AN INVESTMENT MODEL WITH MEAN-FIELD TARGET

HOU DANLIN

M.Phil

The Hong Kong Polytechnic University

2014

The Hong Kong Polytechnic University

Department of Applied Mathematics

An investment model with mean-field target

HOU Danlin

A thesis submitted in partial fulfilment of the requirements for the degree of Master of Philosophy

April 2014

Certificate of Originality

I hereby declare that this thesis is my own work and that, to the best of my knowledge and belief, it reproduces no material previously published or written, nor material that has been accepted for the award of any other degree or diploma, except where due acknowledgement has been made in the text.

____________________________(Signed)

HOU Danlin (Name of student)

Dedicate to my parents.

Abstract

This thesis is concerned with a new compensation model with a performance-based benchmark. In earlier papers about optimal compensation problem, there are relatively few papers engaged in studying optimal investment policy with a given compensation form. Furthermore, the benchmarks in these papers are always assumed to be fixed or have nothing to do with the industry's performance. While the benchmark in this thesis contains an expected form, which represents the industry's performance. Traditional methods will fail in this kind of problem. So we plan to make optimization two times to deal with this changing-target compensation model. Besides, some numerical examples will be given in this thesis to illustrate the solvability of the problem.

Acknowledgements

Fascinatingly, carrying out research is definitely not an isolated activity. I am grateful to several individuals who have supported me in various ways in pursing my MPhil degree, and I would like to hereby acknowledge their assistance.

First and foremost, I wish to express my deep thanks to my supervisor, Dr.XU Zuoquan, for his enlightening guidance, invaluable discussions and insightful ideas throughout the years. What I have benefited most from him is the rigorous and diligent attitude to scientific research.

Furthermore, the indispensable support of my MPhil experience has been the guidance and kindness of my co-supervisor, Dr. Cedric YIU, who has been a constant source of inspiration and encourage.

Especially, I would like to express my heartfelt appreciation to Prof. LI Xun, for his strong encouragement and consideration during my MPhil program. His enthusiasm for mathematical research affects me most.

Finally, I would like to express my special thanks to my parents and my friends for their love, encouragement and support.

Contents

List of Figures

List of Notations

$X(t)$	value of the assets that the manager controls at time t
$P_i(t)$	price of ith asset in the market at time t
$r(t)$	interest rate of the riskless asset
$\mu_i(t)$	appreciation rate of risky asset i
$\sigma_{i,j}$	volatility rates of risky asset i
$B(t)$	excess rate of return vector process
$W(t)$	m-dimensional standard Brownian motion

Chapter 1

Introduction

1.1 Background

In the daily life, almost everyone's salary comprises two parts: basic salary and compensation. Compensation can be a good incentive for managers to exert their efforts and gain more profits for company shareholders. Appropriate compensation for managers is an aspect of corporate governance that is quite controversial to shareholders, institutional activists, and government regulator. Therefore many researches have been inspired to study the appropriate incentive contract, the optimal strategy, the impact the contract has on the manager's risk appetite and so on.

Optimal compensation problem belongs to principal-agent problem (it will be stated as agency problem for short below). In principal-agent relationship, one or more principals will hire one or more agents to perform on the principals' behalf, meanwhile the agents will enjoy some decision-making authority. Agency problem is quite general in the practical world. The principal-agent relationship could arise between any kind of cooperation and employment, like fund manager and investor, employer and employee, CEO and shareholder, even author and co-author etc. Because each party in this relationship will try to maximize his own utility, the agent will behave for the best of his own interests, which may be in the cost of the principal's interests. The conflict between these two parties incurs moral hazard and has inspired plenty of researches. It was Wilson (1965), Ross (1973) and Mirrlees (1976) who began to study agency problem first. Wilson (1965) discussed that diverse individuals in a productive organization would share in the consequence of a single decision. Ross (1973) discussed the conditions when the fee schedule would lead to Pareto Efficiency, and provided some micro foundation for further study. Mirrlees (1976) assumed that implicitly or explicitly every party would exploit the contract for his own interest. Moreover, Mirrlees (1976) studied two models for a productive organization. In the first model, both production and rewards based on performance could be perfectly observed. The second model concentrated on the imperfect observation about the performance. Jensen and Meckling (1976) defined the concept of agency costs, which included "the efforts on the part of the principal to 'control' the behavior of the agent through budget restrictions, compensation policies, operating rules etc". And related it to 'separation and control' issue. Since Holmstrom and Milgrom (1987) shed light on the continuous-time agency model, many researches have been done in the context of Brownian motion, see also Schaettler and Sung (1993), Hellwig and Schmidt (2002), He (2011), Bushman and Smith (2001a), Jin (2002), Haubrich (1994).

Although great achievement has been made on the contracting design in agency problem, few researches are dedicated to the affects the compensation model will have on the agent, in other words, the optimal policy the agent will take with a given compensation model. In Carpenter (2000), the benchmark of the compensation model is some call options of the assets under control. Furthermore, Carpenter (2000) proved that if the agent was given some compensation that he can't hedge, he may not necessarily become more risk-seeking with the option compensation. On the other hand, Grinblatt and Titman (1989) showed that, if a fund manager was compensated with options that he can hedge, he would increase the volatility of the assets as much as possible, which would harm the interests of the investors. These papers designed different forms of compensation model, all of which have specific economic meanings. These papers made continuous implements to various research areas of financial world, thus the relationship between compensation and manager's performance has been much clearer.

This thesis focuses on the relationship between the compensation model with a benchmark depending on the industry's average performance and the investment strategy that manager will take. There exists a mean term, which represents the industry's average performance, in the benchmark of this compensation model. That term makes this problem a mean-field problem and allows us to connect agency problem with mean-field problem. Mean-field problem is a very popular research direction recently, which is initiated by mean-variance model. Actually, as long as there exists a mean term in the cost function or state process of the problem, it belongs to mean-field problem. These years have witnessed a growing number of researches in this field, and a lot of new methods have been developed along with this new research direction. For instance, embedding method in Li and Ng (2000) and Zhou and Li (2000), Lagrangian method in Li et al. (2002), mean-field formulation in Cui et al. (2013). They all provided us with new technics and new perspectives to deal with some related but much more difficult problems. Besides, the problem in this thesis can be briefly stated as maximizing the expected utility of the final wealth, which is an investment problem. For compensation model, as long as the benchmark is based on a given random variable, the problem could be transferred into traditional investment problem. And this kind of problem could be solved with traditional methods, like dynamic programming and martingale method. While in this model, the benchmark is unknown and depends on the investment policy. The problem thus turns out to be time-inconsistant, so the traditional ways to solve investment problem will fail in this situation. That is also the key point of this thesis.

1.2 The Standard Investment Problem

In essence, the optimal policy problem with a given compensation model could be classified into portfolio choice problem or investment problem. The standard investment problem aims at maximizing the utility of terminal wealth, just as the optimal policy problem. There are two traditional methods to solve the problem. Dynamic programming method is introduced by Merton (1969) and Merton (1971), the other martingale method is characterized by Karatzas et al. (1986) and Cox and Huang (1989). The outline of the standard investment problem is as bellow [referring to Carpenter (2000)],

$$\begin{aligned} \max_{X_T} \quad & \mathbf{E}[U(X_T)], \\ \text{s.t.} \quad & \mathbf{E}[\zeta_T X_T] \le x_0, \\ & X_T \ge 0. \end{aligned} \tag{1.1}$$

Here X_T represents the whole wealth at time T, ζ_T denotes the state price, which means the current price of one unit of payoff at time T, and $U : (0, +\infty) \mapsto \mathbb{R}$ is a strictly increasing and strictly concave utility function with $U'(+\infty) = 0$. In dynamic programming method, the optimal solution to standard investment problem (1.1) could be found via solving a more extensive question in below form,

$$\begin{aligned} V(s,x) = & \max_{\pi(\cdot)} \quad \mathbf{E}\left[\int_s^T f(t, X_t, \pi_t)\mathrm{d}t + h(X_T)|\mathcal{F}_s\right], \\ \text{s.t.} \quad X_s = & x. \end{aligned} \tag{1.2}$$

The problem can reduce to solving an Hamilton-Jacobi-Bellman equation (HJB equation) and then the optimal control $\pi.$ will be derived.

Until now papers about compensation model often adopt the following form,

$$\begin{aligned} \max_{X_T} \quad & \mathbf{E}[(U(X_T - A_T)^+)], \\ \text{s.t.} \quad & \mathbf{E}[\zeta_T X_T] = x_0, \end{aligned} \tag{1.3}$$

where A_T is a prior given random variable and is $\mathcal{F}_T$-measurable, X_t should follow the wealth process below,

$$\begin{cases} \mathrm{d}X_t &= (r_t X_t + \pi_t' \sigma_t \theta_t)\mathrm{d}t + \pi_t' \sigma_t \mathrm{d}W_t, \\ X_0 &= x_0, \end{cases}$$

where π_t is the control variable, which also represents the investment strategy of the manager at time t, σ_t is the volatility matrix at time t, r_t represents the interest rate of the riskless asset, and θ_t is the market price of risk at time t. In this kind of problem, set A_T as a target of the investment, then with Backward Stochastic Differential Equation (BSDE) an optimal investment policy π_t^* could be got to achieve this target A_T at time T. Take $X_T - A_T$ as one part and denote it by Y_T, then the original problem could be transferred into the following form,

$$\begin{aligned} \max_{Y_T} \quad & \mathbf{E}[(U(Y_T)^+)], \\ \text{s.t.} \quad & \mathbf{E}[\zeta_T Y_T] = y_0, \end{aligned} \tag{1.4}$$

where Y_t should follow the wealth process below,

$$\begin{cases} \mathrm{d}Y_t &= (r_t Y_t + (\pi_t - \pi_t^*)' \sigma_t \theta_t)\mathrm{d}t + (\pi_t - \pi_t^*)\sigma_t \mathrm{d}W_t, \\ y_0 &= x_0 - \mathbf{E}[\zeta_T A_T]. \end{cases}$$

So problem (1.4) is actually same as the standard investment problem (1.1) and could be solved by standard methods.

While in this thesis, the benchmark in our problem contains an expected term of X_T, which represents the industry's average performance. Because the benchmark is related to the distribution of X_T, the BSDE will fail and the problem couldn't be transferred into the form like problem (1.4). The standard methods no longer apply in our system, and a new approach should be introduced to solve it.

1.3 Compensation Model

Executive compensation has been the subject of extensive prior researches. According to Antle and Smith (1986) and Jensen and Murphy (1999), three primary mechanisms are used to provide executives with compensation and incentives:

- Flow compensation, which is the total of CEO's salary, bonus, new equity grants and other compensation;
- Increase in the value of the CEO's portfolio of stock and options;
- The possibility that the market's assessment of the CEO's human capital will decrease following termination due to poor performance or a change-in-control. For managers below CEO, the possibility of promotion is an additional incentive.

Although stock compensation will give executives more incentives compared to pay and dismissal incentives, executives only hold quite small fractions of their firms' ownership. Jensen and Murphy (1990) studied the pay-performance relation (PPS. dollar-to-dollar measure) of executives, and also found the decline of the pay-performance relation and pay level of executives since 1930s.

Moreover, different country-specific factors could lead to different compensation structures that arise endogenously in those environments [see Bushman and Smith (2001b)]. In some countries, like America, investors are well protected and every firm is required to disclose its material information about its finance and contracts. In these environments, a lot of firms have widely dispersed ownership, and managers only own a small fraction of equity, so equity-based compensation can be a good incentive for managers. While in some other countries, managers and their family may hold most of the company's ownership and explicit equity-based compensation may not be attractive. All these factors result in different types of compensation models, and all of them have certain practical significance.

The design of compensation model is virtually a kind of agency problem, in which the agent enjoys some informational advantage over the principal. This asymmetry of information between agent and principal causes moral hazard. Before the arise of agency problem, it is traditional to proceed researches with the presumption that market will mediate the outcomes efficiently, which actually departs from the reality sometimes. With agency models, we can explain for non-market institutions and contracting forms which received micro-economical theory can not account for. At first, people began to pay attention to moral hazard [see Arrow (1974)], information flows [see Marschak and Radner (1972)] and financial intermediaries in monetary models. Then some people concluded a more general problem from these researches, which is agency problem. Earlier works about agency problem has been done by Wilson (1965), Ross (1973) and Mirrlees (1976). They gave the first further insight into agency problem.

In the beginning, people focused on discrete-time often one-period agency problem, for example, Bhattacharya and Pfleiderer (1985), Dybvig and Spatt (1986), Allen (1990), Stoughton (1993), Heinkel and Stoughton (1994) and so on. Among them, Stoughton (1993) found that the linear contract would lead to under-investment problem of manager, which can be well avoided by quadratic contract. Heinkel and Stoughton (1994) examined the consequences of contracting over multi-period situation.

After several years, Holmstrom and Milgrom (1987) first applied continuous-time model into agency problem in context of Brownian motion, provided important foundation for the researches later. They found the optimal inter-temporal compensation scheme should be a linear function of N independent accounts which represented the aggregate number that observable events occur during the finite time periods. And they assumed that agent got paid at the end of each short time interval, and the agent had exponential utility function. After that, further researches have been done to extend the linear results of Holmstrom and Milgrom (1987). For example, Schaettler and Sung (1993) developed sufficient conditions for validity of first-order method to continuous-time agency model under moral hazard with exponential utility; Hellwig and Schmidt (2002) showed the optimal incentive scheme in discrete-time model of agency problem can converge to the result of Holmstrom and Milgrom (1987); He (2011) solved for optimal contraction problem with private saving, and applied his results into the Leland capital structure model. See also Bushman and Smith (2001a), Jin (2002), Haubrich (1994) and Baiman (1990) for further generalization of Holmstrom and Milgrom (1987). Sannikov (2008) described a new flexible continuous-time model of agency problem when the effort of agent can not be observed directly. While his model will be vulnerable if the agent can save and borrow. Carpenter (2000) used traditional dynamic programming to solve for optimal investment strategy of the manager who was compensated with call options over the assets he controls. Moreover, they found that the option compensation didn't necessarily make the manager more risk-seeking.

Compared to these papers, this thesis engaged in deriving an optimal investment policy with a specific compensation model. Furthermore, the compensation model designed in this thesis has a benchmark which is a linear function of the assets managed. That is the main point different from other papers about compensation models.

1.4 Mean-Field Problem

The theory of mean-field stochastic differential equation (MF-SDE) is first introduced by Kac (1956), who presented a special case MF-SDE, called McKean-Vlasov stochastic differential equation, which was motivated by a stochastic toy model for the Vlasov kinetic equation of plasma. After that, McKean Jr (1966) started the research in mean-field problem. Since then, mean-field problem has been applied in various areas, including engineering, financial management and economics, see also Spohn (1980), Chan (1994), Sznitman (1991), Dawson (1983), Pardoux (1999) and references cited therein. In fact, mean-field problem represents a variety of problems which involve state process as well as their expected values in their cost functions or underlying dynamic systems. And It has been a most popular research direction recently, many research areas are using it to extend their general framework.

Since mean-field problem is rich in content, here we only introduce a special and significant case of the mean-field LQ control problem (MF-LQ control problem), namely mean-variance problem. It is first introduced by Markowitz (1952), which provided a fundamental basis for portfolio construction in a single period. The most important contribution of this paper is that it provides people a method to control their risk within an acceptable level by using the variance to quantify the risk. Markowitz (1952) also found the solution scheme and solved for the optimal portfolio strategy under the assumption that shorting was prohibited. Later Merton (1972) solved for the optimal portfolio strategy when shorting was allowed and covariance matrix was positive definite. Furthermore, Perold (1984) developed a general technique to locate the efficient frontier under the condition that the covariance matrix was nonnegative definite. After that, mean-variance model was extended to multi-period portfolio selection model, see Mossin (1968), Samuelson (1969), Hakansson (1971). The analytical results of Markowitz (1952) and Merton (1972) have been extended into multi-period portfolio selection by Li and Ng (2000) and continuous time portfolio selection by Zhou and Li (2000).

The following equation is an abstract form for the dynamic multi-period mean-variance portfolio selection problem [see Li and Ng (2000)],

$$\begin{aligned} \max \quad & \mathbf{E}(x_T) - \omega \mathbf{Var}(x_T), \\ \text{s.t.} \quad & x_{t+1} = e_t^0 x_0 + \mathbf{P_t}' \mathbf{u_t}, \\ & t = 0, 1 \ldots T-1, \end{aligned} \tag{1.5}$$

where the tradeoff parameter $\omega \in [0, \infty)$, x_t represents the wealth of the investor at the beginning of tth period, e_t^i denotes the random return of ith security at the beginning of the tth period, investment strategy $\mathbf{u_t} = [u_t^0, u_t^1, ..., u_t^n]'$and the ith component of the $\mathbf{u_t}$ represents the amount of the wealth invested in the ith security, and $\mathbf{P_t} = [(e_t^1 - e_t^0), (e_t^2 - e_t^0), ..., (e_t^n - e_t^0)]'$. Below is an abstract form for continuous time mean-variance portfolio selection problem [see Zhou and Li (2000)],

$$\begin{aligned} \min \quad & -\mathbf{E}(x_T) + \mu \mathbf{Var}(x_T), \\ \text{s.t.} \quad & u(\cdot) \in L^2_{\mathcal{F}}(0, T; R^m), \end{aligned} \tag{1.6}$$

where the tradeoff parameter $\mu \in [0, \infty)$, x_t represents the value of the whole portfolio at time t, $u(t)$ denotes the investment policy at time t.

Problem (1.5) and problem (1.6) above are not standard stochastic control problems, because these dynamic optimization problems can't be decomposed by a stage-wise backward recursion, which means they are non-separate in the sense of dynamic programming. The existence of the variance term in the cost function causes the non-separate situation. Since the principle of optimality will fail in this non-separate situation, all the traditional dynamic programming-based optimal stochastic control solution methods no longer apply. In Li and Ng (2000) and Zhou and Li (2000), they managed to achieve the analytical result of the portfolio selection and locate the efficient frontiers by using a new developed method, which is called embedding method. The general principle of the embedding method is that, firstly they try to embed the original problem into a auxiliary problem which is proved to be a stochastic optimal LQ problem, then by using the newly developed general stochastic LQ theory [see Chen et al. (1998)], the auxiliary problem could be solved, and solution to the original problem is found finally via the auxiliary problem's solution. Below is the form the auxiliary problem in Li and Ng (2000),

$$
\begin{aligned}
A(\lambda, \omega): \quad & \max \quad \mathbf{E}\{\lambda x_T - \omega x_T^2\}, \\
\text{s.t.} \quad & x_{t+1} = e_t^0 x_0 + \mathbf{P_t}' \mathbf{u_t}, \\
& t = 0, 1 \ldots T-1,
\end{aligned}
$$

which is a separable LQ stochastic control formulation and can be solved analytically. And the auxiliary problem in Zhou and Li (2000) is as following,

$$
\begin{aligned}
A(\mu, \lambda): \quad & \min \quad \mathbf{E}\{-\lambda x_T + \mu x_T^2\}, \\
\text{s.t.} \quad & u(\cdot) \in L_{\mathcal{F}}^2(0, T; R^m).
\end{aligned}
$$

Later some researches have been done to extend the embedding method, see Zhou and Yin (2003), Bielecki et al. (2005), Li et al. (2002), Lim and Zhou (2002).

Some other methods have been developed to solve the mean-variance problem. For example, Yong (2013) used two Riccati equations to solve the non-separate problem. And Li et al. (2002) came up with Lagrangian formulation. Besides Yong (2013) illustrated some interesting motivation to contain the variance term in the cost function. And framework of mean-field formulation was initially proposed by Cui et al. (2013), under which the mean-variance problem can be solved directly and analytically.

The compensation model in this thesis connects mean-field problem with compensation problem by introducing a mean term of the wealth in the benchmark, which represents the average industry's performance. That makes the problem in this thesis a mean-field problem, and extend the application of mean-field problem into compensation area.

1.5 Summary of Contributions of the Thesis

The original contributions of this thesis are as follows:

- This thesis is the first to introduce a compensation model whose benchmark contains an expected term of the wealth process to represent the average industry's performance. This improvement makes this compensation model more conform to the reality, but also brings us much difficulty in solving it. Besides this thesis is devoted into studying the relationship between the optimal strategy and this given compensation model, which few papers are dedicated to do so.

- Using two-step optimization method, an expression of the optimal final wealth $X(T)$ is derived for the problem.This thesis succeeds to simplify the original random variable optmization problem into a three-dimension scalar optimization problem. And then a numerical experiment is conducted under the assumption that $\rho(T)$ is two-point distributed. The successful settlement of the numerical examples, to a certain extent, shows solvability and rationality of this compensation model. Besides DRRA utility, CRRA utility and HRRA utility are used respectively in these numerical examples, which indicates the generality of the model.

1.6 Organization of the Thesis

The thesis is structured as follows.

- Chap.2 focuses on the formulation and solution of this mean-field target compensation model. Firstly, it illustrates the setups of the whole market that this thesis is based on and the assets are invested in. Specific forms of the asset price processes and wealth process are presented. Secondly, the detailed economic meaning of this compensation model is introduced, which explains the reason why we promote this mean-field target compensation model. Then, a novel two-step optimization method is proposed to solve the problem, and finally an expression of the optimal wealth is derived. All the lemmas and basic theorem are introduced in this chapter.

- Chap.3 is devoted on numerical examples of this compensation model. CRRA utility, DRRA utility and HRRA utility are used respectively in these three examples. With the help of MATLAB, numerical results of these three examples are all achieved. The successful solving of these examples, in some extent, shows the solvability of this compensation model. Besides, figures of the examples will be placed in this chapter.

- Chap.4 concludes the whole thesis and plans for the future work.

Chapter 2

Compensation Model with A Mean-Field Benchmark

Before introducing the compensation model in this thesis, some basic setups will be introduced first. A better understanding of preliminaries and assumptions is of significant help for us to grasp the core content of this chapter. Hereby the manager's preference $U(X)$ and opportunity set will be described, as well as some basic features of the market where the assets will be invested. Various types of assets that exist in this market, price processes of these assets and wealth process the manager follows will all be introduced. After that, the thesis will turn to the construction of the fundamental compensation model, and how to simplify it. This chapter illustrates the formulation of this mean-field compensation model in detail, and explains specifically the economic meaning of every part of this model. Besides, some figures will be provided to show the specific characteristics of the model. With the help of concave envelope, the original problem will be transformed into a more tractable form. Then the final expression of the optimal wealth will be derived finally. Many lemmas and basic theorem will be showed and proved in detail in this chapter.

2.1 Preliminaries and Assumptions

Suppose there are $m+1$ assets which are traded continuously in a complete, arbitrage-free market. One of the assets is riskless, like bond, whose price process $P_0(t)$ follows

$$\begin{cases} \mathrm{d}P_0(t) = r(t)P_0(t)\mathrm{d}t, & t \in [0,T], \\ P_0(0) = p_0, & p_0 > 0, \end{cases}$$

where $r(t) > 0$ is the interest rate of the riskless asset. And it is an $\mathcal{F}_t$-progressively measurable, scalar-valued stochastic process. Other m assets are stocks, whose price processes $P_1(t), ..., P_m(t)$ satisfy

$$\begin{cases} \mathrm{d}P_i(t) = P_i(t)\{\mu_i(t)\mathrm{d}t + \sum\limits_{j=1}^{m} \sigma_{i,j}(t)\mathrm{d}W_j(t)\}, \quad t \in [0,T], \quad i = 1...m, \\ P_i(0) = p_i, \quad p_i > 0, \end{cases}$$

where $W(t)$ denotes an m-dimensional standard Brownian motion defined on some fixed filtered complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$, $\mu_i(t) : [0,T] \times \Omega \to \mathbf{R}^1$ and $\sigma_i(t) : [0,T] \times \Omega \to \mathbf{R}^m$ stand for the appreciation rate and the volatility rates of ith asset, respectively. Furthermore, they are $\mathcal{F}_t$-progressively measurable, scalar-valued stochastic processes. Set the excess rate of return vector process

$$B(t) := (\mu_1(t) - r(t), \cdots, \mu_m(t) - r(t))'$$

and define the positive definite volatility matrix process $\sigma(t) = (\sigma_{i,j}(t))_{m\times m}$.

Assume that the trading of assets takes place continuously in a self-financing fashion and there are no transaction fees. Denote the total value of the assets that the manager controls at time t by $X(t)$, and the initial endowment is $x_0 > 0$ (fixed through the project). Then $X(t)$ follows wealth process [see Karatzas and Shreve (1998)]

$$\begin{cases} \mathrm{d}X(t) &= (r(t)X(t) + \pi(t)'\sigma(t)\theta(t))\mathrm{d}t + \pi(t)'\sigma(t)\mathrm{d}W(t), \\ X(0) &= x_0, \end{cases}$$

where $\pi(t) \equiv (\pi_1(t), ..., \pi_m(t))'$ is the control variable, which also represents the investment strategy of the manager. $\pi_i(t)$ is the total market value of the holdings of asset i at time t, where $i = 1, ..., m$. And $\theta(t) \equiv \sigma(t)^{-1}B(t)$ is the market price of risk. Assume $r(\cdot)$, $\sigma(\cdot)$ and $\theta(\cdot)$ are bounded. We will only consider admissible controls $\pi(\cdot)$, such that $\mathbf{E}\int_0^T |\pi(t)'\sigma(t)|^2 dt < +\infty$ and $X(\cdot) \geq 0$.

2.2 Problem Formulation

At time T, the total basic salary of the manager is denoted by a constant c. Then the wealth of the manager at time T comprises c and the compensation based on some kind of benchmark. In this thesis, compensation is based on the industry's average performance. The benchmark here is assumed to be $a\mathbf{E}[X(T)] + b$, where $\mathbf{E}[X(T)]$ represents average performance of the whole industry. Hence, the utility of the manager at time T is

$$U\left((X(T) - a\mathbf{E}[X(T)] - b)^+ + c\right).$$

Here $U : (0, +\infty) \mapsto \mathbb{R}$ is a (strictly increasing and strictly concave) utility function with $U'(+\infty) = 0$, and $a > 0,\ b \geq 0,\ c > 0$ are scalars. Given the specific compensation form, the manager will choose an investment strategy to maximize his own expected utility. Hence, his objective is to maximize

$$\mathbf{E}\left[U\left((X(T) - a\mathbf{E}[X(T)] - b)^+ + c\right)\right] \tag{2.1}$$

over all possible controls $\pi(\cdot)$ under which the wealth process is nonnegative.

Define the state price density process

$$\begin{cases} \mathrm{d}\rho(t) & = \rho(t)(-r(t)\mathrm{d}t - \theta(t)'\mathrm{d}W(t)), \\ \rho(0) & = 1. \end{cases}$$

Then it can be obtained that

$$\rho(t) = \exp\left(\int_0^t \left(-r(s) - \frac{1}{2}|\theta(s)|^2\right) \mathrm{d}s - \int_0^t \theta(s)' \mathrm{d}W(s)\right).$$

Applying the Itô's lemma, we have

$$\mathrm{d}(\rho(t)X(t)) = (\rho(t)\pi(t)'\sigma(t) - X(t)\rho(t)\theta(t)')\mathrm{d}W(t).$$

It means $\rho(t)X(t)$ is a martingale, and we can obtain $\mathbf{E}[\rho(T)X(T)] = \rho(0)X(0) = x_0$. Denote $\rho = \rho(T)$ and $X = X(T)$ for simplicity. The problem reduces to a static optimization problem

$$\begin{aligned} \max_X \quad & \mathbf{E}[U((X - a\mathbf{E}[X] - b)^+ + c)], \\ \text{s.t.} \quad & \mathbf{E}[\rho X] = x_0, \;\; X \geq 0. \end{aligned} \tag{2.2}$$

If we have solved the above problem and found the optimal X^*, we can use the theory of Backward Stochastic Differential Equation (BSDE) to find a control $\pi(\cdot)$.

2.3 Solving the Problem

Check problem (2.2), we can see that it is hard to solve it by the standard dynamic programming or martingale approach. The objective function of problem (2.2) is not concave and its benchmark is not fixed. So our target is to transform it to a more tractable form. First, let's study a related problem $\mathcal{P}\ (\alpha, \beta)$, which has a fixed benchmark α,

$$\begin{aligned} \max_X \quad & \mathbf{E}[U((X - \alpha)^+ + c)], \\ \text{s.t.} \quad & \mathbf{E}[\rho X] = x_0, \quad \mathbf{E}[X] = \beta, \quad X \geq 0. \end{aligned} \tag{2.3}$$

Remark 2.1. *In particular, we are interested in $\mathcal{P}(a\beta + b,\ \beta)$. Because when $\alpha = a\beta + b = a\mathbf{E}[X] + b$, problem (2.3) is turned into the original problem (2.2).*

To further simplify the model, we let $u(x) = U((x-\alpha)^+ + c)$, then the above problem (2.3) reads

$$\begin{aligned} \max_{X} \quad & \mathbf{E}[u(X)], \\ \text{s.t.} \quad & \mathbf{E}[\rho X] = x_0, \quad \mathbf{E}[X] = \beta, \quad X \geq 0. \end{aligned} \tag{2.4}$$

Remark 2.2. *Now the model gets a quite simple form, and it is seemingly similar with the traditional investment problem. However we should note that u is not a concave function. It is a constant on $(-\infty, \alpha]$ and concave on $[\alpha, +\infty)$.*

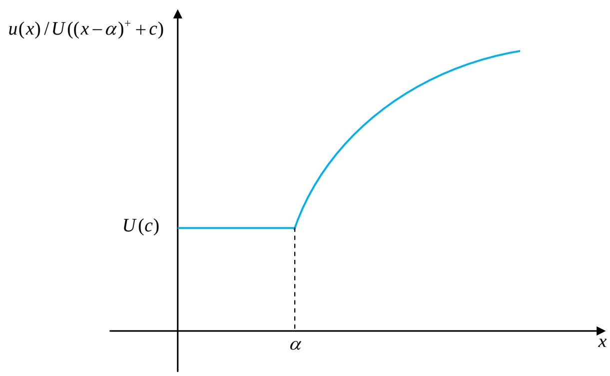

Figure 2.1: x-$u\&U$ plane

Figure 2.1 is a picture of $u\&U$, which is used to illustrate specifically the shape of the function u. It is not concave, so the Lagrangian method can't be applied directly.

To overcome the difficulty, it is important to make the objective function concave first, so the following lemma is introduced.

Lemma 2.1. *Let $\hat{u}$ be the concave envelope of u on $[0, +\infty)$, that is the smallest concave function dominating u on $[0, +\infty)$. Then*

$$\hat{u}(x) = \begin{cases} u'(\underline{x})x + u(0), & 0 \leq x < \underline{x}, \\ u(x), & x \geq \underline{x}, \end{cases} \tag{2.5}$$

where $\underline{x}$ is the unique root of

$$u'(x) = \frac{u(x) - u(0)}{x} \tag{2.6}$$

on $(\alpha, +\infty)$. Moreover, $\hat{u} \in C^1(0, +\infty)$.

Following is a picture of $\hat{u}(x)$. And referring to Aumann and Perles (1965), we

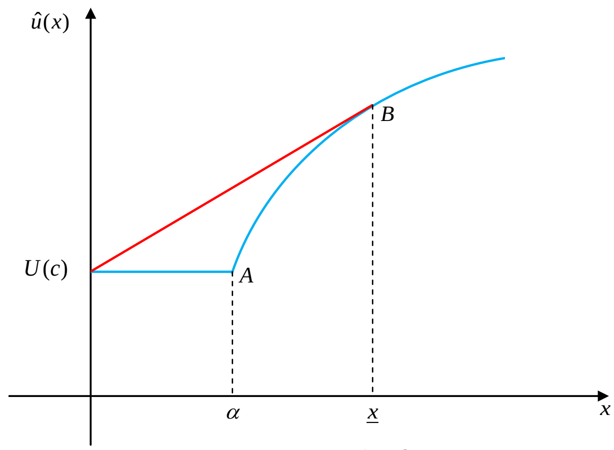

Figure 2.2: x-$\hat{u}$ plane

know that $\hat{u}(x)$ is the smallest concave function domination u on $[0, +\infty)$, which is concave envelope.

Now consider the following concave optimization problem,

$$\begin{aligned} \max_X \quad & \mathbf{E}[\hat{u}(X)], \\ \text{s.t.} \quad & \mathbf{E}[\rho X] = x_0, \quad \mathbf{E}[X] = \beta, \quad X \geq 0. \end{aligned} \tag{2.7}$$

Using Lagrangian method, the result of problem (2.7) is derived in the theorem below.

Theorem 2.1. *The optimal solution of problem (2.7) is given by*

$$X^* = (\hat{u}')^{-1}(\lambda\rho + \mu), \tag{2.8}$$

where λ and μ are determined by

$$\mathbf{E}[\rho X^*] = x_0, \quad \mathbf{E}[X^*] = \beta.$$

And $(\hat{u}')^{-1}$ *is denoted as the left-continuous inverse function of* $\hat{u}'$*, that is*

$$(\hat{u}')^{-1}(x) = \inf\{y \geq 0 : \ \hat{u}'(y) \leq x\} \vee 0. \tag{2.9}$$

Proof. Since the cost function is concave, we can introduce two Lagrangian parameters λ and μ to solve the problem. Then we have

$$\begin{aligned} L(\lambda, \mu) =& \mathbf{E}[\hat{u}(X)] - \lambda(\mathbf{E}[\rho X] - x_0) - \mu(\mathbf{E}[X] - \beta), \\ X \geq& 0. \end{aligned}$$

Differentiate $L(\lambda, \mu)$ corresponding to X, λ and μ, we can derive the following equations,

$$\begin{cases} \frac{\partial L(\lambda,\mu)}{\partial X} = \mathbf{E}[\hat{u}'(X) - \lambda\rho - \mu] = 0, \\ \frac{\partial L(\lambda,\mu)}{\partial \lambda} = \mathbf{E}[\rho X] - x_0 = 0, \\ \frac{\partial L(\lambda,\mu)}{\partial \mu} = \mathbf{E}[X] - \beta = 0. \end{cases}$$

The solution is determined by the equation as below,

$$\hat{u}'(X) = \lambda\rho + \mu, \tag{2.10}$$

where $X \geq 0$. With the equation (2.5), we have following definition for $\hat{u}'(x)$,

$$\hat{u}'(x) = \begin{cases} u'(\underline{x}), & 0 \leq x < \underline{x}, \\ \\ u'(x), & x \geq \underline{x}. \end{cases} \tag{2.11}$$

The characteristics of $\hat{u}'(x)$ can be clearly observed from Figure 2.3. Compared with Figure 2.3 of $U'((x - \alpha)^+ + c)$, we can see that the only different part between these two figures is when $0 < x \leq \underline{x}$.

When $x \geq \underline{x}$, $\hat{u}'(x) = u'(x) = U'((x - \alpha)^+ + c)$ and in this interval u is strictly increasing and strictly concave, with $u'(+\infty) = 0$. So $u'(\underline{x}) \geq u'(x) \geq 0$. With definition (2.11), we know $\hat{u}'(X)$ must be no smaller than 0. So $\lambda\rho + \mu$ should be larger than or equal to 0, which means that $\lambda \geq 0$ and $\mu \geq 0$. Furthermore, $\hat{u}'(X)$ is

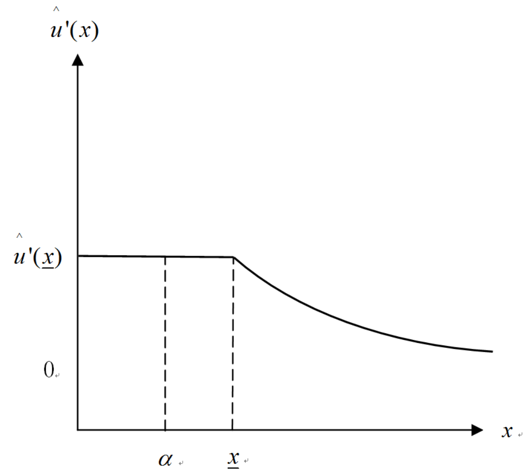

Figure 2.3: x-$\hat{u}'(x)$ plane

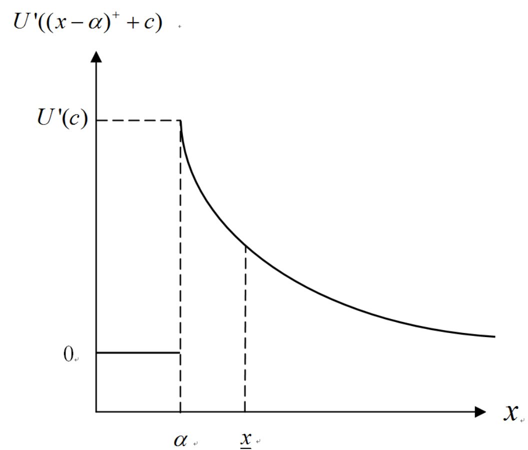

Figure 2.4: x-$U'((x-\alpha)^{+}+c)$ plane

not one-to-one function. To make the inverse function of $\hat{u}'(X)$ reasonable, we define $(\hat{u}')^{-1}(x)$ as 2.9.

Then from (2.10), it is apparent that,

$$X^{*} = (\hat{u}')^{-1}(\lambda\rho + \mu).$$

With the definition of $(\hat{u}')^{-1}(x)$ and the equation (2.11), X^{*} could be expressed in an another way,

$$X^* = \begin{cases} 0 & \lambda\rho + \mu \geq u'(\underline{x}), \\ U'^{-1}(\lambda\rho + \mu) + \alpha - c & \lambda\rho + \mu < u'(\underline{x}). \end{cases} \tag{2.12}$$

□

Lemma 2.2. *One has* $u\left((\hat{u}')^{-1}(x)\right) = \hat{u}\left((\hat{u}')^{-1}(x)\right)$ *for all* $x \geq 0$.

Proof. For any $x \in (u'(+\infty),\ u'(\underline{x}))$, because u' is strictly decreasing, there is a unique $y \in (\underline{x},\ +\infty)$ such that $u'(y) = x$. Because $y \in (\underline{x},\ +\infty)$, by Lemma 2.1, $u(y) = \hat{u}(y)$, $\hat{u}'(y) = u'(y) = x$, $y = (\hat{u}')^{-1}(x)$ and $u((\hat{u}')^{-1}(x)) = u(y) = \hat{u}(y) = \hat{u}((\hat{u}')^{-1}(x))$. For any $x \in [u'(\underline{x}),\ +\infty)$, $(\hat{u}'^{-1})(x) = 0$, so $u((\hat{u}'^{-1}(x))) = u(0) = \hat{u}(0) = \hat{u}((\hat{u}')^{-1}(x))$. □

Theorem 2.2. *The X^* defined in the equation (2.8) is also an optimal solution to problems (2.3) and (2.4).*

Proof. For any feasible solution X of problem (2.7), it is also a feasible solution of the problems (2.4) and (2.3).

Because X^* is an optimal solution of (2.7),

$$\mathbf{E}[u(X)] \leq \mathbf{E}[\hat{u}(X)] \leq \mathbf{E}[\hat{u}(X^*)] = \mathbf{E}[\hat{u}((\hat{u}')^{-1}(\lambda\rho+\mu))] = \mathbf{E}[u((\hat{u}')^{-1}(\lambda\rho+\mu))] = \mathbf{E}[u(X^*)],$$

where the second last equality is due to Lemma 2.2. This proves that X^* is an optimal solution to problems (2.3) and (2.4). □

Now we have solved problems (2.3) and (2.4), in which the benchmark α and expected wealth β are all fixed. However, in problem $\mathcal{P}(\alpha\beta + b,\ \beta)$ that we really concern, β should not be fixed. And both the objective function and the optimal solution X^* depend on β in problem $\mathcal{P}(\alpha\beta + b,\ \beta)$. To solve the original problem (2.2), we need to find an optimal β to maximize its objective function.

Lemma 2.3. *Denote $\upsilon = u'(\underline{x})$. Then $\upsilon \in (0, U'(c))$ and*

$$\underline{x} = \frac{1}{\upsilon}\left(U\left((U')^{-1}(\upsilon)\right) - U(c)\right) := f(\upsilon), \tag{2.13}$$

$$\alpha = \frac{1}{\upsilon}\left(U\left((U')^{-1}(\upsilon)\right) - U(c)\right) - (U')^{-1}(\upsilon) + c := g(\upsilon) + c. \tag{2.14}$$

Moreover, f is one-to-one decreasing mapping from $(0, U'(c))$ to $(0, +\infty)$ and g is one-to-one decreasing mapping from $(0, U'(c))$ to $(-c, +\infty)$.

Proof. Note $\upsilon = u'(\underline{x}) = U'(\underline{x} - \alpha + c)$, $\underline{x} > \alpha$ and U is strictly concave, hence $\upsilon < U'(c)$ and $\underline{x} = (U')^{-1}(\upsilon) + \alpha - c$. On the other hand, $\frac{u(\underline{x}) - u(0)}{\underline{x}} = u'(\underline{x}) = \upsilon$, so $\underline{x} = \frac{1}{\upsilon}(u(\underline{x}) - u(0)) = \frac{1}{\upsilon}(U(\underline{x} - \alpha + c) - U(c)) = \frac{1}{\upsilon}\left(U\left((U')^{-1}(\upsilon)\right) - U(c)\right)$. Moreover, $\alpha = \underline{x} - (U')^{-1}(\upsilon) + c = \frac{1}{\upsilon}\left(U\left((U')^{-1}(\upsilon)\right) - U(c)\right) - (U')^{-1}(\upsilon) + c$. Note $g'(\upsilon) = -\frac{1}{\upsilon^2}\left(U\left((U')^{-1}(\upsilon)\right) - U(c)\right) < 0$, so g is decreasing. Because $g(\upsilon)$ and $(U')^{-1}(\upsilon)$ are both decreasing, $f(\upsilon) = g(\upsilon) + (U')^{-1}(\upsilon)$ is decreasing as well. $\square$

Remark 2.3. *From the above Lemma 2.3 we can use υ as a parameter to determine the relationship between $\underline{x}$ and α. So use υ as a bridge, we can rewrite the equation (2.8).*

Lemma 2.4. *The X^* defined in the equation (2.8) is same as*

$$X^* = \begin{cases} 0, & \text{if } \rho \geq \eta, \\ (U')^{-1}(\lambda(\rho - \eta) + \upsilon) + g(\upsilon), & \text{if } \rho < \eta, \end{cases} \tag{2.15}$$

where υ and g are defined in Lemma 2.3, and constants λ and η are determined by

$$\begin{aligned} \mathbf{E}[\rho X^*] &= \mathbf{E}[\rho (U')^{-1}(\lambda(\rho - \eta) + \upsilon)\mathbf{1}_{\rho<\eta}] + g(\upsilon)\mathbf{E}[\rho \mathbf{1}_{\rho<\eta}] = x_0, \\ \mathbf{E}[X^*] &= \mathbf{E}[(U')^{-1}(\lambda(\rho - \eta) + \upsilon)\mathbf{1}_{\rho<\eta}] + g(\upsilon)\mathbf{P}(\rho < \eta) = \beta. \end{aligned}$$

Proof. If $x \geq u'(\underline{x})$, then $(\hat{u}')^{-1}(x) = 0$ by definition. If $x < u'(\underline{x})$, $(\hat{u}')^{-1}(x) = (u')^{-1}(x)$ by definition. Then

$$X^* = \begin{cases} 0, & \text{if } \lambda\rho + \mu \geq u'(\underline{x}), \\ \\ (u')^{-1}(\lambda\rho + \mu), & \text{if } \lambda\rho + \mu < u'(\underline{x}). \end{cases} \tag{2.16}$$

It is evident that $(u')^{-1}(x) = (U')^{-1}(x) + \alpha - c$ on $(0,\ \hat{u}(\underline{x}))$. From Lemma 2.3, $\upsilon = \hat{u}(\underline{x})$. Hence the equation (2.16) can be turned into

$$X^* = \begin{cases} 0, & \text{if } \lambda\rho + \mu \geq \upsilon, \\ \\ (U')^{-1}(\lambda\rho + \mu) + g(\upsilon), & \text{if } \lambda\rho + \mu < \upsilon, \end{cases} \tag{2.17}$$

Define $\eta = \frac{\upsilon - \mu}{\lambda}$. We can obtain the equation (2.15) immediately. $\square$

The equation (2.15) is actually the result of problem (2.3) with benchmark α. What we are really interested in is the case when $\alpha = a\beta + b = g(\upsilon) + c$. Regarding to problem $\mathcal{P}(\alpha,\ (\alpha - b)/a)$, the above Lemma 2.4 reads $\mathcal{P}(g(\upsilon) + c,\ (g(\upsilon) + c - b)/a)$.

Lemma 2.5. *For problem $\mathcal{P}(g(\upsilon) + c,\ (g(\upsilon) + c - b)/a)$, the optimal solution is*

$$X^* = \left[(U')^{-1}(\lambda(\rho - \eta) + \upsilon) + g(\upsilon)\right] \mathbf{1}_{\rho < \eta}, \tag{2.18}$$

where λ and η are determined by

$$\mathbf{E}[\rho(U')^{-1}(\lambda(\rho - \eta) + \upsilon)\mathbf{1}_{\rho<\eta}] + g(\upsilon)\mathbf{E}[\rho\mathbf{1}_{\rho<\eta}] = x_0, \tag{2.19}$$

$$\mathbf{E}[(U')^{-1}(\lambda(\rho - \eta) + \upsilon)\mathbf{1}_{\rho<\eta}] + g(\upsilon)\mathbf{P}(\rho < \eta) = (g(\upsilon) + c - b)/a. \tag{2.20}$$

The optimal value is

$$\mathbf{E}\left[U((U')^{-1}(\lambda(\rho - \eta) + \upsilon)\mathbf{1}_{\rho<\eta})\right]. \tag{2.21}$$

Now the original random variable optimization problem (2.2) is simplified into a scalar optimization problem in Lemma 2.5, which is much easier than the original one. The target turns to find a υ to maximize the value of (2.21) where λ and η are determined by (2.19) and (2.20). This is a three-dimensional scalar optimization problem. However the objective function is not concave, it is very hard to prove its solvability analytically. So numerical method will be used to solve it in the following section.

Chapter 3

Numerical Examples

The purpose of this chapter is to give simulation on this compensation model. Three examples are used to illustrate how this model works, in which CARA (Constant Absolute Risk Aversion) utility, DARA (Decreasing Absolute Risk Aversion) utility and HARA (Hyperbolic Absolute Risk Aversion) utility are used respectively. The reason that these three utility forms are adopted in this thesis is that they are the most widely used utility forms in both research area and industry practice, which will show the generality of the model. Here the distribution of ρ is characterized as a two-point distribution, and the value of all parameters are assigned. It will follow the same route, when ρ is three-point or four-point distributed. The experiments are done under MATLAB using its Optimization Toolbox, and mostly through *"fmincon"* function. And numerical results are all derived in these three examples, which are also checked to be true. The examples show us the effectiveness and solvability of this compensation model in the above setting.

As showed in Lemma 2.5, the compensation problem in this thesis is actually an optimization problem as below,

$$\max_{\lambda,\upsilon,\mu} \mathbf{E}\left[U((U')^{-1}(\lambda\cdot\rho+\mu)\mathbf{1}_{\lambda\cdot\rho+\mu<\upsilon})\right],$$

$$\text{s.t.}\begin{cases}\mathbf{E}[\rho(U')^{-1}(\lambda\cdot\rho+\mu)\mathbf{1}_{\lambda\cdot\rho+\mu<\upsilon}]+g(\upsilon)\mathbf{E}[\rho\mathbf{1}_{\lambda\cdot\rho+\mu<\upsilon}]=x_0,\\ \mathbf{E}[(U')^{-1}(\lambda\cdot\rho+\mu)\mathbf{1}_{\lambda\cdot\rho+\mu<\upsilon}]+g(\upsilon)\mathbf{P}(\lambda\cdot\rho+\mu<\upsilon)=(g(\upsilon)+c-b)/a.\end{cases}$$

In these three examples, ρ is set to be distributed as following,

$$\rho=\begin{cases}m & p,\\ n & 1-p,\end{cases}$$

where $m>n>0$. To make the problem reasonable, there are some constraints for random variables υ, μ and λ,

$$\begin{cases}\lambda>0,\\ \mu>0,\\ 0<\upsilon<U'(c).\end{cases}$$

3.1 DARA Utility Case

In this case, utility function is set to be $U=\ln(x)$, which is a DARA utility function. Because there exists an indicative function in the objective function, we should solve the optimization problem in three different scopes. The whole problem could be simplified into following optimization problem.

In the first scope,

$$F = \max_{\lambda,\mu,\upsilon} -p\ln(m\lambda+\mu) - (1-p)\ln(n\lambda+\mu),$$

$$\text{s.t.} \quad \begin{cases} \dfrac{pm}{m\lambda+\mu} + \dfrac{(1-p)n}{n\lambda+\mu} + (mp+n(1-p))g(\upsilon) = x_0, \\ \dfrac{p}{m\lambda+\mu} + \dfrac{1-p}{n\lambda+\mu} + g(\upsilon) = \dfrac{g(\upsilon)+c-b}{a}, \\ g(\upsilon) = -\dfrac{1+\ln(c\upsilon)}{\upsilon}, \\ 0 < \upsilon < 1/c, \\ m\lambda+\mu < \upsilon, \\ \lambda,\mu > 0. \end{cases} \tag{3.1}$$

After several transformations, the objective function of problem (3.1) could be simplified to an equation, which is only related to υ. The new objective function is as following,

$$\begin{aligned} F = \max_{\lambda,\mu,\upsilon} & -p\ln\Big(\frac{(mp+n(1-p)+\frac{n}{a}-n)g(\upsilon)+\frac{c-b}{a}n-x_0}{p(n-m)}\Big) \\ & -(1-p)\ln\Big(\frac{x_0-\frac{c-b}{a}m-(mp+n(1-p)+\frac{m}{a}-m)g(\upsilon)}{(1-p)(n-m)}\Big). \end{aligned} \tag{3.2}$$

In the second scope,

$$F = \max_{\lambda,\mu,\upsilon} -(1-p)\ln(n\lambda+\mu),$$

$$\text{s.t.} \quad \begin{cases} \dfrac{(1-p)n}{n\lambda+\mu} + n(1-p)g(\upsilon) = x_0, \\[2ex] \dfrac{1-p}{n\lambda+\mu} + (1-p)g(\upsilon) = \dfrac{g(\upsilon)+c-b}{a}, \\[2ex] g(\upsilon) = -\dfrac{1+\ln(c\upsilon)}{\upsilon}, \\[2ex] m\lambda+\mu > \upsilon, \quad n\lambda+\mu < \upsilon, \\[2ex] 0 < \upsilon < 1/c, \\[2ex] \lambda, \mu > 0. \end{cases} \tag{3.3}$$

With the restriction functions of problem (3.3), υ could be solved directly

$$g(\upsilon) = \frac{a}{n}x_0 + b - c.$$

Then the objective function of problem (3.3) could reduce to

$$F = \max_{\lambda,\mu,\upsilon}(1-p)\{\ln[(\frac{1}{a}+p-1)g(\upsilon) + \frac{c-b}{a}] - \ln(1-p)\}. \tag{3.4}$$

In the third scope where $n\lambda+\mu > \upsilon$, the optimization problem is meaningless.

Set $x_0 = 1$, $a = 0.2$, $b = 0.1$, $c = 0.1$, $m = 0.8$, $n = 0.5$, $p = 0.5$, and substitute them into the above problem. With MATLAB, we got the numerical result of the problem. In the first scope that $m\lambda+\mu < \upsilon$, with $\lambda = 1.0000$, $\mu = 0.1625$, $\upsilon = 1.9664$, objective function (3.2) could achieve the optimal value $F = 0.2250$. Figure 3.1 describes the objective function (3.2), and illustrates the relationship between υ and $F(\upsilon)$. It is easy to see that in the significant interval, (3.2) is a concave function, which means the global maximum exists in this situation. In the second situation that $m\lambda+\mu > \upsilon$, $n\lambda+\mu < \upsilon$, only one admissible υ is got, which is $\upsilon = 1.7946$. The

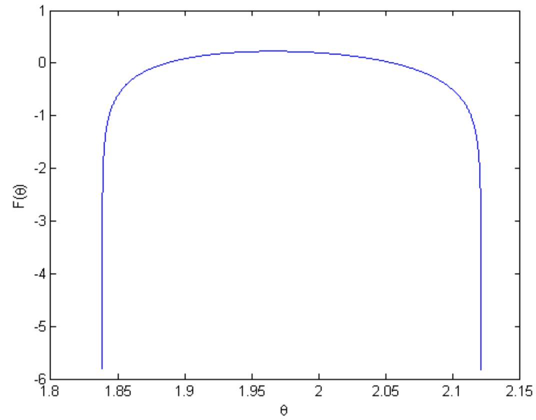

Figure 3.1: υ-$F(\upsilon)$ plane in DARA case

corresponding value of the objective function (3.4) is $F = 0.6405$, which is larger than the optimal value of the objective function in the first situation. In summary, the optimal value of this optimization problem should be $F = 0.6405$, with $\upsilon = 1.7946$, $0.8\lambda + \mu > 1.7946$, $\lambda > 0$, $\mu > 0$.

3.2 CARA Utility Case

In this case, utility function is set to be $U = 1 - e^{-x}$, which is a CARA utility function. Because there exists an indicative function in the objective function, we should solve the optimization problem in three different scopes. The whole problem could be simplified to following optimization problem,

In the first scope,

$$F = \max_{\lambda,\upsilon,\mu} 1-(mp+n(1-p))\lambda-\mu,$$

$$\text{s.t.} \quad \begin{cases} -mp\ln(m\lambda+\mu)-n(1-p)\ln(n\lambda+\mu)+(mp+n(1-p))g(\upsilon)=x_0, \\ -p\ln(m\lambda+\mu)-(1-p)\ln(n\lambda+\mu)+g(\upsilon)=\dfrac{g(\upsilon)+c-b}{a}, \\ g(\upsilon)=-1+\dfrac{1}{\upsilon}e^{-c}+\ln(\upsilon), \\ m\lambda+\mu<\upsilon, \\ 0<\upsilon<e^{-c}, \\ \lambda,\mu>0. \end{cases} \tag{3.5}$$

After several transformations, the objective function of problem (3.5) could be simplified to an equation, which is only related to υ. The new objective function is as following,

$$\begin{aligned} F &= \max_{\upsilon} p(1-\exp(A(\upsilon)))+(1-p)(1-\exp(B(\upsilon))), \\ A(\upsilon) &= -\frac{(mp+n(1-p)+\frac{n}{a}-n)g(\upsilon)+\frac{c-b}{a}n-x_0}{p(n-m)}, \\ B(\upsilon) &= -\frac{x_0-\frac{c-b}{a}m-(mp+n(1-p)+\frac{m}{a}-m)g(\upsilon)}{(1-p)(n-m)}. \end{aligned} \tag{3.6}$$

In the second scope,

$$F = \max_{\lambda,\mu,\upsilon}(1-p)(1-n\lambda-\mu),$$

$$\text{s.t.} \quad \begin{cases} -(1-p)n\ln(n\lambda+\mu)+n(1-p)g(\upsilon)=x_0, \\ -(1-p)\ln(n\lambda+\mu)+(1-p)g(\upsilon)=\dfrac{g(\upsilon)+c-b}{a}, \\ g(\upsilon)=-\dfrac{1+\ln(c\upsilon)}{\upsilon}, \\ m\lambda+\mu>\upsilon, \quad n\lambda+\mu<\upsilon, \\ 0<\upsilon<e^{-c}, \\ \lambda,\mu>0. \end{cases} \tag{3.7}$$

With the restriction functions of problem (3.7), we could solve υ directly

$$g(\upsilon) = \frac{a}{n}x_0 + b - c.$$

Then the objective function of problem (3.7) could reduce to

$$F = \max_{\lambda,\mu,\upsilon}(1-p)(1-\exp(-\frac{x_0}{n(1-p)}+g(\upsilon))). \tag{3.8}$$

In the third scope where $n\lambda+\mu>\upsilon$, the optimization problem is meaningless.

Set $x_0 = 1$, $a = 0.2$, $b = 0.1$, $c = 0.1$, $m = 0.8$, $n = 0.5$, $p = 0.5$, and substitute them into the above problem. With MATLAB, the numerical result of the problem above is got. In the first scope that $m\lambda+\mu<\upsilon$, with $\lambda = 0.3534$, $\mu = 0.0574$, $\upsilon = 0.4097$, objective function (3.6) could achieve optimal value $F = 0.7129$. Figure 3.2 describes the objective function $F(\upsilon)$, and illustrates the relationship between υ and $F(\upsilon)$. It is easy to see that in the significant interval, $F(\upsilon)$ is a concave function, which means the global maximum exists in this situation. In the second situation that $m\lambda+\mu>\upsilon$, $n\lambda+\mu<\upsilon$, we have only one admissible υ, which is

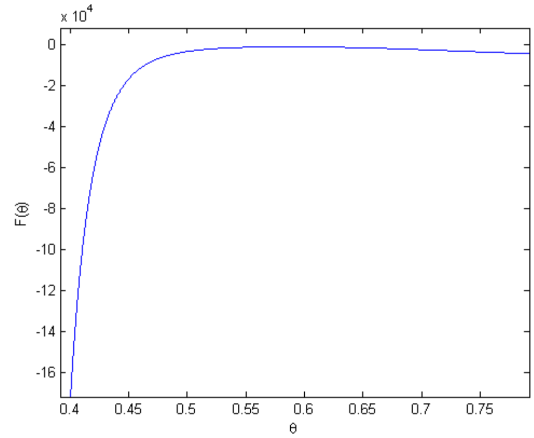

Figure 3.2: υ-$F(\upsilon)$ plane in CARA case

$\upsilon = 0.3838$. The corresponding value of the objective function (3.8) is $F = 0.4863$, which is smaller than the optimal value of the objective function in the first situation. In summary, the optimal value of this optimization problem should be $F = 0.7129$, with $\lambda = 0.3534$, $\mu = 0.0574$, $\upsilon = 0.4097$.

3.3 HARA Utility Case

In this case, utility function is set to be $U = 2\sqrt{x}$, which is a HARA utility function. Because there exists an indicative function in the objective function, we should solve the optimization problem in three different scopes. The whole problem could be simplified to following optimization problem,

In the first scope,

$$
F = \max_{\lambda, \upsilon, \mu} \tfrac{1}{m\lambda+\mu} + \tfrac{1}{n\lambda+\mu},
$$

$$
\text{s.t.} \quad \begin{cases}
\dfrac{mp}{(m\lambda+\mu)^2} + \dfrac{np}{(n\lambda+\mu)^2} + (mp + n(1-p))g(\upsilon) = x_0, \\[2ex]
\dfrac{p}{(m\lambda+\mu)^2} + \dfrac{1-p}{(n\lambda+\mu)^2} + g(\upsilon) = \dfrac{g(\upsilon)+c-b}{a}, \\[2ex]
g(\upsilon) = \dfrac{1}{\upsilon^2} - \dfrac{2\sqrt{c}}{\upsilon}, \\[2ex]
m\lambda + \mu < \upsilon, \\[2ex]
0 < \upsilon < \dfrac{1}{\sqrt{c}}, \lambda, \mu > 0.
\end{cases} \tag{3.9}
$$

After several transformations, the objective function of problem (3.9) could be simplified to an equation, which is only related to variable υ. The new objective function is as following,

$$
\begin{aligned}
F =& \max_{\upsilon} \sqrt{A(\upsilon)} + \sqrt{B(\upsilon)}, \\
A(\upsilon) =& \frac{(mp + n(1-p) + \frac{n}{a} - n)g(\upsilon) + \frac{c-b}{a}n - x_0}{p(n-m)}, \\
B(\upsilon) =& \frac{x_0 - \frac{c-b}{a}m - (mp + n(1-p) + \frac{m}{a} - m)g(\upsilon)}{(1-p)(n-m)}.
\end{aligned} \tag{3.10}
$$

In the second scope,

$$F = \max_{\lambda,\mu,\upsilon} \tfrac{2(1-p)}{n\lambda+\mu},$$

$$\text{s.t.} \quad \begin{cases} \dfrac{n(1-p)}{(n\lambda+\mu)^2} + n(1-p)g(\upsilon) = x_0, \\ \dfrac{1-p}{(n\lambda+\mu)^2} + (1-p)g(\upsilon) = \dfrac{g(\upsilon)+c-b}{a}, \\ g(\upsilon) = \dfrac{1}{\upsilon^2} - \dfrac{2\sqrt{c}}{\upsilon}, \\ m\lambda+\mu > \upsilon, \quad n\lambda+\mu < \upsilon, \\ 0 < \upsilon < \dfrac{1}{\sqrt{c}}, \quad \lambda, \quad \mu > 0. \end{cases} \tag{3.11}$$

With the restriction functions of problem (3.11), we could solve υ directly

$$g(\upsilon) = \frac{a}{n}x_0 + b - c.$$

Then the objective function of problem (3.11) could reduce to

$$F = \max_{\lambda,\mu,\upsilon} 2(1-p)\sqrt{\frac{x_0}{n(1-p)} - g(\upsilon)}. \tag{3.12}$$

In the third scope where $n\lambda+\mu > \upsilon$, the optimization problem is meaningless.

Set $x_0 = 1$, $a = 0.2$, $b = 0.1$, $c = 0.1$, $m = 0.8$, $n = 0.5$, $p = 0.5$, and substitute them into the above problem. With MATLAB, the numerical result of the problem above is got. In the first scope that $m\lambda+\mu < \upsilon$, with $\lambda = 0.8564$, $\mu = 0.3483$, $\upsilon = 1.0334$, objective function achieve the highest point $F = 2.2555$. Figure 3.3 describes the objective function (3.10), and illustrates the relationship between υ and $F(\upsilon)$. It is easy to see that in the significant interval, $F(\upsilon)$ is a concave function, which means the global maximum exists in this problem. In the second situation that $m\lambda+\mu > \upsilon$, $n\lambda+\mu < \upsilon$, only one admissible υ is got, which is $\upsilon = 0.9772$.

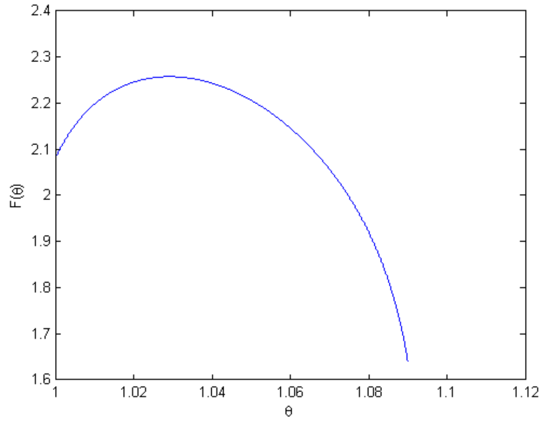

Figure 3.3: υ-$F(\upsilon)$ plane in HARA case

The corresponding value of the objective function (3.12) is $F = 1.8974$, which is smaller than the optimal value of the objective function in the first situation. In summary, the optimal value of this optimization problem should be $F = 2.2555$, with $\lambda = 0.8564$, $\mu = 0.3483$, $\upsilon = 1.0334$.

Chapter 4

Conclusions and Future Work

This chapter draws conclusions on the thesis, and points out some possible research directions related to the work done in this thesis.

4.1 Conclusions

This thesis introduces a new form of compensation model which makes significant changes to the compensation models before. Many works about compensation, like Carpenter (2000), always have their benchmarks fixed or unrelated with the average industry's performance. However fixed benchmark is not prevalent in the practical world, people mostly set benchmarks much more dynamic in consideration of the whole industry and their own conditions. Compared to them, this thesis's benchmark $a\mathbf{E}[X(T)]+b$ is based on the average industry's performance $\mathbf{E}[X(T)]$ and is changing all the time. That makes this model more practical and reasonable. However, this changing benchmark that is related with $X(T)$ also brings us some difficulty in solving the problem.

As the objective function in this model is not concave and also changing with $X(T)$, this thesis is prepared to solve the problem with two steps.

1. Fix the benchmark by making $\mathbf{E}[X(T)]$ equivalent to a parameter β and letting α represent the benchmark. So the problem could be regarded as a normal

compensation problem $\mathcal{P}(\alpha,\ \beta)$ with fixed benchmark. After some transformations, the original random variable optimization problem is turned into a general, concave optimization problem. It could be solved easily with traditional optimization method.

2. Make the benchmark be changing again by turning problem $\mathcal{P}(\alpha,\ \beta)$ into problem $\mathcal{P}(g(v)+c,\ (g(v)+c-b)/a)$ and letting v be a changing parameter. Now the problem is simplified into a three-dimensional scalar optimization problem. It turns out to find an optimal v to maximize the objective function next, which is actually a second-time optimization. And this is what we are doing now. Although the existence of solution to this optimization problem has not been proved analytically yet, some particular examples have been listed, and solved numerically with MATLAB. These examples, in some extent, show the solvability of this model.

4.2 Future Work

Related topics for the future research work are listed below.

1. This thesis only solved the problem in some particular examples numerically with a concrete ρ. Next we will further complete the numerical experiment by using a continuous ρ, which follows Geometric Brownian Motion. Some specific characteristics of the final optimal wealth and objective function will be showed with figures. With the hints numerical results will show us, proving the existence of the solution analytically will be considered. After that, optimal investment strategy π will be achieved with BSDE method.

2. Some other improvements will be made on this model. The introduction of probability distortion function will make the problem closer to real practice [re-

ferring to Wyner and Ziv (1976), Abdellaoui (2000), Gonzalez and Wu (1999) and so on]. When using probability distortion function, events at extremes of the range of the outcomes are likely to be 'overweight'. In such cases, at least some 'intermediate' outcomes, perhaps with the same objective probability, must be underweight. While this characteristic makes anticipated utility more conform to the real situation. According to behavioral finance, people always put more weights on the things that can hardly happen, such as lottery, earthquake and so on. That is why probability distortion function needs to be introduced. And same work will be done as the problem without distortion.

Bibliography

Abdellaoui, M. (2000), "Parameter-free elicitation of utility and probability weighting functions," *Management Science*, 46, 1497–1512.

Allen, F. (1990), "The market for information and the origin of financial intermediation," *Journal of financial intermediation*, 1, 3–30.

Antle, R. and Smith, A. (1986), "An empirical investigation of the relative performance evaluation of corporate executives," *Journal of Accounting Research*, 24, 1–39.

Arrow, K. J. (1974), "A (1970): Essays in the theory of risk-bearing," *Amsterdam und London*, pp. 13–138.

Aumann, R. J. and Perles, M. (1965), "A variational problem arising in economics," *Journal of Mathematical Analysis and Applications*, 11, 488–503.

Baiman, S. (1990), "Agency research in managerial accounting: A second look," *Accounting, Organizations and Society*, 15, 341–371.

Bhattacharya, S. and Pfleiderer, P. (1985), "Delegated portfolio management," *Journal of Economic Theory*, 36, 1–25.

Bielecki, T. R., Jin, H., Pliska, S. R., and Zhou, X. Y. (2005), "Continuous-time mean-variance portfolio selection with bankrutpcy prohibition," *Mathematical Finance*, 15, 213–244.

Bushman, R. M. and Smith, A. J. (2001a), "Financial accounting information and corporate governance," *Journal of accounting and Economics*, 32, 237–333.

Bushman, R. M. and Smith, A. J. (2001b), "Financial accounting information and corporate governance," *Journal of accounting and Economics*, 32, 237–333.

Carpenter, J. N. (2000), "Does option compensation increase managerial risk appetite?" *The Journal of Finance*, 55, 2311–2331.

Chan, T. (1994), "Dynamics of the McKean-Vlasov equation," *The Annals of Probability*, pp. 431–441.

Chen, S., Li, X., and Zhou, X. Y. (1998), "Stochastic linear quadratic regulators with indefinite control weight costs," *SIAM Journal on Control and Optimization*, 36, 1685–1702.

Cox, J. C. and Huang, C.-f. (1989), "Optimal consumption and portfolio policies when asset prices follow a diffusion process," *Journal of economic theory*, 49, 33–83.

Cui, X., Li, X., and Li, D. (2013), "Unified framework of mean-field formulations for optimal multi-period mean-variance portfolio selection," *arXiv preprint arXiv:1303.1064*.

Dawson, D. A. (1983), "Critical dynamics and fluctuations for a mean-field model of cooperative behavior," *Journal of Statistical Physics*, 31, 29–85.

Dybvig, P. and Spatt, C. (1986), "Agency and the market for mutual fund managers: The principle of preference similarity," Tech. rep., Working paper, Graduate School of Industrial Administration, Carnegie Mellon University.

Gonzalez, R. and Wu, G. (1999), "On the shape of the probability weighting function," *Cognitive psychology*, 38, 129–166.

Grinblatt, M. and Titman, S. (1989), "Adverse risk incentives and the design of performance-based contracts," *Management Science*, 35, 807–822.

Hakansson, N. H. (1971), "Multi-period mean-variance analysis: Toward a general theory of portfolio choice*," *The Journal of Finance*, 26, 857–884.

Haubrich, J. G. (1994), "Risk aversion, performance pay, and the principal-agent problem," *Journal of Political Economy*, 102, 258.

He, Z. (2011), "A model of dynamic compensation and capital structure," *Journal of Financial Economics*, 100, 351–366.

Heinkel, R. and Stoughton, N. M. (1994), "The dynamics of portfolio management contracts," *Review of Financial Studies*, 7, 351–387.

Hellwig, M. F. and Schmidt, K. M. (2002), "Discrete-time approximations of the Holmström–Milgrom Brownian–Motion model of intertemporal incentive provision," *Econometrica*, 70, 2225–2264.

Holmstrom, B. and Milgrom, P. (1987), "Aggregation and linearity in the provision of intertemporal incentives," *Econometrica: Journal of the Econometric Society*, pp. 303–328.

Jensen, M. and Murphy, K. (1999), "CEO incentives: it's not how much you pay, but how," .

Jensen, M. C. and Meckling, W. H. (1976), "Theory of the firm: Managerial behavior, agency costs and ownership structure," *Journal of financial economics*, 3, 305–360.

Jensen, M. C. and Murphy, K. J. (1990), "Performance pay and top-management incentives," *Journal of political economy*, pp. 225–264.

Jin, L. (2002), "CEO compensation, diversification, and incentives," *Journal of Financial Economics*, 66, 29–63.

Kac, M. (1956), "Foundations of kinetic theory," in *Proc. 3rd Berkeley symp. Math. Stat. Prob., J. Neyman, ed. Univ. of California*, vol. 3, pp. 171–197.

Karatzas, I. and Shreve, S. E. (1998), *Methods of mathematical finance*, vol. 39, Springer.

Karatzas, I., Lehoczky, J. P., Sethi, S. P., and Shreve, S. E. (1986), "Explicit solution of a general consumption/investment problem," *Mathematics of Operations Research*, 11, 261–294.

Li, D. and Ng, W.-L. (2000), "Optimal dynamic portfolio selection: multiperiod mean-variance formulation," *Mathematical Finance*, 10, 387–406.

Li, X., Zhou, X. Y., and Lim, A. E. (2002), "Dynamic mean-variance portfolio selection with no-shorting constraints," *SIAM Journal on Control and Optimization*, 40, 1540–1555.

Lim, A. E. and Zhou, X. Y. (2002), "Mean-variance portfolio selection with random parameters in a complete market," *Mathematics of Operations Research*, 27, 101–120.

Markowitz, H. (1952), "Portfolio selection*," *The journal of finance*, 7, 77–91.

Marschak, J. and Radner, R. (1972), "Economic theory of teams," .

McKean Jr, H. (1966), "A class of Markov processes associated with nonlinear parabolic equations," *Proceedings of the National Academy of Sciences of the United States of America*, 56, 1907.

Merton, R. C. (1969), "Lifetime portfolio selection under uncertainty: The continuous-time case," *The review of Economics and Statistics*, 51, 247–257.

Merton, R. C. (1971), "Optimum consumption and portfolio rules in a continuous-time model," *Journal of economic theory*, 3, 373–413.

Merton, R. C. (1972), "An analytic derivation of the efficient portfolio frontier," *Journal of financial and quantitative analysis*, 7, 1851–1872.

Mirrlees, J. A. (1976), "The optimal structure of incentives and authority within an organization," *The Bell Journal of Economics*, pp. 105–131.

Mossin, J. (1968), "Optimal multiperiod portfolio policies," *The Journal of Business*, 41, 215–229.

Pardoux, É. (1999), "BSDEs, weak convergence and homogenization of semilinear PDEs," in *Nonlinear analysis, differential equations and control*, pp. 503–549, Springer.

Perold, A. F. (1984), "Large-scale portfolio optimization," *Management science*, 30, 1143–1160.

Ross, S. A. (1973), "The economic theory of agency: The principal's problem," .

Samuelson, P. A. (1969), "Lifetime portfolio selection by dynamic stochastic programming," *The Review of Economics and Statistics*, 51, 239–246.

Sannikov, Y. (2008), "A continuous-time version of the principal-agent problem," *The Review of Economic Studies*, 75, 957–984.

Schaettler, H. and Sung, J. (1993), "The first-order approach to the continuous-time principal–agent problem with exponential utility," *Journal of Economic Theory*, 61, 331–371.

Spohn, H. (1980), "Kinetic equations from Hamiltonian dynamics: Markovian limits," *Reviews of Modern Physics*, 52, 569.

Stoughton, N. M. (1993), "Moral hazard and the portfolio management problem," *The journal of finance*, 48, 2009–2028.

Sznitman, A.-S. (1991), "Topics in propagation of chaos," in *Ecole d'Eté de Probabilités de Saint-Flour XIX—1989*, pp. 165–251, Springer.

Wilson, R. (1965), "On the theory of syndicates," .

Wyner, A. D. and Ziv, J. (1976), "The rate-distortion function for source coding with side information at the decoder," *Information Theory, IEEE Transactions on*, 22, 1–10.

Yong, J. (2013), "Linear-Quadratic optimal control problems for mean-field stochastic differential equations—time-consistent solutions," *arXiv preprint arXiv:1304.3964*.

Zhou, X. Y. and Li, D. (2000), "Continuous-time mean-variance portfolio selection: A stochastic LQ framework," *Applied Mathematics and Optimization*, 42, 19–33.

Zhou, X. Y. and Yin, G. (2003), "Markowitz's mean-variance portfolio selection with regime switching: A continuous-time model," *SIAM Journal on Control and Optimization*, 42, 1466–1482.