, the theory of stochastic games, the optimal control problems, the discrete event systems models (see [2,4,22,23,42,43,68] for different applications). A fundamental problem about a topical mapping f is the existence and uniqueness of the eigenvalue and its corresponding eigenvector, which is to

$$\text{find } x \in \mathbb{R}^n \text{ and } \lambda \in \mathbb{R} \text{ such that } f(x) = \lambda + x. \tag{4.5}$$

If such a $\lambda \in \mathbb{R}$ exists in (4.5), we call it an *eigenvalue* of f, and call $x \in \mathbb{R}^n$ an *eigenvector* of f associated with λ. In the applications of discrete event systems, the eigenvalue provides the output, and eigenvectors provide stationary schedules. In stochastic control, the eigenvalue provides the optimal remuneration in unit time, and eigenvectors provide stationary remuneration [2]. Batap [4] established the max version of the Perron-Frobenius theorem. Recently, the nonlinear Perron-Frobenius theorem about topical mapping were obtained by Gaubert and Gunawardena [22].

Let $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n; x \geq \theta\}$ and $\mathbb{R}^n_{++} = \{x \in \mathbb{R}^n; x > \theta\}$. For $x \in \mathbb{R}^n$ and a strictly nonnegative tensor $\mathcal{A}$ of order m and dimension n, let

$$f(x) = \frac{1}{m-1} \log(\mathcal{A}(\exp(x))^{m-1}), \tag{4.6}$$

where $\log(x) = (\log(x_1), \cdots, \log(x_n))^T$ and $\exp(x) = (\exp(x_1), \cdots, \exp(x_n))^T$. It is easy to see that $f(x)$ is a topical mapping. Furthermore, $y \in \mathbb{R}^n$ is an additive eigenvector of f with eigenvalue $\lambda \in \mathbb{R}$ if and only if $\exp(y) \in \mathbb{R}^n_{++}$ is an H-eigenvector of $\mathcal{A}$ in Qi's definition with H-eigenvalue $\exp((m-1)\lambda)$:

$$\mathcal{A}(\exp(y))^{m-1} = \exp((m-1)\lambda)(\exp(y))^{[m-1]}. \tag{4.7}$$

Note that (additive) eigenvectors of $f(x)$ connect bijectively to the (multiplicative) H-eigenvectors of $\mathcal{A}$ all of whose components are positive. The word eigenvector (eigenvalue) will be used in both contexts; the reader should have no difficulty inferring the right meaning.

Note further that a nonnegative tensor $\mathcal{A}$ connects to a topical mapping f if and only if $\mathcal{A}$ is strictly nonnegative, i.e., $F_{\mathcal{A}}(x) = (\mathcal{A}x^{m-1})^{[\frac{1}{m-1}]} > \theta$ for all $x > \theta$, which is defined by Hu, Huang and Qi [26]. Moveover, they established the Perron-Frobenius theorem for such a tensor.

The classical Perron-Frobenius theorem was extended to matrix pairs (A, B) (or matrix pencils) by many mathematical researchers. For instance, see Mangasarian [44], Mehrmann,Nabben and Virnik [45], Mehrmann, Olesky, Phan, Van den Driessche [46] for various studies.

Motivated by the eigenvalue theory of the tensor pairs $(\mathcal{A}, \mathcal{B})$ as well as matrix pairs (A, B), spontaneously, we introduce the concept of the eigenvalue to the topical mapping pairs (f, g).

Let f, g be two topical mappings on $\mathbb{R}^n$. If there exists $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^n$ such that

$$f(x) = \lambda + g(x), \tag{4.8}$$

we call λ the *g-eigenvalue* of f, and call $x \in \mathbb{R}^n$ the *g-eigenvector* of f associated with λ.

In this chapter, our purpose is to establish the existence and uniqueness of the eigenvalue with a corresponding eigenvector for the topical mapping pairs (f, g) if some orbits of f, g is bounded in the Hilbert semi-norm. That is, the nonlinear Perron-Frobenius property is extended to topical mapping pairs (f, g). Our approach is motivated by the study of the tensor pairs, the matrix pairs, and the coincidence points problem (fixed point problem) about nonlinear mappings, see Ref. [4, 13, 65, 73, 74]. It is based on the construction of eigenvalue for topical mappings as these were introduced in the context [21–23]. By particularizing the mapping g or f, some other methods and related results can be derived from our main theorems. In particular, taking $f(x) = \frac{1}{m-1}\log(\mathcal{A}(\exp(x))^{m-1})$ and $g(x) = \frac{1}{m-1}\log(\mathcal{B}(\exp(x))^{m-1})$, our results reduces to the Perron-Frobenius property of the eigenvalue problem for strictly nonnegative tensor pairs $(\mathcal{A}, \mathcal{B})$ without the requirement of the tensor inversion, i.e., the existence and uniqueness of positive eigenvalue with a corresponding positive eigenvector for strictly nonnegative tensor pairs $(\mathcal{A}, \mathcal{B})$; in the case that $f(x) = \log(A(\exp(x)))$ and $g(x) = \log(B(\exp(x)))$, we obtain that the Perron-Frobenius property for matrix pairs (A, B).

The rest of the chapter is organized as follows. In Section 4.2, we introduce the notion of eigenvalue for topical mapping pairs (f, g) as well as some related concepts, and show several nice properties of topical mapping pairs (f, g) such as the Collatz-Wielandt min-max type property, the independence of $\overline{\chi}(f, g)$ and so on, and give some lemmas and results. In Section 4.3, we study the existence and uniqueness of the eigenvalue with a corresponding eigenvector for the topical mapping pairs (f, g). In Section 4.4, as an application, the nonlinear Perron-Frobenius property of the nonnegative tensor pairs $(\mathcal{A}, \mathcal{B})$ (or matrix pairs (A, B)) is provided without the calculation of the tensor (matrix) inversion. Some related results derived from our main theorems and some concluding remarks can be found also in this section.

4.2 Properties of topical mapping pairs

Let $a, b \in \mathbb{R}$. We write $a \vee b$ and $a \wedge b$ indicating $\max\{a, b\}$ and $\min\{a, b\}$, respectively. Following Gunawardena, Keane [21] and Gaubert, Gunawardena [22], we also define $t, b : \mathbb{R}^n \to \mathbb{R}$ as ("top function") $t(x) = x_1 \vee x_2 \vee \cdots \vee x_n$, and ("bottom function") $b(x) = x_1 \wedge x_2 \wedge \cdots \wedge x_n$. Clearly, $b(x) = -t(-x)$, and both of which are topical mappings. It is easy to see that the supremum norm (or l^∞ norm) on $\mathbb{R}^n$ can be written as

$$\|x\| = t(x) \vee (-b(x)) \tag{4.9}$$

and

$$\|x\|_H = t(x) - b(x) \tag{4.10}$$

defines a semi-norm on $\mathbb{R}^n$, which is referred to as the *Hilbert semi-norm.* $d_H(x, y) = \|\log(x) - \log(y)\|_H$ gives the Hilbert projective metric on $\mathbb{R}^n_{++}$ while $\|x\|$ can define the Thompson's "part" metric on $\mathbb{R}^n_{++}$. For more details of Hilbert projective metric and Thompson's metric, see Nussbaum [50–52].

For a topical mapping f on $\mathbb{R}^n$, using (4.2) and (4.3), we easily obtain the following nonexpansiveness properties (see [21–23]): for all $x, y \in \mathbb{R}^n$,

$$\|f(x) - f(y)\| \leq \|x - y\|; \tag{4.11}$$

$$\|f(x) - f(y)\|_H \leq \|x - y\|_H; \tag{4.12}$$

$$t(f(x) - f(y)) \leq t(x - y). \tag{4.13}$$

For two topical mappings f, g on $\mathbb{R}^n$ with $fg = gf$, if x is a g-eigenvector of f with associated g-eigenvalue λ, i.e., $(\lambda, x) \in \mathbb{R} \times \mathbb{R}^n$ is a solution of (4.8),

$$f(x) = g(x) + \lambda,$$

then by the homogeneity (4.2) of f, g, we easily obtain

$$f^k(x) = g^k(x) + k\lambda,$$

and hence

$$b(\frac{f^k(x) - g^k(x)}{k}) = t(\frac{f^k(x) - g^k(x)}{k}) = \lambda. \tag{4.14}$$

That is, if g-eigenvalue λ of f exists, then it must be

$$\lambda = \lim_{k\to\infty} b(\frac{f^k(x) - g^k(x)}{k}) = \lim_{k\to\infty} t(\frac{f^k(x) - g^k(x)}{k}).$$

Let $y \in \mathbb{R}^n$ be any given. Then we have $f(y) = g(y) + (f(y) - g(y))$, and so

$$f(y) \leq g(y) + t(f(y) - g(y)).$$

It follows from the additive homogeneity (4.2) and increasing (4.3) of f, g that

$$f^2(y) \leq g(f(y)) + t(f(y) - g(y)) \leq g^2(y) + 2t(f(y) - g(y)).$$

Therefore, we have

$$f^k(y) \leq g^k(y) + kt(f(y) - g(y)),$$

and hence,

$$b(\frac{f^k(y) - g^k(y)}{k}) \leq t(\frac{f^k(y) - g^k(y)}{k}) \leq t(f(y) - g(y)). \tag{4.15}$$

A comparison between (4.14) and (4.15), one can not help asking: for two topical mappings f, g, whether or not the following two limits exist for each $y \in \mathbb{R}^n$,

$$\lim_{k\to\infty} b(\frac{f^k(y) - g^k(y)}{k}) \text{ and } \lim_{k\to\infty} t(\frac{f^k(y) - g^k(y)}{k});$$

if such limits exist, whether or not it is equal to some g-eigenvalue λ of f, and both limits are independent of $y \in \mathbb{R}^n$.

Next we go and try to answer (partially) the above questions with the aid of the following classical lemma.

Lemma 4.2.1 (Krengel [38, Lemma 5.1]) *If $\{a^k\}_{k\geq 1}$ is a subadditive sequence of real numbers, i.e.,*

$$a^{k+j} \leq a^k + a^j \text{ for all } k, j \in \mathbb{N},$$

where $\mathbb{N}$ is the set of all positive integer. Then $\lim_{n\to\infty} \frac{a^k}{k} = \gamma := \inf_{k\in\mathbb{N}} \frac{a^k}{k}$.

Following the nonexpansiveness property (4.13) and (4.11) of two topical mappings f, g together with Lemma 4.2.1, we have the following.

Proposition 4.2.2 *Let f, g be two topical mappings on $\mathbb{R}^n$ with $f(g(y)) = g(f(y))$ for some $y \in \mathbb{R}^n$. Then the following limits all exist and*

$$\lim_{k\to\infty} \|\frac{f^k(y) - g^k(y)}{k}\| = \inf\{\|\frac{f^k(y) - g^k(y)}{k}\|;\ k \in \mathbb{N}\}, \tag{4.16}$$

$$\lim_{k\to\infty} b(\frac{f^k(y) - g^k(y)}{k}) = \sup\{b(\frac{f^k(y) - g^k(y)}{k});\ k \in \mathbb{N}\}, \tag{4.17}$$

$$\lim_{k\to\infty} t(\frac{f^k(y) - g^k(y)}{k}) = \inf\{t(\frac{f^k(y) - g^k(y)}{k});\ k \in \mathbb{N}\}. \tag{4.18}$$

Furthermore, they are independent of $y \in \mathbb{R}^n$, only depend on f, g.

Proof. First we show (4.16). Let $a^k = \|\frac{f^k(y)-g^k(y)}{k}\|$. It follows from the nonexpansiveness property (4.11) of f, g along with $f(g(y)) = g(f(y))$ that

$$\begin{aligned}
a^{k+j} &= \|\frac{f^{k+j}(y) - f^k g^j(y) + f^k g^j(y) - g^{k+j}(y)}{k+j}\| \\
&\leq \frac{\|f^{k+j}(y) - f^k g^j(y)\|}{k+j} + \frac{\|f^k g^j(y) - g^{k+j}(y)\|}{k+j} \\
&\leq \|\frac{f^j(y) - g^j(y)}{j}\| + \|\frac{f^k(y) - g^k(y)}{k}\| \\
&= a^k + a^j.
\end{aligned}$$

An application of Lemma 4.2.1 yields the desired result.

Similarly, from the nonexpansiveness property (4.13) of f, g, it is easy to see (4.18). Since $t(\frac{g^k(y)-f^k(y)}{k}) = -b(\frac{f^k(y)-g^k(y)}{k})$, then we obtain easily (4.17).

Next we prove the independentness. It follows from the nonexpansiveness property (4.11) of f, g that for all $x \in \mathbb{R}^n$,

$$\begin{aligned}
\|(f^k(x) - g^k(x)) - (f^k(y) - g^k(y))\| &\leq \|f^k(x) - f^k(y)\| + \|g^k(y) - g^k(x)\| \\
&\leq 2\|x - y\|,
\end{aligned}$$

then we have

$$\|\frac{f^k(x) - g^k(x)}{k} - \frac{f^k(y) - g^k(y)}{k}\| \leq \frac{2\|x - y\|}{k}.$$

Therefore,

$$\lim_{k\to\infty} \|\frac{(f^k(x) - g^k(x))}{k} - \frac{(f^k(y) - g^k(y))}{k}\| = 0,$$

which means that

$$\lim_{k\to\infty} \|\frac{f^k(x) - g^k(x)}{k}\| = \lim_{k\to\infty} \|\frac{f^k(y) - g^k(y)}{k}\|$$

That is, the limit (4.16) is independent of $y \in \mathbb{R}^n$. Using the same proof technique, the limit (4.18) is independent of $y \in \mathbb{R}^n$. By duality, the independentness of the limit (4.17) is obtained easily also. ■

Since both (4.17) and (4.18) are only dependent on f, g, the concept of cycle-time (vector) in Gunawardena and Keane [21] may be extended to topical mapping pairs

(f, g). We call the value of (4.17) and (4.18) the *generalized lower cycle-time* and *generalized upper cycle-time* of (f, g), respectively denoted by $\underline{\chi}(f, g)$ and $\overline{\chi}(f, g)$, i.e.,

$$\underline{\chi}(f, g) = \lim_{k\to\infty} b(\frac{f^k(y) - g^k(y)}{k}) \text{ and } \overline{\chi}(f, g) = \lim_{k\to\infty} t(\frac{f^k(y) - g^k(y)}{k}).$$

We call

$$\chi(f, g) = \lim_{k\to\infty} \frac{f^k(y) - g^k(y)}{k}$$

the *generalized cycle-time vector* of (f, g). When $g = I$, identity mapping on $\mathbb{R}^n$, we still call them *lower cycle-time of f, cycle-time vector of f, upper cycle-time of f*, denoted respectively by $\underline{\chi}(f)$, $\chi(f)$, $\overline{\chi}(f)$. Clearly,

$$\underline{\chi}(f, g)e \leq \chi(f, g) \leq \overline{\chi}(f, g)e,$$

where $e = (1, 1, \cdots, 1)^T$;

whenever $f_1(x) \leq f_2(x)$ for all $x \in \mathbb{R}$, then $\overline{\chi}(f_1, g) \leq \overline{\chi}(f_2, g)$;

whenever $g_1(x) \leq g_2(x)$ for all $x \in \mathbb{R}$, then $\overline{\chi}(f, g_1) \geq \overline{\chi}(f, g_2)$.

Proposition 4.2.3 *Let f, g be two topical mappings on $\mathbb{R}^n$ with $f(g(x)) = g(f(x))$ for some $x \in \mathbb{R}^n$. Then for any $s \in \mathbb{N}$, we have*

$$\overline{\chi}(f^s, g^s) = s\overline{\chi}(f, g) \text{ and } \underline{\chi}(f^s, g^s) = s\underline{\chi}(f, g). \tag{4.19}$$

Proof. We only show the first equation. Another is similar, we omit it. It follows from the definition of $\overline{\chi}(f, g)$ that $\forall\varepsilon > 0, \exists k_0 \in \mathbb{N}$ such that

$$f^k(x) - g^k(x) \leq t(f^k(x) - g^k(x)) \leq k(\overline{\chi}(f, g) + \varepsilon) \text{ for all } k \geq k_0.$$

Following the homogeneity (4.2) and monotonicity (4.3) of f^s, g^s, we have

$$(f^k)^s(x) \leq (g^k)^s(x) + ks(\overline{\chi}(f, g) + \varepsilon) \text{ for all } k \geq k_0,$$

and hence

$$\frac{(f^s)^k(x) - (g^s)^k(x)}{k} \leq s(\overline{\chi}(f, g) + \varepsilon) \text{ for all } k \geq k_0.$$

Therefore, $\overline{\chi}(f^s, g^s) \leq s\overline{\chi}(f, g)$.

Similarly, $\forall \varepsilon > 0, \exists k_1 \in \mathbb{N}$ such that for all $k \geq k_1$

$$(f^s)^k(x) - (g^s)^k(x) \leq t(f^s)^k(x) - (g^s)^k(x)) \leq k(\overline{\chi}(f^s, g^s) + \varepsilon).$$

Then we have

$$f^{ks}(x) \leq g^{ks}(x) + k(\overline{\chi}(f^s, g^s) + \varepsilon),$$

and so $(ks = l)$

$$f^l(x) - g^l(x) \leq \frac{l}{s}(\overline{\chi}(f^s, g^s) + \varepsilon).$$

Therefore,

$$\frac{1}{s}\overline{\chi}(f^s, g^s) \geq \overline{\chi}(f, g) = \lim_{l \to \infty} t(\frac{f^l(x) - g^l(x)}{l}).$$

The desired result follows. ■

However, the existence of $\underline{\chi}(f, g)$ and $\overline{\chi}(f, g)$ depends on the fact that $f(g(x)) = g(f(x))$ for some $x \in \mathbb{R}^n$. Hence, one may ask:

Problem 4 *Do the generalized lower cycle-time and generalized upper cycle-time of non-commutative topical mapping pairs (f, g) exist? or Can the condition "$f(g(x)) = g(f(x))$ for some $x \in \mathbb{R}^n$" be removed or weakened in Proposition 4.2.1?*

Next we introduce the notions of the generalized super-eigenspace, generalized sub-eigenspace and generalized slice space for the topical mapping pairs (f, g), and furthermore, give the like Collatz-Wielandt property of (f, g). The *generalized super-eigenspace* $S^\lambda(f, g)$ associated to λ, *generalized sub-eigenspace* $S_\lambda(f, g)$ associated to λ, and *generalized slice space* $S^\lambda_\mu(f, g)$ associated to λ and μ, are defined respectively by

$$S^\lambda(f, g) = \{x \in \mathbb{R}^n;\ f(x) \leq \lambda + g(x)\},$$

$$S_\lambda(f, g) = \{x \in \mathbb{R}^n;\ f(x) \geq \lambda + g(x)\},$$

$$S_\mu^\lambda(f,g) = \{x \in \mathbb{R}^n;\ \mu + g(x) \le f(x) \le \lambda + g(x)\}.$$

It is obvious from (4.2) and (4.3) that all such spaces are invariant subsets whenever f and g commutate with each other. When $g = I$, the identity mapping on $\mathbb{R}^n$, we still call them *super-eigenspace* of f associated to λ, *sub-eigenspace* of f associated to λ, *slice space* of f associated to λ and μ, denoted respectively by $S^\lambda(f)$, $S_\lambda(f)$, $S_\mu^\lambda(f)$.

Let $\Lambda(f,g) \subset \mathbb{R}$ denote the set of those λ for which the corresponding generalized super-eigenspace is nonempty:

$$\Lambda(f,g) = \{\lambda \in \mathbb{R}; S^\lambda(f,g) \ne \emptyset\}.$$

Clearly, for any topical mapping pairs (f,g) on $\mathbb{R}^n$ and any $\lambda, \mu \in \mathbb{R}$,

$$S^\lambda(f,g) \cap S^\lambda(g) \subset S^\lambda(f);$$

$$S^{\lambda+\mu}(f,g) = S^\lambda(f-\mu, g);$$

$$\lambda \le \mu \Rightarrow S^\lambda(f,g) \subset S^\mu(f,g).$$

Now we give the like Collatz-Wielandt formula of the topical mapping pairs (f,g).

Proposition 4.2.4 *Let f, g be two topical mappings on $\mathbb{R}^n$ with $fg = gf$. Then*

$$\overline{\chi}(f,g) \le \inf_{x \in \mathbb{R}^n} t(f(x) - g(x)) = \inf \Lambda(f,g). \tag{4.20}$$

Proof. Let $a = \inf_{x\in\mathbb{R}^n} t(f(x) - g(x))$ and $b = \inf \Lambda(f,g)$. Then $\forall \varepsilon > 0, \exists y \in \mathbb{R}^n$ such that

$$f(y) - g(y) \le t(f(y) - g(y)) \le a + \varepsilon.$$

Then we have $f(y) \le g(y) + a + \varepsilon$, and hence $S^{a+\varepsilon}(f,g) \ne \emptyset$. Therefore, $b = \inf \Lambda(f,g) \le a + \varepsilon$. Since ε is arbitrary, then $b \le a$.

Following the definition of $b = \inf \Lambda(f, g)$, for all $\varepsilon > 0$, there exists $\lambda \in \Lambda(f, g)$ such that

$$\lambda \leq b + \varepsilon.$$

This implies that $S^{\lambda}(f, g) \subset S^{b+\varepsilon}(f, g)$, and so $S^{b+\varepsilon}(f, g) \neq \emptyset$. Then taking $x \in S^{b+\varepsilon}(f, g)$, and hence $f(x) - g(x) \leq b + \varepsilon$. Thus,

$$a = \inf_{y \in \mathbb{R}^n} t(f(y) - g(y)) \leq t(f(x) - g(x)) \leq b + \varepsilon.$$

Since ε is arbitrary, then $b \geq a$. So $a = b$.

Now we show $\overline{\chi}(f, g) \leq \inf_{x \in \mathbb{R}^n} t(f(x) - g(x))$. In fact, since $f(y) \leq g(y) + t(f(y) - g(y))$, then $f^k(y) \leq g^k(y) + kt(f(y) - g(y))$, and hence,

$$\overline{\chi}(f, g) = \lim_{k \to \infty} t(\frac{f^k(y) - g^k(y)}{k}) \leq t(f(y) - g(y)).$$

Since $\overline{\chi}(f, g)$ is independent of $y \in \mathbb{R}^n$, then the desired result follows. ■

From this proposition, we have a conjecture on $\overline{\chi}(f, g)$.

Conjecture $\overline{\chi}(f, g) = \inf_{x \in \mathbb{R}^n} t(f(x) - g(x)).$

4.3 Eigenvalue problem of topical mapping pairs

It follows from (4.14) that the g-eigenvalue of f must be $\overline{\chi}(f, g)$ if such a g-eigenvalue exists for two commutative topical mappings f, g. In particular, the g-eigenvalue of f is unique by Proposition 4.2.2. Then, what conditions would make a topical mapping f possess g-eigenvector. In this section, we will try to deal with this question. We extend the results of Gaubert and Gunawardena [22, Theorem 9] to topical mapping pairs (f, g).

Theorem 4.3.1 *Let f, g be two topical mappings on $\mathbb{R}^n$ with $fg = gf$. Assume that some orbits $\{f^k(y)\}$ and $\{g^k(y)\}$ of f, g are bounded in the Hilbert semi-norm as $k \to \infty$. If there exists $x \in \mathbb{R}^n$ such that $g(x) \geq x$, then f has a g-eigenvector z associated to the g-eigenvalue $\overline{\chi}(f, g)$.*

Proof. Let $F = f - \overline{\chi}(f, g)$. Then F is a topical mapping and

$$\overline{\chi}(F, g) = \lim_{l \to \infty} t(\frac{f^l(x) - l\overline{\chi}(f, g) - g^l(x)}{l}) = 0.$$

Following Proposition 4.2.3, we have $\overline{\chi}(F^k, g^k) = k\overline{\chi}(F, g) = 0$ for each $k \in \mathbb{N}$, and hence

$$\underline{\chi}(F^k, g^k) \leq \overline{\chi}(F^k, g^k) = 0.$$

From Proposition 4.2.4, it follows that for all $k \in \mathbb{N}$,

$$t(F^k(x) - g^k(x)) \geq \overline{\chi}(F^k, g^k) = 0. \tag{4.21}$$

By symmetry,

$$b(F^k(x) - g^k(x)) \leq \underline{\chi}(F^k, g^k) \leq 0. \tag{4.22}$$

Since both $f^k(y)$ and $g^k(y)$ are bounded in the Hilbert semi-norm, then so is $\{f^k(y) - g^k(y)\}$, and hence for all $k \in \mathbb{N}$ and some $M' > 0$,

$$\|F^k(y) - g^k(y)\|_H = \|f^k(y) - g^k(y)\|_H \leq M'.$$

By (4.12), we have

$$\begin{aligned} &\|(F^k(y) - g^k(y)) - (F^k(x) - g^k(x))\|_H \\ &\leq \|F^k(y) - F^k(x)\|_H + \|g^k(x) - g^k(y)\|_H \\ &\leq 2\|x - y\|_H, \end{aligned}$$

and hence for all $k \in \mathbb{N}$ and some $M > 0$

$$\max\{\|F^k(x)\|_H, \|g^k(x)\|_H, \|F^k(x) - g^k(x)\|_H\} \leq M. \tag{4.23}$$

It follows from (4.21) and (4.22) that $-b(F^k(x) - g^k(x) \geq 0$ and

$$\begin{aligned}\|F^k(x) - g^k(x)\| &= t(F^k(x) - g^k(x)) \vee (-b(F^k(x) - g^k(x))) \\ &\leq t(F^k(x) - g^k(x)) + (-b(F^k(x) - g^k(x)) \\ &= \|F^k(x) - g^k(x)\|_H \leq M.\end{aligned}$$

Thus, $\|F^k(x) - g^k(x)\|$ is bounded. Since $g(x) \geq x$ by the assumption, then

$$\cdots \geq g^{k+1}(x) \geq g^k(x) \geq \cdots \geq g^2(x) \geq g(x) \geq x, \tag{4.24}$$

and so $b(g^k(x))$ is bounded from below as $k \to \infty$. As $\|g^k(x)\|_H = t(g^k(x)) - b(g^k(x))$ remains bounded by (4.23), then $\|g^k(x)\|$ is bounded as $k \to \infty$, and hence, so is $\|F^k(x)\|$.

Next we iteratively defined a sequence $\{x^k\}$ to find some coincidence points of mappings F, g, and further such a point is a g-eigenvector of f.

$$\begin{aligned}x^1 &= g(x), \\ x^2 &= F(x^1) = F(g(x)), \\ x^3 &= g(x^2) = g(F(g(x))) = F(g^2(x)), \\ &\vdots \\ x^{2k} &= F(x^{2k-1}) = F^k(g^k(x)), \\ x^{2k+1} &= g(x^{2k}) = F^k(g^{k+1}(x)), \\ &\vdots\end{aligned} \tag{4.25}$$

Then we have

$$\|x^{2k} - F^k(x)\| = \|F^k(g^k(x)) - F^k(x)\| \leq \|g^k(x) - x\|$$

and

$$\|x^{2k+1} - F^k(x)\| = \|F^k(g^{k+1}(x)) - F^k(x)\| \leq \|g^{k+1}(x) - x\|.$$

This together with the boundedness of both $\|g^k(x)\|$ and $\|F^k(x)\|$ imply that the sequence $\{x^k\}$ is bounded in the supremum norm as $k \to \infty$. Thus, we can choose $u, v \in \mathbb{R}^n$ such that

$$u = \lim_{l\to\infty} \bigwedge_{k\geq l} x^{2k} \quad \text{and} \quad v = \lim_{l\to\infty} \bigwedge_{k\geq l} x^{2k+1}$$

where $\wedge x^k = (\wedge x_1^k, \wedge x_2^k, \cdots, \wedge x_n^k)$. It follows from (4.24) and the monotonicity of g, F that

$$x^{2k} = F^k(g^k(x)) \leq F^k(g^{k+1}(x)) = x^{2k+1} \text{ for all } k \in \mathbb{N},$$

which means $u \leq v$. By continuity and monotonicity of g, F, we have

$$g(u) = \lim_{l\to\infty} g(\bigwedge_{k\geq l} x^{2k}) \leq \lim_{l\to\infty} \bigwedge_{k\geq l} g(x^{2k}) = \lim_{l\to\infty} \bigwedge_{k\geq l} x^{2k+1} = v$$

and

$$F(v) = \lim_{l\to\infty} F(\bigwedge_{k\geq l} x^{2k-1}) \leq \lim_{l\to\infty} \bigwedge_{k\geq l} F(x^{2k-1}) = \lim_{l\to\infty} \bigwedge_{k\geq l} x^{2k} = u$$

From the monotonicity of g along with the fact that $g(u) \leq v$ and $u \leq v$, it follows that

$$g(F(v)) \leq g(u) \leq v \text{ and } F(v) \leq u \leq v. \tag{4.26}$$

Let $G = gF$. Using the same iterative technique as (4.25), we can define a sequence $\{y^k\}$,

$$\begin{aligned} y^1 = F(v), y^2 &= G(y^1), y^3 = F(y^2), \\ &\cdots\cdots, \\ y^{2k} &= G(y^{2k-1}) = F^k(G^k(v)), \\ y^{2k+1} &= F(y^{2k}) = F^{k+1}(G^k(v)), \\ &\cdots\cdots. \end{aligned}$$

By the monotonicity of F, G and (4.26), we obtain

$$\begin{aligned} y^{2k+2} &= G(y^{2k+1}) = F^{k+1}(G^k(G(v))) \\ &\leq F(F^k(G^k(v))) = F(y^{2k}) = y^{2k+1} = G^k(F^k(F(v))) \\ &\leq F^k(G^k(v)) = G(F^k(G^{k-1}(v))) = G(y^{2k-1}) = y^{2k}. \end{aligned}$$

That is, for all $k \in \mathbb{N}$,

$$y^{2k+2} \leq y^{2k+1} \leq y^{2k}, \text{ and hence } y^{k+1} \leq y^k.$$

This implies that the sequence $\{y^k\}$ is nonincreasing as $k \to \infty$. Since both $\|g^k(x)\|$ and $\|F^k(x)\|$ is bounded as $k \to \infty$ and

$$\|g^k(x) - g^k(v)\| \leq \|x - v\| \text{ and } \|F^k(x) - F^k(v)\| \leq \|x - v\|,$$

then $\|g^k(v)\|$, $\|F^k(v)\|$ and $\|G^k(v)\|(= \|F^k(g^k(v))\|)$ are bounded also. Thus, the sequence $\{y^k\}$ is nonincreasing and bounded in the supremum norm as $k \to \infty$. This implies that $\{y^k\}$ must converge to a point in $\mathbb{R}^n$, say z. By continuity of G, F, we have

$$F(z) = \lim_{k\to\infty} F(y^{2k}) = \lim_{k\to\infty} y^{2k+1} = z$$

and

$$g(F(z)) = G(z) = \lim_{k\to\infty} G(y^{2k-1}) = \lim_{k\to\infty} y^{2k} = z.$$

As a result,

$$g(z) = g(F(z)) = z = F(z).$$

Consequently, $F(z) = f(z) - \overline{\chi}(f, g) = g(z)$, as required. ■

Corollary 4.3.2 *Let f, g be two topical mappings on $\mathbb{R}^n$ with $fg = gf$. Assume that some generalized super-eigenspace $S^\lambda(f, g)$ is non-empty and bounded in the Hilbert semi-norm. If there exists $x \in \mathbb{R}^n$ such that $g(x) \geq x$, then f has a g-eigenvector z associated to the g-eigenvalue $\overline{\chi}(f, g)$.*

Proof. Since $S^\lambda(f, g)$ is non-empty and bounded in the Hilbert semi-norm, we may take $y \in S^\lambda(f, g)$, i.e.,

$$f(y) \leq g(y) + \lambda.$$

By the assumption $fg = gf$ and (4.2) and (4.3), we have for all $k \in \mathbb{N}$

$$f(f^k(y)) = f^k(f(y)) \leq f^k(g(y) + \lambda) = g(f^k(y)) + \lambda$$

and

$$f(g^k(y)) = g^k(f(y)) \leq g^k(g(y) + \lambda) = g(g^k(y)) + \lambda.$$

Thus, $f^k(y), g^k(y) \in S^\lambda(f, g)$ for all $k \in \mathbb{N}$, and so they are bounded in the Hilbert semi-norm as $k \to \infty$. By Theorem 4.3.1, the desired result follows. ■

Corollary 4.3.3 *Let f, g be two topical mappings on $\mathbb{R}^n$ with $fg = gf$. Assume that some super-eigenspaces $S^\lambda(f)$ and $S^\mu(g)$ are non-empty and bounded in the Hilbert* *semi-norm. If there exists $x \in \mathbb{R}^n$ such that $g(x) \geq x$, then f has a g-eigenvector z associated to the g-eigenvalue $\overline{\chi}(f, g)$.*

Proof. Since $S^\lambda(f)$ and $S^\mu(g)$ are non-empty and bounded in the Hilbert semi-norm, we may choose $y \in S^\lambda(f)$ and $y^* \in S^\mu(g)$, i.e.,

$$f(y) \leq y + \lambda \text{ and } g(y^*) \leq y^* + \mu.$$

By (4.2) and (4.3), we have for all $k \in \mathbb{N}$

$$f(f^k(y)) = f^k(f(y)) \leq f^k(y + \lambda) = f^k(y) + \lambda$$

and

$$g(g^k(y^*)) = g^k(g(y^*)) \leq g^k(y * +\mu) = g^k(y*) + \mu.$$

Thus, $f^k(y) \in S^\lambda(f)$ and $g^k(y^*) \in S^\mu(g)$ for all $k \in \mathbb{N}$, and so they are bounded in the Hilbert semi-norm as $k \to \infty$. Since

$$\|g^k(y^*) - g^k(y)\|_H \leq \|y^* - y\|_H,$$

$\{g^k(y)\}$ is bounded in the Hilbert semi-norm as $k \to \infty$ also. By Theorem 4.3.1, the desired result follows. ■

Let $g(x) = x$ for all $x \in \mathbb{R}^n$. Then we have the following which is one of the most important results in Gaubert and Gunawardena [22].

Corollary 4.3.4 (Gaubert and Gunawardena [22, Theorem 9]) *Let f be a topical mappings on $\mathbb{R}^n$. If some orbits of f are bounded in the Hilbert semi-norm, then f has an eigenvector in $\mathbb{R}^n$ associated to a unique eigenvalue $\overline{\chi}(f)$.*

4.4 Generalized $H-$eigenvalue of nonnegative tensors

For an m-order n-dimensional tensor $\mathcal{A}$, when $(\mathcal{A}x^{m-1})^{[\frac{1}{m-1}]}$ can be well defined, let

$$F_{\mathcal{A}}(x) = (\mathcal{A}x^{m-1})^{[\frac{1}{m-1}]}.$$

$\mathcal{A}$ is called *strictly nonnegative* if $F_{\mathcal{A}}(x) > \theta$ for all $x > \theta$, which is first defined by Hu, Huang and Qi [26]. They also showed that each weakly irreducible nonnegative tensor is strictly nonnegative.

Let an m-order n-dimensional tensor $\mathcal{A}$ be nonnegative. $\mathcal{A}$ is called *reducible*, if there exists a nonempty proper index subset $\mathcal{N} \in \{1, 2, \cdots, n\}$ such that

$$a_{i_1 i_2 \cdots i_m} = 0 \text{ for all } i_1 \in \mathcal{N}, \text{ for all } i_2, i_3, \cdots, i_m \notin \mathcal{N}.$$

If $\mathcal{A}$ is not reducible, then we call it *irreducible*.

The directed graph of $\mathcal{A}$, $G(\mathcal{A}) = (V, E(\mathcal{A}))$, where V is its vertex set $\{1, 2, \cdots, n\}$ and $E(\mathcal{A})$ is the set of its all edges (the edge $(i, j) \in E(\mathcal{A})$ if and only if $a_{ii_2 \cdots i_m} > 0$ for some $i_k = j, k = 2, 3, \cdots, m$). An m-order n-dimensional tensor $\mathcal{A}$ is called *weakly irreducible* if $G(\mathcal{A})$ is strongly connected.

The notion of irreducible tensor is first introduced by Lim [40]. Chang, Pearson, Zhang [11] adopted this notion in their subsequent work, and obtained the Perron-Frobenius theorem for such tensor pairs $(\mathcal{A}, \mathcal{B})$ which involves the calculation of the inverse of $\mathcal{B}$. The concept of weakly irreducible tensor is first introduced by Friedland, Gauber, Han [19], and the Perron-Frobenius theorem was proved for such a class of tensor. They also showed that each irreducible nonnegative tensor is weakly irreducible. Yang and Yang [70] gave the following definition of weakly irreducible tensor, which directly reveals the relationship between the weakly irreducible tensor and irreducible tensor, and also showed the equivalence of this definition and Friedland, Gauber, Han's.

An m-order n-dimensional nonnegative tensor $\mathcal{A}$ is called *weakly reducible*, if there exists a nonempty proper index subset $\mathcal{N} \in \{1, 2, \cdots, n\}$ such that

$$a_{i_1 i_2 \cdots i_m} = 0 \text{ for all } i_1 \in \mathcal{N}, \text{ for some } i_j \notin \mathcal{N}, j = 2, 3, \cdots, m.$$

If $\mathcal{A}$ is not weakly reducible, then we call it *weakly irreducible*.

Theorem 4.4.1 *Let $\mathcal{A}, \mathcal{B}$ be two weakly irreducible and nonnegative tensors with same order m and same dimension n and $F_{\mathcal{A}}(F_{\mathcal{B}}(y)) = F_{\mathcal{B}}(F_{\mathcal{A}}(y))$ for some $y \in \mathbb{R}^n_{++}$. Then for each $x \in \mathbb{R}^n$, two limits*

$$\lim_{k\to\infty} \max_{i=1,\cdots,n} \left(\frac{\log(F^k_{\mathcal{A}}(\exp(x))) - \log(F^k_{\mathcal{B}}(\exp(x)))}{k} \right)_i \tag{4.27}$$

$$\lim_{k\to\infty} \min_{i=1,\cdots,n} \left(\frac{\log(F^k_{\mathcal{A}}(\exp(x))) - \log(F^k_{\mathcal{B}}(\exp(x)))}{k} \right)_i \tag{4.28}$$

exists and only depends on $\mathcal{A}, \mathcal{B}$, where $(x)_i = x_i$ for $x = (x_1, x_2, \cdots, x_n)^T$.

Proof. Let $f(x) = \log(F_{\mathcal{A}}(\exp(x)))$ and $g(x) = \log(F_{\mathcal{B}}(\exp(x)))$. Then both f and g are topical mappings and

$$f^2(x) = \log(F_{\mathcal{A}}(\exp(\log(F_{\mathcal{A}}(\exp(x)))))) = \log(F^2_{\mathcal{A}}(\exp(x))),$$

and hence, the following can be done in the same manner,

$$f^k(x) = \log(F^k_{\mathcal{A}}(\exp(x))) \text{ and } g^k(x) = \log(F^k_{\mathcal{B}}(\exp(x))).$$

Let $v = \log(y)$. It follows from the fact that $F_{\mathcal{A}}(F_{\mathcal{B}}(y)) = F_{\mathcal{B}}(F_{\mathcal{A}}(y))$ that

$$\begin{aligned} f(g(v)) =& \log(F_{\mathcal{A}}(\exp(\log(F_{\mathcal{B}}(\exp(v)))))) = \log(F_{\mathcal{A}}(F_{\mathcal{B}}(\exp(\log(y))))) \\ =& \log(F_{\mathcal{A}}(F_{\mathcal{B}}(y))) = \log(F_{\mathcal{B}}(F_{\mathcal{A}}(y))) \\ =& \log(F_{\mathcal{B}}(F_{\mathcal{A}}(\exp(\log(y))))) = \log(F_{\mathcal{B}}(\exp(\log(F_{\mathcal{A}}(\exp(v)))))) \\ =& g(f(v)). \end{aligned}$$

From Proposition 4.2.2, the desired conclusion follows. ∎

Since both the limits (4.27) and (4.28) only depend on $\mathcal{A}, \mathcal{B}$, as already mentioned (see page 55), we respectively call the value of (4.27) and (4.28) the *generalized lower cycle-time* and *generalized upper cycle-time* of $\mathcal{A}, \mathcal{B}$, denoted by $\underline{\chi}(\mathcal{A}, \mathcal{B})$ and $\overline{\chi}(\mathcal{A}, \mathcal{B})$, i.e.,

$$\underline{\chi}(\mathcal{A}, \mathcal{B}) = \lim_{k\to\infty} \min_{i=1,\cdots,n} \left(\frac{\log(F_{\mathcal{A}}^k(\exp(x))) - \log(F_{\mathcal{B}}^k(\exp(x)))}{k} \right)_i$$
$$\overline{\chi}(\mathcal{A}, \mathcal{B}) = \lim_{k\to\infty} \max_{i=1,\cdots,n} \left(\frac{\log(F_{\mathcal{A}}^k(\exp(x))) - \log(F_{\mathcal{B}}^k(\exp(x)))}{k} \right)_i .$$

We also call

$$\chi(\mathcal{A}, \mathcal{B}) = \lim_{k\to\infty} \frac{\log(F_{\mathcal{A}}^k(\exp(x))) - \log(F_{\mathcal{B}}^k(\exp(x)))}{k}$$

the *generalized cycle-time vector* of $(\mathcal{A}, \mathcal{B})$. When $F_{\mathcal{A}}F_{\mathcal{B}} = F_{\mathcal{B}}F_{\mathcal{A}}$, it is easy to obtain the following result from Proposition 4.2.4.

Theorem 4.4.2 *Let $\mathcal{A}, \mathcal{B}$ be two weakly irreducible and nonnegative tensors with same order m and same dimension n with $F_{\mathcal{A}}(F_{\mathcal{B}}) = F_{\mathcal{B}}(F_{\mathcal{A}})$. Then*

$$\overline{\chi}(\mathcal{A}, \mathcal{B}) \le \inf_{x\in\mathbb{R}^n} \max_{i=1,\cdots,n} \left(\log(F_{\mathcal{A}}(\exp(x))) - \log(F_{\mathcal{B}}(\exp(x)))\right)_i . \tag{4.29}$$

We now show the Perron-Frobenius property for nonnegative tensor pairs $(\mathcal{A}, \mathcal{B})$ without the requirement of the tensor inversion, which is referred to as an immediate conclusions of Theorem 4.3.1.

Theorem 4.4.3 *Let $\mathcal{A}, \mathcal{B}$ be two weakly irreducible and nonnegative tensors with order m and dimension n and $F_{\mathcal{A}}F_{\mathcal{B}} = F_{\mathcal{B}}F_{\mathcal{A}}$. If $\exists x > \theta$ such that $\mathcal{B}x^{m-1} \ge x^{[m-1]}$, then $\mathcal{A}$ has a unique positive $\mathcal{B}$-eigenvalue* $\exp((m-1)\lambda)$ *with a corresponding positive $\mathcal{B}$-eigenvector, where*

$$\lambda = \overline{\chi}(\mathcal{A}, \mathcal{B}) = \lim_{k\to\infty} \max_{i=1,\cdots,n} \left(\frac{\log(F_{\mathcal{A}}^k(\exp(x))) - \log(F_{\mathcal{B}}^k(\exp(x)))}{k} \right)_i .$$

Proof. Let $f(x) = \log(F_{\mathcal{A}}(\exp(x)))$ and $g(x) = \log(F_{\mathcal{B}}(\exp(x)))$. Similarly to Theorem 4.4.1, we also have $fg = gf$, and both directed graph $G(f)$ and $G(g)$ are strongly connected by Friedland, Gauber, Han [19, Lemma 3.2]. It follows from Gaubert and Gunawardena [22, Theorem 10] that both $S^\lambda(f)$ and $S^\lambda(g)$ are bounded in the Hilbert semi-norm. From the assumption $\mathcal{B}x^{m-1} \geq x^{[m-1]}$, taking $x^* = \log(x)$, we have

$$\begin{aligned} g(x^*) = &\ \log(\mathcal{B}(\exp(x^*))^{m-1})^{[\frac{1}{m-1}]} = \log(\mathcal{B}x^{m-1})^{[\frac{1}{m-1}]} \\ \geq &\ \log(x^{[m-1]})^{[\frac{1}{m-1}]} = \log(x) = x^*. \end{aligned}$$

By Corollary 4.3.3, there exists $z \in \mathbb{R}^n$ such that $f(z) = g(z) + \lambda$, where $\lambda = \overline{\chi}(f, g) = \overline{\chi}(\mathcal{A}, \mathcal{B})$. That is,

$$\mathcal{A}(\exp(z))^{m-1} = \exp((m-1)\lambda)\mathcal{B}(\exp(z))^{m-1}.$$

Let $v = \exp(z)$ and $\mu = \exp((m-1)\lambda)$. Clearly, $\mu > 0$ and $v > 0$. This yields the desired result. ■

When $m = 2$, we have the Perron-Frobenius property for nonnegative matrix pairs.

Corollary 4.4.4 *Let A, B be two $n \times n$ nonnegative and irreducible matrices with $AB = BA$. If $\exists x > \theta$ such that $Bx \geq x$, then the nonnegative matrix pairs (A, B) has a unique positive eigenvalue* $\exp(\overline{\chi}(A, B))$ *with a corresponding positive eigenvector, where*

$$\overline{\chi}(A, B) = \lim_{k\to\infty} \max_{i=1,\cdots,n} \left(\frac{\log(A^k(\exp(x))) - \log(B^k(\exp(x)))}{k} \right)_i. \tag{4.30}$$

Remark 4.4.1 *In Corollary 4.4.4, if $B = E$, the unit matrix, then it is not difficult to derive the classical Perron-Frobenius theorem of nonnegative matrix. In Theorem 4.4.3, if $\mathcal{B} = \mathcal{I}$, the unit tensor, then the nonlinear Perron-Frobenius theorem (Friedland, Gauber, Han [19, theorem 3.2]) of higher order nonnegative tensor can be reached.*

Remark 4.4.2 *Theorem 4.3.1 actually provides a method for constructing g-eigenvector of f, while g-eigenvalue of f may be obtained by calculating the generalized upper cycle-time $\overline{\chi}(f, g)$ (Proposition 4.2.2),*

$$\overline{\chi}(f, g) = \lim_{k\to\infty} t(\frac{f^k(y) - g^k(y)}{k}) \quad \textit{for all } y \in \mathbb{R}^n.$$

Since $\overline{\chi}(f, g)$ is independent of the initial value $y \in \mathbb{R}^n$, we may take some special point to make it easier for computing. For example, the initial value $y = \theta$ or $e = (1, 1, \cdots, 1)^T$ or the unit vector, etc.

Remark 4.4.3 *Reviewing the proof of Theorem 4.3.1 again, it is not difficult to see that the conclusions still hold if g in the hypothesis, "there exists $x \in \mathbb{R}^n$ such that $g(x) \geq x$", is replaced by f. On the other hand, we also have the following questions: is such a condition necessary? Can it be removed or be replaced by other more general condition to meet the same conclusion? So, we have to work hard for further study.*

Remark 4.4.4 *The commutativity of topical mapping pairs (f, g) seems not to be necessary for the existence of eigenvalue-eigenvector. The following is an example that a non-commutative topical mapping pairs may possess eigenvalue-eigenvector.*

Example 4.4.5 *Consider the topical mappings $f, g : \mathbb{R}^3 \to \mathbb{R}^3$*

$$f(x) = \begin{pmatrix} x_2 \vee x_3 \\ (x_1 \vee x_2) \wedge x_3 \\ (x_2 \vee x_3) \wedge x_1 \end{pmatrix}, \qquad g(x) = \begin{pmatrix} x_2 \vee x_1 \\ x_3 \vee x_2 \\ x_1 \vee x_3 \end{pmatrix}.$$

It is easy to see that $fg \neq gf$. However, 0 is an g-eigenvalue of f, i.e., $f(x) = g(x)$ for all $x_1 = x_2 = x_3$. Furthermore, for $x = (1, 2, 3)$, $g(x) \geq x$.

Chapter 5

Pareto eigenvalue of higher order tensors

5.1 Introduction

Let $\mathcal{A}$ be an m-order n-dimensional tensor. For an element $x = (x_1, x_2, \cdots, x_n)^T \in \mathbb{R}^n$ or $\mathbb{C}^n$, $\mathcal{A}x^m$ is defined by

$$\mathcal{A}x^m = \sum_{i_1,i_2,\cdots,i_m=1}^{n} a_{i_1 i_2 \cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m}; \tag{5.1}$$

$\mathcal{A}x^{m-1}$ is a vector in $\mathbb{R}^n$ (or $\mathbb{C}^n$) with its ith component defined by

$$(\mathcal{A}x^{m-1})_i = \sum_{i_2,\cdots,i_m=1}^{n} a_{i i_2 \cdots i_m} x_{i_2} \cdots x_{i_m} \text{ for } i = 1, 2, \ldots, n. \tag{5.2}$$

An m-order n-dimensional tensor $\mathcal{A}$ is said to be *symmetric* if its entries $a_{i_1 \cdots i_m}$ are invariant for any permutation of the indices. Clearly, each m-order n-dimensional symmetric tensor $\mathcal{A}$ defines a homogeneous polynomial $\mathcal{A}x^m$ of degree m with n variables and vice versa.

Given an m-order n-dimensional symmetric tensor $\mathcal{A}$, we consider a constrained optimization problem of the form:

$$\begin{aligned}
&\min \frac{1}{m}\mathcal{A}x^m \\
&s.t.\ x^T x^{[m-1]} = 1 \qquad (5.3)\\
&\quad x \in \mathbb{R}^n_+.
\end{aligned}$$

Then the Lagrange function of the problem (5.3) is given clearly by

$$L(x,\lambda,y) = \frac{1}{m}\mathcal{A}x^m + \frac{1}{m}\lambda(1 - x^T x^{[m-1]}) - x^T y \qquad (5.4)$$

where $x, y \in \mathbb{R}^n_+$, $\frac{\lambda}{m} \in \mathbb{R}$ is the Lagrange multiplier of the equality constraint and y is the Lagrange multiplier of non-negative constraint. So the solution x of the problem (5.3) satisfies the following conditions:

$$\mathcal{A}x^{m-1} - \lambda x^{[m-1]} - y = \theta \qquad (5.5)$$

$$1 - x^T x^{[m-1]} = 0 \qquad (5.6)$$

$$x^T y = 0 \qquad (5.7)$$

$$x, y \in \mathbb{R}^n_+, \qquad (5.8)$$

where $\theta = (0, 0, \cdots, 0)^T$. The equation (5.6) means that $\sum\limits_{i=1}^{n} x_i^m = 1$. It follows from the equations (5.5), (5.7) and (5.8) that

$$\begin{aligned}
x^T y = x^T \mathcal{A}x^{m-1} - \lambda x^T x^{[m-1]} = 0 \\
x \geq \theta, \mathcal{A}x^{m-1} - \lambda x^{[m-1]} = y \geq \theta,
\end{aligned}$$

and hence,

$$\begin{cases}
\mathcal{A}x^m = \lambda x^T x^{[m-1]} \\
\mathcal{A}x^{m-1} - \lambda x^{[m-1]} \geq \theta \\
x \geq \theta.
\end{cases} \qquad (5.9)$$

Following Qi [59] ($H-$eigenvalue of the tensor $\mathcal{A}$) and Seeger [67] (Pareto eigenvalue of the matrix A), for a m-order n-dimensional tensor $\mathcal{A}$, a real number λ is called *Pareto* $H-$*eigenvalue* of the tensor $\mathcal{A}$ if there exists a non-zero vector $x \in \mathbb{R}^n$ satisfying the system (5.9). The non-zero vector x is called a *Pareto $H-$eigenvector* of the tensor $\mathcal{A}$ associated to λ.

Similarly, for given an m-order n-dimensional symmetric tensor $\mathcal{A}$, we consider another constrained optimization problem of the form ($m \geq 2$):

$$\begin{aligned} &\min \frac{1}{m}\mathcal{A}x^m \\ &s.t.\ x^Tx = 1 \\ &\quad\ x \in \mathbb{R}^n_+. \end{aligned} \tag{5.10}$$

Obviously, when $x \in \mathbb{R}^n$, $x^Tx = 1$ if and only if $(x^Tx)^{\frac{m}{2}} = 1$. The corresponding Lagrange function may be written in the form

$$L(x, \mu, y) = \frac{1}{m}\mathcal{A}x^m + \frac{1}{m}\mu(1 - (x^Tx)^{\frac{m}{2}}) - x^Ty.$$

So the solution x of the problem (5.10) satisfies the conditions:

$$\mathcal{A}x^{m-1} - \mu(x^Tx)^{\frac{m}{2}-1}x - y = \theta,\ 1 - (x^Tx)^{\frac{m}{2}} = 0,\ x^Ty = 0,\ x, y \in \mathbb{R}^n_+.$$

Then we also have $\sum\limits_{i=1}^{n} x_i^2 = 1$ and

$$\begin{cases} \mathcal{A}x^m = \mu(x^Tx)^{\frac{m}{2}} \\ \mathcal{A}x^{m-1} - \mu(x^Tx)^{\frac{m}{2}-1}x \geq \theta \\ x \geq \theta. \end{cases} \tag{5.11}$$

Following Qi [59] ($Z-$eigenvalue of the tensor $\mathcal{A}$) and Seeger [67] (Pareto eigenvalue of the matrix A), for an m-order n-dimensional tensor $\mathcal{A}$, a real number μ is said to be *Pareto $Z-$eigenvalue* of the tensor $\mathcal{A}$ if there is a non-zero vector $x \in \mathbb{R}^n$ satisfying the system (5.11). The non-zero vector x is called a *Pareto $Z-$eigenvector* of the tensor $\mathcal{A}$ associated to μ.

So the constrained optimization problem (5.3) and (5.10) of homogeneous polynomial may be respectively solved by means of the Pareto H-eigenvalue (5.9) and Pareto

$Z-$eigenvalue (5.11) of the corresponding tensor. It will be an interesting work to compute the Pareto H-eigenvalue ($Z-$eigenvalue) of a higher order tensor.

When $m = 2$, both Pareto $H-$eigenvalue and Pareto $Z-$eigenvalue of the m-order n-dimensional tensor obviously changes into Pareto eigenvalue of the symmetric matrix A, i.e., μ is called *Pareto eigenvalue* of the symmetric matrix A if there exists a non-zero element $x \in \mathbb{R}^n$ (or $\mathbb{C}^n$) such that

$$\begin{cases} Ax^2 = \mu x^T x \\ Ax - \mu x \geq \theta \\ x \geq \theta. \end{cases} \tag{5.12}$$

The concept of Pareto eigenvalue is first introduced and used by Seeger [67] for studying the equilibrium processes defined by linear complementarity conditions. For more details, also see Hiriart-Urruty and Seeger [25].

In this chapter, we will study the properties of the Pareto H-eigenvalue ($Z-$eigenvalue) of a higher order tensor $\mathcal{A}$. It will be proved that a real number λ is Pareto H-eigenvalue ($Z-$eigenvalue) of $\mathcal{A}$ if and only if λ is H^{++}-eigenvalue (Z^{++}-eigenvalue) of some $|\mathcal{N}|$-dimensional principal sub-tensor of $\mathcal{A}$ with corresponding $H-$eigenvector ($Z-$eigenvector) w and

$$\sum_{i_2,\cdots,i_m \in \mathcal{N}} a_{ii_2\cdots i_m} w_{i_2} w_{i_3} \cdots w_{i_m} \geq 0 \text{ for } i \in \{1, 2, \cdots, n\} \setminus \mathcal{N}.$$

So we may calculate some Pareto H-eigenvalue ($Z-$eigenvalue) of a higher order tensor by means of H^{++}-eigenvalue (Z^{++}-eigenvalue) of the lower dimensional tensors. What's more, we will show that

$$\min_{\substack{x \geq 0 \\ \|x\|_m = 1}} \mathcal{A}x^m = \min\{\mu; \mu \text{ is Pareto } H\text{-eigenvalue of } \mathcal{A}\} \tag{5.13}$$

$$\min_{\substack{x \geq 0 \\ \|x\|_2 = 1}} \mathcal{A}x^m = \min\{\mu; \mu \text{ is Pareto } Z\text{-eigenvalue of } \mathcal{A}\}. \tag{5.14}$$

Therefore, we may solve the constrained minimization problem for homogeneous polynomial and test the (strict) copositivity of a symmetric tensor $\mathcal{A}$ with the help of computing the Pareto H-eigenvalue (or Pareto Z-eigenvalue) of a symmetric tensor. As a corollary, a symmetric tensor $\mathcal{A}$ is copositive if and only if every Pareto H-eigenvalue ($Z-$eigenvalue) of $\mathcal{A}$ is non-negative and $\mathcal{A}$ is strictly copositive if and only if every Pareto H-eigenvalue ($Z-$eigenvalue) of $\mathcal{A}$ is positive.

5.2 Preliminaries and Basic facts

In 1952, Motzkin [47] introduced the concept of copositive matrices, which is an important concept in applied mathematics and graph theory. A real symmetric matrix A is said to be

- *copositive* if $x \geq \theta$ implies $x^T Ax \geq 0$;
- *strictly copositive* if $x \geq \theta$ and $x \neq \theta$ implies $x^T Ax > 0$.

Recently, Qi [63] extended this concept to the higher order symmetric tensors and obtained its some nice properties as ones of copositive matrices. Let $\mathcal{A}$ be a real symmetric tensor of order m and dimension n. $\mathcal{A}$ is said to be

- *copositive* if $\mathcal{A}x^m \geq 0$ for all $x \in \mathbb{R}^n_+$;
- *strictly copositive* if $\mathcal{A}x^m > 0$ for all $x \in \mathbb{R}^n_+ \setminus \{\theta\}$.

Let $\|\cdot\|$ denote any norm on $\mathbb{R}^n$. Now we give the equivalent definition of (strict) copositivity of a symmetric tensor in the sense of any norm on $\mathbb{R}^n$.

Lemma 5.2.1 *Let $\mathcal{A}$ be a symmetric tensor of order m and dimension n. Then*

(i) $\mathcal{A}$ is copositive if and only if $\mathcal{A}x^m \geq 0$ for all $x \in \mathbb{R}^n_+$ with $\|x\| = 1$;

(ii) $\mathcal{A}$ *is strictly copositive if and only if* $\mathcal{A}x^m > 0$ *for all* $x \in \mathbb{R}^n_+$ *with* $\|x\| = 1$*;*

(iii) $\mathcal{A}$ *is strictly copositive if and only if* $\mathcal{A}$ *is copositive and the fact that* $\mathcal{A}x^m = 0$ *for* $x \in \mathbb{R}^n_+$ *implies* $x = \theta$*.*

Proof. (i) When $\mathcal{A}$ is copositive, the conclusion is obvious. Conversely, take $x \in \mathbb{R}^n_+$. If $\|x\| = 0$, then it follows that $x = \theta$, and hence $\mathcal{A}x^m = 0$. If $\|x\| > 0$, then let $y = \frac{x}{\|x\|}$. We have $\|y\| = 1$ and $x = \|x\|y$, and so

$$\mathcal{A}x^m = \mathcal{A}(\|x\|y)^m = \|x\|^m \mathcal{A}y^m \geq 0.$$

Therefore, $\mathcal{A}x^m \geq 0$ for all $x \in \mathbb{R}^n_+$, as required.

Similarly, (ii) is easily proved.

(iii) Let $\mathcal{A}$ be strictly copositive. Clearly, $\mathcal{A}$ is copositive. Suppose there exists $x_0 \in \mathbb{R}^n_+$ and $x_0 \neq \theta$ such that $\mathcal{A}x_0^m = 0$, which contradicts the strict copositivity of $\mathcal{A}$. Conversely, if $x \neq \theta$ and $x \in \mathbb{R}^n_+$, then $\mathcal{A}x^m \neq 0$. Since $\mathcal{A}$ is copositive, $\mathcal{A}x^m > 0$. The conclusion follows. ■

Let $\mathcal{A}$ be an m-order n-dimensional symmetric tensor. A number $\lambda \in \mathbb{C}$ is called an *eigenvalue of* $\mathcal{A}$ if there exists a nonzero vector $x \in \mathbb{C}^n$ satisfying

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]}, \tag{5.15}$$

where $x^{[m-1]} = (x_1^{m-1}, \cdots, x_n^{m-1})^T$, and call x an *eigenvector* of $\mathcal{A}$ associated with the eigenvalue λ. We call such an eigenvalue *H-eigenvalue* if it is real and has a real eigenvector x, and call such a real eigenvector x an *H-eigenvector*. These concepts were first introduced by Qi [59] to the higher order symmetric tensor, and the existence of the eigenvalues and its some application were studied also. Lim [40] independently introduced these concept and obtained the existence results using the variational approach.

A number $\mu \in \mathbb{C}$ is said to be an *E-eigenvalue of* $\mathcal{A}$ if there exists a nonzero vector $x \in \mathbb{C}^n$ such that

$$\mathcal{A}x^{m-1} = \mu x (x^T x)^{\frac{m-2}{2}}. \tag{5.16}$$

Such a nonzero vector $x \in \mathbb{C}^n$ is called an *E-eigenvector* of $\mathcal{A}$ associated with μ, If x is real, then μ is also real. In this case, μ and x are called a *Z-eigenvalue* of $\mathcal{A}$ and a *Z-eigenvector* of $\mathcal{A}$ (associated with μ), respectively. Qi [59–61] first introduced and used these concepts and showed that if $\mathcal{A}$ is regular, then a complex number is an E-eigenvalue of higher order symmetric tensor if and only if it is a root of the corresponding E-characteristic polynomial. Also see Hu and Qi [27], Hu, Huang, Ling and Qi [28], Li, Qi and Zhang [39] for more details.

In homogeneous polynomial $\mathcal{A}x^m$ defined by (5.1), if we let some (but not all) x_i be zero, then we have a homogeneous polynomial with fewer variables, which defines a lower dimensional tensor. We call such a lower dimensional tensor a *principal sub-tensor* of $\mathcal{A}$. That is, An m-order r-dimensional tensor $\mathcal{B}$ is called *principal sub-tensor* of an m-order n-dimensional tensor $\mathcal{A}$ if there is a set $\mathcal{N}$ that composed of r elements of $\{1, 2, \cdots, n\}$ such that $\mathcal{B}$ consists of r^m entries of $\mathcal{A} = (a_{i_1 \cdots i_m})$ and

$$\mathcal{B} = (a_{i_1 \cdots i_m}), \text{ for all } i_1, i_2, \cdots, i_m \in \mathcal{N}.$$

The concept were first introduced and used by Qi [59] to the higher order symmetric tensor.

Recently, Qi [62] introduced and used the following concepts for studying the properties of hypergraph. An H-eigenvalue λ of $\mathcal{A}$ is called

- *H^+-eigenvalue of* $\mathcal{A}$, if its H-eigenvector $x \in \mathbb{R}^n_+$;
- *H^{++}-eigenvalue of* $\mathcal{A}$, if its H-eigenvector $x \in \mathbb{R}^n_{++}$.

Similarly, we introduce the concepts of Z^+-eigenvalue and Z^{++}-eigenvalue. An Z-eigenvalue μ of $\mathcal{A}$ is said to be

- *Z^+-eigenvalue of* $\mathcal{A}$, if its Z-eigenvector $x \in \mathbb{R}^n_+$;
- *Z^{++}-eigenvalue of* $\mathcal{A}$, if its Z-eigenvector $x \in \mathbb{R}^n_{++}$.

5.3 Pareto H-eigenvalue and Pareto Z-eigenvalue

Let $\mathcal{N}$ be a subset of the index set $\{1, 2, \cdots, n\}$ and $\mathcal{A}$ be a tensor of order m and dimension n. We denote the principal sub-tensor of $\mathcal{A}$ by $\mathcal{A}^{\mathcal{N}}$ which is obtained by homogeneous polynomial $\mathcal{A}x^m$ for all $x = (x_1, x_2, \cdots, x_n)^T$ with $x_i = 0$ for $i \in \{1, 2, \cdots, n\}\backslash\mathcal{N}$. The symbol $|\mathcal{N}|$ denotes the cardinality of $\mathcal{N}$. So, $\mathcal{A}^{\mathcal{N}}$ is a tensor of order m and dimension $|\mathcal{N}|$ and the principal sub-tensor $\mathcal{A}^{\mathcal{N}}$ is just $\mathcal{A}$ itself when $\mathcal{N} = \{1, 2, \cdots, n\}$.

Theorem 5.3.1 *Let $\mathcal{A}$ be a m-order and n-dimensional tensor. A real number λ is Pareto H-eigenvalue of $\mathcal{A}$ if and only if there exists a nonempty subset $\mathcal{N} \subseteq \{1, 2, \cdots, n\}$ and a vector $w \in \mathbb{R}^{|\mathcal{N}|}$ such that*

$$\mathcal{A}^{\mathcal{N}} w^{m-1} = \lambda w^{[m-1]}, \quad w \in \mathbb{R}^{|\mathcal{N}|}_{++} \tag{5.17}$$

$$\sum_{i_2, \cdots, i_m \in \mathcal{N}} a_{ii_2\cdots i_m} w_{i_2} w_{i_3} \cdots w_{i_m} \geq 0 \text{ for } i \in \{1, 2, \cdots, n\} \setminus \mathcal{N} \tag{5.18}$$

In such a case, the vector $y \in \mathbb{R}^{|\mathcal{N}|}_{+}$ defined by

$$y_i = \begin{cases} w_i, & i \in \mathcal{N} \\ 0, & i \in \{1, 2, \cdots, n\} \setminus \mathcal{N} \end{cases} \tag{5.19}$$

is a Pareto H-eigenvector of $\mathcal{A}$ associated to the real number λ.

Proof. First we show the necessity. Let the real number λ be a Pareto H-eigenvalue of $\mathcal{A}$ with a corresponding Pareto H-eigenvector y. Then by the definition (5.9) of the Pareto H-eigenvalue, the Pareto H-eigenpairs (λ, y) may be rewritten in the form

$$\begin{aligned} y^T(\mathcal{A}y^{m-1} - \lambda y^{[m-1]}) &= 0 \\ \mathcal{A}y^{m-1} - \lambda y^{[m-1]} &\geq \theta \\ y &\geq \theta \end{aligned} \tag{5.20}$$

and hence

$$\sum_{i=1}^{n} y_i(\mathcal{A}y^{m-1} - \lambda y^{[m-1]})_i = 0 \tag{5.21}$$

$$(\mathcal{A}y^{m-1} - \lambda y^{[m-1]})_i \geq 0, \quad \text{for } i = 1, 2, \ldots, n \tag{5.22}$$

$$y_i \geq 0, \quad \text{for } i = 1, 2, \ldots, n. \tag{5.23}$$

Combining the equation (5.21) with (5.22) and (5.23), we have

$$y_i(\mathcal{A}y^{m-1} - \lambda y^{[m-1]})_i = 0, \quad \text{for all } i \in \{1, 2, \ldots, n\}. \tag{5.24}$$

Take $\mathcal{N} = \{i \in \{1, 2, \ldots, n\}; y_i > 0\}$. Let the vector $w \in \mathbb{R}^{|\mathcal{N}|}$ be defined by

$$w_i = y_i \text{ for all } i \in \mathcal{N}.$$

Clearly, $w \in \mathbb{R}^{|\mathcal{N}|}_{++}$. Combining the equation (5.24) with the fact that $y_i > 0$ for all $i \in \mathcal{N}$, we have

$$(\mathcal{A}y^{m-1} - \lambda y^{[m-1]})_i = 0, \quad \text{for all } i \in \mathcal{N},$$

and so

$$\mathcal{A}^{\mathcal{N}} w^{m-1} = \lambda w^{[m-1]}, \quad w \in \mathbb{R}^{|\mathcal{N}|}_{++}.$$

It follows from the equation (5.22) and the fact that $y_i = 0$ for all $i \in \{1, 2, \cdots, n\} \setminus \mathcal{N}$ that

$$(\mathcal{A}y^{m-1})_i \geq 0, \quad \text{for all } i \in \{1, 2, \cdots, n\} \setminus \mathcal{N}.$$

By the definition (5.2) of $\mathcal{A}y^{m-1}$, the conclusion (5.18) holds.

Now we show the sufficiency. Suppose that there exists a nonempty subset $\mathcal{N} \subseteq \{1, 2, \cdots, n\}$ and a vector $w \in \mathbb{R}^{|\mathcal{N}|}$ satisfying (5.17) and (5.18). Then the vector y defined by (5.19) is a non-zero vector in $\mathbb{R}^{|\mathcal{N}|}_{+}$ such that (λ, y) satisfying (5.20). The desired conclusion follows. ∎

Using the same proof techniques as that of Theorem 5.3.1 with appropriate changes in the inequalities or equalities ($y^{[m-1]}$ is replaced by $(y^T y)^{\frac{m-2}{2}} y$ and so on). We can obtain the following conclusions about the Pareto Z-eigenvalue of $\mathcal{A}$.

Theorem 5.3.2 *Let $\mathcal{A}$ be a m-order and n-dimensional tensor. A real number μ is Pareto Z-eigenvalue of $\mathcal{A}$ if and only if there exists a nonempty subset $\mathcal{N} \subseteq \{1, 2, \cdots, n\}$ and a vector $w \in \mathbb{R}^{|\mathcal{N}|}$ such that*

$$\mathcal{A}^{\mathcal{N}} w^{m-1} = \mu (w^T w)^{\frac{m-2}{2}} w, \quad w \in \mathbb{R}_{++}^{|\mathcal{N}|} \tag{5.25}$$

$$\sum_{i_2, \cdots, i_m \in \mathcal{N}} a_{ii_2 \cdots i_m} w_{i_2} w_{i_3} \cdots w_{i_m} \geq 0 \text{ for } i \in \{1, 2, \cdots, n\} \setminus \mathcal{N} \tag{5.26}$$

In such a case, the vector $y \in \mathbb{R}_{+}^{|\mathcal{N}|}$ defined by

$$y_i = \begin{cases} w_i, & i \in \mathcal{N} \\ 0, & i \in \{1, 2, \cdots, n\} \setminus \mathcal{N} \end{cases} \tag{5.27}$$

is a Pareto Z-eigenvector of $\mathcal{A}$ associated to the real number μ.

Following Theroem 5.3.1 and 5.3.2, the following results are obvious.

Corollary 5.3.3 *Let $\mathcal{A}$ be a m-order and n-dimensional tensor. If a real number λ is Pareto H-eigenvalue (Z-eigenvalue) of $\mathcal{A}$, then λ is H^{++}-eigenvalue (Z^{++}-eigenvalue, respectively) of some $|N|$-dimensional principal sub-tensor of $\mathcal{A}$.*

Since the definition of H^+-eigenvalue (Z^+-eigenvalue) λ of $\mathcal{A}$ means that $\mathcal{A}x^{m-1} - \lambda x^{[m-1]} = 0$ ($\mathcal{A}x^{m-1} - \lambda(x^T x)^{\frac{m}{2}-1} x = 0$, respectively) for some non-zero vector $x \geq 0$, the following conclusions are trivial.

Proposition 5.3.4 *Let $\mathcal{A}$ be a m-order and n-dimensional tensor. Then*

(i) each H^+-eigenvalue (Z^+-eigenvalue) of $\mathcal{A}$ is its Pareto H-eigenvalue (Z-eigenvalue, respectively);

(ii) the Pareto H-eigenvalues (Z-eigenvalues) of a diagonal tensor $\mathcal{A}$ coincide with its diagonal entries. In particular, a n-dimensional and diagonal tensor may have at most n distinct Pareto H-eigenvalues (Z-eigenvalues).

It follows from the above results that some Pareto H-eigenvalue ($Z-$eigenvalue) of a higher order tensor may be calculated by means of H^{++}-eigenvalue (Z^{++}-eigenvalue, respectively) of the lower dimensional tensors.

Example 5.3.5 *Let $\mathcal{A}$ be a 4-order and 2-dimensional tensor. Suppose that* $a_{1111} = 1, a_{2222} = 2$*,* $a_{1122} + a_{1212} + a_{1221} = -1$*,* $a_{2121} + a_{2112} + a_{2211} = -2$*, and other* $a_{i_1 i_2 i_3 i_4} = 0$*. Then*

$$\mathcal{A}x^4 = x_1^4 + 2x_2^4 - 3x_1^2 x_2^2$$

$$\mathcal{A}x^3 = \begin{pmatrix} x_1^3 - x_1 x_2^2 \\ 2x_2^3 - 2x_1^2 x_2 \end{pmatrix}$$

When $\mathcal{N} = \{1, 2\}$, the principal sub-tensor $\mathcal{A}^{\mathcal{N}}$ is just $\mathcal{A}$ itself. $\lambda_1 = 0$ is a H^{++}-eigenvalue of $\mathcal{A}$ with a corresponding eigenvector $x^{(1)} = (\frac{\sqrt[4]{8}}{2}, \frac{\sqrt[4]{8}}{2})^T$, and so it follows from Theorem 5.3.1 that $\lambda_1 = 0$ is a Pareto H-eigenvalue with Pareto H-eigenvector $x^{(1)} = (\frac{\sqrt[4]{8}}{2}, \frac{\sqrt[4]{8}}{2})^T$.

$\lambda_2 = 0$ is a Z^{++}-eigenvalue of $\mathcal{A}$ with a corresponding eigenvector $x^{(2)} = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})^T$, and so it follows from Theorem 5.3.2 that $\lambda_2 = 0$ is a Pareto Z-eigenvalue of $\mathcal{A}$ with Pareto Z-eigenvector $x^{(2)} = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})^T$.

When $\mathcal{N} = \{1\}$, the 1-dimensional principal sub-tensor $\mathcal{A}^{\mathcal{N}} = 1$. Obviously, $\lambda_3 = 1$ is both H^{++}-eigenvalue and Z^{++}-eigenvalue of $\mathcal{A}^{\mathcal{N}}$ with a corresponding eigenvector $w = 1$ and $a_{2111}w^3 = 0$, and hence it follows from Theorem 5.3.1 and 5.3.2 that $\lambda_3 = 1$ is both Pareto H-eigenvalue and Pareto Z-eigenvalue of $\mathcal{A}$ with a corresponding eigenvector $x^{(3)} = (1, 0)^T$.

Similarly, when $\mathcal{N} = \{2\}$, the 1-dimensional principal sub-tensor $\mathcal{A}^{\mathcal{N}} = 2$. Clearly, $\lambda_4 = 2$ is both H^{++}-eigenvalue and Z^{++}-eigenvalue of $\mathcal{A}^{\mathcal{N}}$ with a corresponding eigenvector $w = 1$ and $a_{1222}w^3 = 0$, and so $\lambda_4 = 2$ is both Pareto H-eigenvalue and Pareto Z-eigenvalue of $\mathcal{A}$ with a corresponding eigenvector $x^{(4)} = (0, 1)^T$.

Example 5.3.6 *Let $\mathcal{A}$ be a 3-order and 2-dimensional tensor. Suppose that* $a_{111} = 1, a_{222} = 2$, $a_{122} = a_{212} = a_{221} = \frac{1}{3}$, *and* $a_{112} = a_{121} = a_{211} = -\frac{2}{3}$. *Then*

$$\mathcal{A}x^3 = x_1^3 + x_1x_2^2 - 2x_1^2x_2 + 2x_2^3$$

$$\mathcal{A}x^2 = \begin{pmatrix} x_1^2 + \dfrac{1}{3}x_2^2 - \dfrac{4}{3}x_1x_2 \\ 2x_2^2 + \dfrac{2}{3}x_1x_2 - \dfrac{2}{3}x_1^2 \end{pmatrix}$$

When $\mathcal{N} = \{1\}$, the 1-dimensional principal sub-tensor $\mathcal{A}^{\mathcal{N}} = 1$. Obviously, $\lambda_1 = 1$ is both H^{++}-eigenvalue and Z^{++}-eigenvalue of $\mathcal{A}^{\mathcal{N}}$ with a corresponding eigenvector $w = 1$ and $a_{211}w^2 = -\frac{2}{3} < 0$, and so $\lambda_1 = 1$ is neither Pareto H-eigenvalue nor Pareto Z-eigenvalue of $\mathcal{A}$.

When $\mathcal{N} = \{2\}$, the 1-dimensional principal sub-tensor $\mathcal{A}^{\mathcal{N}} = 2$. Clearly, $\lambda_2 = 2$ is both H^{++}-eigenvalue and Z^{++}-eigenvalue of $\mathcal{A}^{\mathcal{N}}$ with a corresponding eigenvector $w = 1$ and $a_{122}w^2 = \frac{1}{3} > 0$, and so $\lambda_2 = 2$ is both Pareto H-eigenvalue and Pareto Z-eigenvalue of $\mathcal{A}$ with a corresponding eigenvector $x^{(2)} = (0, 1)^T$. But $\lambda = 2$ is neither H^+-eigenvalue nor Z^+-eigenvalue of $\mathcal{A}$.

Remark 5.3.1 *The Example 5.3.6 reveals that a Pareto H-eigenvalue (Z-eigenvalue) of a tensor $\mathcal{A}$ may not be its H^+-eigenvalue (Z^+-eigenvalue) even when $\mathcal{A}$ is symmetric.*

5.4 Pareto eigenvalue and Constrained minimization

Let $\mathcal{A}$ be a symmetric tensor of order m and dimension n and $\|x\|_k = (|x_1|^k + |x_2|^k + \cdots + |x_n|^k)^{\frac{1}{k}}$ for $k \geq 1$. Denote by $e^{(i)} = (e_1^{(i)}, e_2^{(i)}, \cdots, e_n^{(i)})^T$ the ith unit vector in $\mathbb{R}^n$, i.e.,

$$e_j^{(i)} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad \text{for } i, j \in \{1, 2, \cdots, n\}.$$

We consider the constrained minimization problem

$$\gamma(\mathcal{A}) = \min\{\mathcal{A}x^m;\ x \geq \theta \text{ and } \|x\|_m = 1\}, \tag{5.28}$$

Theorem 5.4.1 *Let $\mathcal{A}$ be a m-order and n-dimensional symmetric tensor. If*

$$\lambda(\mathcal{A}) = \min\{\lambda; \lambda \textit{ is Pareto } H\textit{-eigenvalue of } \mathcal{A}\},$$

then $\gamma(\mathcal{A}) = \lambda(\mathcal{A})$.

Proof. Let λ be a Pareto H-eigenvalue of $\mathcal{A}$. Then there exists a non-zero vector $y \in \mathbb{R}^n$ such that

$$\mathcal{A}y^m = \lambda y^T y^{[m-1]},\ y \geq \theta,$$

and so

$$\mathcal{A}y^m = \lambda \sum_{i=1}^{n} y_i^m = \lambda \|y\|_m^m \text{ and } \|y\|_m > 0. \tag{5.29}$$

Then we have

$$\lambda = \mathcal{A}(\frac{y}{\|y\|_m})^m \text{ and } \|\frac{y}{\|y\|_m}\|_m = 1.$$

From (5.28), it follows that $\gamma(\mathcal{A}) \leq \lambda$. Since λ is arbitrary, we have

$$\gamma(\mathcal{A}) \leq \lambda(\mathcal{A}).$$

Now we show $\gamma(\mathcal{A}) \geq \lambda(\mathcal{A})$. Let $S = \{x \in \mathbb{R}^n; x \geq 0 \text{ and } \|x\|_m = 1\}$. It follows from the continuity of the homogeneous polynomial $\mathcal{A}x^m$ and the compactness of the set S that there exists a $v \in S$ such that

$$\gamma(\mathcal{A}) = \mathcal{A}v^m,\ v \geq \theta,\ \|v\|_m = 1. \tag{5.30}$$

Let $g(x) = \mathcal{A}x^m - \gamma(\mathcal{A})x^T x^{[m-1]}$ for all $x \in \mathbb{R}^n$. We claim that for all $x \geq \theta$, $g(x) \geq 0$. Suppose not, then there exists non-zero vector $y \geq 0$ such that

$$g(y) = \mathcal{A}y^m - \gamma(\mathcal{A}) \sum_{i=1}^{n} y_i^m < 0,$$

and hence $\gamma(\mathcal{A}) \leq \mathcal{A}(\frac{y}{\|y\|_m})^m < \gamma(\mathcal{A})$, a contradiction. Thus we have

$$g(x) = \mathcal{A}x^m - \gamma(\mathcal{A})x^T x^{[m-1]} \geq 0 \text{ for all } x \in \mathbb{R}^n_+. \tag{5.31}$$

For each $i \in \{1, 2, \cdots, n\}$, we define a one-variable function

$$f(t) = g(v + te^{(i)}) \text{ for all } t \in \mathbb{R}^1.$$

Clearly, $f(t)$ is continuous and $v + te^{(i)} \in \mathbb{R}^n_+$ for all $t \geq 0$. It follows from (5.30) and (5.31) that

$$f(0) = g(v) = 0 \text{ and } f(t) \geq 0 \text{ for all } t \geq 0.$$

From the necessary conditions of extremum of one-variable function, it follows that the right-hand derivative $f'_+(0) \geq 0$, and hence

$$\begin{aligned} f'_+(0) = (e^{(i)})^T \nabla g(v) =& m(e^{(i)})^T (\mathcal{A}v^{m-1} - \gamma(\mathcal{A})v^{[m-1]}) \\ =& m(\mathcal{A}v^{m-1} - \gamma(\mathcal{A})v^{[m-1]})_i \geq 0. \end{aligned}$$

So we have

$$(\mathcal{A}v^{m-1} - \gamma(\mathcal{A})v^{[m-1]})_i \geq 0, \text{ for } i \in \{1, 2, \cdots, n\}.$$

Therefore, we obtain

$$f(0) = g(v) = \mathcal{A}v^m - \gamma(\mathcal{A})v^T v^{[m-1]} = 0 \tag{5.32}$$

$$\mathcal{A}v^{m-1} - \gamma(\mathcal{A})v^{[m-1]} \geq \theta \tag{5.33}$$

$$v \geq \theta$$

Namely, $\gamma(\mathcal{A})$ is a Pareto H-eigenvalue of $\mathcal{A}$, and hence $\gamma(\mathcal{A}) \geq \lambda(\mathcal{A})$, as required. ■

It follows from the proof of the inquality $\gamma(\mathcal{A}) \geq \lambda(\mathcal{A})$ in Theorem 5.4.1 that $\gamma(\mathcal{A})$ is a Pareto H-eigenvalue of $\mathcal{A}$, which implies the existence of Pareto H-eigenvalue of a symmetric tensor $\mathcal{A}$.

Theorem 5.4.2 *If a m-order and n-dimensional tensor $\mathcal{A}$ is symmetric, then $\mathcal{A}$ has at least one Pareto H-eigenvalue* $\gamma(\mathcal{A}) = \min\limits_{\substack{x\geq 0\\ \|x\|_m=1}} \mathcal{A}x^m$.

Since $(x^Tx)^{\frac{m}{2}} = \|x\|_2^m$, using the same proof techniques as that of Theorem 5.4.1 with appropriate changes in the inequalities or equalities ($x^Tx^{[m-1]}$ and $y^{[m-1]}$ are respectively replaced by $(x^Tx)^{\frac{m}{2}}$ and $(y^Ty)^{\frac{m-2}{2}}y$. We can obtain the following conclusions about the Pareto Z-eigenvalue of a symmetric tensor $\mathcal{A}$.

Theorem 5.4.3 *Let $\mathcal{A}$ be a m-order and n-dimensional symmetric tensor. Then $\mathcal{A}$ has at least one Pareto Z-eigenvalue* $\mu(\mathcal{A}) = \min\limits_{\substack{x\geq 0\\ \|x\|_2=1}} \mathcal{A}x^m$. *What's more,*

$$\mu(\mathcal{A}) = \min\{\mu; \mu \text{ is Pareto } Z\text{-eigenvalue of } \mathcal{A}\}. \tag{5.34}$$

As the immediate conclusions of the above results together with Lemma 5.2.1, it is easy to obtain the following results about the copositive (strictly copositive) tensor $\mathcal{A}$.

Corollary 5.4.4 *Let $\mathcal{A}$ be a m-order and n-dimensional symmetric tensor. Then*

(a) $\mathcal{A}$ always has Pareto H-eigenvalue. $\mathcal{A}$ is copositive (strictly copositive) if and only if all of its Pareto H-eigenvalues are nonnegative (positive, respectively).

(b) $\mathcal{A}$ always has Pareto Z-eigenvalue. $\mathcal{A}$ is copositive (strictly copositive) if and only if all of its Pareto Z-eigenvalues are nonnegative (positive, respectively).

Now we give an example for solving the constrained minimization problem for homogeneous polynomial and testing the (strict) copositivity of a symmetric tensor $\mathcal{A}$ with the help of the above results.

Example 5.4.5 *Let $\mathcal{A}$ be a 4-order and 2-dimensional tensor. Suppose that $a_{1111} = a_{2222} = 1$, $a_{1112} = a_{1211} = a_{1121} = a_{2111} = t$, and other $a_{i_1i_2i_3i_4} = 0$. Then*

$$\mathcal{A}x^4 = x_1^4 + x_2^4 + 4tx_1^3x_2$$

$$\mathcal{A}x^3 = \begin{pmatrix} x_1^3+3tx_1^2x_2 \\ x_2^3+tx_1^3 \end{pmatrix}$$

When $\mathcal{N} = \{1, 2\}$, the principal sub-tensor $\mathcal{A}^{\mathcal{N}}$ is just $\mathcal{A}$ itself. $\lambda_1 = 1 + \sqrt[4]{27}t$ is H^{++}-eigenvalue of $\mathcal{A}$ with a corresponding eigenvector $x^{(1)} = (\sqrt[4]{\frac{3}{4}}, \sqrt[4]{\frac{1}{4}})^T$. Then it follows from Theorem 5.3.1 or Proposition 5.3.4 that $\lambda_1 = 1 + \sqrt[4]{27}t$ is Pareto H-eigenvalues with Pareto H-eigenvector $x^{(1)} = (\sqrt[4]{\frac{3}{4}}, \sqrt[4]{\frac{1}{4}})^T$.

When $\mathcal{N} = \{1\}$, the 1-dimensional principal sub-tensor $\mathcal{A}^{\mathcal{N}} = 1$. Obviously, $\lambda_2 = 1$ is both H^{++}-eigenvalue and Z^{++}-eigenvalue of $\mathcal{A}^{\mathcal{N}}$ with a corresponding eigenvector $w = 1$ and $a_{2111}w^3 = t$. Then when $t > 0$, it follows from Theorem 5.3.1 and 5.3.2 that $\lambda_2 = 1$ is both Pareto H-eigenvalue and Pareto Z-eigenvalue of $\mathcal{A}$ with a corresponding eigenvector $x^{(2)} = (1, 0)^T$; when $t < 0$, $\lambda_2 = 1$ is neither Pareto H-eigenvalue nor Pareto Z-eigenvalue of $\mathcal{A}$.

Similarly, when $\mathcal{N} = \{2\}$, the 1-dimensional principal sub-tensor $\mathcal{A}^{\mathcal{N}} = 1$. Clearly, $\lambda_3 = 1$ is both H^{++}-eigenvalue and Z^{++}-eigenvalue of $\mathcal{A}^{\mathcal{N}}$ with a corresponding eigenvector $w = 1$ and $a_{1222}w^3 = 0$, and so $\lambda_3 = 1$ is both Pareto H-eigenvalue and Pareto Z-eigenvalue of $\mathcal{A}$ with a corresponding eigenvector $x^{(3)} = (0, 1)^T$.

So the following conclusions are easily obtained:

(i) Let $t < -\frac{1}{\sqrt[4]{27}}$. Then $\lambda_1 = 1 + \sqrt[4]{27}t < 0$ and $\lambda_3 = 1$ are Pareto H-eigenvalues of $\mathcal{A}$ with Pareto H-eigenvectors $x^{(1)} = (\sqrt[4]{\frac{3}{4}}, \sqrt[4]{\frac{1}{4}})^T$ and $x^{(3)} = (0, 1)^T$, respectively. It follows from Theorem 5.4.1 and 5.4.2 that

$$\gamma(\mathcal{A}) = \min_{\substack{x \geq 0 \\ \|x\|_4 = 1}} \mathcal{A}x^4 = \min\{\lambda_1, \lambda_3\} = 1 + \sqrt[4]{27}t < 0.$$

The polynomial $\mathcal{A}x^4$ attains its minimum value at $x^{(1)} = (\sqrt[4]{\frac{3}{4}}, \sqrt[4]{\frac{1}{4}})^T$. It follows from Corollary 5.4.4 that $\mathcal{A}$ is not copositive.

(ii) Let $t = -\frac{1}{\sqrt[4]{27}}$. Then $\lambda_1 = 1 + \sqrt[4]{27}t = 0$ and $\lambda_3 = 1$ are Pareto H-eigenvalues of $\mathcal{A}$ with Pareto H-eigenvectors $x^{(1)} = (\sqrt[4]{\frac{3}{4}}, \sqrt[4]{\frac{1}{4}})^T$ and $x^{(3)} = (0, 1)^T$, respectively. It follows from Theorem 5.4.1 and 5.4.2 that

$$\gamma(\mathcal{A}) = \min_{\substack{x \geq 0 \\ \|x\|_4 = 1}} \mathcal{A}x^4 = \min\{\lambda_1, \lambda_3\} = 0.$$

The polynomial $\mathcal{A}x^4$ attains its minimum value at $x^{(1)} = (\sqrt[4]{\frac{3}{4}}, \sqrt[4]{\frac{1}{4}})^T$. It follows from Corollary 5.4.4 that $\mathcal{A}$ is copositive.

(iii) Let $0 > t > -\frac{1}{\sqrt[4]{27}}$. Clearly, $0 < 1 + \sqrt[4]{27}t < 1$. Then $\lambda_1 = 1 + \sqrt[4]{27}t$ and $\lambda_3 = 1$ are Pareto H-eigenvalues of $\mathcal{A}$. It follows from Theorem 5.4.1 and 5.4.2 that

$$\gamma(\mathcal{A}) = \min_{\substack{x \geq 0 \\ \|x\|_4 = 1}} \mathcal{A}x^4 = \min\{\lambda_1, \lambda_3\} = 1 + \sqrt[4]{27}t > 0.$$

The polynomial $\mathcal{A}x^4$ attains its minimum value at $x^{(1)} = (\sqrt[4]{\frac{3}{4}}, \sqrt[4]{\frac{1}{4}})^T$. It follows from Corollary 5.4.4 that $\mathcal{A}$ is strictly copositive.

(iv) Let $t = 0$. Then $\lambda_1 = \lambda_2 = \lambda_3 = 1$ are Pareto H-eigenvalues of $\mathcal{A}$ with Pareto H-eigenvectors $x^{(1)} = (\sqrt[4]{\frac{3}{4}}, \sqrt[4]{\frac{1}{4}})^T$ and $x^{(2)} = (1, 0)^T$ and $x^{(3)} = (0, 1)^T$, respectively. It follows from Theorem 5.4.1 and 5.4.2 that

$$\gamma(\mathcal{A}) = \min_{\substack{x \geq 0 \\ \|x\|_4 = 1}} \mathcal{A}x^4 = \min\{\lambda_1, \lambda_2, \lambda_3\} = 1 > 0.$$

The polynomial $\mathcal{A}x^4$ attains its minimum value at $x^{(1)} = (\sqrt[4]{\frac{3}{4}}, \sqrt[4]{\frac{1}{4}})^T$ or $x^{(2)} = (1, 0)^T$ or $x^{(3)} = (0, 1)^T$. It follows from Corollary 5.4.4 that $\mathcal{A}$ is strictly copositive.

(v) Let $t > 0$. Then $\lambda_1 = 1 + \sqrt[4]{27}t$ and $\lambda_2 = \lambda_3 = 1$ are Pareto H-eigenvalues of $\mathcal{A}$ with Pareto H-eigenvectors $x^{(1)} = (\sqrt[4]{\frac{3}{4}}, \sqrt[4]{\frac{1}{4}})^T$ and $x^{(2)} = (1, 0)^T$ and $x^{(3)} = (0, 1)^T$, respectively. It follows from Theorem 5.4.1 and 5.4.2 that

$$\gamma(\mathcal{A}) = \min_{\substack{x \geq 0 \\ \|x\|_4 = 1}} \mathcal{A}x^4 = \min\{\lambda_1, \lambda_2, \lambda_3\} = 1 > 0.$$

The polynomial $\mathcal{A}x^4$ attains its minimum value at $x^{(2)} = (1, 0)^T$ or $x^{(3)} = (0, 1)^T$. It follows from Corollary 5.4.4 that $\mathcal{A}$ is strictly copositive.

Chapter 6

Conclusions and suggestions for further research

In this thesis, we develop and extend some spectral properties of matrices to higher order tensors. Our studied methods are mainly to defined the proper operators based on the definition of eigenvalue of higher order tensors, and then to study some nonlinear spectral properties of such operators. So, the corresponding properties of higher order tensor are proved. We also introduce the concept of Pareto H-eigenvalue and Pareto Z-eigenvalue to higher order tensors and study their properties by means of the constrained minimization problem of homogeneous polynomial.

In Chapter 2, we obtain the Fredholm alternative theorems of the eigenvalue (included E-eigenvalue, H-eigenvalue, Z-eigenvalue) of a higher order tensor $\mathcal{A}$, and prove some relationship between the Gelfand formula and the spectral radius for the spectra induced by such several classes of eigenvalues of a higher order tensor.

In Chapter 3, we discuss the existence and uniqueness of the positive eigenvalue-eigenvector for a class of nonlinear and positively homogeneous mappings in a Banach space by means of the Edelstein Contraction Theorem. We successfully turn the eigenvalue problem of a class of nonnegative tensor into the fixed point problem of the Edelstein Contraction with respect to Hilbert's projective metric, and then by some fixed point theory, we obtain some solution of the eigenvalue problem of such a class of nonnegative tensors. Furthermore, we find an iteration sequence which strongly converges to positive eigenvalue of such a tensor.

In Chapter 4, we introduce the notion of eigenvalue to the additively homogeneous mapping pairs (f, g), and discuss the existence and uniqueness of such a eigenvalue, and also obtain the Collatz-Wielandt min-max type property of such a class of mapping pairs. As an application, we obtain the iteration sequence for finding generalized $H-$eigenvalue of the nonnegative tensor pairs $(\mathcal{A}, \mathcal{B})$.

In Chapter 5, we introduce the concepts of Pareto H-eigenvalue and Pareto Z-eigenvalue of higher order tensor for studying constrained minimization problem. Furthermore, we study the existence of Pareto H-eigenvalue (Pareto Z-eigenvalue) of a symmetric tensor and establish that the minimum Pareto H-eigenvalue (or Pareto Z-eigenvalue) of a symmetric tensor exactly is the minimum value of constrained minimization problem of homogeneous polynomial deduced by such a tensor. In particular, a symmetric tensor $\mathcal{A}$ is copositivie if and only if all the Pareto H-eigenvalue ($Z-$eigenvalue) of $\mathcal{A}$ are nonnegative.

The following topics are worth of serious consideration.

- In Chapter 2, we define the spectrum of T (Page 15), and give some relationship between the Gelfand formula and the spectral radius (Theorem 2.4.3, 2.4.6). The following topics subject to further research: whether the above spectral radius is exactly equivalent to the largest eigenvalue of $T \in \mathcal{CH}(\mathbb{C}^n)$ or some subclass of operators in $\mathcal{CH}(\mathbb{C}^n)$. In particular, for a higher order tensor $\mathcal{A}$, is the following equations hold?

$$\max\{|\lambda|^{\frac{1}{m-1}}; \lambda \text{ is eigenvalue of } \mathcal{A}\} = \lim_{k\to\infty} \|F_{\mathcal{A}}^k\|^{\frac{1}{k}}$$

and

$$\max\{|\mu|; \mu \text{ is } E\text{-eigenvalue of } \mathcal{A}\} = \lim_{k\to\infty} \|T_{\mathcal{A}}^k\|^{\frac{1}{k}}.$$

- In Chapter 3, the eigenvalue problem of a primitive and nonnegative tensor $\mathcal{A}$ may be viewed as the fixed point problem of the Edelstein Contraction with respect to Hilbert's projective metric (Theorem 3.4.1). However, it isn't still known whether or not a strictly positive operator $F_{\mathcal{A}}$ (or equivalently, a strictly nonnegative tensor) has the same properties. For a tensor, not necessary nonnegative, whether or not we may obtain similar conclusions.

- In Chapter 4, the existence and uniqueness of generalized $H-$eigenvalue of the nonnegative tensor pairs $(\mathcal{A}, \mathcal{B})$ are showed under the condition $F_{\mathcal{A}}F_{\mathcal{B}} = F_{\mathcal{B}}F_{\mathcal{A}}$ (Theorem 4.4.3 or 4.3.1). Clearly, the condition of the commutativity is very strong, then it is greatly significant to such a condition is removed or be replaced by other more general condition to meet the same conclusion.

- In Chapter 5, the concepts of Pareto H-eigenvalue (Page 70, the system (5.9)) and Pareto Z-eigenvalue (Page 71, the system (5.11)) are introduced and used to study the properties of a higher order tensor. Then we will continue to study how to define Pareto H-spectra (or Pareto Z-spectra) and dicuss its some properties such as the Pareto spectral radius. Of course, it will be very interesting to show the number of Pareto H-eigenvalue (or Pareto Z-eigenvalue) of a tensor and to construct an algorithm that converges to Pareto H-eigenvalue (or Pareto Z-eigenvalue) of a tensor.

Index

Bibliography

[1] J. Appell, E.D. Pascale and A. Vignoli,*Nonlinear Spectral Theory*, Walter De Gruyter GmbH & Co. KG, 10785 Berlin, Germany (April 2004).

[2] M. Akiav and S. Gaubert, *Spectral theorem for convex monotone homogeneous maps, and ergodic control*, Nonlinear Analysis, 52(2003) 637-679.

[3] R.B. Bapat, D.D. Olesky, P. Van den Driessche, *Perron-Frobenius theory for a generalized eigenproblem*, Linear and Multilinear Algebra, 40(1995) 141-152.

[4] R.B. Bapat, *A max version of the Perron-Frobenius theorem*, Linear Algebra Appl., 275/276(1998) 3-18.

[5] G.D. Birkhoff, O.D. Kellogg, *Invariant points in function space*, Trans. Amer. Math. Soc., 23(1922) 96-115.

[6] G. Birkhoff, *Extensions of Jentzch,s Theorem*, Trans. Amer. Math. Soc. 85(1957) 219-227.

[7] P.J. Bushell, *On the projective contration ratio for positive linear mappings*, J. London Math. Soc., 6(1973) 256-258.

[8] P.J. Bushell, *Hilbert's Metric and Positive Contraction Mappings in a Banacb Space*, Arch. Rat. Mech. Anal., 52(1973) 330-338.

[9] P.J. Bushell, *The Cayley-Hilbert metric and positive operators*, Linear Algebra Appl., 84(1986) 271-280.

[10] K.C. Chang, *A nonlinear Krein Rutman theorem*, J. Sys. Sci. Com. 22(4)(2009) 542-554.

[11] K.C. Chang, K. Pearson, and T. Zhang, *Perron-Frobenius theorem for nonnegative tensors*, Commun. Math. Sci., 6(2008) 507-520.

[12] K.C. Chang, K. Pearson, and T. Zhang, *Primitivity, The Convergence of the NQZ Method, And the Largest Eigenvalue for Nonnegative Tensors*, SIAM. J. Matrix Anal. & Appl., 32(3)(1011), 806-819.

[13] K.C. Chang, K. Pearson, T. Zhang, *On eigenvalue problems of real symmetric tensors*, J. Math. Anal. Appl., 350(2009) 416-422.

[14] K.C. Chang, K. Pearson, and T. Zhang, *Some variational principles for Z-eigenvalues of nonnegative tensors*, Linear Algebra Appl. 438(2013) 4166-4182.

[15] K. Deimling, *Nonlinear Functional Analysis*, 1988 Springer-Verlag New York, Inc.

[16] M. Edelstein, *On fixed and periodic points under contractive mappings* J. London Math. Soc., 37(1962) 74-79.

[17] W. Feng, J.R.L. Webb, *Surjectivity results for nonlinear mappings without oddness conditions*, Comment. Math. Univ. Carolin., 38(1)(1997) 15-28

[18] S. Fucik, J. Necas, J. Soucek, V. Soucek, *Spectral Analysis of Nonlinear Operators*, Lecture Notes in Mathematics 346, Springer-Verlag, Berlin, Heidelberg, New York, 1973.

[19] S. Friedland, S. Gauber and L. Han, *Perron-Frobenius theorem for nonnegative multilinear forms and and extensions*, Linear Algebra Appl., 438(2)(2013), 738-749.

[20] A. Granas, J. Dugundji, *Fixed point theory*, Springer-Verlag, New York. 2003.

[21] J. Gunawardena and M. Keane, *On the existence of cycle times for some nonexpansive maps*, Technical Report HPL-BRIMS-95-003, Hewlett-Packard Labs, 1995.

[22] S. Gaubert and J. Gunawardena, *The Perron-Frobenius theorem for homogeneous, monotone functions*, Trans. Amer. Math. Soc., 356(2004), 4931-4950.

[23] J. Gunawardena, *From max-plus algebra to nonexpansive mappings: a nonlinear theory for discrete event systems*, Theoretical Computer Science, 293(2003) 141-167.

[24] D. Hilbert. Über die gerade Linie als kürzeste Verbindung zweier Punkte? Math. Ann. 46(1895) 91–96.

[25] J.B. Hiriart-Urruty and A. Seeger, *A variational approach to copositive matrices*, SIAM Review, 52(4)(2010), 593-629.

[26] S. Hu, Z. Huang and L. Qi, *Finding the Spectral radius of a nonnegative tensor*, Department of Applied Mathematics, The Hong Kong Polytechnic University, December, 2010. arXiv:1111.2138v1 [math.NA], 9 Nov 2011

[27] S. Hu and L. Qi, *E-characteristic polynomial of a tensor of dimension two*, Appl. Math. Letters, 26(2013) 225-231.

[28] S. Hu, Z. Huang, C. Ling and L. Qi, *On Determinants and Eigenvalue Theory of Tensors*, J. Symbolic Comput., 50 (2013) 508-531.

[29] S. Hu and L. Qi, *Algebraic connectivity of an even uniform hypergraph*, J. Comb Optim, 24(4)(2012), 564-579.

[30] E. Haynsworth and A.J. Hoffman, *Two remarks on copositive matrices*, Linear Algebra Appl. 2(1969) 387-392.

[31] M.J. Huang, C.Y. Huang and T.M. Tsai, *Applications of Hilbert,s projective metric to a class of positive nonlinear operators* Linear Algebra Appl., 413(2006) 202-211

[32] M.J. Huang, C.Y. Huang and T.M. Tsai, *Eigenvalue problems and fixed point theorems for a class of positive nonlinear operators*, Math. Z., 257(2007)581-595.

[33] C.R. Johnson and R. Reams, *Spectral theory of copositive matrices*, Linear Algebra Appl., 395(2005) 275-281.

[34] E. Kohlberg and J.W. Pratt, *The contraction mapping approach to the Perron-Frobenius theorem: Why Hilbert's metric?*, Math. Opera. Research, 7(1982) 198-210.

[35] E. Kohlberg, *The Perron-Frobenius theorem without additivity*, J. Math. Economics, 10(1982) 299-303.

[36] U. Krause, *Perron's stability theorem for nonlinear mappings*, J. Math. Economics, 15(1986) 275-282.

[37] W. Kaplan, *A test for copositive matrices*, Linear Algebra Appl., 313(2000) 203-206.

[38] U. Krengel, *Ergodic theorems*(De Gruyter Studies in Mathematics 6)(with a supplement by Antoine Brunel), Walter de Gruyter & Co (July 1985), Berlin.

[39] A.M. Li, L. Qi and B. Zhang, *E-characteristic polynomials of tensors*, Communications in Mathematical Sciences, 11(2013) 33-53.

[40] L.H. Lim, *Singular values and eigenvalues of tensors: A variational approach*, in: Proc. 1st IEEE International workshop on computational advances of multi-tensor adaptive processing, Dec. 13-15, 2005, pp. 129-132.

[41] R. Mahadevan, *A note on a non-linear Krein-Rutman theorem*, Nonlinear Analysis, 67(11)(2007), 3084-3090.

[42] Max-Plus Working Group, *Max-plus algebra and applications to system theory and optimal control* In Proceedings of the International Congress of Mathematicians, Zürich, 1994. Birkhäuser, 1995.

[43] J. Mallet-Pareta and R.D. Nussbaum, *A basis theorem for a class of max-plus eigenproblems*, J. Differential Equations, 189(2003) 616-639.

[44] O.L. Mangasarian, *Perron-Frobenius properties of $Ax-\lambda Bx$*, J. Math. Anal. Appl., 36(1971) 86-102.

[45] V. Mehrmann, R. Nabben and E. Virnik, *Generalisation of the Perron-Frobenius theory to matrix pencils*, Linear Algebra Appl., 428(2008) 20-38.

[46] V. Mehrmann, D. Olesky, T. Phan, P. Van den Driessche, *Relations between Perron-Frobenius results for matrix pencils*, Linear Algebra Appl., 287(1998) 257-269.

[47] T.S. Motzkin, *Copositive Quadratic forms*, National Bureau of Standards Report, 1818 (1952) 11-12.

[48] D.H. Martin, *Copositlve matrices and definiteness of quadratic forms subject to homogeneous linear inequality constraints*, Linear Algebra Appl., 35(1981) 227-258.

[49] I. Meghea, *Some results of the Fredholm Alternative type for operators of* $\lambda J_\varphi - S$ *form with applications*, U.P.B. Sci. Bull., Series A, 72(4)(2010) 21-32.

[50] R.D. Nussbaum, *Convexity and log convexity for the spectral radius*, Linear Algebra Appl., 73(1986) 59-122.

[51] R.D. Nussbaum, *Hilbert's projective metric and iterated nonlinear maps*, Memoirs of the AMS, 75(391), 1988.

[52] R.D. Nussbaum, *Iterated nonlinear maps and Hilbert's projective metric, II*, Memoirs of the AMS, 79(401), 1989.

[53] M. Ng, L. Qi and G. Zhou, *Finding the largest eigenvalue of a non-negative tensor*, SIAM J. Matrix Anal. Appl., 31 (2009) 1090-1099.

[54] G. Ni, L. Qi, F. Wang and Y. Wang, *The degree of the E-characteristic polynomial of an even order tensor*, J. Math. Anal. Appl., 329(2007) 1218-1229.

[55] T. Ogiwara, *Nonlinear Perron-Frobenius problem for order-preserving mappings, I*, Proc. Japan Acad. Ser. A Math. Sci., 69(8) (1993), 312–316.

[56] T. Ogiwara, *Nonlinear Perron-Frobenius problem for order-preserving mappings, II. applications*, Proc. Japan Acad. Ser. A Math. Sci., 69(8)(1993), 317–321.

[57] A.J.B. Potter, *Applications of Hilbert's projective metric to certain classes of non-homogeneous operators*, Quart. J. Math. Oxford, 28(1977) 93-99.

[58] L. Ping and F. Y. Yu, *Criteria for Copositive Matrices of Order Four*, Linear Algebra Appl., 194(1993) 109-124.

[59] L. Qi, *Eigenvalues of a real supersymmetric tensor*, J. Symbolic Comput., 40(2005) 1302-1324.

[60] L. Qi, *Eigenvalues and invariants of tensors*, J. Math. Anal. Appl., 325(2007) 1363-1377.

[61] L. Qi, *Rank and eigenvalues of a supersymmetric tensor, the multivariate homogeneous polynomial and the algebraic hypersurface it defines*, J. Symbolic Comput., 41(2006) 1309-1327.

[62] L. Qi, *H^+-eigenvalues of Laplacian and signless Laplacian tensors*, Preprint, Department of Applied Mathematics, The Hong Kong Polytechnic University, November, 2012. arXiv:1303.2186 [math.SP] 9 Mar 2013

[63] L. Qi, *Symmetric Nonnegative Tensors and Copositive Tensors*, 439(1)(2013) 228-238.

[64] L. Qi, *Symmetric nonnegative tensors and copositive tensors*, Linear Algebra Appl., 439(2013) 228-238.

[65] Y. Song, *Coincidence points of weakly compatible mappings*, Bull. Kor. Math. Soc., 45(2008), 607-614

[66] J. Schauder, *Der Fixpunktsatz in Funktionalroaumen*, Studia Math. 2(1930) 171-180.

[67] A. Seeger, *Eigenvalue analysis of equilibrium processes defined by linear complementarity conditions*, Linear Algebra Appl., 292(1999), 1-14.

[68] J. M. Vincent, *Some ergodic results on stochastic iterative discrete event systems*, Discrete Event Dynamic Systems, 7(1997), 209-233.

[69] Y. Yang and Q. Yang, *Further Results for Perron-Frobenius Theorem for Nonnegative Tensors*, SIAM J. Matrix Anal. Appl., 31(5)(2010) 2517-2530.

[70] Q. Yang and Y. Yang, *Further Results for Perron-Frobenius Theorem for Nonnegative Tensors II*, SIAM J. Matrix Anal. Appl., 32(4)(2011) 1236-1250.

[71] L. Zhang and L. Qi, *Linear convergence of an algorithm for computing the largest eigenvalue of a nonnegative tensor*, Numerical Linear Algebra with Applications, 19(2012) 830-841.

[72] L. Zhang, L. Qi and Y. Xu, *Linear convergence of the LZI algorithm for weakly positive tensors*, J. Compu. Math., 30(2012) 24-33.

[73] Q. Zhang and Y. Song, *Fixed point theory for generalized $\varphi-$weak contractions*, Appl. Math. Letters, 22(1)(2009) 75-78.

[74] T. Zhang, *Existence of real eigenvalues of real tensors*, Nonlinear Analysis, 74(8)(2011) 2862-2868.