MATHEMATICAL STUDIES ON SOME MODELS ARISING IN CHEMOTAXIS AND MAGNETOHYDRODYNAMIC TURBULENCE

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Mathematical Studies on Some Models Arising in Chemotaxis and Magnetohydrodynamic Turbulence

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______________________________(Signed)

JIN Haiyang (Name of student)

Dedicate to my family.

Abstract

This thesis is mainly focused on the theoretical studies on some models arising in chemotaxis and magnetohydrodynamic turbulence. The main results of this thesis consist of the following three parts.

1. A quasilinear parabolic volume-filling chemotaxis model with critical sensitivity in two dimensions is considered. In this study, a threshold number is explicitly found such that the solution exists globally with uniform-in-time bound or blows up if the initial cell mass is less than or greater than this number. Furthermore we determine the blowup time is infinite under certain conditions on the decay rate of the chemotactic sensitivity.

2. We consider the initial-boundary value problem of the attraction-repulsion Keller-Segel (ARKS) chemotaxis model describing the quorum effect in chemotaxis and the aggregation of Microglia in the central nervous system in Alzhemer's disease. First, we study the asymptotic behavior of solutions to the ARKS chemotaxis model in one dimension, where we obtain the uniform-in-time boundedness of solutions and prove that the model possesses a global attractor. For a special case where the attractive and repulsive chemical signals have the same degradation rate, we show that the solution converges to a stationary solution algebraically as time tends to infinity if the attraction dominates. In two dimensional spaces, we show that if the repulsion dominates over attraction, then the global classical solutions exist with uniform-in time bound for large initial data. Moreover we present a Lyapunov

function at the first time for the irreducible three-component attraction-repulsion chemotaxis model which plays a central role to obtain our results.

3. We establish the asymptotic nonlinear stability of solutions to the Cauchy problem of a strongly coupled Burgers system arising in magnetohydrodynamic (MHD) turbulence. We show that, as time tends to infinity, the solutions of the Cauchy problem converge to constant states or rarefaction waves with large initial data, or viscous shock waves with arbitrarily large amplitude, where the precise asymptotic behavior depends on the relationship between the left and right end states of the initial value. Our results confirm the existence of shock waves (or turbulence) numerically found in the literature.

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Contents

Chapter 1

Introduction

For the living organisms, an essential feature is the ability to sense signals in the environment and change their movement accordingly. A response to an external stimulus is generally called taxis. There are many different tactical responses such as chemotaxis, galvanotaxis and phototaxis. In this thesis, I will focus on chemotactical movement of mobile species, which can lead to various different pattern formation. Chemotaxis can be either positive or negative depending on whether it is toward or away from the external signal. The substances that lead to positive chemotaxis are chemoattractant and those leading to negative chemotaxis are so called repellents. Mathematical analysis of the chemotaxis phenomena has become more and more important in understanding these complex processes. Theoretical and mathematical modelling of chemotaxis dates to the pioneering works of Patlak in the 1950s [78] and Keller and Segel in the 1970s [49, 50]. Cell aggregation is one of the characteristic consequences of chemotaxis. This phenomenon has been shown to lead to finite time blowup under certain formulations of the model, a sequence of elegant works has been devoted to determining whether blow-up occurs or global solution exists. Part of this thesis will be focused on the study of the global existence and blow-up of solution to the chemotaxis models with volume-filling effect and quorum sensing effect.

The three-dimensional motion of compressible magnetohydrodynamic (MHD) equations were proposed [7, 13, 53] to describe the dynamics of MHD fluid such as macroscopic plasma motions and dynamic process in the outer core of the earth, however, which are too complicated to investigate small scale structure of the MHD turbulence by direct numerical simulations. To remedy this defect, a new one-dimensional MHD-Burgers system was derived in [24, 103]. This new system is the simplest possible system allowing energy transfer between the fluid and magnetic field excitations. In this thesis, asymptotic behavior of solutions of the new one-dimensional MHD-Burgers system will be investigated.

1.1 Main Results of the Thesis

In this thesis, I will focus on the theoretical studies on some models arising in chemotaxis and magnetohydrodynamic turbulence. The organization of this thesis is as follows.

In the rest of chapter 1, the motivations and main results of our studies will be given. The known results related to the models studied in the thesis will be introduced along the presentation.

Chapter 2 deals with a quasilinear parabolic volume-filling chemotaxis model with critical sensitivity in two dimensions. The chemotaxis models with volume-filling effect were initially proposed by Painter and Hillen [30, 77]. The basic assumption of the volume-filling effect is that cells have a finite volume and can not move into regions which are already filled by other cells. The global existence and asymptotic behavior of solutions as well as pattern formation have been studied in the literature [30, 44, 79, 93, 100, 101, 105] under the assumption that there is a maximal density U_{max} of cells at which chemotaxis vanishes. For the case that there is no value of u at which chemotaxis is switched off (i.e., chemotaxis vanishes as $u \to \infty$), only few results are known. First, the stationary state and global dynamics of such type of volume-filling chemotaxis model were established in [56, 106] in one dimension. In the higher dimensions, the global-in-time weak solutions were obtained in [14, 16] with the cell kinetics or the chemotactic sensitivity decaying fast enough. In chapter 2, I will study the volume-filling chemotaxis model as in [56, 106] with critical sensitivity in two dimensions. A threshold number of cell mass is explicitly found such that the solution exists globally with uniform-in-time bound if the initial cell mass is less than this number and blows up in finite or infinite time if the initial cell mass is greater than this number. Furthermore we determine the blowup time is infinite under certain conditions on the decay rate of the chemotactic sensitivity.

In chapter 3, I will study the global dynamics of the attraction-repulsion Keller-Segel (ARKS) chemotaxis model. A striking feature of the classical attractive Keller-Segel system is the finite-time blowup of solutions in two dimensions when the initial cell mass is larger than a threshold number (see the details in the section 1.2.1). The ARKS chemotaxis model was first proposed in [63], which has been mathematically studied in the literature [60, 63, 77, 85]. In this chapter, I will study the ARKS chemotaxis model further in different aspects. First, I explore the asymptotic dynamics of the ARKS model in one dimension [46], which improves the results of [60] by deriving a uniform-in-time bound for solutions and furthermore prove that the model possesses a global attractor. Second, if repulsion prevails over attraction, the globally bounded classical solutions exist for large initial data will be obtained in two dimensions. Moreover, a Lyapunov function is obtained at the first time for the irreducible three-component ARKS model which plays a central role to obtain our results.

In chapter 4, I will study the asymptotic nonlinear stability of solutions to the Cauchy problem of a strongly coupled Burgers system arising in MHD turbulence [24, 103]. Based on the theory of conservation laws, the nonlinear stability of constant states or rarefaction waves with large initial data, or viscous shock waves with arbitrarily large amplitude will be established [47].

Chapter 5 briefly summarize the results obtained in this thesis and present some research problems that I will pursue in the future.

1.2 Introduction of the Models

1.2.1 Keller-Segel Chemotaxis Model

To describe the motion of cells toward the gradient of a substance called chemoattractant, the following chemotaxis model was first proposed by Keller and Segel [49]

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u - \chi\phi(u)\nabla v), & x \in \Omega, t > 0 \\ \tau v_t = \Delta v + \alpha u - \beta v, & x \in \Omega, t > 0 \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.2.1}$$

where $u(x,t)$ denotes the density of the cells population and $v(x,t)$ represents the concentration of the chemoattractant, Ω is a bounded domain of $\mathbb{R}^n (n \geqslant 1)$, $\frac{\partial}{\partial \nu}$ denotes outward normal derivatives on $\partial\Omega$, $\chi,\ \alpha,\ \beta$ are given positive constants. τ is a constant equal to 0 or 1 justifying whether the change of chemicals is stationary or dynamical in time. In this subsection, I will briefly review some results concerning the blowup or global existence of solutions to the chemotaxis model (1.2.1).

If $D(u) = 1,\ \phi(u) = u$, model (1.2.1) was called the minimal or classical chemotaxis model, which has been extensively studied in various aspects. It was first conjectured by Nanjundiah [73] that the aggregation of cells may eventually lead to singularities. Moreover due to the conservation of cell mass, the singularities can only be of δ-function type. This phenomenon was called chemotactic collapse or blowup. Based on the numerical computations for the steady state, Childress and Percus [10, 11] pointed out that the singular behavior of the solution is a phenomenon depending on the space dimension, and showed that the singularity behavior was not possible in one dimension. While in higher dimensions ($n \geqslant 2$), they confirmed Nanjundiah's argument that blowup can occur and argued that chemotactic blowup requires a threshold number of cell mass in two dimensions. Subsequently, a sequence of elegant works on the critical mass problem of the classical chemotaxis model have been established. For the parabolic-elliptic case ($\tau = 0$), by substituting the second equation to the first equation and then constructing a radially symmetric lower solution for the first equation of model (1.2.1), Jäger and Luckhaus [42] proved that there exists a radially symmetric solution can blow up for suitable initial mass $\int_\Omega u_0(x)dx$ in two dimensions. Precisely, they showed that there exists a critical number m_0 such that if $\int_\Omega u_0(x)dx < m_0$, the solution exists globally in time, and the solution blows up in a finite time if $\int_\Omega u_0(x)dx > m_0$. After Jäger and Luckhaus' paper in 1992s, the next progress was made by Nagai in [68], in which the critical number m_0 was identified to be $8\pi/\alpha\chi$. He showed that blowup cannot occur if $n = 1$, or if $n = 2$ and Ω is a ball and $u_0(x)$ is radially symmetric such that $\int_\Omega u_0(x)dx < 8\pi/\alpha\chi$, whereas blowup occurs if $\int_\Omega u_0(x)dx > 8\pi/\alpha\chi$. Global existence and blowup results for nonradial solutions or for general domain Ω can also found in references [6, 68, 72, 71, 70, 69, 81]. Coming to the full parabolic-parabolic chemotaxis model ($\tau = 1$), Osaki and Yagi [76] showed that the solution of model (1.2.1) exists globally in time and converges to a stationary solution as $t \to \infty$ in one dimension. In two dimensional spaces, the critical mass phenomenon has been found. First, if $\int_\Omega u_0(x)dx < 4\pi/\alpha\chi$, it was proved in [6, 27, 34, 72] that the solutions exist globally in time with uniform-in-time bound. If $\int_\Omega u_0(x)dx > 4\pi/\alpha\chi$, then there exists initial data such that the corresponding solution blows up either in finite or infinite time [34, 38, 80, 29, 35, 36]. Specially, if $4\pi/\alpha\chi < \int_\Omega u_0(x)dx < 8\pi/\alpha\chi$, then the corresponding solution blows up at the boundary of Ω either in finite or infinite time. Here, we should point out that the proof of global existence or blow up of solutions was based on the existence of a Lyapounv functional of chemotaxis model (1.2.1). Furthermore, the low energy initial data can been constructed such that the corresponding solution of chemotaxis model (1.2.1) blows up, and we shall call this method as 'low energy method' in this thesis (see chapter 2 for details). However, we can not confirm that whether the blowup occurs in finite time or in infinite time by using the low energy method. The only finite time blowup result was obtained in [29] by using the asymptotic expansion method, where it was shown that that there exists a radially symmetric solution of model (1.2.1) which blows up in finite time. However this result only refers to one single unbounded solution, hence leaving open the possibility that finite-time aggregation might be a non-generic, unstable phenomenon. Recently, the low energy method has been successfully used in [96, 97, 98] to established the finite time blowup of solutions independent of the size of initial mass in dimensions $n \geqslant 3$ for the full parabolic chemotaxis model. At last, we should point out that for the general case, the global existence and blowup of solutions in higher dimensions have been studied in a large body of works [19, 17, 39, 86, 99, 97]. It was shown that the ratio $\frac{\phi(u)}{D(u)} \propto u^\theta$ for $u > 1$ plays an essential role. If $\theta < \frac{2}{n}$, it has been proved that the solutions globally exist with uniform-in-time bound, whereas if $\theta > \frac{2}{n}$ for each initial mass $\int_\Omega u_0(x)dx > 0$, there exists finite time blow-up solution. For the critical sensitivity case $\theta = \frac{2}{n}$, it was suspected (not confirmed) that there is a critical mass m_0 beyond which solutions blow up and below which solutions globally exist.

The blowup results reflect the initial 'self-aggregation' and the existence of blowup solutions is of interest mathematically. Since the blowup is an extreme case, a large number of ideas were proposed to modify the classical Keller-Segel chemotaxis model such that global bounded solutions of modified models are admitted, see a review article [31] and recent development in [12, 32]. In this thesis, I will study the dynamics of the chemotaxis models with volume-filling effect [14, 56, 77, 106] and attraction-repulsion mechanism [60, 61, 63, 77, 85], which are two important mechanisms developed to regularize the classical chemotaxis model.

1.2.2 MHD-Burgers Model

To depict the dynamics of magnetohydrodynamic (MHD) fluid such as macroscopic plasma motions and the dynamic process in the out core of the earth, the following three dimensional compressible MHD equations were proposed in [7, 13, 53]

$$\begin{cases} \frac{\partial \rho}{\partial t} + (\mathbf{v}\cdot\nabla)\rho + \rho\nabla\cdot\mathbf{v} = 0, \\ \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p - \frac{1}{\rho\lambda_0}\mathbf{u}\times(\nabla\times\mathbf{u}) + \frac{\eta}{\rho}\Delta\mathbf{v} + \frac{1}{\rho}(\zeta + \frac{1}{3}\eta), \\ \frac{\partial \mathbf{u}}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{u}) + \mu\Delta\mathbf{u}, \\ \nabla\cdot\mathbf{u} = 0, \end{cases} \tag{1.2.2}$$

where $\mathbf{v}$ denotes the velocity, $\mathbf{u}$ stands for the magnetic field, p is the pressure and λ_0 is the magnetic permeability of the vacuum, η and ζ are the viscosity coefficients, and μ is the magnetic diffusivity. Since the MHD equations (1.2.2) are too complicated to investigate the small scale structure of the MHD turbulence even by numerical simulations, it is necessary to build a simpler model which, however, still contains essential features of the MHD turbulence. For this reason, a new one dimensional MHD Burgers system was established in [24, 103] by assuming the following conditions:

(i) Turbulence field depends on one-dimensional space variable x and time t.

(ii) Velocity field has only the x-component as $\mathbf{v}(x,t) = v(x,t)i$.

(iii) Magnetic field has only the y-component as $\mathbf{u}(x,t) = u(x,t)j$.

(iv) Density ρ is put constant ρ_0 in the equations for the velocity and the magnetic fields.

(v) The pressure term $\rho^{-1}\frac{\partial p}{\partial x}$ is neglected.

With suitable scalings (see [24, 103] for details), the system (1.2.2) can be transformed into the following MHD Burgers system

$$\begin{cases} u_t + (uv)_x = Du_{xx}, \\ v_t + \left(\frac{1}{2}u^2 + \frac{1}{2}v^2\right)_x = \mu v_{xx}. \end{cases} \tag{1.2.3}$$

In this model, the turbulence is represented by an ensemble of Alfvenic shock waves on a homogeneous density background. The one-dimensional Burgers-model analog of MHD is by far the simplest set of nonlinear, coupled partial differential equations with symmetries and conservation laws akin to those in three dimensional MHD system. It was also shown in [24] that the MHD-Burgers system (1.2.3) is the simplest possible system allowing energy transfer between the fluid and magnetic field excitations. Furthermore, the dissipation terms and wavelike propagation are similar to the three dimensional MHD system. Therefore, the study of the one-dimensional MHD-Burgers system can provide some insight into the three dimensional MHD system. Moreover system (1.2.3) may also be used to model the opposite limit of a fluid-dominated (i.e., unmagnetized) system with arbitrary density variations reacting to an adiabatic pressure [24]. For more applications of (1.2.3), we refer the readers to [25, 54, 89]. Using the Elsässer variables $e^{\pm} = v \pm u$, system (1.2.3) is transformed into

$$\frac{\partial e^{\pm}}{\partial t} + \frac{\partial}{\partial x}\frac{(e^{\pm})^2}{2} = \frac{\mu + D}{2}\frac{\partial^2 e^{\pm}}{\partial x^2} + \frac{\mu - D}{2}\frac{\partial^2 e^{\mp}}{\partial x^2}. \tag{1.2.4}$$

If $D = \mu$, then e^- and e^+ do not interact each other and system (1.2.3) can be reduced to two independent viscous Burgers equations for e^+ and e^-, respectively. For this special case a shock type solution was obtained. The nontrivial case $D \neq \mu$ reveals more interesting interactions between the fluid and the magnetic field [24]. In this thesis, I will study the asymptotic behavior of solutions of system (1.2.3), which confirm the numerical results in the literature [24, 103] about the existence of shock waves.

Chapter 2

Volume-Filling Chemotaxis Model

2.1 Introduction

In this chapter, we will study the following volume-filling chemotaxis model

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u - \chi\phi(u)\nabla v), & x \in \Omega, t > 0 \\ v_t = \Delta v + \alpha u - \beta v, & x \in \Omega, t > 0 \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{2.1.1}$$

where $D(u)$ and $\phi(u)$ satisfy following relations

$$D(u) = q(u) - uq'(u),\ \phi(u) = uq(u),\ u \geqslant 0, \tag{2.1.2}$$

where $q(u)$ denotes the probability that the particle attains a position (x, t) if the density of cells at this position equals u. Such kind of model was first proposed by Hillen and Painter [77] based on a biased random. The first version of the volume-filling chemotaxis model has been studied in [30] under the assumption that there is a maximal density of cells at which chemotaxis vanishes. Based on the investigation of the biology that stands behind assumptions they put on the model in [30], it was suggested considering the second version of the volume-filling chemotaxis model in [77], in this case there is no value of u at which chemotaxis is switched off (i.e., $q(u) > 0$ and $q(u) \to 0$ as $u \to \infty$). In this chapter, we will study the second version of the volume-filling chemotaxis model further. More precisely, we consider an interesting example of $q(u) = (1+u)^{-\lambda}(\lambda > 0)$. Then from (2.1.2) we can deduce that

$$D(u) = \frac{1+(1+\lambda)u}{(1+u)^{\lambda+1}}, \quad \phi(u) = \frac{u}{(1+u)^{\lambda}}. \tag{2.1.3}$$

The system (2.1.1) with (2.1.3) has been studied in different aspects. The global existence of solutions and stationary state were investigated in [56, 106], which exclude the possible of blowup of solutions in one dimension ($n = 1$). If $n \geqslant 3$, it was shown in [97] that there exist unbounded solutions may blow up in finite or infinite time. Furthermore, when $n \geqslant 3$ it was proved that the unbounded solutions blow up in infinite time for $\lambda > n$ in [17]. The case $n = 2$ corresponds to the critical sensitivity. Hence it is nature to consider whether there is a critical mass m_0 beyond which solutions blow up and below which solutions globally exist. In this chapter, we will study this critical mass problem of system (2.1.1) with (2.1.3) in two dimensions. Based on the existence of Lyapunov function, we find a threshold number $\frac{4\pi(1+\lambda)}{\alpha\chi}$ such that the solution exists globally with uniform-in-time bound if $\int_\Omega u_0 dx < \frac{4\pi(1+\lambda)}{\alpha\chi}$ and blows up in finite or infinite time if $\int_\Omega u_0 dx > \frac{4\pi(1+\lambda)}{\alpha\chi}$. Furthermore, if $\lambda > 1$ we construct global-in-time solutions admitting infinite-time blowup. We notice that the critical mass phenomenon has also been studied recently in [18] for chemotaxis model (2.1.1) with (2.1.3). There are two major differences between [18] and our studies: (1) [18] proves the existence of blowup solutions under the assumption that the initial data u_0 and v_0 are radially symmetric and the domain is a ball, hence the blowup point is only the origin, however, in our studies, we consider the blowup results without the radially symmetric assumptions on the initial data and domain, and the solution may blow up on the boundary. (2) In our studies, the transformation (2.4.63) is essentially used such that we can find some initial data with large negative energy in the space $\mathcal{D} \equiv \{f \in W^{1,\infty}(\Omega) | \int_\Omega f dx = 0\}$ in which the corresponding solution blows up in finite or infinite time (see details in Section 2.4.3). To claim the solution belongs to the space $\mathcal{D}$, the transformation (2.4.63) has to be used. However, the paper [18] considers the solution in $\mathcal{D}$ without introducing the transformation (2.4.63). This is an error from our understanding. Our results in this thesis correct this error. Hence the above two differences can be viewed as the supplement of the paper [18]. Before concluding of this section, we introduce some notations. Hereafter, c_i denotes a generic constant, which may change from one section to another, where $i = 1, 2, 3, \cdots$.

2.2 Preliminaries

First, we give a lemma to be used for the estimates of solutions in the sequel. This lemma was proposed in [39, Lemma 4.1] and improved in [51, Lemma 1].

Lemma 2.1 ([51])**.** *Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary. Assume $0 \leqslant v_0 \in W^{1,\infty}(\Omega)$. Moreover*

$$\|u\|_{L^s} \leqslant C, \quad \text{for all} \quad t \in (0, T).$$

Then there exists some constant C_q such that for every $t \in (0, T)$ and $s < n$, the solution of (2.1.1) *satisfies*

$$\|v\|_{W^{1,q}} \leqslant C_q \tag{2.2.4}$$

where $q < \frac{ns}{n-s}$. If $s = n$, (2.2.4) *holds for all $q < \infty$, and if $s > n$,* (2.2.4) *is true with $q = \infty$. Here C and C_q are positive constants independent of t.*

Lemma 2.2 ([22])**.** *Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Assume $1 \leqslant p < n$ and $u \in W^{1,p}(\Omega)$. Then $u \in L^{p^*}(\Omega)$, with the estimate*

$$\|u\|_{L^{p^*}} \leqslant C\|u\|_{W^{1,p}}, \tag{2.2.5}$$

where $p^* = \frac{np}{n-p}$ and the constant C depends only on p, n and Ω.

The following inequalities will be used frequently.

Lemma 2.3 ([72]). *Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary. Then for any $\varepsilon > 0$, there exists a positive constant C_ε such that*

$$\|u\|_{L^3} \leqslant \varepsilon \|\nabla u\|_{L^2}^{\frac{2}{3}} \|u \ln u\|_{L^1}^{\frac{1}{3}} + C_\varepsilon(\|u \ln u\|_{L^1} + \|u\|_{L^1}^{\frac{1}{3}}). \tag{2.2.6}$$

Lemma 2.4 ([26]). *Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary. Let l and k be any integers satisfying $0 \leqslant l < k$, and let $1 \leqslant q, r \leqslant \infty$, and $p \in \mathbb{R}^+$, $\frac{l}{k} \leqslant a \leqslant 1$ such that*

$$\frac{1}{p} - \frac{l}{n} = a\left(\frac{1}{q} - \frac{k}{n}\right) + (1-a)\frac{1}{r}. \tag{2.2.7}$$

Then, for any $u \in W^{k,q}(\Omega) \cap L^r(\Omega)$, there exist two constants c_1 and c_2 depending only on Ω, q, k, r and n such that the following inequalities holds:

$$\|D^l u\|_{L^p} \leqslant c_1 \|D^k u\|_{L^q}^a \|u\|_{L^r}^{1-a} + c_2 \|u\|_{L^r}, \tag{2.2.8}$$

with the following exception: if $1 < q < \infty$ and $k - l - \frac{n}{q}$ is a nonnegative integer, then (2.2.7) holds only for a satisfying $\frac{l}{k} \leqslant a < 1$.

For the special case $l = 0, k = 1$ and $q = 2$, we may employ the inequality $(X+Y)^2 \leqslant 2(X^2 + Y^2)$ for any $X, Y \in \mathbb{R}$, and obtain the following inequality

$$\|u\|_{L^p}^2 \leqslant c_3(\|\nabla u\|_{L^2}^{2a} \|u\|_{L^r}^{2(1-a)} + \|u\|_{L^r}^2), \quad \frac{n}{p} = a\left(\frac{n}{2} - 1\right) + \frac{n}{r}(1-a). \tag{2.2.9}$$

Lemma 2.5 ([87]). *Suppose $y(t) \geqslant 0$ and satisfies*

$$\begin{cases} \frac{dy}{dt} + Ay^\rho \leqslant B, & t > 0, \\ y(0) = y_0, \end{cases} \tag{2.2.10}$$

where $\rho > 0, A > 0$ and $B \geqslant 0$. Then for any $t > 0$, we have

$$y(t) \leqslant \max\left(y_0, \left(\frac{B}{A}\right)^{\frac{1}{\rho}}\right). \tag{2.2.11}$$

Lemma 2.6 ([72]). *Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary. Then for any $\varepsilon > 0$ there exist a constant C_ε depending on ε and Ω such that*

$$\int_\Omega \exp |u| dx \leqslant C_\varepsilon \exp\left\{ \left(\frac{1}{8\pi} + \varepsilon \right) \|\nabla u\|_{L^2}^2 + \frac{1}{|\Omega|} \|u\|_{L^1} \right\}. \tag{2.2.12}$$

Lemma 2.7 ([22]). *Let $\eta(\cdot)$ be a nonnegative, absolutely continuous function on $[0, \infty)$ satisfying the differential inequality $\eta'(t) + l\eta(t) \leqslant w(t)$, where l is a constant and $w(t)$ is a nonnegative continuous function on $[0, \infty)$. Then, one has*

$$\eta(t) \leqslant \left(\eta(0) + \int_0^t e^{l\tau} w(\tau) d\tau \right) e^{-lt}. \tag{2.2.13}$$

2.3 Boundedness for Subcritical Mass

In this section, we will consider the boundedness of solutions of the chemotaxis model (2.1.1) with (2.1.3). We have the following theorem on the global existence of solutions.

Theorem 2.1. *Let Ω be a bounded domain with smooth boundary in $\mathbb{R}^2$. Assume $0 \leqslant (u_0, v_0) \in W^{1,\infty}(\Omega) \times W^{1,\infty}(\Omega)$. If $\int_\Omega u_0(x) dx < \frac{(1+\lambda)4\pi}{\alpha\chi}$, then there exists a unique pair (u, v) of nonnegative bounded functions belongs to $C^0(\bar{\Omega} \times [0, \infty)) \cap C^{2,1}(\bar{\Omega} \times (0, \infty))$ which solves (2.1.1) with (2.1.3) classically. Furthermore, there exists a constant C independent of t such that*

$$\|u(\cdot, t)\|_{L^\infty} + \|v(\cdot, t)\|_{L^\infty} \leqslant C. \tag{2.3.14}$$

First, we consider the local existence of classical solutions to system (2.1.1) with (2.1.3), which can be proved by the standard parabolic regularity theory and an appropriate fixed point framework.

Lemma 2.8 ([86]). *Assume that $0 \leqslant (u_0, v_0) \in W^{1,\infty}(\Omega) \times W^{1,\infty}(\Omega)$. Then there exist $T_{max} \in (0, \infty]$ and a unique pair (u, v) of nonnegative functions from $C^0(\bar{\Omega} \times$*

$[0, T_{max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max}))$ *solving* (2.1.1) *with* (2.1.3) *classically in* $\Omega \times (0, T_{max})$. *Moreover*

$$if\ T_{max} < \infty,\ then\ \sup_{t \geqslant 0}(\|u(t)\|_{L^\infty} + \|v(t)\|_{L^\infty}) \to \infty \quad as \quad t \nearrow T_{max}.$$

The following important property on mass can be easily derived.

Lemma 2.9. *Let* (u, v) *be the solution of the system* (2.1.1) *with* (2.1.3). *Then we have*

$$\|u(\cdot, t)\|_{L^1} = \|u_0\|_{L^1} \equiv M \tag{2.3.15}$$

and

$$\|v(\cdot, t)\|_{L^1} = \frac{\alpha}{\beta}\|u_0\|_{L^1} - \left(\frac{\alpha}{\beta}\|u_0\|_{L^1} - \|v_0\|_{L^1}\right) e^{-\beta t}. \tag{2.3.16}$$

Proof. Integrating the first and second equations of (2.1.1) over Ω, the lemma is obtained immediately by the boundary conditions. □

From Lemma 2.8, we know that there exist a small positive number τ_0 and a constant M_1 depending on the initial data, Ω and τ_0 such that the solution of (2.1.1) and (2.1.3) satisfy $\sup_{0 \leqslant t \leqslant \tau_0} \|u\|_{L^\infty} \leqslant M_1$. If we divide the time interval $[0, T_{max})$ into two parts: $[0, \tau_0]$ and $[\tau_0, T_{max})$, to complete the proof of Theorem 2.1, we only need to prove that there exists a constant M_2 independent of t such that $\sup_{t \geqslant \tau_0} \|u\|_{L^\infty} \leqslant M_2$. Next, we will prove this fact by using the Lyapunov function and the Moser-like procedure.

2.3.1 Uniform Lower Bound of the Lyapunov Functional

We can verify that the system (2.1.1) with (2.1.3) has the following Lyapunov function

$$F(t) = \int_\Omega \Big(u \ln u + \lambda(1 + u)\ln(1 + u) - \chi uv + \frac{\chi}{2\alpha}(|\nabla v|^2 + \beta v^2)\Big)\, dx. \tag{2.3.17}$$

Lemma 2.10. *The solution of* (2.1.1) *with* (2.1.3) *satisfies*

$$\frac{dF(t)}{dt} + \frac{\chi}{\alpha}\|v_t\|_{L^2}^2 + \int_\Omega \frac{u}{(1+u)^\lambda}(\nabla \ln(u(1+u)^\lambda) - \chi\nabla v)^2 dx = 0, \tag{2.3.18}$$

where $F(t)$ *is defined by* (2.3.17).

Proof. Multiplying the first equation of (2.1.1) by $\ln u + \lambda\ln(1+u) - \chi v$ and integrating by parts, we have

$$\begin{aligned}
&\int_\Omega u_t(\ln u + \lambda\ln(1+u) - \chi v)dx \\
&= \int_\Omega \nabla\cdot\left(\frac{1+(1+\lambda)u}{(1+u)^{\lambda+1}}\nabla u - \chi\frac{u}{(1+u)^\lambda}\nabla v\right)(\ln u + \lambda\ln(1+u) - \chi v)dx \\
&= -\int_\Omega \frac{u}{(1+u)^\lambda}(\nabla\ln(u(1+u)^\lambda) - \chi\nabla v)^2 dx,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega (u\ln u + \lambda(1+u)\ln(1+u) - \chi uv)dx + \chi\int_\Omega v_t u dx \\
&+ \int_\Omega \frac{u}{(1+u)^\lambda}(\nabla\ln(u(1+u)^\lambda) - \chi\nabla v)^2 dx = 0.
\end{aligned} \tag{2.3.19}$$

Multiplying the second equation of (2.1.1) by v_t, we obtain

$$\int_\Omega v_t u dx = \frac{d}{dt}\int_\Omega \left(\frac{\beta}{2\alpha}v^2 + \frac{1}{2\alpha}|\nabla v|^2\right)dx + \frac{1}{\alpha}\int_\Omega v_t^2 dx. \tag{2.3.20}$$

The combination of (2.3.19) and (2.3.20) implies (2.3.18). □

Lemma 2.11. *Let* $F(t)$ *be defined by* (2.3.17). *If* $\int_\Omega u_0 dx < \frac{(1+\lambda)4\pi}{\alpha\chi}$, *then there exist two constants* C_1 *and* C_2 *independent of* t *such that*

$$F(t) \geqslant -C_1 \quad \text{and} \quad \|u\ln u\|_{L^1} \leqslant C_2. \tag{2.3.21}$$

Proof. From (2.3.17), we have

$$F(t) \geqslant (1+\lambda)\int_\Omega \left(u\ln u - \frac{\chi+\delta}{1+\lambda}uv\right)dx + \frac{\chi}{2\alpha}\int_\Omega \left(|\nabla v|^2+\beta v^2\right)dx + \delta\int_\Omega uvdx$$
$$= (1+\lambda)\int_\Omega \left(u\ln u - u\ln e^{\frac{\chi+\delta}{1+\lambda}v}\right)dx + \frac{\chi}{2\alpha}\int_\Omega \left(|\nabla v|^2+\beta v^2\right)dx + \delta\int_\Omega uvdx$$
$$= -(1+\lambda)\int_\Omega u\ln\frac{e^{\frac{\chi+\delta}{1+\lambda}v}}{u}dx + \frac{\chi}{2\alpha}\int_\Omega \left(|\nabla v|^2+\beta v^2\right)dx + \delta\int_\Omega uvdx. \tag{2.3.22}$$

Since $-\ln z$ is a convex function for all $z \geqslant 0$ and $\int_\Omega \frac{u}{M}dx = 1$, then using the Jensen's inequality, we obtain

$$-\ln\left\{\frac{1}{M}\int_\Omega e^{\frac{\chi+\delta}{1+\lambda}v}dx\right\} = -\ln\int_\Omega \frac{e^{\frac{\chi+\delta}{1+\lambda}v}}{u}\frac{u}{M}dx$$
$$\leqslant \int_\Omega\left(-\ln\frac{e^{\frac{\chi+\delta}{1+\lambda}v}}{u}\right)\frac{u}{M}dx \tag{2.3.23}$$
$$= -\frac{1}{M}\int_\Omega u\left(\ln\frac{e^{\frac{\chi+\delta}{1+\lambda}v}}{u}\right)dx.$$

The combination of (2.3.22) and (2.3.23) implies that

$$F(t) \geqslant -(1+\lambda)M\ln\left\{\frac{1}{M}\int_\Omega e^{\frac{\chi+\delta}{1+\lambda}v}dx\right\} + \frac{\chi}{2\alpha}\int_\Omega|\nabla v|^2dx + \frac{\chi\beta}{2\alpha}\int_\Omega v^2dx + \delta\int_\Omega uvdx. \tag{2.3.24}$$

Using the Trudinger-Moser inequality (2.2.12) and the condition that $\|v\|_{L^1} \leqslant c_1$ (see (2.3.16)), we can obtain two constants c_2 and c_3 depending on ε such that

$$\int_\Omega e^{\frac{\chi+\delta}{1+\lambda}v}dx \leqslant c_2 e^{\left(\frac{1}{8\pi}+\varepsilon\right)\frac{(\chi+\delta)^2}{(1+\lambda)^2}\|\nabla v\|^2_{L^2} + \frac{\chi+\delta}{|\Omega|(1+\lambda)}\|v\|_{L^1}} \leqslant c_3 e^{\left(\frac{1}{8\pi}+\varepsilon\right)\frac{(\chi+\delta)^2}{(1+\lambda)^2}\|\nabla v\|^2_{L^2}}. \tag{2.3.25}$$

Substituting (2.3.25) into (2.3.24), we can find a constant $c_4 = (1+\lambda)M\ln\frac{c_3}{M}$ such that

$$F(t) \geqslant \left(\frac{\chi}{2\alpha} - \left(\frac{1}{8\pi}+\varepsilon\right)\frac{(\chi+\delta)^2M}{(1+\lambda)}\right)\int_\Omega|\nabla v|^2dx + \frac{\chi\beta}{2\alpha}\int_\Omega v^2dx + \delta\int_\Omega uvdx - c_4. \tag{2.3.26}$$

Since $M = \int_\Omega u_0 dx < \frac{(1+\lambda)4\pi}{\alpha\chi}$, we can choose $\varepsilon > 0$ and $\delta > 0$ small enough such that $\frac{\chi}{2\alpha} - \left(\frac{1}{8\pi} + \varepsilon\right) \frac{(\chi+\delta)^2 M}{(1+\lambda)} > 0$. Then from (2.3.26) we have

$$F(0) \geqslant F(t) \geqslant \frac{\chi\beta}{2\alpha} \int_\Omega v^2 dx + \delta \int_\Omega uvdx - c_4 \geqslant -c_4. \tag{2.3.27}$$

From (2.3.27), we have $F(t) \geqslant -c_4$ and $\delta \int_\Omega uvdx \leqslant F(0) + c_4$. Hence using (2.3.17), we can derive that

$$\begin{aligned}(\lambda+1)\int_\Omega u \ln u dx &\leqslant F(t) + \chi \int_\Omega uvdx - \frac{\chi}{2\alpha}\int_\Omega \left(|\nabla v|^2 + \beta v^2\right) dx \\ &\leqslant F(t) + \chi \int_\Omega uvdx \leqslant \left(1 + \frac{\chi}{\delta}\right) F(0) + \frac{\chi c_4}{\delta}.\end{aligned} \tag{2.3.28}$$

Then the proof of the lemma is completed. □

2.3.2 Boundedness of $\|v\|_{W^{1,\infty}}$

Lemma 2.12. *If $\int_\Omega u_0(x)dx < \frac{(1+\lambda)4\pi}{\alpha\chi}$, then there exists a constant C_1 independent of t such that*

$$\int_0^t \|v_t\|_{L^2}^2 d\tau \leqslant C_1. \tag{2.3.29}$$

Furthermore, if $\lambda \geqslant 1$, we can find a constant $C_2 > 0$ depending on τ_0 such that

$$\|v_t\|_{L^2} \leqslant C_2 \quad \text{for all} \quad t \geqslant \tau_0 > 0. \tag{2.3.30}$$

Proof. Integrating (2.3.18) on $[0,t]$ and using Lemma 2.11, we have

$$\begin{aligned}&\int_0^t \int_\Omega \frac{u}{(1+u)^\lambda}(\nabla \ln(u(1+u)^\lambda) - \chi\nabla v)^2 dx d\tau + \frac{\chi}{\alpha}\int_0^t \|v_t\|_{L^2}^2 d\tau \\ &= F(0) - F(t) \leqslant F(0) + c_1,\end{aligned}$$

which implies (2.3.29) and

$$\int_0^t \int_\Omega \frac{u}{(1+u)^\lambda}(\nabla \ln(u(1+u)^\lambda) - \chi\nabla v)^2 dx d\tau \leqslant c_2. \tag{2.3.31}$$

By the Hölder's inequality and the first equation of (2.1.1), we obtain

$$\begin{aligned}
\|u_t\|_{(H^1)'} &= \sup_{\psi\in H^1} \frac{(u_t,\psi)}{\|\psi\|_{H^1}} \\
&\leqslant \left\| \frac{u}{(1+u)^\lambda}(\nabla \ln(u(1+u)^\lambda) - \chi\nabla v)\right\|_{L^2} \\
&= \left(\int_\Omega \left(\frac{u}{(1+u)^\lambda}\right)^2 (\nabla \ln(u(1+u)^\lambda) - \chi\nabla v)^2 dx\right)^{\frac{1}{2}} \\
&\leqslant \left(\int_\Omega \frac{u}{(1+u)^\lambda}(\nabla \ln(u(1+u)^\lambda) - \chi\nabla v)^2 dx\right)^{\frac{1}{2}},
\end{aligned} \tag{2.3.32}$$

where we have used $\frac{u}{(1+u)^\lambda} < 1$ for $\lambda \geqslant 1$. The combination of (2.3.31) and (2.3.32) gives

$$\int_0^t \|u_t\|^2_{(H^1)'} d\tau \leqslant \int_0^t \int_\Omega \frac{u}{(1+u)^\lambda}(\nabla \ln(u(1+u)^\lambda) - \chi\nabla v)^2 dx d\tau \leqslant c_2. \tag{2.3.33}$$

We differentiate the second equation of (2.1.1) with respect to t, then multiply it by v_t to obtain

$$\frac{1}{2}\frac{d}{dt}\|v_t\|^2_{L^2} + \|\nabla v_t\|^2_{L^2} + \beta\|v_t\|^2_{L^2} = \alpha\int_\Omega u_t v_t dx \leqslant \varepsilon\|v_t\|^2_{H^1} + c_3\|u_t\|^2_{(H^1)'},$$

where c_3 depends on ε. Letting $\varepsilon < \min\{1,\beta\}$, we have

$$\frac{d}{dt}\|v_t\|^2_{L^2} \leqslant 2c_3\|u_t\|^2_{(H^1)'}. \tag{2.3.34}$$

Integrating (2.3.34) with respect to t over $[\tau_0, t]$, and using (2.3.33), we have

$$\|v_t\|^2_{L^2} \leqslant \|v_t(\tau_0)\|^2_{L^2} + 2c_3\int_{\tau_0}^t \|u_t\|^2_{(H^1)'} d\tau \leqslant \|v_t(\tau_0)\|^2_{L^2} + 2c_2c_3, \tag{2.3.35}$$

which implies (2.3.30). The proof of the lemma is completed. □

Lemma 2.13. *Assume* $\int_\Omega u_0(x)dx < \frac{(1+\lambda)4\pi}{\alpha\chi}$. *Then there exists a positive constant* C *depending on* τ_0 *such that* $\|1+u\|_{L^{2+\lambda}} \leqslant C$ *for all* $t \geqslant \tau_0 > 0$.

Proof. Multiplying the first equation of (2.1.1) by $(1+u)^{1+\lambda}$, and integrating the equation over Ω, we obtain

$$\begin{aligned}
&\frac{1}{\lambda+2}\frac{d}{dt}\int_\Omega (u+1)^{\lambda+2}dx + (\lambda+1)\int_\Omega \frac{1+u+\lambda u}{1+u}|\nabla u|^2 dx \\
&= (\lambda+1)\chi\int_\Omega u\nabla u\nabla v dx \\
&= -\frac{(\lambda+1)\chi}{2}\int_\Omega u^2\Delta v dx \\
&= -\frac{(\lambda+1)\chi}{2}\int_\Omega u^2(v_t+\beta v-\alpha u)dx \\
&\leqslant -\frac{(\lambda+1)\chi}{2}\int_\Omega u^2 v_t dx + \frac{\alpha\chi(\lambda+1)}{2}\int_\Omega u^3 dx.
\end{aligned} \tag{2.3.36}$$

By Hölder's inequality and the Gagliardo-Nirenberg inequality we have

$$\begin{aligned}
-\frac{(\lambda+1)\chi}{2}\int_\Omega |u^2 v_t|dx &\leqslant \frac{(\lambda+1)\chi}{2}\|v_t\|_{L^2}\|u\|_{L^4}^2 \\
&\leqslant c_1\|v_t\|_{L^2}\left(\|\nabla u\|_{L^2}^{\frac{1}{2}}\|u\|_{L^2}^{\frac{1}{2}} + \|u\|_{L^2}\right)^2 \\
&\leqslant c_2\|v_t\|_{L^2}\left(\|\nabla u\|_{L^2}\|u\|_{L^2} + \|u\|_{L^2}^2\right) \\
&\leqslant \frac{\lambda}{2}\|\nabla u\|_{L^2}^2 + c_3\left(\|v_t\|_{L^2}^2 + \|v_t\|_{L^2}\right)\|u\|_{L^2}^2.
\end{aligned} \tag{2.3.37}$$

Substituting (2.2.6) and (2.3.37) into (2.3.36) and using (2.3.21), we obtain that

$$\begin{aligned}
&\frac{d}{dt}\|u+1\|_{L^{\lambda+2}}^{\lambda+2} + (\lambda+1)\|\nabla u\|_{L^2}^2 \\
&\leqslant \frac{\lambda}{2}\|\nabla u\|_{L^2}^2 + c_3\left(\|v_t\|_{L^2}^2 + \|v_t\|_{L^2}\right)\|u\|_{L^2}^2 + \frac{\alpha\chi(\lambda+1)}{2}\int_\Omega u^3 dx \\
&\leqslant \frac{\lambda}{2}\|\nabla u\|_{L^2}^2 + c_3\left(\|v_t\|_{L^2}^2 + \|v_t\|_{L^2}\right)\|u\|_{L^2}^2 + \frac{\lambda}{2}\|\nabla u\|_{L^2}^2 + c_4
\end{aligned}$$

which yields

$$\frac{d}{dt}\|u+1\|_{L^{\lambda+2}}^{\lambda+2}+\|\nabla u\|_{L^2}^2\leqslant c_3\left(\|v_t\|_{L^2}^2+\|v_t\|_{L^2}\right)\|u\|_{L^2}^2+c_4. \tag{2.3.38}$$

By the Gagliardo-Nirenberg inequality we have

$$\begin{aligned}\|u+1\|_{L^{\lambda+2}}^{\lambda+2}&\leqslant c_5\left(\|\nabla u\|_{L^2}^{\frac{\lambda+1}{\lambda+2}}\|u+1\|_{L^1}^{\frac{1}{\lambda+2}}+\|u+1\|_{L^1}\right)^{\lambda+2}\\&\leqslant c_6(\|\nabla u\|_{L^2}^{1+\lambda}\|u+1\|_{L^1}+\|u+1\|_{L^1}^{\lambda+2}).\end{aligned} \tag{2.3.39}$$

If $0<\lambda<1$, applying the Young's inequality to (2.3.39), we have

$$\|u+1\|_{L^{\lambda+2}}^{\lambda+2}\leqslant\|\nabla u\|_{L^2}^2+c_7\|u+1\|_{L^1}^{\frac{2}{1-\lambda}}+c_6\|u+1\|_{L^1}^{\lambda+2},$$

which implies

$$\|\nabla u\|_{L^2}^2\geqslant\|u+1\|_{L^{\lambda+2}}^{\lambda+2}-c_7\|u+1\|_{L^1}^{\frac{2}{1-\lambda}}-c_6\|u+1\|_{L^1}^{\lambda+2}. \tag{2.3.40}$$

Substituting (2.3.40) into (2.3.38), and using the inequality $\|u+1\|_{L^1}=\|u\|_{L^1}+|\Omega|$, we have

$$\begin{aligned}&\frac{d}{dt}\|u+1\|_{L^{\lambda+2}}^{\lambda+2}+\|u+1\|_{L^{\lambda+2}}^{\lambda+2}\\&\leqslant c_3\left(\|v_t\|_{L^2}^2+\|v_t\|_{L^2}\right)\|u\|_{L^2}^2+c_7\|u+1\|_{L^1}^{\frac{2}{1-\lambda}}+c_6\|u+1\|_{L^1}^{\lambda+2}+c_4\\&\leqslant c_3\left(\|v_t\|_{L^2}^2+\|v_t\|_{L^2}\right)\|u+1\|_{L^{\lambda+2}}^{\lambda+2}+c_8\\&\leqslant c_3\|v_t\|_{L^2}^2\|u+1\|_{L^{\lambda+2}}^{\lambda+2}+\left(\frac{1}{2}+\frac{c_3^2\|v_t\|_2^2}{2}\right)\|u+1\|_{L^{\lambda+2}}^{\lambda+2}+c_8\\&\leqslant\frac{2c_3+c_3^2}{2}\|v_t\|_{L^2}^2\|u+1\|_{L^{\lambda+2}}^{\lambda+2}+\frac{1}{2}\|u+1\|_{L^{\lambda+2}}^{\lambda+2}+c_8,\end{aligned}$$

which yields

$$\begin{aligned}&\frac{d}{dt}\|u+1\|_{L^{\lambda+2}}^{\lambda+2}+\frac{1}{2}\|u+1\|_{L^{\lambda+2}}^{\lambda+2}\\&\leqslant\frac{2c_3+c_3^2}{2}\|v_t\|_{L^2}^2\|u+1\|_{L^{\lambda+2}}^{\lambda+2}+c_8.\end{aligned} \tag{2.3.41}$$

Upon integration and using (2.3.29), we infer that

$$\|u+1\|_{L^{\lambda+2}} \leqslant c_9 \quad \text{for all} \quad t \geqslant 0 \text{ and } 0<\lambda<1. \tag{2.3.42}$$

Next we consider the case $\lambda \geqslant 1$. Let $y(t) = \|u+1\|_{L^{\lambda+2}}^{\lambda+2}$, then from (2.3.39), we have

$$\begin{aligned} y^{\frac{2}{\lambda+1}} &\leqslant c_6^{\frac{2}{\lambda+1}} (\|\nabla u\|_{L^2}^{1+\lambda} \|u+1\|_{L^1} + \|u+1\|_{L^1}^{\lambda+2})^{\frac{2}{\lambda+1}} \\ &\leqslant c_{10}(\|\nabla u\|_{L^2}^2 + 1). \end{aligned} \tag{2.3.43}$$

Substituting (2.3.43) into (2.3.38), we have

$$\begin{aligned} y'(t) + \frac{1}{c_{10}} y^{\frac{2}{\lambda+1}} &\leqslant c_3 \left(\|v_t\|_{L^2}^2 + \|v_t\|_{L^2}\right) \|u\|_{L^2}^2 + c_4 + 1 \\ &\leqslant c_{11} \left(\|v_t\|_{L^2}^2 + \|v_t\|_{L^2}\right) y^{\frac{2}{\lambda+2}} + c_4 + 1, \end{aligned} \tag{2.3.44}$$

where we have used the inequality

$$\begin{aligned} \|u\|_{L^2}^2 \leqslant \int_\Omega (u+1)^2 dx &\leqslant \left(\int_\Omega (u+1)^{\lambda+2} dx\right)^{\frac{2}{\lambda+2}} \left(\int_\Omega dx\right)^{\frac{\lambda}{\lambda+2}} \\ &= |\Omega|^{\frac{\lambda}{\lambda+2}} \|u+1\|_{L^{\lambda+2}}^2 \\ &= |\Omega|^{\frac{\lambda}{\lambda+2}} y^{\frac{2}{\lambda+2}}. \end{aligned}$$

For $t \geqslant \tau_0$ and $\lambda \geqslant 1$, using (2.3.30) and (2.3.44), we have

$$y'(t) + \frac{1}{c_{10}} y^{\frac{2}{\lambda+1}} \leqslant c_{13} y^{\frac{2}{\lambda+2}} + c_4 + 1 \leqslant \frac{1}{2c_{10}} y^{\frac{2}{\lambda+1}} + c_{14},$$

which yields that

$$y'(t) + \frac{1}{2c_{10}} y^{\frac{2}{\lambda+1}} \leqslant c_{14}. \tag{2.3.45}$$

Applying Lemma 2.5 to (2.3.45), we obtain that

$$\|u+1\|_{L^{\lambda+2}}^{\lambda+2} = y(t) \leqslant \max\left(y_{\tau_0}, (2c_{10}c_{14})^{\frac{1+\lambda}{2}}\right) \leqslant c_{15} \quad \text{for all} \quad t \geqslant \tau_0 > 0 \quad \text{and} \quad \lambda \geqslant 1. \tag{2.3.46}$$

For all $t \geqslant \tau_0 > 0$ and $\lambda > 0$, the combination of (2.3.42) and (2.3.46) implies that there exists a constant c_{16} depending on τ_0 such that

$$\|u+1\|_{L^{\lambda+2}} \leqslant c_{16}.$$

Then we complete the proof of the lemma. □

Next, we will show the boundedness of $\|v\|_{W^{1,\infty}}$.

Lemma 2.14. *Assume* $\int_\Omega u_0(x)dx < \frac{(1+\lambda)4\pi}{\alpha\chi}$. *Then there exists a constant* $C > 0$ *depending on* τ_0 *such that*

$$\|v\|_{W^{1,\infty}} \leqslant C \quad \text{for all} \quad t \geqslant \tau_0. \tag{2.3.47}$$

Proof. From Lemma 2.13, we obtain for all $t \geqslant \tau_0 > 0$, there exist a constant c_1 depending on τ_0 such that

$$\|u\|_{L^{\lambda+2}} \leqslant \|u+1\|_{L^{\lambda+2}} \leqslant c_1. \tag{2.3.48}$$

Since $\lambda + 2 > 2$ for $\lambda > 0$, using Lemma 2.1, we obtain (2.3.47) directly. □

2.3.3 Boundedness of $\|u\|_{L^\infty}$

We are now in a position to prove Theorem 2.1.

Proof of Theorem 2.1. Using Lemma 2.8, we only need to show that there exists a constant M_2 depending only on the initial date, Ω and τ_0, such that for all $t \geqslant \tau_0 > 0$

$$\|u\|_{L^\infty} \leqslant M_2. \tag{2.3.49}$$

To prove (2.3.49), we will use the Moser-Alikakos iteration procedure as in [3]. First, we define $(p_k)_{k\in\mathbb{N}}$ recursively by setting

$$p_k := 2p_{k-1} + \lambda, \quad k \geqslant 1 \text{ and } p_0 = 2 + \lambda. \tag{2.3.50}$$

Then, we can easily derive that $p_k = (p_0+\lambda)2^k - \lambda = (1+\lambda)2^{k+1} - \lambda$ and $p_k \geqslant 2+\lambda$ for all $k \geqslant 0$. Moreover, there exist two constants $c_1 > 0$ and $c_2 > 0$ such that

$$c_1 \cdot 2^k \leqslant p_k \leqslant c_2 \cdot 2^k \quad \text{for all } k \geqslant 0. \tag{2.3.51}$$

Multiplying the first equation of (2.1.1) by $(u+1)^{p_k-1}$, integrating the result over Ω and using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\frac{1}{p_k-1}\frac{d}{dt}\int_\Omega (u+1)^{p_k}dx \\
&= -(p_k-1)\int_\Omega \frac{1+u+\lambda u}{u+1}(u+1)^{p_k-2-\lambda}|\nabla u|^2 dx \\
&\quad + \chi(p_k-1)\int_\Omega (u+1)^{p_k-2-\lambda} u\nabla u\cdot\nabla v dx \\
&\leqslant -(p_k-1)\int_\Omega (u+1)^{p_k-2-\lambda}|\nabla u|^2 dx + \chi(p_k-1)\int_\Omega u^{p_k-1-\lambda}|\nabla u||\nabla v| dx \\
&\leqslant -\frac{p_k-1}{2}\int_\Omega (u+1)^{p_k-2-\lambda}|\nabla u|^2 dx + c_3(p_k-1)\int_\Omega (u+1)^{p_k-\lambda}dx,
\end{aligned} \tag{2.3.52}$$

where we have used Lemma 2.14 and the fact that $\frac{1+u+\lambda u}{u+1} > 1$. Here $c_3 > 0$ which, like $c_4, c_5, \cdots$ below, may depend on τ_0 but not on t, T and k. If we let $w = (u+1)^{\frac{p_k-\lambda}{2}}$, then from (2.3.52), we have

$$\begin{aligned}
&\frac{1}{p_k-1}\frac{d}{dt}\int_\Omega (u+1)^{p_k}dx \\
&\leqslant -\frac{p_k-1}{2}\int_\Omega (u+1)^{p_k-2-\lambda}|\nabla (u+1)|^2 dx + c_3(p_k-1)\int_\Omega (u+1)^{p_k-\lambda}dx, \\
&= -\frac{2(p_k-1)}{(p_k-\lambda)^2}\int_\Omega |\nabla (u+1)^{\frac{p_k-\lambda}{2}}|^2 dx + c_3(p_k-1)\int_\Omega (u+1)^{p_k-\lambda}dx \\
&= -\frac{2(p_k-1)}{(p_k-\lambda)^2}\int_\Omega |\nabla w|^2 dx + c_3(p_k-1)\int_\Omega w^2 dx,
\end{aligned}$$

which yields

$$\begin{aligned}\frac{d}{dt}\int_\Omega (u+1)^{p_k}dx &\leqslant -\frac{2(p_k-1)^2}{(p_k-\lambda)^2}\int_\Omega |\nabla w|^2dx + c_3p_k^2\int_\Omega w^2dx \\ &\leqslant -c_4\int_\Omega |\nabla w|^2dx + c_3p_k^2\int_\Omega w^2dx,\end{aligned} \tag{2.3.53}$$

where we have use the fact that $\frac{2(p_k-1)^2}{(p_k-\lambda)^2} = \frac{(2p_{k-1}+\lambda-1)^2}{2p_{k-1}^2} \geqslant c_4$ for all $c_4 \in (0,1]$. Using the Gagliardo-Nirenberg inequality and the fact that $(X+Y)^d \leqslant 2^d(X^d+Y^d)$ for all $X \geqslant 0$ and $Y \geqslant 0$, we have

$$\begin{aligned}c_3p_k^2\int_\Omega w^2dx &\leqslant c_5p_k^2\|\nabla w\|_{L^2}\|w\|_{L^1} + c_5p_k^2\|w\|_{L^1}^2 \\ &\leqslant \frac{c_4}{2}\|\nabla w\|_{L^2}^2 + \frac{1}{2c_4}(c_5p_k^2\|w\|_{L^1})^2 + c_5p_k^2\|w\|_{L^1}^2 \\ &= \frac{c_4}{2}\|\nabla w\|_{L^2}^2 + \frac{c_5^2}{2c_4}p_k^4\|w\|_{L^1}^2 + c_5p_k^2\|w\|_{L^1}^2 \\ &\leqslant \frac{c_4}{2}\|\nabla w\|_{L^2}^2 + c_6p_k^4\|w\|_{L^1}^2,\end{aligned} \tag{2.3.54}$$

where $c_6 = \frac{c_5^2}{2c_4} + c_5$. Substituting (2.3.54) into (2.3.53), we have

$$\frac{d}{dt}\int_\Omega (u+1)^{p_k}dx \leqslant -\frac{c_4}{2}\|\nabla w\|_{L^2}^2 + c_6p_k^4\|w\|_{L^1}^2. \tag{2.3.55}$$

Using the Gagliardo-Nirenberg inequality, we can find a constant $\theta = \frac{p_k+\lambda}{2p_k}$ such that

$$\begin{aligned}\int_\Omega (u+1)^{p_k}dx = \int_\Omega w^{\frac{2p_k}{p_k-\lambda}} &\leqslant c_7\|\nabla w\|_{L^2}^{\theta\frac{2p_k}{p_k-\lambda}}\|w\|_{L^1}^{(1-\theta)\frac{2p_k}{p_k-\lambda}} + c_7\|w\|_{L^1}^{\frac{2p_k}{p_k-\lambda}} \\ &\leqslant c_7\left(\|\nabla w\|_{L^2}^{\frac{2p_k}{p_k-\lambda}} + \|w\|_{L^1}^{\frac{2p_k}{p_k-\lambda}}\right) + c_7\|w\|_{L^1}^{\frac{2p_k}{p_k-\lambda}} \\ &= c_7\|\nabla w\|_{L^2}^{\frac{2p_k}{p_k-\lambda}} + 2c_7\|w\|_{L^1}^{\frac{2p_k}{p_k-\lambda}}.\end{aligned} \tag{2.3.56}$$

Now thanks to the easily verified elementary inequality $(X-Y)^d \geqslant 2^{-d}X^d - Y^d$ valid whenever $d > 0$ and $0 \leqslant Y \leqslant X$, we infer that

$$
\begin{aligned}
\|\nabla w\|_{L^2}^2 &\geqslant \left[\frac{1}{c_7}\int_\Omega (u+1)^{p_k}dx - 2\left(\int_\Omega (u+1)^{p_{k-1}}dx\right)^{\frac{2p_k}{p_k-\lambda}}\right]^{\frac{p_k-\lambda}{p_k}} \\
&\geqslant (2c_7)^{-\frac{p_k-\lambda}{p_k}}\left(\int_\Omega (u+1)^{p_k}dx\right)^{\frac{p_k-\lambda}{p_k}} - 2^{\frac{p_k-\lambda}{p_k}}\left(\int_\Omega (u+1)^{p_{k-1}}dx\right)^2 \qquad (2.3.57) \\
&\geqslant \frac{1}{2c_7}\left(\int_\Omega (u+1)^{p_k}dx\right)^{\frac{p_k-\lambda}{p_k}} - 2\left(\int_\Omega (u+1)^{p_{k-1}}dx\right)^2,
\end{aligned}
$$

here we used the conditions $2c_7 > 1$ and $\frac{p_k-\lambda}{p_k} < 1$. Then substituting (2.3.57) into (2.3.53), we have the following ordinary differential inequality

$$
\begin{aligned}
\frac{d}{dt}\int_\Omega (u+1)^{p_k}dx &\leqslant -\frac{c_4}{4c_7}\left(\int_\Omega (u+1)^{p_k}dx\right)^{\frac{p_k-\lambda}{p_k}} + (c_4 + c_6p_k^4)\left(\int_\Omega (u+1)^{p_{k-1}}dx\right)^2 \\
&\leqslant -c_8\left(\int_\Omega (u+1)^{p_k}dx\right)^{\frac{p_k-\lambda}{p_k}} + c_9p_k^4\left(\int_\Omega (u+1)^{p_{k-1}}dx\right)^2,
\end{aligned}
\qquad (2.3.58)
$$

where $c_8 = \frac{c_4}{4c_7}$ and $c_9 = c_4 + c_6$. Letting $\gamma_k = \int_\Omega (u+1)^{p_k}dx$, then from (2.3.58), we have

$$
\frac{d\gamma_k}{dt} + c_8\gamma_k^{\frac{p_k-\lambda}{p_k}} \leqslant c_9p_k^4\gamma_{k-1}^2. \qquad (2.3.59)
$$

Applying Lemma 2.5 to (2.3.59) and letting $\delta_k = \frac{\lambda}{2p_{k-1}} < \frac{1}{2}$, then we have

$$
\begin{aligned}
\gamma_k \leqslant \max\left\{\gamma_k(\tau_0), \left(\frac{c_9p_k^4\gamma_{k-1}^2}{c_8}\right)^{\frac{p_k}{p_k-\lambda}}\right\} &= \max\left\{\gamma_p(\tau_0), \left(\frac{c_9p_k^4}{c_8}\right)^{(1+\frac{\lambda}{2p_{k-1}})}\gamma_{k-1}^{2(1+\frac{\lambda}{2p_{k-1}})}\right\} \\
&\leqslant \max\left\{\gamma_k(\tau_0), \left(\frac{c_9}{c_8}\right)^{\frac{3}{2}}p_k^6\gamma_{k-1}^{2(1+\delta_k)}\right\} \\
&\leqslant \max\left\{\gamma_k(\tau_0), c_{10}(2^6)^k\gamma_{k-1}^{2(1+\delta_k)}\right\}, \\
&\leqslant \max\left\{\gamma_k(\tau_0), b^k\gamma_{k-1}^{2(1+\delta_k)}\right\},
\end{aligned}
\qquad (2.3.60)
$$

where we have used $p_k \leqslant c_2 2^k$ in (2.3.51) and $c_{10} = \left(\frac{c_9}{c_8}\right)^{\frac{3}{2}} c_2^6$ and $b = 2^6 c_{10}$. Now in the case when $\gamma_k \leqslant \gamma_k(\tau_0)$ for infinitely many $k \in \mathbb{N}$, we immediately conclude that (2.3.49) holds. Otherwise we may assume upon increasing p_0 if necessary that

$$\gamma_k \leqslant b^k \gamma_{k-1}^{2(1+\delta_k)} \text{ for all } k \geqslant 1. \tag{2.3.61}$$

Using the induction (2.3.61) and $c_1 2^k \leqslant p_k$ in (2.3.51), we have

$$\begin{aligned}\gamma_k^{\frac{1}{p_k}} &\leqslant \left\{ b^{\sum\limits_{j=1}^{k} j \cdot \prod\limits_{i=j+1}^{k} 2(1+\delta_i)} \cdot \gamma_0^{\prod\limits_{i=1}^{k} 2(1+\delta_i)} \right\}^{\frac{1}{c_1 2^k}} \\ &= b^{\frac{1}{c_1} \cdot \sum\limits_{j=1}^{k} j \cdot 2^{-j} \cdot \prod\limits_{i=j+1}^{k} (1+\delta_i)} \cdot \gamma_0^{\frac{1}{c_1} \cdot \prod\limits_{i=1}^{k} (1+\delta_i)}\end{aligned}$$

which implies

$$\|u+1\|_{p_k} \leqslant b^{\frac{1}{c_1} \cdot \sum\limits_{j=1}^{k} j \cdot 2^{-j} \cdot \prod\limits_{i=j+1}^{k} (1+\delta_i)} \cdot \gamma_0^{\frac{1}{c_1} \cdot \prod\limits_{i=1}^{k} (1+\delta_i)}. \tag{2.3.62}$$

From (2.3.51), we have $c_1 2^k \leqslant 2p_{k-1} \leqslant c_2 2^k$, hence

$$\delta_k = \frac{\lambda}{2p_{k-1}} \leqslant \frac{\lambda}{c_1} 2^{-k} \text{ for all } k \geqslant 1,$$

which implies $\sum\limits_{i=1}^{\infty} \delta_i$ converges, hence $\prod\limits_{i=1}^{\infty} (1+\delta_i)$ is finite. Moreover $\sum\limits_{j=1}^{\infty} j \cdot 2^{-j} < \infty$ and $\gamma_0 = \int_\Omega (u+1)^{p_0} dx = \int_\Omega (u+1)^{2+\lambda} dx \leqslant c_{11}$, since $p_k \to \infty$ as $k \to \infty$. Hence, using (2.3.62) we have $\|u+1\|_{L^\infty} \leqslant c_{12}$, then (2.3.49) is obtained directly. Hence the proof of the theorem is completed. □

From Theorem 2.1, we see that a necessary condition for the blowup of solutions of (2.1.1) is that $\int_\Omega u_0(x) dx > \frac{4\pi(1+\lambda)}{\alpha\chi}$. It is nature to ask whether this condition is a sufficient condition. We will study this problem in the next section.

2.4 Blowup for Supercritical Mass

2.4.1 Main Results and Key Steps

Theorem 2.2. *Let Ω be a bounded domain with smooth boundary in $\mathbb{R}^2$. Then for any $M > \frac{(1+\lambda)4\pi}{\alpha\chi}$ and $M \notin \left\{m\frac{(1+\lambda)4\pi}{\alpha\chi} | m \in \mathbb{N}^+\right\}$, there exist initial data $0 \leqslant (u_0, v_0) \in W^{1,\infty}(\Omega) \times W^{1,\infty}(\Omega)$ with $\int_\Omega u_0(x)dx = M$ such that the solution component u of (2.1.1) blows up in finite or infinite time. Moreover if $\lambda > 1$, the blow-up time is infinite.*

First we introduce the transformation

$$\tilde{v} = \chi(v - \bar{v}), \quad \bar{v} = \frac{1}{|\Omega|}\int_\Omega v dx. \tag{2.4.63}$$

Substituting (2.4.63) into (2.1.1), and dropping the tildes for convenience, we get a transformed version of the Keller-Segel model (2.1.1) as follows

$$\begin{cases} u_t = \nabla \cdot \left(\frac{1+u+\lambda u}{(u+1)^{\lambda+1}} \nabla u - \frac{u}{(1+u)^\lambda} \nabla v \right), & x \in \Omega, t > 0 \\ v_t = \Delta v + \alpha\chi(u - \bar{u}) - \beta v, & x \in \Omega, t > 0 \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \\ \int_\Omega u dx = M, \quad \int_\Omega v dx = 0. \end{cases} \tag{2.4.64}$$

The corresponding Lyapunov functional of the transformed system (2.4.64) is

$$E(t) = \int_\Omega \left(u \ln u + \lambda(1+u)\ln(1+u) - uv + \frac{1}{2\alpha\chi}(|\nabla v|^2 + \beta v^2) \right) dx. \tag{2.4.65}$$

Using the same argument of deriving (2.3.22), we have

$$\begin{aligned} E(t) &= E(u(t), v(t)) \\ &\geqslant (\lambda+1)\int_\Omega u \ln u dx - \int_\Omega uv dx + \frac{1}{2\alpha\chi}\int_\Omega (|\nabla v|^2 + \beta v^2) dx \\ &\geqslant -(1+\lambda)M \ln\left\{\frac{1}{M}\int_\Omega e^{\frac{v}{1+\lambda}} dx\right\} + \frac{1}{2\alpha\chi}\left(\int_\Omega |\nabla v|^2 dx + \beta\int_\Omega v^2 dx\right) \\ &= \mathcal{E}(v). \end{aligned} \tag{2.4.66}$$

The corresponding stationary solutions of the system (2.4.64) satisfy the following equations

$$\begin{cases} -\Delta v + \beta v = \alpha\chi(u - \bar{u}), & x \in \Omega, \\ u(u+1)^\lambda = \sigma e^v, & x \in \Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \\ \int_\Omega u dx = M, \quad \int_\Omega v dx = 0, \end{cases} \tag{2.4.67}$$

where $\sigma = \frac{\int_\Omega u(u+1)^\lambda dx}{\int_\Omega e^v dx}$ is a positive constant. Using the Lyapunov functional $E(u, v)$, we have the following properties on the stationary solutions of system (2.4.64).

Lemma 2.15. *Suppose that (u, v) is a global and bounded solution of* (2.4.64)*. Then there exist a sequence of times $t_k \to \infty$ and nonnegative functions $u_\infty, v_\infty \in C^2(\bar{\Omega})$ such that $u(\cdot, t_k) \to u_\infty$, $v(\cdot, t_k) \to v_\infty$ in $C^2(\bar{\Omega})$ and*

$$\begin{cases} \nabla \ln(u_\infty(1+u_\infty)^\lambda) - \nabla v_\infty = 0, & x \in \Omega \\ \Delta v_\infty + \alpha\chi(u_\infty - \bar{u}) - \beta v_\infty = 0, & x \in \Omega, \\ \frac{\partial u_\infty}{\partial \nu} = \frac{\partial v_\infty}{\partial \nu} = 0, & x \in \partial\Omega, \\ \int_\Omega u_\infty dx = M, \quad \int_\Omega v_\infty dx = 0, \end{cases} \tag{2.4.68}$$

as well as

$$E(u_\infty, v_\infty) \leqslant E(u_0, v_0). \tag{2.4.69}$$

Proof. The lemma can be proved with the similar argument in [97], hence we omit the details for convenience. □

Next we are devoted to proving Theorem 2.2 by using the idea as in [35, 38, 97]. The plan is to find a lower bounded for the energy of all conceivable steady states and then prove that there exist solutions having energy below this bound, that cannot be bounded since otherwise they should approach some steady states with a forbidden energy. Here 'energy' is measured in term of the Lyapunov functional $E(u, v)$. More precisely, the proof of Theorem 2.2 will be carried out by the following three steps.

Step 1 (i.e., Lemma 2.16). Under the assumptions that $M > \frac{(1+\lambda)4\pi}{\alpha\chi}$ and $M \notin \left\{ m\frac{(1+\lambda)4\pi}{\alpha\chi} \middle| m \in \mathbb{N}^+ \right\}$, we find a constant $K > 0$ such that all of stationary solutions of (2.4.64) satisfy

$$E(u, v) \geqslant -K. \tag{2.4.70}$$

Step 2 (i.e., Lemma 2.19). We show that if $M > \frac{(1+\lambda)4\pi}{\alpha\chi}$, there exist a sequence $(v_\varepsilon)_{\varepsilon>0} \subset \mathcal{D} \equiv \{f \in W^{1,\infty}(\Omega) | \int_\Omega f dx = 0\}$ such that $\mathcal{E}(v_\varepsilon) \to -\infty$ and $\int_\Omega |\nabla v_\varepsilon|^2 dx \to \infty$ as $\varepsilon \to 0$. Then, there exists a ε_0 such that $(u_{\varepsilon_0}, v_{\varepsilon_0})$ can be chosen as an initial data of system (2.4.64). Moreover for such initial data $(u_0, v_0) := (u_{\varepsilon_0}, v_{\varepsilon_0})$, we can prove that

$$E(u_0, v_0) < -K. \tag{2.4.71}$$

Step 3 (i.e., Lemma 2.20). For the initial data chosen in step 2, we conclude that the corresponding solution pair (u, v) of (2.4.64) has to blow up in finite or infinite time. Otherwise, using Lemma 2.15, we have $E(u_\infty, v_\infty) \leqslant E(u_0, v_0)$, where the solution pair (u_∞, v_∞) is the stationary solution of (2.4.64). Then the combination of the results in step 1 and step 2 implies $-K \leqslant E(u_\infty, v_\infty) \leqslant E(u_0, v_0) < -K$, which is a contradiction.

2.4.2 Lower Bound for Steady-State Energy

This subsection is to find a lower bound for the value of $E(u, v)$ for all the solutions of (2.4.67). The result can be stated as follow.

Lemma 2.16. *Let Ω be a bounded domain with smooth boundary in $\mathbb{R}^2$. Suppose that $\int_\Omega u_0(x)dx > \frac{(1+\lambda)4\pi}{\alpha\chi}$ and $\int_\Omega u_0(x)dx \neq m\frac{(1+\lambda)4\pi}{\alpha\chi}$ for some $m \in \mathbb{N}^+$, then there exists a constant $K > 0$ such that*

$$\mathcal{E}(v) \geqslant -K \tag{2.4.72}$$

holds for all the solutions of (2.4.67).

Proof. If there is not a constant K such that (2.4.72) holds true with the assumptions $\int_\Omega u_0(x)dx > \frac{(1+\lambda)4\pi}{\alpha\chi}$ and $\int_\Omega u_0(x)dx \neq m\frac{(1+\lambda)4\pi}{\alpha\chi}$ for some $m \in \mathbb{N}^+$, then using the Lyapunov functional $\mathcal{E}(v)$ in (2.4.66), we can claim that there exists a solution sequence $(v_k)_{k\in\mathbb{N}}$ of (2.4.67) such that as $k \to \infty$

$$\int_\Omega e^{\frac{v_k}{1+\lambda}}dx \to \infty \quad \text{and} \quad \max_{x\in\bar{\Omega}} v_k(x) \to \infty. \tag{2.4.73}$$

Let $v_k^* = v_k + \frac{\alpha\chi}{\beta}\bar{u}$, then system (2.4.67) is transformed into the following equations

$$\begin{cases} -\Delta v_k^* + \beta v_k^* = \alpha\chi u_k^*, & x \in \Omega, \\ u_k^*(1+u_k^*)^\lambda = \sigma_k e^{v_k^*}, & x \in \Omega, \\ \frac{\partial v_k^*}{\partial\nu} = \frac{\partial u_k^*}{\partial\nu} = 0, & x \in \partial\Omega, \\ \int_\Omega u_k^* dx = M, \quad \int_\Omega v_k^* dx = \frac{\alpha\chi M}{\beta}. \end{cases} \tag{2.4.74}$$

From the second equation of (2.4.74), we can derive that there exists a sequence $0 < (\mu_k)_{k\in\mathbb{N}}$ such that

$$\mu_k e^{\frac{v_k^*}{1+\lambda}} - 1 \leqslant u_k^* \leqslant \mu_k e^{\frac{v_k^*}{1+\lambda}} \tag{2.4.75}$$

Using (2.4.73), we have $\int_\Omega e^{\frac{v_k^*}{1+\lambda}}dx \to \infty$. Furthermore, (2.4.75) implies that

$$\mu_k \leqslant \frac{\int_\Omega (u_k^*+1)dx}{\int_\Omega e^{\frac{v_k^*}{1+\lambda}}dx} = \frac{M+|\Omega|}{\int_\Omega e^{\frac{v_k^*}{1+\lambda}}dx} \to 0 \quad \text{as} \quad k \to \infty. \tag{2.4.76}$$

By employing a similar argument [38, 90], we can show that there exists a subsequence of $(u_k^*)_{k\in\mathbb{N}}$ (denoted by $(u_k^*)_{k\in\mathbb{N}}$ again for simplicity) such that for some $m \in \mathbb{N}^+$

$$\int_\Omega u_k^* dx \to m\frac{(1+\lambda)4\pi}{\alpha\chi}, \quad \text{as } k \to \infty, \tag{2.4.77}$$

which contradict the assumption that $M \neq m\frac{(1+\lambda)4\pi}{\alpha\chi}$ since $\int_\Omega u_k^* dx = M$. Then the proof of the lemma is completed. □

Next, we will give the details for the proof of (2.4.77). First, we define the blowup set $\mathcal{S}$ as follows:

$$\mathcal{S} := \left\{x \in \bar{\Omega} : \exists \;\; x_k \to x \text{ such that } v_k^*(x_k) \to \infty \text{ as } k \to \infty\right\}. \tag{2.4.78}$$

Then from (2.4.73), we know that $card\, \mathcal{S} \geqslant 1$, where $card\, \mathcal{S}$ stands for the cardinality of set $\mathcal{S}$. Furthermore, we have the following lemma.

Lemma 2.17. *Assume the blowup set $\mathcal{S}$ is defined as* (2.4.78)*. Then one has $1 \leqslant card\; \mathcal{S} < \infty$.*

Proof. Since $(u_k^*)_{k\in\mathbb{N}}$ is bounded in $L^1(\Omega)$, then using the Prokhorov's theorem we may extract a subsequence (still denoted $(u_k^*)_{k\in\mathbb{N}}$ for simplicity) such that u_k^* converges in the sense of measure on Ω to some nonnegative bounded measure η, i.e.

$$\int_\Omega u_k^* \psi dx \to \int_\Omega \psi d\eta, \tag{2.4.79}$$

for every $\psi \in C_0^\infty(\Omega)$. As in [8, 38, 90], we call $x_0 \in \bar{\Omega}$ a δ-regular point if there is a function $\psi \in C_0^\infty(\Omega)$, $0 \leqslant \psi \leqslant 1$, with $\psi = 1$ in a neighborhood of x_0 such that

$$\int_\Omega \psi d\mu < \frac{4\pi(1+\lambda)}{\alpha\chi(1+3\delta)}. \tag{2.4.80}$$

Let$\sum(\delta)$ be the set of points which are not δ-regular points in $\bar{\Omega}$. Clearly $x_0 \in \sum(\delta)$ if and only if $\eta(\{x_0\}) \geqslant \frac{4\pi(1+\lambda)}{\alpha\chi(1+3\delta)}$. Since η is a bounded measure with $\int_\Omega d\eta = M$, it follows that the elements of $\sum(\delta)$ are finite and

$$card \sum(\delta) \leqslant \frac{\alpha\chi M(1+3\delta)}{4\pi(1+\lambda)}. \tag{2.4.81}$$

Using the similar argument as in [38, 90], we state the following two claims without proof for convenience.

(i). If x_0 is a δ-regular point, then $(v_k^*)_{k\in\mathbb{N}}$ is uniformly bounded in $L^\infty(\bar{\Omega}\cap B_{R_0}(x_0))$ for some $R_0>0$.

(ii). $\mathcal{S}=\sum(\delta)$.

Hence the combination of (2.4.81) and the claim (ii) implies $1\leqslant card\ \mathcal{S}=card\ \sum(\delta)<\infty$. Then we complete the proof of the lemma. $\square$

Due to $1\leqslant card\ \mathcal{S}<\infty$, without loss of generality, we assume $\mathcal{S}=\{p_1,\cdots,p_N\}$. We decompose $\mathcal{S}$ into a boundary blowup set $\mathcal{S}_1=\mathcal{S}\cap\partial\Omega$ and an interior blowup set $\mathcal{S}_2=\mathcal{S}\cap\Omega$. Let

$$\sigma_j^k(r)=\alpha\chi\int_{B_r(p_j)}u_k^*dx, \tag{2.4.82}$$

where $r>0$ is a small constant. Then we can derive the following properties on $\sigma_j^k(r)$ by using a similar argument in [90, Lemma 3.4] and [38, Lemma 3].

Lemma 2.18.

$$\lim_{r\to 0}\lim_{k\to\infty}\sigma_j^k(r)=\begin{cases}4(1+\lambda)\pi, & p_j\in\mathcal{S}_1,\\ 8(1+\lambda)\pi, & p_j\in\mathcal{S}_2.\end{cases} \tag{2.4.83}$$

Proof. Without loss of generality, we assume the blowup point $p_j=0$. Let $U_r=B_r(0)\cap\bar{\Omega}$. Assume the function w_k is a solution of the following problem

$$\begin{cases}\Delta w-\beta w=0, & x\in U_r,\\ \frac{\partial w}{\partial\nu}=\frac{\partial v_k^*}{\partial\nu}, & x\in\partial U_r.\end{cases} \tag{2.4.84}$$

It is easy to see that $w_k=O(1)$ in $C^2(U_r)$ since $|\frac{\partial v_k^*}{\partial\nu}|\leqslant C$ on ∂U_r. As in [38, 90], we let $h_k=(v_k^*-w_k)/\sigma_j^k(r)$, then $h_k\to G(\cdot,0)$ in $C^2_{loc}(B_r(0)\cap\bar{\Omega}/\{0\})$ (see [21, Lemma 2.6]), where $G(\cdot,0)$ satisfies

$$\begin{cases}-\Delta G+\beta G=\delta_0, & x\in U_r,\\ \frac{\partial G}{\partial\nu}=0, & x\in\partial U_r,\end{cases}$$

where δ_0 denotes the Dirac measure on U_r giving unit mass to the point 0. The regular part of $G(x,0)$ is defined depending on whether the blowup point 0 lies in the domain or on its boundary as

$$H(x,0) = \begin{cases} G(x,0) + \frac{1}{\pi}\ln|x|, & \text{if } \ 0 \in \partial U_r, \\ G(x,0) + \frac{1}{2\pi}\ln|x|, & \text{if } \ 0 \in U_r, \end{cases} \tag{2.4.85}$$

where $H(\cdot,0) \in C^{1,\alpha}(\bar{U}_r)$ (see [20]). Hence, for $x \in \bar{U}_r$ it holds that

$$G(x,0) = \begin{cases} -\frac{1}{\pi}\ln|x| + O(1), & \text{if } \ 0 \in \partial U_r, \\ -\frac{1}{2\pi}\ln|x| + O(1), & \text{if } \ 0 \in U_r. \end{cases} \tag{2.4.86}$$

Using the condition $w_k = O(1)$ in $C^2(U_r)$, we have

$$v_k^*(x) = \begin{cases} -\frac{\sigma_j^k(r)}{\pi}\ln|x| + O(1), & \text{if } \ 0 \in \partial U_r, \\ -\frac{\sigma_j^k(r)}{2\pi}\ln|x| + O(1), & \text{if } \ 0 \in U_r \end{cases} \tag{2.4.87}$$

in $C^1(\partial U_r)$(here $O(1)$ may depend on r but is uniform in k). First, we consider the case that blowup point $0 \in \mathcal{S}_1$, hence $v_k^*(x) = -\frac{\sigma_j^k(r)}{\pi}\ln|x| + O(1)$. For the equation

$$\Delta w - \beta w + f(w) = 0, \quad x \in U \subset \mathbb{R}^2,$$

we have the following Pohozaev's identity

$$\begin{aligned} &\int_U (-\beta w^2 + 2F(w))dx \\ &= \int_{\partial U} \left[(x\cdot\nabla w)\frac{\partial w}{\partial \nu} - (x\cdot\nu)\frac{|\nabla w|^2}{2} + x\cdot\nu\left(-\beta\frac{w^2}{2} + F(w)\right)\right] dS, \end{aligned} \tag{2.4.88}$$

where $F(w) = \int_0^w f(s)ds$ ([90]). Applying (2.4.88) to the first equation of (2.4.74) on U_r, then one has

$$\begin{aligned} &\int_{U_r} \left(-\beta(v_k^*)^2 + 2F(v_k^*)\right) dx \\ &= \int_{\partial U_r} \left[(x\cdot\nabla v_k^*)\frac{\partial v_k^*}{\partial \nu} - (x\cdot\nu)\frac{|\nabla v_k^*|^2}{2} + x\cdot\nu\left(-\beta\frac{(v_k^*)^2}{2} + F(v_k^*)\right)\right] dS. \end{aligned} \tag{2.4.89}$$

Next, we will estimate all the terms on both sides of (2.4.89). First, using the elliptic estimate and the fact $\|u_k^*\|_{L^1} = M$, we have $\|v_k^*\|^2_{W^{1,4/3}} \leqslant C$. Hence, we have the following estimate

$$\int_{U_r} (v_k^*)^2 dx \leqslant \left(\int_{U_r} 1dx\right)^{\frac{1}{2}} \left(\int_{U_r} (v_k^*)^4\right)^{\frac{1}{2}} = O(r\|v_k^*\|^2_{L^4}) = O(r\|v_k^*\|^2_{W^{1,4/3}}) = O(r). \tag{2.4.90}$$

Next we estimate the term $\int_{U_r} F(v_k^*)dx$. Letting $f(v_k^*) = \alpha\chi u_k^*$ and using (2.4.75), we have

$$F(v_k^*) \geqslant \alpha\chi(1+\lambda)\mu_k(e^{\frac{v_k^*}{1+\lambda}} - 1) - \alpha\chi v_k^* \geqslant \alpha\chi(1+\lambda)u_k^* - \alpha\chi(1+\lambda)\mu_k - \alpha\chi v_k^*, \tag{2.4.91}$$

and

$$F(v_k^*) \leqslant \alpha\chi(1+\lambda)\mu_k(e^{\frac{v_k^*}{1+\lambda}} - 1) \leqslant \alpha\chi(1+\lambda)(1+u_k^*) - \alpha\chi(1+\lambda)\mu_k. \tag{2.4.92}$$

The combination of (2.4.91) and (2.4.92) implies that

$$\alpha\chi(1+\lambda)u_k^* - \alpha\chi(1+\lambda)\mu_k - \alpha\chi v_k^* \leqslant F(v_k^*) \leqslant \alpha\chi(1+\lambda)(1+u_k^*) - \alpha\chi(1+\lambda)\mu_k. \tag{2.4.93}$$

Integrating (2.4.93) over U_r, one has

$$(1+\lambda)\sigma_j^k(r) - \alpha\chi(1+\lambda)\int_{U_r} \mu_k dx - \alpha\chi\int_{U_r} v_k^* dx \leqslant \int_{U_r} F(v_k^*)dx, \tag{2.4.94}$$

and

$$\int_{U_r} F(v_k^*)dx \leqslant (1+\lambda)\sigma_j^k(r) + \alpha\chi(1+\lambda)\int_{U_r} dx - \alpha\chi(1+\lambda)\int_{U_r} \mu_k dx. \tag{2.4.95}$$

Using (2.4.94), (2.4.95) and noting the facts

$$\int_{U_r} \mu_k dx = O(\mu_k r^2), \quad \int_{U_r} v_k^* dx = O(r), \quad \int_{U_r} dx = O(r^2), \tag{2.4.96}$$

we have the following estimate

$$(1+\lambda)\sigma_j^k(r) - O(\mu_k r^2) - O(r) \leqslant \int_{U_r} F(v_k^*)dx \leqslant (1+\lambda)\sigma_j^k(r) + O(r^2) - O(\mu_k r^2). \tag{2.4.97}$$

Using the equalities (2.4.87) and $\frac{\partial v_k^*}{\partial \nu} = \nu \cdot \nabla v_k^*$, we have

$$\begin{aligned}
\int_{\partial U_r} (x\cdot \nabla v_k^*)\frac{\partial v_k^*}{\partial \nu} dS &= -\int_{\partial U_r} \frac{\sigma_j^k(r)}{\pi}\nu\cdot\nabla v_k^* dS \\
&= -\int_{\partial U_r} \frac{\sigma_j^k(r)}{\pi}\nu\cdot\nabla v_k^* dS \\
&= \left(\frac{\sigma_j^k(r)}{\pi}\right)^2 \int_{\partial U_r} \left(\frac{1}{r} + O(1)\right) dS \\
&= \left(\frac{\sigma_j^k(r)}{\pi}\right)^2 (\pi + O(r)).
\end{aligned} \tag{2.4.98}$$

$$\int_{\partial U_r} (x\cdot\nu)\frac{|\nabla v_k^*|^2}{2} dS = \int_{\partial U_r} \frac{1}{2r}\left(\frac{\sigma_j^k(r)}{\pi}\right)^2 dS = \left(\frac{\sigma_j^k(r)}{\pi}\right)^2 \left(\frac{\pi}{2} + O(r)\right). \tag{2.4.99}$$

Using $v_k^* \in C^1(\partial U_r)$ and (2.4.91), we have

$$\int_{\partial U_r} (x\cdot\nu)(v_k^*)^2 dS = O(r). \tag{2.4.100}$$

$$\int_{\partial U_r} (x\cdot\nu)F(v_k^*)dS = O(r\mu_k \max_{x\in\partial U_r} e^{\frac{v_k^*}{1+\lambda}}) = O(\mu_k r). \tag{2.4.101}$$

Substituting (2.4.90), (2.4.97)-(2.4.101) into (2.4.89), and letting $k \to \infty$ first and then $\gamma \to 0$, we can obtain that

$$2(1+\lambda)\lim_{r\to 0}\lim_{k\to\infty} \sigma_j^k(r) = \frac{1}{\pi^2}\frac{\pi}{2}(\lim_{r\to 0}\lim_{k\to\infty}\sigma_j^k(r))^2,$$

which implies

$$\lim_{r\to 0}\lim_{k\to\infty}\sigma_j^k(r) = 4(1+\lambda)\pi. \tag{2.4.102}$$

For the case $0 \in \mathcal{S}_2$, then $v_k^*(x) = -\frac{\sigma_j^k(r)}{2\pi}\ln|x| + O(1)$. Hence, we can obtain the same estimates except that

$$\int_{\partial U_r}(x\cdot\nabla v_k^*)\frac{\partial v_k^*}{\partial\nu}dS = \left(\frac{\sigma_j^k(r)}{2\pi}\right)^2(2\pi + O(r)), \tag{2.4.103}$$

and

$$\int_{\partial U_r}(x\cdot\nu)\frac{|\nabla v_k^*|^2}{2}dS = \left(\frac{\sigma_j^k(r)}{2\pi}\right)^2(\pi + O(r)). \tag{2.4.104}$$

Then using the Pohozaev's inequality again, we have

$$2(1+\lambda)\lim_{r\to0}\lim_{k\to\infty}\sigma_j^k(r) = \frac{1}{4\pi^2}\pi(\lim_{r\to0}\lim_{k\to\infty}\sigma_j^k(r))^2,$$

which yields

$$\lim_{r\to0}\lim_{k\to\infty}\sigma_j^k(r) = 8(1+\lambda)\pi. \tag{2.4.105}$$

Then we complete the proof of the lemma. □

Next we will prove (2.4.77) to complete the proof of Lemma 2.16 by using the Lemma 2.18.

Proof of (2.4.77). Using the definition of $\sigma_j^k(r)$, we have that

$$\alpha\chi\lim_{k\to\infty}\int_\Omega u_k^*dx = \alpha\chi\sum_{j=1}^N\lim_{k\to\infty}\int_{B_r(p_j)}u_k^*dx = \sum_{j=1}^N\lim_{r\to0}\lim_{k\to\infty}\sigma_j^k(r). \tag{2.4.106}$$

Hence, the combination of (2.4.106) and Lemma 2.18 gives (2.4.77). Then the proof of Lemma 2.16 is completed. □

2.4.3 Initial Data With Large Negative Energy

In this subsection, we assert that there exist some initial data with supercritical mass having energy below any prescribed bound. To attain the aim, we look for a sequence $(v_\varepsilon)_{\varepsilon\geqslant 0}$ satisfying $\int_\Omega v_\varepsilon(x)dx = 0$ such that $\lim\limits_{\varepsilon\to 0}\mathcal{E}(v_\varepsilon) = -\infty$. From [9], we know that the functions

$$\phi_\varepsilon(x) = \ln\left(\frac{8\varepsilon^2}{(\varepsilon^2+\pi|x|^2)^2}\right),$$

$\varepsilon > 0$, are solution of the following system

$$\begin{cases} -\Delta\phi(x) = e^{\phi(x)}, & x \in \mathbb{R}^2, \\ \int_{\mathbb{R}^2} e^{\phi(x)}dx < \infty. \end{cases} \tag{2.4.107}$$

We note that $\phi_\varepsilon(x) \to -\infty$ for all $x \neq 0$ and $\phi_\varepsilon(0) \to \infty$ as $\varepsilon \to 0$. As in [35], we choose the sequence $(v_\varepsilon)_{\varepsilon\geqslant 0}$ with

$$\begin{aligned} v_\varepsilon(x) &= (1+\lambda)\left(\phi_\varepsilon(x) - \frac{1}{|\Omega|}\int_\Omega \phi_\varepsilon(x)dx\right) \\ &= (1+\lambda)\left[\ln\left(\frac{\varepsilon^2}{(\varepsilon^2+\pi|x|^2)^2}\right) - \frac{1}{|\Omega|}\int_\Omega \ln\left(\frac{\varepsilon^2}{(\varepsilon^2+\pi|x|^2)^2}\right)dx\right], \end{aligned} \tag{2.4.108}$$

as our candidate to obtain the properties $\lim\limits_{\varepsilon\to 0}\mathcal{E}(v_\varepsilon) = -\infty$ with supercritical mass.

Lemma 2.19. *Assume $M > \frac{(1+\lambda)4\pi}{\alpha\chi}$ and the sequence $(v_\varepsilon)_{\varepsilon\geqslant 0}$ is defined by* (2.4.108)*. Then as $\varepsilon \to 0$, we have*

$$\mathcal{E}(v_\varepsilon) \to -\infty \quad \text{and} \quad \int_\Omega |\nabla v_\varepsilon|^2 dx \to \infty. \tag{2.4.109}$$

Proof. From (2.4.108), we have

$$\frac{1}{2\chi\alpha}\int_\Omega |\nabla v_\varepsilon|^2dx = \frac{16(1+\lambda)^2\pi^2}{2\chi\alpha}\int_\Omega \frac{x^2}{(\varepsilon^2+\pi x^2)^2}dx = \frac{8(1+\lambda)^2\pi^2}{\chi\alpha}\int_\Omega \frac{x^2}{(\varepsilon^2+\pi x^2)^2}dx. \tag{2.4.110}$$

Substituting $y = \frac{x}{\varepsilon}$, we obtain that

$$\frac{1}{2\alpha\chi}\|\nabla v_\varepsilon\|_2^2 = \frac{8(1+\lambda)^2\pi^2}{\chi\alpha}\int_{\Omega_\varepsilon} \frac{|y|^2}{(1+\pi|y|^2)^2}dy, \tag{2.4.111}$$

where $\Omega_\varepsilon = \{y|\varepsilon y \in \Omega\}$. Applying the polar coordinates around original point $0 \in \partial\Omega$ to (2.4.111), we obtain

$$\begin{aligned}
\frac{\chi}{2\alpha}\|\nabla v_\varepsilon\|_{L^2}^2 &= \frac{8(1+\lambda)^2\pi^2}{\chi\alpha}\int_{\Omega_\varepsilon}\frac{|y|^2}{(1+\pi|y|^2)^2}dy \\
&= \frac{8(1+\lambda)^2\pi^2}{\chi\alpha}\int_0^\pi\int_0^{\frac{R}{\varepsilon}}\frac{r^3}{(1+\pi r^2)^2}drd\theta \\
&= \frac{8(1+\lambda)^2\pi^3}{\chi\alpha}\int_0^{\frac{1}{\varepsilon}}\frac{r^3}{(1+\pi r^2)^2}dr + \frac{8(1+\lambda)^2\pi^3}{\chi\alpha}\int_{\frac{1}{\varepsilon}}^{\frac{R}{\varepsilon}}\frac{r^3}{(1+\pi r^2)^2}dr \\
&= \frac{8(1+\lambda)^2\pi^3}{\chi\alpha}I_1 + \frac{8(1+\lambda)^2\pi^3}{\chi\alpha}I_2,
\end{aligned} \tag{2.4.112}$$

where R denote the maximum distance between original point and the pole. First, we can estimate I_1 as follows

$$\begin{aligned}
I_1 = \frac{1}{2}\int_0^{\frac{1}{\varepsilon}}\frac{r^2}{(1+\pi r^2)^2}dr^2 &= \frac{1}{2}\int_0^{\frac{1}{\varepsilon^2}}\frac{z}{(1+\pi z)^2}dz \\
&= \frac{1}{2\pi}\int_0^{\frac{1}{\varepsilon^2}}\frac{1+\pi z-1}{(1+\pi z)^2}dz \\
&= \frac{1}{2\pi}\int_0^{\frac{1}{\varepsilon^2}}\frac{1}{1+\pi z}dz - \frac{1}{2\pi}\int_0^{\frac{1}{\varepsilon^2}}\frac{1}{(1+\pi z)^2}dz \\
&= \frac{1}{2\pi^2}\ln\left(\frac{\varepsilon^2+\pi}{\varepsilon^2}\right) + \frac{1}{2\pi^2}\frac{\varepsilon^2}{\varepsilon^2+\pi} - \frac{1}{2\pi^2}.
\end{aligned} \tag{2.4.113}$$

Similarly, we can obtain the following estimates of I_2

$$\begin{aligned}
I_2 &= \frac{1}{2}\int_{\frac{1}{\varepsilon}}^{\frac{R}{\varepsilon}}\frac{r^2}{(1+\pi r^2)^2}dr^2 \\
&= \frac{1}{2\pi}\int_{\frac{1}{\varepsilon^2}}^{\frac{R^2}{\varepsilon^2}}\frac{1}{1+\pi z}dz - \frac{1}{2\pi}\int_{\frac{1}{\varepsilon^2}}^{\frac{R^2}{\varepsilon^2}}\frac{1}{(1+\pi z)^2}dz \\
&= \frac{1}{2\pi^2}\ln\left(\frac{\varepsilon^2+\pi R^2}{\varepsilon^2+\pi}\right) + \frac{1}{2\pi^2}\frac{\varepsilon^2}{\varepsilon^2+\pi R^2} - \frac{1}{2\pi^2}\frac{\varepsilon^2}{\varepsilon^2+\pi}.
\end{aligned} \tag{2.4.114}$$

Substituting (2.4.113) and (2.4.114) into (2.4.112), we have

$$\begin{aligned}
&\frac{1}{2\alpha\chi}\|\nabla v_\varepsilon\|_{L^2}^2 \\
&= \frac{4(1+\lambda)^2\pi}{\alpha\chi}\left(\ln\frac{1}{\varepsilon^2} + \ln(\varepsilon^2+\pi R^2) - 1 + \frac{\varepsilon^2}{\varepsilon^2+\pi R^2}\right) \\
&= \frac{8(1+\lambda)^2\pi}{\alpha\chi}\ln\frac{1}{\varepsilon} + O_1(1),
\end{aligned} \tag{2.4.115}$$

where $|O_1(1)| \leqslant C$ as $\varepsilon \to 0$. Since

$$\begin{aligned}
v_\varepsilon^2 &= (1+\lambda)^2\left(\ln(\varepsilon^2+\pi|x|^2)^2 - \frac{1}{|\Omega|}\int_\Omega \ln(\varepsilon^2+\pi|x|^2)^2 dx\right)^2 \\
&= (1+\lambda)^2\left[\left(\ln(\varepsilon^2+\pi|x|^2)^2\right)^2 - \frac{2}{|\Omega|}\ln(\varepsilon^2+\pi|x|^2)^2\int_\Omega \ln(\varepsilon^2+\pi|x|^2)^2 dx\right] \\
&\quad + \frac{(1+\lambda)^2}{|\Omega|^2}\left(\int_\Omega \ln(\varepsilon^2+\pi|x|^2)^2 dx\right)^2,
\end{aligned} \tag{2.4.116}$$

then we can deduce

$$\begin{aligned}
&\frac{\beta}{2\alpha\chi}\int_\Omega v_\varepsilon^2 dx \\
&= \frac{\beta(1+\lambda)^2}{2\alpha\chi}\int_\Omega (\ln(\varepsilon^2+\pi|x|^2)^2)^2 dx - \frac{\beta(1+\lambda)^2}{2\alpha\chi|\Omega|}\left(\int_\Omega \ln(\varepsilon^2+\pi|x|^2)^2 dx\right)^2 \\
&= O_2(1),
\end{aligned} \tag{2.4.117}$$

where $|O_2(1)| \leqslant C$ as $\varepsilon \to 0$. Using (2.4.108), we have the estimates

$$\int_\Omega e^{\frac{v_\varepsilon}{1+\lambda}} dx = |\Omega| e^{-\frac{1}{|\Omega|}\int_\Omega \ln\left(\frac{\varepsilon^2}{(\varepsilon^2+\pi|x|^2)^2}\right)dx}\int_\Omega \left(\frac{\varepsilon^2}{(\varepsilon^2+\pi|x|^2)^2}\right) dx,$$

and

$$\ln\int_\Omega e^{\frac{v_\varepsilon}{1+\lambda}} dx = \ln|\Omega| + \ln\int_\Omega \left(\frac{\varepsilon^2}{(\varepsilon^2+\pi|x|^2)^2}\right) dx - \frac{1}{|\Omega|}\int_\Omega \ln\left(\frac{\varepsilon^2}{(\varepsilon^2+\pi|x|^2)^2}\right) dx,$$

which imply

$$
\begin{aligned}
& -(1+\lambda)M\ln\left\{\frac{1}{M}\int_\Omega e^{\frac{v_\varepsilon}{1+\lambda}}dx\right\} \\
& = -(1+\lambda)M\left(\ln\frac{1}{M}+\ln\int_\Omega e^{\frac{v_\varepsilon}{1+\lambda}}dx\right) \\
& = -(1+\lambda)M\left(\ln\frac{|\Omega|}{M}+\ln\int_\Omega\left(\frac{\varepsilon^2}{(\varepsilon^2+\pi|x|^2)^2}\right)dx-\frac{1}{|\Omega|}\int_\Omega\ln\left(\frac{\varepsilon^2}{(\varepsilon^2+\pi|x|^2)^2}\right)dx\right) \\
& = \frac{(1+\lambda)M}{|\Omega|}\int_\Omega\ln\varepsilon^2dx+\frac{(1+\lambda)M}{|\Omega|}\int_\Omega\ln(\varepsilon^2+\pi|x|^2)^2dx \\
& \quad -(1+\lambda)M\ln\left[\frac{|\Omega|}{M}\int_\Omega\left(\frac{\varepsilon^2}{(\varepsilon^2+\pi|x|^2)^2}\right)dx\right] \\
& = 2(1+\lambda)M\ln\varepsilon+O_3(1)
\end{aligned}
\tag{2.4.118}
$$

where $|O_3(1)|\leqslant C$ as $\varepsilon\to 0$. Then the combination of (2.4.115), (2.4.117) and (2.4.118) implies

$$
\mathcal{E}(v_\varepsilon)\leqslant\left(\frac{8(1+\lambda)^2\pi}{\alpha\chi}-2(1+\lambda)M\right)\ln\frac{1}{\varepsilon}+O(1), \tag{2.4.119}
$$

where $O(1)=O_1(1)+O_2(1)+O_3(1)$ and $|O(1)|\leqslant C$ as $\varepsilon\to 0$. Then (2.4.119) leads to the assertion of the lemma. □

Remark 2.1. In the proof of Lemma 2.19, we assume that the blowup point $0\in\partial\Omega$. If the blowup point $0\in\Omega$, then we have the same estimates except that $\frac{1}{2\alpha\chi}\|\nabla v_\varepsilon\|_{L^2}^2=\frac{16(1+\lambda)^2\pi}{\alpha\chi}\ln\frac{1}{\varepsilon}+O_1(1)$ by using the polar coordinates around original point $0\in\Omega$. Hence, we have

$$
\mathcal{E}(v_\varepsilon)\leqslant\left(\frac{16(1+\lambda)^2\pi}{\alpha\chi}-2(1+\lambda)M\right)\ln\frac{1}{\varepsilon}+O(1), \tag{2.4.120}
$$

which implies that $\mathcal{E}(v_\varepsilon)\to-\infty$ as $\varepsilon\to 0$ if $M>\frac{8(1+\lambda)\pi}{\alpha\chi}$.

Lemma 2.20. *Assume the conditions in Lemma* 2.16 *are satisfied. Then there exists initial data* (u_0, v_0) *such that the corresponding solution of* (2.4.64) *blows up in finite or infinite time.*

Proof. Let $K > 0$ be the constant in Lemma 2.16. Now let us choose a ε_0 arbitrary but fixed, and

$$v_{\varepsilon_0} = (1+\lambda)\left[\ln\left(\frac{\varepsilon_0^2}{(\varepsilon_0^2+\pi|x|^2)^2}\right) - \frac{1}{|\Omega|}\int_\Omega \ln\left(\frac{\varepsilon_0^2}{(\varepsilon_0^2+\pi|x|^2)^2}\right)dx\right].$$

such that

$$\mathcal{E}(v_{\varepsilon_0}) < -K - (\lambda + \ln\sigma_{\varepsilon_0})M - C_{\varepsilon_0},$$

where C_{ε_0} is a constant depending on ε_0 and will be defined later. The existence of appropriate ε_0 is a direct consequence of Lemma 2.19. We can check that $v_{\varepsilon_0} \in W^{1,\infty}(\Omega)$. And we choose u_{ε_0} satisfying

$$u_{\varepsilon_0}(1+u_{\varepsilon_0})^\lambda = \sigma_{\varepsilon_0}e^{v_{\varepsilon_0}}, \qquad \int_\Omega u_{\varepsilon_0}(x)dx = M. \tag{2.4.121}$$

Collecting (2.4.65) and (2.4.66) and using the fact that $v_{\varepsilon_0} \in W^{1,\infty}(\Omega)$, we obtain

$$\begin{aligned} E(u_{\varepsilon_0}, v_{\varepsilon_0}) - \mathcal{E}(v_{\varepsilon_0}) &= \int_\Omega (u_{\varepsilon_0}\ln u_{\varepsilon_0} + \lambda(1+u_{\varepsilon_0})\ln(1+u_{\varepsilon_0}) - u_{\varepsilon_0}v_{\varepsilon_0})\,dx \\ &\quad + (1+\lambda)M\ln\left\{\frac{1}{M}\int_\Omega e^{\frac{v_{\varepsilon_0}}{1+\lambda}}dx\right\} \\ &\leqslant \lambda\int_\Omega \ln(1+u_{\varepsilon_0})dx + \ln\sigma_{\varepsilon_0}\int_\Omega u_{\varepsilon_0}dx + C_{\varepsilon_0} \\ &\leqslant \lambda\int_\Omega u_{\varepsilon_0}dx + \ln\sigma_{\varepsilon_0}\int_\Omega u_{\varepsilon_0}dx + C_{\varepsilon_0} \\ &\leqslant (\lambda + \ln\sigma_{\varepsilon_0})M + C_{\varepsilon_0}. \end{aligned} \tag{2.4.122}$$

which implies

$$E(u_{\varepsilon_0}, v_{\varepsilon_0}) \leqslant \mathcal{E}(v_{\varepsilon_0}) + (\lambda + \ln\sigma_{\varepsilon_0})M + C_{\varepsilon_0} < -K. \tag{2.4.123}$$

Hence, we can define $(u_0, v_0) = (u_{\varepsilon_0}, v_{\varepsilon_0})$ as the initial data, and the corresponding solution of chemotaxis model (2.4.65) has to blow up in finite time or infinite time. Otherwise, if the corresponding solution (u, v) of (2.4.65) is global in time and bounded in $L^\infty(\Omega \times (0, \infty))$, then from Lemma 2.15, we have $-K \leqslant \mathcal{E}(u_\infty) \leqslant E(u_\infty, v_\infty) \leqslant E(u_0, v_0) < -K$, which is a contradiction. Then the proof of this lemma is completed. □

2.4.4 Infinite Time Blowup

Next, we show that if $\lambda > 1$, the blowup time is infinite. First, we show that if $\lambda > 1$, the solution of (2.1.1) globally exists without the smallness assumption on the initial data.

Lemma 2.21. *Let Ω be a bounded domain with smooth boundary in $\mathbb{R}^2$. Assume $0 \leqslant (u_0, v_0) \in W^{1,\infty}(\Omega) \times W^{1,\infty}(\Omega)$ and $\lambda > 1$. Then there exists a constant $C > 0$ depending on T such that the solution of* (2.1.1) *satisfies*

$$\|u\|_{L^\infty} \leqslant C. \tag{2.4.124}$$

Proof. Multiplying the first equation of (2.1.1) by u and integrating it over Ω to obtain that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx + \int_\Omega \frac{|\nabla u|^2}{(u+1)^\lambda} dx &= \chi \int_\Omega \frac{u}{(u+1)^\lambda} \nabla u \cdot \nabla v dx \\ &\leqslant \frac{1}{2}\int_\Omega \frac{|\nabla u|^2}{(u+1)^\lambda} dx + \frac{\chi^2}{2}\int_\Omega u^{2-\lambda}|\nabla v|^2 dx. \end{aligned} \tag{2.4.125}$$

Multiplying the second equation of (2.1.1) by Δv and integrating the result with respect to x, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla v|^2 dx + \int_\Omega |\Delta v|^2 dx + \frac{\beta}{2}\int_\Omega |\nabla v|^2 dx &= -\alpha \int_\Omega \Delta v u dx \\ &\leqslant \frac{1}{2}\int_\Omega |\Delta v|^2 + \frac{\alpha^2}{2}\int_\Omega u^2 dx. \end{aligned} \tag{2.4.126}$$

The combination of (2.4.125) and (2.4.126) gives

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega (u^2+|\nabla v|^2)dx + \int_\Omega \frac{|\nabla u|^2}{(u+1)^\lambda}dx + \beta\int_\Omega |\nabla v|^2 dx + \int_\Omega |\Delta v|^2 dx \\
&\leqslant \chi^2\left(\int_\Omega (u^{2-\lambda})^p dx\right)^{\frac{1}{p}}\left(\int_\Omega |\nabla v|^{2q}dx\right)^{\frac{1}{q}} + \alpha^2\int_\Omega u^2 dx \\
&\leqslant \chi^2\|u^{2-\lambda}\|_{L^p}\|\nabla v\|^2_{L^{2q}} + \alpha^2\|u\|^2_{L^2},
\end{aligned} \tag{2.4.127}$$

where the Hölder's inequality has been used and $\frac{1}{p}+\frac{1}{q}=1$. Using Lemma 2.1 and $\|u\|_{L^1}\leqslant c_1$, we can deduce that

$$\|\nabla v\|_{L^r}\leqslant c_2, \quad \text{for all} \quad r<2. \tag{2.4.128}$$

Using the Gagliardo-Nirenberg inequality, we can choose $\theta = 1-\frac{r}{2q}$ such that

$$\|\nabla v\|^2_{L^{2q}} \leqslant c_3\|\Delta v\|^{2\theta}_{L^2}\|\nabla v\|^{2(1-\theta)}_{L^r} + c_4\|\nabla v\|^2_{L^r} \leqslant c_5\left(\|\Delta v\|^{2-\frac{r}{q}}_{L^2}+1\right), \tag{2.4.129}$$

which implies

$$\begin{aligned}
\chi^2\|u^{2-\lambda}\|_{L^p}\|\nabla v\|^2_{L^{2q}} &\leqslant c_5\chi^2\|u^{2-\lambda}\|_{L^p}\left(\|\Delta v\|^{2-\frac{r}{q}}_{L^2}+1\right) \\
&\leqslant \varepsilon\|\Delta v\|^{2q-r}_{L^2} + c_6\|u^{2-\lambda}\|^p_{L^p} + c_7.
\end{aligned} \tag{2.4.130}$$

Now we let $r=\frac{4}{\lambda}-2<2$, which requires $\lambda>1$, and choose p,q such that

$$2q-r=2, \quad (2-\lambda)p=2, \tag{2.4.131}$$

then from (2.4.127) and (2.4.130), we have

$$\begin{aligned}
\frac{d}{dt}(\|u\|^2_{L^2}+\|\nabla v\|^2_{L^2}) \leqslant (c_6+\alpha^2)\|u\|^2_{L^2} + c_7 &\leqslant (c_6+\alpha^2)\left(\|u\|^2_{L^2}+\|\nabla v\|^2_{L^2}\right)+c_7 \\
&= c_8\left(\|u\|^2_{L^2}+\|\nabla v\|^2_{L^2}\right)+c_7,
\end{aligned}$$

which implies

$$\|u\|^2_{L^2} \leqslant \|u_0\|^2_{L^2} + \|\nabla v_0\|^2_{L^2} + \frac{c_7}{c_8}e^{c_8 T}, \tag{2.4.132}$$

by using the Gronwall's inequality. Then using Lemma 2.1 and (2.4.132), we can find a constant c_9 depending on T such that

$$\|\nabla v\|_{L^p} \leqslant c_9, \text{ for all } 1 \leqslant p < \infty. \tag{2.4.133}$$

Multiplying the first equation of (2.1.1) by $(1+u)^{1+\lambda}$, integrating the equation over Ω, and using (2.4.132) and (2.4.133), we obtain

$$\begin{aligned}
&\frac{1}{\lambda+2}\frac{d}{dt}\int_\Omega (u+1)^{\lambda+2}dx + (\lambda+1)\int_\Omega \frac{1+u+\lambda u}{1+u}|\nabla u|^2 dx \\
&= (\lambda+1)\chi\int_\Omega u\nabla u\nabla v dx \\
&\leqslant \int_\Omega |\nabla u|^2 dx + \frac{(\lambda+1)^2\chi^2}{4}\left(\int_\Omega u^{\lambda+2}dx\right)^{\frac{2}{\lambda+2}}\left(\int_\Omega |\nabla v|^{\frac{2(\lambda+2)}{\lambda}}dx\right)^{\frac{\lambda}{\lambda+2}} \\
&\leqslant \int_\Omega |\nabla u|^2 dx + \frac{1}{\lambda+2}\int_\Omega (u+1)^{\lambda+2}dx + c_{10},
\end{aligned} \tag{2.4.134}$$

where c_{10} depends on T. Using (2.4.134) and the fact $\frac{1+u+\lambda u}{1+u} > 1$, we obatin

$$\frac{d}{dt}\int_\Omega (u+1)^{\lambda+2}dx \leqslant \int_\Omega (u+1)^{\lambda+2}dx + (\lambda+2)c_{10},$$

which implies $\|u+1\|_{L^{\lambda+2}} \leqslant c_{11}$ by using the Gronwall's inequality, where c_{11} depends on T. Then we can find a constant c_{12} depending on T such that $\|\nabla v\|_{L^\infty} \leqslant c_{12}$ by using Lemma 2.1 and $\|u\|_{L^{\lambda+2}} \leqslant \|u+1\|_{L^{\lambda+2}} \leqslant c_{11}$. Hence carrying out the Moser-Alikakos iteration procedure, we obtain a constant c_{13} depending on T such that $\|u(\cdot,t)\|_{L^\infty} \leqslant c_{13}$ for all $t \in (0,T)$. Then the proof of this lemma is completed. □

Next, we give the proof of Theorem 2.2.

Proof of Theorem 2.2. The blowup result has been proved in Lemma 2.20. The combination of Lemma 2.8 and the fact $\|u\|_{L^\infty} \leqslant C(T)$ gives the existence of a unique global-in-time solution of (2.1.1) for any initial data with $\lambda > 1$. Hence if the solution blows up, it has to blow up at infinite time. □

Chapter 3

Attraction-Repulsion Keller-Segel Chemotaxis Model

3.1 Introduction

In this chapter, we consider the following attraction-repulsion chemotaxis system

$$\begin{cases} u_t = \Delta u - \nabla \cdot (\chi u \nabla v) + \nabla \cdot (\xi u \nabla w), & x \in \Omega, \quad t > 0, \\ \tau_1 v_t = \Delta v + \alpha u - \beta v, & x \in \Omega, \quad t > 0, \\ \tau_2 w_t = \Delta w + \gamma u - \delta w, & x \in \Omega, \quad t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x), \ v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{3.1.1}$$

where $\chi, \xi, \alpha, \gamma > 0$ and $\beta, \delta \geqslant 0$, Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, ν denotes the unit outward normal vector to the boundary $\partial\Omega$. The model (3.1.1) was proposed in [63] to describe how the combination of chemicals might interact to produce aggregates of cells. A documented example is the motion of *Microglia* in the central nervous system (CNS) in Alzhemer's disease (AD) which is affected by the interaction of chemoattractant(e.g., β-amyloid) and chemorepellent(e.g., TNF-α) which are secreted by *Microglia*, where the concentrations of *Microglia*, chemoattractant and chemorepellent are denoted by $u(x,t), v(x,t)$ and $w(x,t)$ in the model (3.1.1) respectively. The positive parameters χ and ξ are called the chemosensitivity coefficients, and $\chi, \beta, \gamma, \delta > 0$ are chemical production and depredation rates. τ_1, τ_2

are constants equal to 0 or 1 justifying whether the change of chemicals is stationary or dynamical in time. It is noted that the model (3.1.1) was also introduced in the paper [77] to interpret the quorum sensing effect in the chemotactic movement.

Proposed first by Keller and Segel [49], the classical (attractive) chemotaxis model was a system of two partial differential equations (i.e. the first two equations of (3.1.1) with $\xi = 0$) which possesses an apparent Lyapunov functional. This particular structure motivated a vast amount of mathematical studies in the past (see review articles [37, 31, 95]) and recent studies [12, 32, 86, 96, 98], where most of works were focused at whether the solution blows up or not (see some early works in [38, 68, 69] in this area). On the other hand, for the repulsive Keller-Segel model (i.e. the coupling of first and third equations of (3.1.1) with $\chi = 0$), a Laypunoval function (which was different from that of the attractive Keller-Segel model) was found in [15] which leads to the global existence of classical solutions in two dimensions and weak solutions in three and four dimensions. Compared to the classical Keller-Segel model, the three-component system of ARKS model (3.1.1) is much harder to analyze. In one dimensional space the linear stability analysis has been done in the work [63]. Furthermore Liu and Wang [60] has studied the global existence of classical solution and the stationary solution. Meanwhile, the time-periodic orbits has been found recently in [61] by employing the local and global Hopf bifurcation theory. Due to the lack of an apparent Lyapunov functional, no progress has been made for higher dimensional space until a recent work by Tao and Wang [85] where the main contribution has three folds: (1) when $\tau_1 = \tau_2 = 0$, the parameter regime of global boundedness and blowup of solutions was successfully identified by the Moser iteration method, which reveals the competing effect of attraction and repulsion plays a central role in determining the dynamics of solutions. (2) when $\tau_1 = \tau_2 = 1$ and $\beta = \delta$, numerous clean transformation were introduced to reduce the ARKS model (3.1.1) to the classical chemotaxis model so that the existing mathematical techniques (like Lyapunov functional) and results can be employed to derive various behaviors of solutions; (3) when $\tau_1 = \tau_2 = 1$ and $\beta \neq \delta$, an entropy inequality was provided to establish the time dependent global boundedness of solutions when the initial mass $\int_\Omega u_0 dx$ is small and repulsion prevails (i.e. $\xi\gamma - \chi\alpha > 0$).

The study of [85] leaves two evident gaps in the case of $\tau_1 = \tau_2 = 1$ and $\beta \neq \delta$: (a) existence of global solutions with uniform-in-time boundedenss or with large data of initial value u_0 if the repulsion dominates; (b) behavior of solutions if the attraction prevails. All the past and current methods (e.g. see [38, 68, 69, 96, 98]) of proving the blowup of solutions of the attractive Keller-Segel model essentially depends on the existence of a Lyapunov functional. It appears to be hopeless at present due to the failure of finding a Lyapunov functional to establish the blowup of solutions for the case where the attraction prevails (i.e. $\xi\gamma - \chi\alpha < 0$).

In this chapter, we first consider the asymptotic behavior of the ARKS model in one dimension. Furthermore, we remove the the smallness assumption on the initial mass $\int_\Omega u_0(x)dx$ for the global existence of solutions with uniform-in-time bound in two dimensions, which substantially improves the results of [85, Theorem 2.7]. In particulary, for the case $\tau_1 = 1, \tau_2 = 0, \beta \neq \delta$, the ARKS model (3.1.1) is irreducible to a two-component chemotaxis model and we succeed in finding a Lyapunov functional to prove the uniform-in-time boundedness of solutions, which was not found in [85]. As we know, it is the first result that presents a Lyapunov functional for an irreducible three component attraction-repulsion chemotaxis model. Moreover, the Lyapunov functional for an irreducible three component attraction-repulsion chemotaxis model may be useful in establishing the blowup of solutions for the case where the attraction prevails, this problem will be pursued in future.

Before proceeding our main results, we give the following local existence theorem of the solutions to system (3.1.1), which was proved in [60, 85].

Lemma 3.1. *Assume that* $0 \leqslant (u_0, \tau_1 v_0, \tau_2 w_0) \in [W^{1,\infty}(\Omega)]^3$. *Then there exist* $T_{max} \in (0, \infty]$ *and a unique triple* (u, v, w) *of nonnegative functions from* $C^0(\bar{\Omega} \times [0, T_{max}); \mathbb{R}^3) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max}); \mathbb{R}^3)$ *solving* (3.1.1) *classically in* $\Omega \times (0, T_{max})$. *Moreover*

$$\text{if } T_{max} < \infty, \quad \text{then} \quad \|u(\cdot, t)\|_{L^\infty} \to \infty \quad \text{as} \quad t \nearrow T_{max}. \tag{3.1.2}$$

3.2 Asymptotic Behavior in One Dimension

In one dimension, $\Omega = I = (a, b)$ is a bounded open interval in $\mathbb{R}$ and ∂I denotes the boundary of the interval I. When $\tau_1 = \tau_2 = 1$, then the system (3.1.1) becomes

$$\begin{cases} u_t = u_{xx} - (\chi u v_x)_x + (\xi u w_x)_x, & x \in I, t > 0, \\ v_t = v_{xx} + \alpha u - \beta v, & x \in I, t > 0, \\ w_t = w_{xx} + \gamma u - \delta w, & x \in I, t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial I, t > 0, \\ u(x,0) = u_0(x), v(x,0) = v_0(x), w(x,0) = w_0(x), & x \in I. \end{cases} \tag{3.2.3}$$

3.2.1 Uniform-in-time Bound of Solutions

First, we address the global existence of classical solutions to system (3.2.3) which improve the results in [60] by deriving a uniform-in-time bound for solutions.

Theorem 3.1. *Let* $u_0 \in H^1(I), (v_0, w_0) \in [H^2(I)]^2$. *Then the system* (3.2.3) *has a unique global classical solution* $(u, v, w) \in C^0(\bar{I} \times [0, \infty); \mathbb{R}^3) \cap C^{2,1}(\bar{I} \times (0, \infty); \mathbb{R}^3)$ *such that* $u, v, w \geqslant 0$ *if* $u_0, v_0, w_0 \geqslant 0$.

Theorem 3.1 is a consequence of local existence theorem (Lemma 3.1) and the a *priori* estimates (Propositions 3.1) by the continuation argument.

Proposition 3.1 (A *priori* estimates)**.** *Let* $u_0 \in H^1(I), (v_0, w_0) \in [H^2(I)]^2$ *and* (u, v, w) *be a solution of* (3.2.3). *Then for any* $T > 0$, *there exists a constant* $C > 0$ *such that* (u, v, w) *satisfies*

$$\|u(t)\|_{H^1} + \|(v, w)(t)\|_{H^2} \leqslant C, \quad \text{for all } 0 < t \leqslant T. \tag{3.2.4}$$

Next we are devoted to proving the Proposition 3.1. The Proposition 3.1 will be verified by the following two lemmas.

Lemma 3.2. *Let the assumptions in Proposition 3.1 hold. If* (u, v, w) *is a solution of* (3.2.3)*, then for any* $T > 0$*, there is a constant* C *independent of* T *such that the following inequality holds for each* $0 < t \leqslant T$

$$\|u(t)\|_{L^2}^2 + \|(v,w)(t)\|_{H^1}^2 \leqslant C(\|(v_0,w_0)\|_{H^1}^2 + \|u_0\|_{L^2}^2 + \|u_0\|_{L^1}^2). \tag{3.2.5}$$

Proof. First the result of [76, Eq. (4.3)] gives that $\|(v,w)(t)\|_{H^1}^2 \leqslant C(\|(v_0,w_0)\|_{H^1}^2 + \|u_0\|_{L^2}^2)$. Hence it remains to derive that

$$\|u(t)\|_{L^2}^2 \leqslant C(\|u_0\|_{L^2}^2 + \|(v_0,w_0)\|_{H^1}^2). \tag{3.2.6}$$

Multiplying the first equation of (3.2.3) by u and integrating the resulting equation with respect to x over I gives rise to

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_I u^2 dx + \int_I u_x^2 dx &= \chi \int_I u v_x u_x dx - \xi \int_I u w_x u_x dx \\ &\leqslant \int_I \chi^2 (u v_x)^2 dx + \int_I \xi^2 (u w_x)^2 dx + \frac{1}{2}\int_I u_x^2 dx, \end{aligned} \tag{3.2.7}$$

where we have used the Young's inequality

$$ab \leqslant \varepsilon a^p + (\varepsilon p)^{-q/p} q^{-1} b^q, \text{ for any } a,\ b \geqslant 0,\ \varepsilon > 0,\ 1 < p,\ q < \infty,\ \frac{1}{p} + \frac{1}{q} = 1. \tag{3.2.8}$$

Applying (2.2.9) and $\|u\|_{L^1} = \|u_0\|_{L^1} =: M$ to (3.2.7) and using the Hölder's inequality as well as the Young's inequality, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_I u^2 dx + \frac{1}{2}\int_I u_x^2 dx \\ &\leqslant \chi^2 \|u\|_{L^4}^2 \|v_x\|_{L^4}^2 + \xi^2 \|u\|_{L^4}^2 \|w_x\|_{L^4}^2 \\ &\leqslant C(\|u_x\|_{L^2}\|u\|_{L^1} + \|u\|_{L^1}^2)(\|v_{xx}\|_{L^2}^{\frac{1}{2}} \|v_x\|_{L^2}^{\frac{3}{2}} + \|v_x\|_{L^2}^2 + \|w_{xx}\|_{L^2}^{\frac{1}{2}} \|w_x\|_{L^2}^{\frac{3}{2}} + \|w_x\|_{L^2}^2) \\ &\leqslant \frac{1}{8}(\|u_x\|_{L^2}^2 + \|v_{xx}\|_{L^2}^2 + \|w_{xx}\|_{L^2}^2) + C(\|v_x\|_{L^2}^6 + \|w_x\|_{L^2}^6). \end{aligned} \tag{3.2.9}$$

Multiplying the second equation of (3.2.3) by $-v_{xx}$, the third equation by $-w_{xx}$, and adding them, then integrating the resulting equation with respect to x, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_I (v_x^2 + w_x^2)dx + \int_I (v_{xx}^2 + w_{xx}^2)dx + \beta\int_I v_x^2 dx + \delta\int_I w_x^2 dx \\
&= -\alpha\int_I u v_{xx} dx - \gamma\int_I u w_{xx} dx \qquad (3.2.10)\\
&\leqslant \frac{1}{2}\int_I (v_{xx}^2 + w_{xx}^2)dx + \frac{\alpha^2+\gamma^2}{2}\int_I u^2 dx,
\end{aligned}$$

where the Cauchy-Schwarz inequality has been used. Using (2.2.9) and (3.2.8), we have

$$\int_I u^2 dx \leqslant C(\|u_x\|_{L^2}^{\frac{2}{3}}\|u\|_{L^1}^{\frac{4}{3}} + \|u\|_{L^1}^2) \leqslant \frac{1}{4(\alpha^2+\gamma^2+2)}\|u_x\|_{L^2}^2 + C\|u\|_{L^1}^2. \quad (3.2.11)$$

Adding (3.2.9), (3.2.10) and (3.2.11), applying (3.2.5) to the resulting inequality, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_I (u^2 + v_x^2 + w_x^2)dx + \int_I (u_x^2 + v_{xx}^2 + w_{xx}^2)dx + \int_I (u^2 + \beta v_x^2 + \delta w_x^2)dx \\
&\leqslant C(1 + \|v_x\|_{L^2}^6 + \|w_x\|_{L^2}^6) \leqslant C,
\end{aligned} \qquad (3.2.12)$$

where $C > 0$ depends on $\|(v_0, w_0)\|_{H^1} + \|u_0\|_{L^1}$. Solving (3.2.12) yields (3.2.6). Then the proof of Lemma 3.2 is completed.

□

Lemma 3.3. *Let the assumptions in Proposition 3.1 hold, and (u, v, w) be a solution of* (3.2.3)*. Then for any $0 < t \leqslant T$, there exists a constant $C > 0$ independent of T such that the solution satisfies that*

$$\|u_x(t)\|_{L^2}^2 + \|(v_{xx}, w_{xx})(t)\|_{L^2}^2 \leqslant C(\|u_0\|_{H^1}^2 + \|(v_0, w_0)\|_{H^2}^2), \qquad (3.2.13)$$

and

$$\begin{aligned}
&\int_0^t \left(\|u_{xx}(s)\|_{L^2}^2 + \|(v_{xxx}, w_{xxx})(s)\|_{L^2}^2\right) ds \\
&\leqslant C\left(\|u_0\|_{H^1}^2 + \|(v_0, w_0)\|_{H^2}^2 + t(\|u_0\|_{L^2}^2 + \|(v_0, w_0)\|_{H^1}^2)\right),
\end{aligned} \qquad (3.2.14)$$

Proof. Multiplying the first equation of system (3.2.3) by $-u_{xx}$, and integrating the resulting equation with respect to x yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_I u_x^2 dx + \int_I u_{xx}^2 dx \\
&= \chi\int_I (uv_x)_x u_{xx} dx - \xi\int_I (uw_x)_x u_{xx} dx \\
&\leqslant \frac{1}{2}\int_I u_{xx}^2 dx + 2(\chi^2+\xi^2)\int_I \left[u^2(v_{xx}^2+w_{xx}^2)+u_x^2(v_x^2+w_x^2)\right] dx,
\end{aligned} \tag{3.2.15}$$

where the Young's inequality (3.2.8) has been used. Applying (2.2.8) to v_x, one has

$$\|v_x\|_{L^\infty}^2 \leqslant C(\|v_{xxx}\|_{L^2}^{\frac{1}{2}}\|v_x\|_{L^2}^{\frac{3}{2}}+\|v_x\|_{L^2}^2). \tag{3.2.16}$$

Similarly, we obtain

$$\|w_x\|_{L^\infty}^2 \leqslant C(\|w_{xxx}\|_{L^2}^{\frac{1}{2}}\|w_x\|_{L^2}^{\frac{3}{2}}+\|w_x\|_{L^2}^2). \tag{3.2.17}$$

The combination of (3.2.16) and (3.2.17) with (3.2.8) gives

$$\begin{aligned}
&2(\chi^2+\xi^2)\int_I u_x^2(v_x^2+w_x^2)dx \\
&\leqslant 2(\chi^2+\xi^2)\|u_x\|_{L^2}^2(\|v_x\|_{L^\infty}^2+\|w_x\|_{L^\infty}^2) \\
&\leqslant C\|u_x\|_{L^2}^2(\|v_{xxx}\|_{L^2}^{\frac{1}{2}}\|v_x\|_{L^2}^{\frac{3}{2}}+\|w_{xxx}\|_{L^2}^{\frac{1}{2}}\|w_x\|_{L^2}^{\frac{3}{2}}+\|(v_x,w_x)\|_{L^2}^2) \\
&\leqslant C(\|u_{xx}\|_{L^2}\|u\|_{L^2}+\|u\|_{L^2}^2)(\|v_{xxx}\|_{L^2}^{\frac{1}{2}}\|v_x\|_{L^2}^{\frac{3}{2}}+\|w_{xxx}\|_{L^2}^{\frac{1}{2}}\|w_x\|_{L^2}^{\frac{3}{2}}+\|(v_x,w_x)\|_{L^2}^2) \\
&\leqslant \frac{1}{8}(\|u_{xx}\|_{L^2}^2+\|v_{xxx}\|_{L^2}^2+\|w_{xxx}\|_{L^2}^2)+C\|u\|_{L^2}^4(\|v_x\|_{L^2}^6+\|w_x\|_{L^2}^6+1).
\end{aligned} \tag{3.2.18}$$

Using (2.2.9) and (3.2.8), one has

$$\begin{aligned}
&2(\chi^2+\xi^2)\int_I u^2(v_{xx}^2+w_{xx}^2)dx \\
&\leqslant C(\|u_{xx}\|_{L^2}^{\frac{1}{2}}\|u\|_{L^2}^{\frac{3}{2}}+\|u\|_{L^2}^2)(\|v_{xxx}\|_{L^2}\|v_x\|_{L^2}+\|w_{xxx}\|_{L^2}\|w_x\|_{L^2}+\|(v_x,w_x)\|_{L^2}^2) \\
&\leqslant \frac{1}{8}(\|u_{xx}\|_{L^2}^2+\|v_{xxx}\|_{L^2}^2+\|w_{xxx}\|_{L^2}^2)+C\|u\|_{L^2}^6(\|v_x\|_{L^2}^4+\|w_x\|_{L^2}^4+1).
\end{aligned} \tag{3.2.19}$$

Therefore substituting (3.2.18) and (3.2.19) back to (3.2.15) gives

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_I u_x^2 dx + \frac{1}{2}\int_I u_{xx}^2 dx \\
&\leqslant \frac{1}{4}(\|u_{xx}\|_{L^2}^2 + \|v_{xxx}\|_{L^2}^2 + \|w_{xxx}\|_{L^2}^2) + C(\|u_0\|_{L^2}^2 + \|(v_0, w_0)\|_{H^1}^2),
\end{aligned}
\tag{3.2.20}
$$

where Lemma 3.2 and (3.2.8) have been used.

Differentiating the second and third equations of (3.2.3) with respect to x once, we have

$$
\begin{cases}
v_{tx} = v_{xxx} + \alpha u_x - \beta v_x, \\
w_{tx} = w_{xxx} + \gamma u_x - \delta w_x.
\end{cases}
\tag{3.2.21}
$$

Multiplying the first equation of (3.2.21) by $-v_{xxx}$, the second by $-w_{xxx}$, and adding them, we end up with the following results after integrating the resulting equation with respect to x

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_I (v_{xx}^2 + w_{xx}^2)dx + \int_I (v_{xxx}^2 + w_{xxx}^2)dx + \beta\int_I v_{xx}^2 dx + \delta\int_I w_{xx}^2 dx \\
&= -\alpha\int_I u_x v_{xxx} dx - \gamma\int_I u_x w_{xxx} dx \\
&\leqslant \frac{1}{2}\int_I (v_{xxx}^2 + w_{xxx}^2)dx + \frac{\alpha^2+\gamma^2}{2}\int_I u_x^2 dx,
\end{aligned}
\tag{3.2.22}
$$

where we have used the Cauchy-Schwarz inequality and

$$
-\int_I v_{xt} v_{xxx} dx - \int_I w_{xt} w_{xxx} dx = \frac{1}{2}\frac{d}{dt}\int_I (v_{xx}^2 + w_{xx}^2)dx
$$

Noting (2.2.8) with $n = 1$ and (3.2.8) entails that

$$
\|u_x\|_{L^2}^2 \leqslant C(\|u_{xx}\|_{L^2}\|u\|_{L^2} + \|u\|_{L^2}^2) \leqslant \frac{1}{4(\alpha^2+\gamma^2+2)}\|u_{xx}\|_{L^2}^2 + C\|u\|_{L^2}^2. \tag{3.2.23}
$$

Then combining (3.2.6), (3.2.20), (3.2.22) and (3.2.23) yields that

$$\frac{1}{2}\frac{d}{dt}\int_I (u_x^2 + v_{xx}^2 + w_{xx}^2)dx + \frac{1}{4}\int_I (u_{xx}^2 + v_{xxx}^2 + w_{xxx}^2)dx + \int_I (u_x^2 + \beta v_{xx}^2 + \delta w_{xx}^2)dx$$

$$\leqslant \int_I u_x^2 dx + \frac{\alpha^2+\gamma^2}{2}\int_I u_x^2 dx + C(\|u_0\|_{L^2}^2 + \|(v_0,w_0)\|_{H^1}^2)$$

$$\leqslant \frac{1}{8}\|u_{xx}\|_{L^2}^2 + C(\|u_0\|_{L^2}^2 + \|(v_0,w_0)\|_{H^1}^2),$$

which implies that

$$\frac{d}{dt}\int_I (u_x^2 + v_{xx}^2 + w_{xx}^2)dx + \int_I (u_{xx}^2 + v_{xxx}^2 + w_{xxx}^2)dx + \int_I (u_x^2 + \beta v_{xx}^2 + \delta w_{xx}^2)dx$$

$$\leqslant C(\|u_0\|_{L^2}^2 + \|(v_0,w_0)\|_{H^1}^2).$$

Therefore it follows that

$$\|u_x(t)\|_{L^2}^2 + \|(v_{xx},w_{xx})(t)\|_{L^2}^2 \leqslant C(\|u_0\|_{H^1}^2 + \|(v_0,w_0)\|_{H^2}^2),$$

and

$$\int_0^t \left(\|u_{xx}(s)\|_{L^2}^2 + \|(v_{xxx},w_{xxx})(s)\|_{L^2}^2\right) ds$$

$$\leqslant C\left(\|u_0\|_{H^1}^2 + \|(v_0,w_0)\|_{H^2}^2 + t(\|u_0\|_{L^2}^2 + \|(v_0,w_0)\|_{H^1}^2\right)$$

which completes the proof of Lemma 3.3. □

With the above results in hand, we are in a position to prove Theorem 3.1.

Proof of Theorem 3.1. Based on the estimates obtained in Lemma 3.2, Lemma 3.3 and Sobolev embedding $H^1 \hookrightarrow L^\infty$, we have for any $T > 0$

$$\sup_{0<t<\min\{T_0,T\}} \|(u,v,w)\|_{L^\infty} \leqslant C(\|u_0\|_{H^1} + \|(v_0,w_0)\|_{H^2}),$$

which combined with Lemma 3.1 gives the existence of a unique nonnegative global in time solution. The regularity of the solution is obtained by the standard parabolic regularity argument (see [85] for details). The proof of Theorem 3.1 is finished.

3.2.2 Existence of Global Attractor

Next, we consider the large time behavior of the solution of (3.2.3). Define

$$\mathscr{X} = \{(u, v, w) \in H^1(I) \times H^2(I) \times H^2(I) | u \geqslant 0, v \geqslant 0, w \geqslant 0\}.$$

From Theorem 3.1, we know that for any initial function $U_0 = (u_0, v_0, w_0) \in \mathscr{X}$, the system (3.2.3) has a unique solution $U(t; U_0) = (u, v, w)$ for all $t > 0$. Hence, we can define a dynamical system $(\{S(t)_{t\geqslant 0}\}, \mathscr{X})$ by a nonlinear C^0 semigroup $S(t) : \mathscr{X} \to \mathscr{X}$ by

$$S(t)U_0 = U(t; U_0),$$

such that

$$S(0) = Identity, \quad S(t)S(s) = S(s)S(t) = S(s + t), \quad S(t)U_0 \text{ is continuous in } U_0 \text{ and } t.$$

The definition of a global attractor is presented below.

Definition 3.1 ([87]). *We say that $\mathscr{A} \subset \mathscr{X}$ is a global attractor for the semigroup $\{S(t)_{t\geqslant 0}\}$ if $\mathscr{A}$ is a compact attractor that attracts the bounded sets of $\mathscr{X}$.*

A useful concept associated with global attractor is the absorbing set as defined below.

Definition 3.2 ([87]). *Let $\mathscr{B}$ be a subset of $\mathscr{X}$ and $\mathscr{U}$ an open set containing $\mathscr{B}$. We say that $\mathscr{B}$ is absorbing in $\mathscr{U}$ if the orbit of any bounded set of $\mathscr{U}$ enters into $\mathscr{B}$ after a certain time:*

$$\forall\ \mathscr{B}_0 \subset \mathscr{U}, \quad \mathscr{B}_0 \text{ bounded}, \quad \exists\ t_{\mathscr{B}_0} \text{ such that } S(t)\mathscr{B}_0 \subset \mathscr{B}, \ \forall\ t \geqslant t_{\mathscr{B}_0}.$$

Then we have the following result.

Theorem 3.2. *The dynamical system $(\{S(t)_{t\geqslant 0}\}, \mathscr{X})$ possesses a global attractor.*

To study the large time behavior of solutions of (3.2.3), we need higher-order energy estimates.

Lemma 3.4. *Let* $u_0 \in H^2(I), (v_0, w_0) \in [H^3(I)]^2$. *Let* (u, v, w) *be a solution of* (3.2.3). *Then for any* $T > 0$, *it holds that for any* $0 < t \leqslant T$

$$\|u_{xx}(t)\|_{L^2}^2 + \|(v_{xxx}, w_{xxx})(t)\|_{L^2}^2 \leqslant C(\|u_0\|_{H^2}^2 + \|(v_0, w_0)\|_{H^3}^2). \tag{3.2.24}$$

Proof. We differentiate the first equation of (3.2.3) with respect to x, multiply the resulting equation by $-u_{xxx}$, and then integrate the product in x. Finally, we end up with

$$\begin{aligned}\frac{d}{dt}\int_I \frac{u_{xx}^2}{2}dx + \int_I u_{xxx}^2 dx &= \chi \int_I (uv_x)_{xx} u_{xxx} dx - \xi \int_I (uw_x)_{xx} u_{xxx} dx \\ &\leqslant \frac{1}{2}\int_I u_{xxx}^2 dx + \int_I \left(\chi^2 \left|(uv_x)_{xx}\right|^2 + \xi^2 \left|(uw_x)_{xx}\right|^2\right) dx.\end{aligned} \tag{3.2.25}$$

Using (3.2.8), Lemma 3.2 and Lemma 3.3, as well as the Sobolev embedding $H^1 \hookrightarrow L^\infty$, we have

$$\begin{aligned}&\int_I \left(\chi^2 \left|(uv_x)_{xx}\right|^2 + \xi^2 \left|(uw_x)_{xx}\right|^2\right) dx \\ &\leqslant C\int_I \left[u_{xx}^2(v_x^2 + w_x^2) + u_x^2(v_{xx}^2 + w_{xx}^2) + u^2(v_{xxx}^2 + w_{xxx}^2)\right] dx \\ &\leqslant C\left(\|(v_x, w_x)\|_{L^\infty}^2 \|u_{xx}\|_{L^2}^2 + \|u_x\|_{L^\infty}^2 \|(v_{xx}, w_{xx})\|_{L^2}^2 + \|u\|_{L^\infty}^2 \|(v_{xxx}, w_{xxx})\|_{L^2}^2\right) \\ &\leqslant C(\|u_{xx}\|_{L^2}^2 + \|v_{xxx}\|_{L^2}^2 + \|w_{xxx}\|_{L^2}^2 + \|(v_{xx}, w_{xx})\|_{L^2}^2) \\ &\leqslant C(\|u_x\|_{L^2}\|u_{xxx}\|_{L^2} + \|v_{xx}\|_{L^2}\|v_{xxxx}\|_{L^2} + \|w_{xx}\|_{L^2}\|w_{xxxx}\|_{L^2}) \\ &\quad + C(\|u_x\|_{L^2}^2 + \|(v_{xx}, w_{xx})\|_{L^2}^2) \\ &\leqslant \frac{1}{8}(\|u_{xxx}\|_{L^2}^2 + \|v_{xxxx}\|_{L^2}^2 + \|w_{xxxx}\|_{L^2}^2) + C(\|u_x\|_{L^2}^2 + \|(v_{xx}, w_{xx})\|_{L^2}^2).\end{aligned} \tag{3.2.26}$$

Combining (3.2.25) and (3.2.26) gives

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_I u_{xx}^2dx+\frac{1}{2}\int_I u_{xxx}^2dx\\&\leqslant\frac{1}{8}(\|u_{xxx}\|_{L^2}^2+\|v_{xxxx}\|_{L^2}^2+\|w_{xxxx}\|_{L^2}^2)+C(\|u_x\|_{L^2}^2+\|(v_{xx},w_{xx})\|_{L^2}^2).\end{aligned}\tag{3.2.27}$$

Differentiating the second and third equations of (3.2.3) with respect to x three times, one derives that

$$\begin{cases}v_{txxx}=v_{xxxxx}+\alpha u_{xxx}-\beta v_{xxx},\\ w_{txx}=w_{xxxxx}+\gamma u_{xxx}-\delta w_{xxx}.\end{cases}\tag{3.2.28}$$

Multiplying the first equation of (3.2.28) by v_{xxx}, the second by w_{xxx}, and adding them, integrating the results yields that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_I(v_{xxx}^2+w_{xxx}^2)dx+\int_I(v_{xxxx}^2+w_{xxxx}^2)dx+\beta\int_I v_{xxx}^2dx+\delta\int_I w_{xxx}^2dx\\&=-\alpha\int_I u_{xx}v_{xxxx}dx-\gamma\int_I u_{xx}w_{xxxx}dx\\&\leqslant\frac{1}{2}\int_I(v_{xxxx}^2+w_{xxxx}^2)dx+\frac{\alpha^2+\gamma^2}{2}\int_I u_{xx}^2dx,\end{aligned}\tag{3.2.29}$$

where we have used the facts $v_{xxx}=w_{xxx}=0$ on ∂I which can be derive from (3.2.21) and the boundary conditions $u_x=v_x=w_x=0$ on ∂I. Furthermore (2.2.8) with $n=1$ and (3.2.8) entail that

$$\int_I u_{xx}^2dx\leqslant C(\|u_{xxx}\|_{L^2}\|u_x\|_{L^2}+\|u_x\|_{L^2}^2)\leqslant\frac{1}{4(\alpha^2+\gamma^2+2)}\|u_{xxx}\|_{L^2}^2+C\|u_x\|_{L^2}^2.\tag{3.2.30}$$

Jointing (3.2.27), (3.2.29) and (3.2.30) yields that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_I(u_{xx}^2+v_{xxx}^2+w_{xxx}^2)dx+\frac{1}{2}\int_I(u_{xxx}^2+v_{xxxx}^2+w_{xxxx}^2)dx\\&\quad+\int_I(u_{xx}^2+\beta v_{xxx}^2+\delta w_{xxx}^2)dx\\&\leqslant\frac{1}{4}\|u_{xxx}\|_{L^2}^2+\frac{1}{8}\|(v_{xxxx},w_{xxxx})\|_{L^2}^2+C(\|u_x\|_{L^2}^2+\|(v_{xx},w_{xx})\|_{L^2}^2),\end{aligned}$$

which implies

$$\frac{d}{dt}\int_I (u_{xx}^2 + v_{xxx}^2 + w_{xxx}^2)dx + \int_I (u_{xxx}^2 + v_{xxxx}^2 + w_{xxxx}^2)dx$$
$$+ \int_I (u_{xx}^2 + \beta v_{xxx}^2 + \delta w_{xxx}^2)dx \tag{3.2.31}$$
$$\leqslant C(\|u_x\|_{L^2}^2 + \|(v_{xx}, w_{xx})\|_{L^2}^2).$$

Then the application of Lemma 3.3 and the Gronwall's inequality to (3.2.31) gives

$$\|u_{xx}(t)\|_{L^2}^2 + \|(v_{xxx}, w_{xxx})(t)\|_{L^2}^2 \leqslant C(\|u_0\|_{H^2}^2 + \|(v_0, w_0)\|_{H^3}^2).$$

Thus the proof of Lemma 3.4 is completed. □

Next, we derive the estimates of $\|u(t)\|_{H^2}$ and $\|(v,w)(t)\|_{H^3}$ for $(u_0, v_0, w_0) \in H^1(I) \times H^2(I) \times H^2(I)$.

Proposition 3.2. *Let $u_0 \in H^1(I)$, $(v_0, w_0) \in [H^2(I)]^2$ and (u, v, w) be the global solution obtained in Theorem* 3.1. *Then we have the following estimate*

$$\|u(t)\|_{H^2}^2 + \|(v,w)(t)\|_{H^3}^2 \leqslant C\left(\frac{1}{t} + \|u_0\|_{H^1}^2 + \|(v_0, w_0)\|_{H^2}^2\right). \tag{3.2.32}$$

Proof. Using (3.2.13) and integrating (3.2.31) in the interval $[s,t]$, we have

$$\|u_{xx}(t)\|_{L^2}^2 + \|(v_{xxx}, w_{xxx})(t)\|_{L^2}^2$$
$$\leqslant C\left(\|u_{xx}(s)\|_{L^2}^2 + \|(v_{xxx}, w_{xxx})(s)\|_{L^2}^2 + \|u_0\|_{H^1}^2 + \|(v_0, w_0)\|_{H^2}^2\right), \ 0 < s < t. \tag{3.2.33}$$

Furthermore the integration of (3.2.33) with respect to s over $(0,t)$ gives

$$t\left(\|u_{xx}(t)\|_{L^2}^2 + \|(v_{xxx}, w_{xxx})(t)\|_{L^2}^2\right)$$
$$\leqslant C\int_0^t (\|u_{xx}(s)\|_{L^2}^2 + \|(v_{xxx}, w_{xxx})(s)\|_{L^2}^2)ds + Ct\left(\|u_0\|_{H^1}^2 + \|(v_0, w_0)\|_{H^2}^2\right), \ 0 < s < t. \tag{3.2.34}$$

Applying (3.2.14) to (3.2.34) yields that

$$\|u_{xx}(t)\|_{L^2}^2 + \|(v_{xxx}, w_{xxx})(t)\|_{L^2}^2 \leqslant C\left(\frac{1}{t}+1\right)(\|u_0\|_{H^1}^2 + \|(v_0, w_0)\|_{H^2}^2). \quad (3.2.35)$$

The combination of Lemma 3.2, Lemma 3.3 and (3.2.35) gives

$$\begin{aligned}
&\|u(t)\|_{H^2}^2 + \|(v,w)(t)\|_{H^3}^2 \\
&\leqslant C\left(\frac{1}{t}+1\right)(\|u_0\|_{H^1}^2 + \|(v_0, w_0)\|_{H^2}^2) + C(\|u_0\|_{H^1}^2 + \|(v_0, w_0)\|_{H^2}^2) \\
&\leqslant C\left(\frac{1}{t}+1\right)(\|u_0\|_{H^1}^2 + \|(v_0, w_0)\|_{H^2}^2).
\end{aligned}$$

Then we complete the proof of Proposition 3.2. □

As a consequence of Proposition 3.2, we have the following result.

Proposition 3.3. *For each bounded ball* $B_r = \{(u_0, v_0, w_0) \in \mathscr{X} \big| \|u_0\|_{H^1}^2 + \|(v_0, w_0)\|_{H^2}^2 \leqslant r\}$*, there exists a time* t_r *depending on* B_r *such that for any* $U_0 \in B_r$*, it has that*

$$\sup_{t \geqslant t_r} \sup_{U_0 \in B_r} \|S(t)U_0\|_{H^2 \times H^3 \times H^3} \leqslant C,$$

where $C > 0$ *is a constant.*

Next, we are devoted to proving Theorem 3.2. First we present a result in [87].

Lemma 3.5 ([87])**.** *Assume that for some subset* $\mathscr{B} \subset \mathscr{X}, \mathscr{B} \neq \varnothing$*, and for some* $t_0 > 0$*, the set* $\cup_{t \geqslant t_0} S(t)\mathscr{B}$ *is relatively compact in* $\mathscr{X}$*. Then* $\omega(\mathscr{B})$ *is nonempty, compact and invariant.*

We are now in a position to prove Theorem 3.2.

Proof of Theorem 3.2. Define the set

$$\mathscr{B} = \{(u, v, w) \in H^2(I) \times H^3(I) \times H^3(I) \big| \|u(t)\|_{H^2}^2 + \|(v, w)(t)\|_{H^3}^2 \leqslant C\} \cap \mathscr{X},$$

where C is a constant appearing in Proposition 3.3. By the Sobolev imbedding theorem, it follows that $\mathscr{B}$ is a compact subset of $\mathscr{X}$. From Proposition 3.3, we know that for any bounded subset $B_r \subset \mathscr{X}$, there is a time t_r such that $\bigcup_{t \geqslant t_r} S(t)B_r \subset \mathscr{B}$. Hence $\mathscr{B}$ is a compact absorbing set for $(\{S(t)_{t\geqslant 0}\}, \mathscr{X})$. Using [87, Theorem 1.1], we conclude that $\mathscr{A} = \omega(\mathscr{B})$ is a global attractor of the dynamical system $(\{S(t)_{t\geqslant 0}\}, \mathscr{X})$. By Lemma 3.5, this global attractor is nonempty, compact and invariant in $\mathscr{X}$. Then the proof of Theorem 3.2 is completed.

3.2.3 Convergence to Stationary Solution

In this subsection, we explore the asymptotical behavior of solution for a special case $\beta = \delta$. First noticing that the integration of the first equation of (3.2.3) in x entails that the cell preserves the mass:

$$\|u(t)\|_{L^1} = \|u_0\|_{L^1} =: M \tag{3.2.36}$$

where $M > 0$ is a prescribed constant denoting the cell mass. Therefore the stationary solution $(U, V, W)(x)$ of (3.2.3) satisfies

$$\begin{cases} 0 = U_{xx} - (\chi U V_x)_x + (\xi U W_x)_x, & x \in I, \\ 0 = V_{xx} + \alpha U - \beta V, & x \in I, \\ 0 = W_{xx} + \gamma U - \delta W, & x \in I, \\ U_x = V_x = W_x = 0, & x \in \partial I, \\ \int_I U(x)dx = M, & x \in I. \end{cases} \tag{3.2.37}$$

When $\beta = \delta$ and $\xi\gamma - \chi\alpha \geqslant 0$ (i.e., repulsion dominates), the results of [85, Proposition 2.3 and Proposition 2.4] showed that (3.2.37) has a unique constant solution $(\overline{u}_0, \frac{\alpha}{\beta}\overline{u}_0, \frac{\gamma}{\beta}\overline{u}_0)$ where $\overline{u}_0 := M/|I|$, and the solution of (3.2.3) approaches this constant solution exponentially as time goes to infinity in two dimension. When $\beta = \delta$ and $\xi\gamma - \chi\alpha < 0$ (i.e., attraction dominates), the existence of non-constant solution (U, V, W) has been established in [85, Proposition 2.3], whereas the asymptotical behavior of the solution to (3.2.3) has not been obtained for this case. In this subsection, we shall explore this question and show that the solution of (3.2.3) converges to a solution of (3.2.37) algebraically as time tends to infinity in one dimension.

Theorem 3.3. *Let $u_0 \in H^1(I), (v_0, w_0) \in [H^2(I)]^2$. If $\beta = \delta$ and $\xi\gamma - \chi\alpha < 0$, then the global solution (u, v, w) of (3.2.3) converges to a stationary solution $(U(x), V(x), W(x))$ in $[H^1(I)]^3$ as time tends to infinity. Moreover, there exist a $\theta \in (0, \frac{1}{2})$ and a positive constant C such that for all $t \geqslant 0$, it holds that*

$$\|u(x,t) - U(x)\|_{H^1} + \|v(x,t) - V(x)\|_{H^1} + \|w(x,t) - W(x)\|_{H^1} \leqslant C(1+t)^{-\theta/(1-2\theta)}.$$

Next, we are devoted to proving Theorem 3.3. If $\beta = \delta$ and $\xi\gamma - \chi\alpha < 0$, we set

$$s := \chi v - \xi w. \tag{3.2.38}$$

Substituting (3.2.38) into (3.2.3), we have

$$\begin{cases} u_t = u_{xx} - (us_x)_x, & x \in I, t > 0, \\ s_t = s_{xx} + (\chi\alpha - \xi\gamma)u - \beta s, & x \in I, t > 0, \\ u_x = s_x = 0, & x \in \partial I, t > 0, \\ u(x,0) = u_0(x), s(x,0) = \chi v_0(x) - \xi w_0(x) := s_0(x), & x \in I. \end{cases} \tag{3.2.39}$$

Due to the conservation of cell mass (3.2.36), the corresponding stationary problem of system (3.2.39) is

$$\begin{cases} 0 = U_{xx} - (US_x)_x, & x \in I, \\ 0 = S_{xx} + (\chi\alpha - \xi\gamma)U - \beta S, & x \in I, \\ U_x = S_x = 0, & x \in \partial I, \\ \int_I U(x) = M, & x \in I. \end{cases} \tag{3.2.40}$$

Notice that the non-constant stationary steady state solution (U, S) of (3.2.40) have been established in [85, Proposition 2.3] when $\xi\gamma - \chi\alpha < 0$. By Theorem 3.1 and the Minkowski inequality, we have the following estimates on the solution of (3.2.39).

Lemma 3.6. *Let* $u_0 \in H^1(I), (v_0, w_0) \in [H^2(I)]^2$. *Then problem* (3.2.39) *has a global classical solution* $(u, s) \in C^0(\bar{I} \times [0, \infty); \mathbb{R}^2) \cap C^{2,1}(\bar{I} \times (0, \infty); \mathbb{R}^2)$ *such that*

$$\|u(t)\|_{H^1}^2 + \|s(t)\|_{H^2}^2 \leqslant C. \tag{3.2.41}$$

It is well-known (e.g., see [37]) that if $s \geqslant 0$, the system (3.2.39) has a Lyapunov functional

$$E(u, s) = \int_I \left\{ u \ln u + \frac{1}{2(\chi\alpha - \xi\gamma)}(s_x^2 + \beta s^2) - us \right\} dx \tag{3.2.42}$$

satisfying

$$\frac{d}{dt}E(u(t), s(t)) + \int_I u\left[(\ln u - s)_x\right]^2 dx + \frac{1}{\chi\alpha - \xi\gamma}\int_I s_t^2 dx = 0. \tag{3.2.43}$$

However we should underline that the initial condition $s_0(x) = \chi v_0(x) - \xi w_0(x)$ may be negative in principle and hence the non-negativity of the solution component s can not be guaranteed. Fortunately the second and third terms of (3.2.43) do not depend on the sign of s and hence (3.2.42) is still a Lyapunov functional of (3.2.39) for any $s \in \mathbb{R}$. However the sign of s will affect the lower bound of the Lyapunov functional. Since in one dimension, the solution (u, s) is uniformly bounded in time, we can easily find a lower bound for the Lyapunov functional (3.2.42) as given below.

Lemma 3.7. *For* $(u_0, s_0) \in H^1(I) \times H^2(I)$, *the Lyapunov functional (3.2.42) satisfies*

$$E(u, s) \geqslant -C - \frac{|I|}{e} \quad \text{for any } t > 0$$

where C *is a positive constant.*

Proof. Employing (3.2.36), (3.2.41) and Sobolev embedding $H^1 \hookrightarrow L^\infty$, we have

$$\int_I usdx \leqslant \|s\|_{L^\infty} \|u_0\|_{L^1} \leqslant C. \tag{3.2.44}$$

Substituting (3.2.44) into (3.2.42), and using $u \ln u \geqslant -\frac{1}{e}$ for all $u > 0$, we obtain that

$$E(u, s) \geqslant -C - \frac{|I|}{e} \quad \text{for any } t > 0$$

which completes the proof. □

If (u, s) is a global classical solution of (3.2.39), we introduce the ω-limit set

$$\omega[u, s] := \left\{ (U, S) | \ \exists (t_n) \uparrow \infty, \ \text{s.t.} \ \lim_{n \to \infty} (u, s)(t_n) = (U, S) \text{ in } C^1(\bar{I}) \right\}. \tag{3.2.45}$$

Then based on the Lyapunov functional and the LaSalle invariant principle, it can be concluded (see also [23, Eq. (3.23)]) that

$$\omega[u, s] := \{(U, S) | (U, S) \text{ solves } (3.2.40)\} \tag{3.2.46}$$

and there exists E_∞ such that for any stationary solution $(U, S) \in \omega[u, s]$, there holds

$$E(U, S) = E_\infty = \inf_{t > 0} E(u(t), s(t)) = \lim_{t \to \infty} E(u(t), s(t)). \tag{3.2.47}$$

Furthermore we can solve the first equation of (3.2.40) and obtain that

$$U(x) = \lambda e^{S(x)}$$

with λ being a positive constant. Hence we have

$$\inf_{x \in I} U(x) \geqslant \lambda > 0 \text{ for all } (U, S) \in \omega[u, s].$$

Thanks to (3.2.45), (3.2.46) and (3.2.47), we may assume without loss of generality that

$$\inf_{x \in I} u(x, t) \geqslant \lambda > 0 \text{ for all } t > 0.$$

Using the results of [23, section 5], we have the following result.

Lemma 3.8. *Let $(u(t), s(t))$ be a solution of system (3.2.39) and $(U, S) \in \omega[u, s]$ be a stationary solution of (3.2.39). Then there exists a constant $C_0 > 0$ such that for some $t \geqslant t_*$, it holds that*

$$E(u, s) - E_\infty \leqslant C_0 \left\{ \int_I \left(u\left[(\ln u - s)_x\right]^2 + \frac{1}{\chi\alpha - \xi\gamma} s_t^2 \right) dx \right\}^{\frac{1}{2(1-\theta)}}, \qquad (3.2.48)$$

where $\theta \in (0, \frac{1}{2})$ and t_* is a time such that when $t \geqslant t_*$ one has

$$\|u(t) - U\|_{L^2} + \|s(t) - S\|_{H^1} < \varepsilon.$$

Proof of Theorem 3.3. The convergence of the solution (u, s) to (U, S) follows from the results of [23] directly. Next we derive the convergence rate announced in the Theorem 3.3 based on an idea of [44, 43, 107]. First note that

$$E(u(t), s(t)) - E_\infty = \int_t^\infty \left\{ \int_I \left(u\left[(\ln u - s)_x\right]^2 + \frac{1}{\chi\alpha - \xi\gamma} s_t^2 \right) dx \right\} d\tau \geqslant 0.$$

Combining (3.2.43) and (3.2.48), we have for any $t \geqslant t_*$

$$\frac{d}{dt}(E(u(t), s(t)) - E_\infty) + \frac{1}{C_0^{2(1-\theta)}}(E(u(t), s(t)) - E_\infty)^{2(1-\theta)} \leqslant 0, \qquad (3.2.49)$$

and

$$\frac{d}{dt}(E(u(t), s(t)) - E_\infty)^\theta + \frac{\theta}{C_0^{1-\theta}} \left\{ \int_I \left(u\left[(\ln u - s)_x\right]^2 + \frac{1}{\chi\alpha - \xi\gamma} s_t^2 \right) dx \right\}^{1/2} \leqslant 0. \qquad (3.2.50)$$

The inequality (3.2.49) entails that

$$E(u(t), s(t)) - E_\infty \leqslant C(1 + t)^{-1/(1-2\theta)} \text{ for all } t > 0, \qquad (3.2.51)$$

for some constant $C > 0$, where we should point out that for $t < t_*$ the term on the left hand of (3.2.51) is bounded by a constant depending only on initial data.

Then integrating the inequality (3.2.50) with respect to time in (t, ∞) leads to

$$\frac{\theta}{C_0^{1-\theta}} \int_t^\infty \left\{ \int_I \left(u \left[(\ln u - s)_x \right]^2 + \frac{1}{\chi\alpha - \xi\gamma} s_t^2 \right) dx \right\}^{1/2} d\tau \leqslant (E(u(t), s(t)) - E_\infty)^\theta$$
$$\leqslant C(1+t)^{-\theta/(1-2\theta)}. \tag{3.2.52}$$

From the first equation of (3.2.39) we have $\langle u_t, h \rangle = -\langle u(\ln u - s)_x, h_x \rangle$, where $\langle f, g \rangle = \int_I fg dx$, which implies

$$\|u_t\|_{(H^1)'} \leqslant \left(\int_I u^2 \left[(\ln u - s)_x \right]^2 dx \right)^{\frac{1}{2}} \leqslant C \left(\int_I u \left[(\ln u - s)_x \right]^2 dx \right)^{\frac{1}{2}}, \tag{3.2.53}$$

where $(H^1)'$ denotes the dual of H^1 and we have used $\|u\|_{L^\infty} \leqslant C$. Hence the inequalities (3.2.52) and (3.2.53) entail that

$$\|s(t) - S\|_{L^2} \leqslant \int_t^\infty \|s_t\|_{L^2} \, d\tau \leqslant C(1+t)^{-\theta/(1-2\theta)}, \tag{3.2.54}$$

and

$$\|u(t) - U\|_{(H^1)'} \leqslant \int_t^\infty \|u_t\|_{(H^1)'} \, d\tau \leqslant C(1+t)^{-\theta/(1-2\theta)}. \tag{3.2.55}$$

Define

$$A_0 f := -\frac{d^2 f}{dx^2} \text{ for } f \in D(A) \cap H_0,$$

where $D(A) = \{f(x) | f \in W^{2,2}(I), f_x|_{\partial I} = 0\}$ and $H_0 = \left\{ f(x) | f \in L^2(I), \int_I f(x) dx = 0 \right\}$. Noting that A_0 is a positive linear operator, for any $r \in \mathbb{R}$, we can define its powers A_0^r (see [5, 4, 52] for details).

Letting $\phi = u - U$ and $\psi = s - S$, using (3.3.107) and (3.3.111), we have

$$\begin{cases} \phi_t = (\phi_x - u\psi_x - \phi S_x)_x, \\ \psi_t = \psi_{xx} + (\chi\alpha - \xi\gamma)\phi - \beta\psi. \end{cases} \tag{3.2.56}$$

Multiplying the first equation in (3.2.56) by $\phi, A_0^{-1}\phi$ and $A_0^{-1}\phi_t$ respectively, and integrating by parts, then applying the Young's inequality (3.2.8), we end up with

$$\frac{1}{2}\frac{d}{dt}\|\phi\|_{L^2}^2 + \|\phi_x\|_{L^2}^2 \leqslant \varepsilon\|\phi_x\|_{L^2}^2 + C_\varepsilon(\|\psi_x\|_{L^2}^2 + \|\phi\|_{L^2}^2), \tag{3.2.57}$$

$$\frac{1}{2}\frac{d}{dt}\|A_0^{-1/2}\phi\|_{L^2}^2 + \|\phi\|_{L^2}^2 \leqslant \varepsilon(\|\phi\|_{L^2}^2 + \|\psi_x\|_{L^2}^2) + C_\varepsilon\|\partial_x A_0^{-1}\phi\|_{L^2}^2, \tag{3.2.58}$$

and

$$\frac{1}{2}\frac{d}{dt}\|\phi\|_{L^2}^2 + \|A_0^{-1/2}\phi_t\|_{L^2}^2 \leqslant \varepsilon\|A_0^{-1/2}\phi_t\|_{L^2}^2 + C_\varepsilon(\|\psi_x\|_{L^2}^2 + \|\phi\|_{L^2}^2) \tag{3.2.59}$$

where the boundedness of u and S_x have been used. Integrating the second equation in (3.2.30) multiplied by ψ,$-\psi_{xx}$ and ψ_t respectively, we have by the Young's inequality (3.2.8)

$$\frac{1}{2}\frac{d}{dt}\|\psi\|_{L^2}^2 + \|\psi_x\|_{L^2}^2 + \beta\|\psi\|_{L^2}^2 = \langle(\chi\alpha - \xi\gamma)\phi, \psi\rangle \leqslant \varepsilon\|\psi\|_{H^1} + C_\varepsilon\|\phi\|_{(H^1)'}, \tag{3.2.60}$$

$$\frac{1}{2}\frac{d}{dt}\|\psi_x\|_{L^2}^2 + \|\psi_{xx}\|_{L^2}^2 + \beta\|\psi_x\|_{L^2}^2 \leqslant \varepsilon\|\psi_x\|_{L^2}^2 + C_\varepsilon\|\phi_x\|_{L^2}^2, \tag{3.2.61}$$

and

$$\frac{1}{2}\frac{d}{dt}(\|\psi_x\|_{L^2}^2 + \beta\|\psi\|_{L^2}^2) + \|\psi_t\|_{L^2}^2 \leqslant \varepsilon\|\psi_t\|_{L^2}^2 + C_\varepsilon\|\phi\|_{L^2}^2. \tag{3.2.62}$$

Differentiating (3.2.56) with respect to t and noticing that $\phi_t = u_t$, we have

$$\begin{cases} \phi_{tt} = \phi_{xxt} - (\phi_t\psi_x + u\psi_{xt} + \phi_t S_x)_x, \\ \psi_{tt} = \psi_{xxt} + (\chi\alpha - \xi\gamma)\phi_t - \beta\psi_t. \end{cases} \tag{3.2.63}$$

Multiplying the first equation in (3.2.63) by $A_0^{-1}\phi_t$ and noticing that $\partial_x A_0^{-1}\phi_t =$

$\phi_x - u\psi_x - \phi S_x = 0$ for $x \in \partial I$, we have

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|A_0^{-1/2}\phi_t\|_{L^2}^2 + \|\phi_t\|_{L^2}^2 &= \langle \phi_t\psi_x + u\psi_{xt} + \phi_t S_x, \partial_x A_0^{-1}\phi_t\rangle \\
&= \langle \phi_t\psi_x + \phi_t S_x, \partial_x A_0^{-1}\phi_t\rangle + \langle u\psi_{xt}, \partial_x A_0^{-1}\phi_t\rangle \\
&\leqslant \varepsilon\,\|\phi_t\|_{L^2}^2 + C_\varepsilon\|\partial_x A_0^{-1}\phi_t\|_{L^2}^2 - \langle u\psi_t, \phi_t\rangle - \langle u_x\psi_t, \partial_x A_0^{-1}\phi_t\rangle \\
&\leqslant \varepsilon\,\|\phi_t\|_{L^2}^2 + C_\varepsilon(\|\psi_t\|_{L^2}^2 + \|\partial_x A_0^{-1}\phi_t\|_{L^2}^2)
\end{aligned} \tag{3.2.64}
$$

where the boundedness of ψ_x, S_x, u and u_x has been used. We multiply the second equation in (3.2.63) by ψ_t and integrate over I to obtain

$$
\frac{1}{2}\frac{d}{dt}\,\|\psi_t\|_{L^2}^2 + \|\psi_{xt}\|_{L^2}^2 + \beta\,\|\psi_t\|_{L^2}^2 \leqslant \varepsilon\,\|\psi_t\|_{L^2}^2 + C_\varepsilon\,\|\phi_t\|_{L^2}^2\,. \tag{3.2.65}
$$

Set $y(t) = \|A_0^{-1/2}\phi\|_{L^2}^2 + \|\phi\|_{L^2}^2 + \|A_0^{-1/2}\phi_t\|_{L^2}^2 + \|\psi_x\|_{L^2}^2 + \|\psi\|_{L^2}^2 + \|\psi_t\|_{L^2}^2$. Noting $\|\partial_x A_0^{-1}\phi\|_{L^2}^2 = \|A_0^{-1/2}\phi\|_{L^2}^2 \leqslant C\,\|\phi\|_{(H^1)'}^2$ and letting ε small, we deduce from inequalities (3.2.57)-(3.2.62) and (3.2.64)-(3.2.65) that

$$
\frac{d}{dt}y(t) + ky(t) \leqslant C(\|\phi\|_{(H^1)'}^2 + \|\psi\|_{L^2}^2) \leqslant C(1+t)^{-2\theta/(1-2\theta)},
$$

for some $k > 0$, where (3.2.54) and (3.2.55) have been used. Solving above inequality yields that

$$
y(t) \leqslant y(0)e^{-kt} + Ce^{-kt}\int_0^t e^{ks}(1+s)^{-2\theta/(1-2\theta)}ds. \tag{3.2.66}
$$

Note that the last term of (3.2.66) can be estimated as

$$
\begin{aligned}
\int_0^t e^{ks}(1+s)^{-2\theta/(1-2\theta)}ds &= \int_0^{t/2} e^{ks}(1+s)^{-2\theta/(1-2\theta)}ds + \int_{t/2}^t e^{ks}(1+s)^{-2\theta/(1-2\theta)}ds \\
&\leqslant e^{\frac{k}{2}t}\int_0^{t/2}(1+s)^{-2\theta/(1-2\theta)}ds + (1+t/2)^{-2\theta/(1-2\theta)}\int_{t/2}^t e^{ks}ds \\
&\leqslant Ce^{\frac{k}{2}t}(1+t/2)^{-2\theta/(1-2\theta)+1} + \frac{1}{k}e^{kt}(1+t/2)^{-2\theta/(1-2\theta)}.
\end{aligned}
$$

Then substituting above inequality into (3.2.66) gives rise to

$$y(t) \leqslant C(1+t)^{-2\theta/(1-2\theta)} \tag{3.2.67}$$

which implies that

$$\|u(t)-U\|_{L^2} + \|s(t)-S\|_{H^1} \leqslant C(1+t)^{-\theta/(1-2\theta)}. \tag{3.2.68}$$

To derive the decay rates for v and w, we subtract (3.2.40) from (3.2.39) and obtain that

$$\begin{cases} (v-V)_t = (v-V)_{xx} + \alpha(u-U) - \beta(v-V), \\ (w-W)_t = (w-W)_{xx} + \gamma(u-U) - \delta(w-W). \end{cases} \tag{3.2.69}$$

By the Duhamel principle, $v(t)-V$ can be represented in the form of

$$v(t)-V = e^{-t(A+\beta)}(v_0-V) + \alpha\int_0^t e^{-(t-s)(A+\beta)}(u(s)-U)ds. \tag{3.2.70}$$

Noting that $\int_I (u(s)-U)ds = 0$ which allows us to use the inequality $\|e^{-tA}f\|_{L^p} \leqslant C\|f\|_{L^p}$ for any $f \in L^p$ such that $\int_I f dx = 0$, we have

$$\begin{aligned} \|v(t)-V\|_{L^2} &\leqslant Ce^{-\beta t} + C\int_0^t e^{-\beta(t-s)} \|u(s)-U\|_{L^2} ds \\ &\leqslant Ce^{-\beta t} + C\int_0^t e^{-\beta(t-s)}(1+s)^{-\theta/(1-2\theta)} ds \\ &\leqslant C\left(e^{-\beta t} + \int_0^t e^{-\beta s}(1+t-s)^{-\theta/(1-2\theta)} ds\right) \\ &\leqslant C(1+t)^{-\theta/(1-2\theta)}, \end{aligned} \tag{3.2.71}$$

where we have used the inequality

$$\int_0^t (1+t-s)^{-\kappa} e^{-\rho s} ds \leqslant C(1+t)^{-\kappa} \text{ for any } \kappa, \rho > 0$$

which was proved in [94, Lemma 4.4]. Similarly, we can prove that the convergence of w satisfies

$$\|w(t)-W\|_{L^2} \leqslant C(1+t)^{-\theta/(1-2\theta)}.$$

By the first equation of (3.2.56) and the classic elliptic regularity theory, using (3.2.67) we have

$$\begin{aligned}\|u_x - U_x\|_{L^2}^2 = \|\phi_x\|_{L^2}^2 &= \|A_0^{1/2}\phi\|_{L^2}^2 \\ &\leqslant C\left(\|A_0^{-1/2}\phi_t\|_{L^2}^2 + \|A_0^{-1/2}(u\psi_x + \phi S_x)_x\|_{L^2}^2\right) \\ &\leqslant C\left(\|A_0^{-1/2}\phi_t\|_{L^2}^2 + \|u\psi_x + \phi S_x\|_{L^2}^2\right) \\ &\leqslant Cy(t) \leqslant C(1+t)^{-2\theta/(1-2\theta)}.\end{aligned} \tag{3.2.72}$$

The combination of (3.2.68) and (3.2.72) gives that

$$\|u(t) - U\|_{H^1} \leqslant C(1+t)^{-\theta/(1-2\theta)}. \tag{3.2.73}$$

Letting $\varphi = v_x - V_x$ and using the first equation of (3.2.70), one has

$$\varphi_t = \varphi_{xx} + \alpha(u_x - U_x) - \beta\varphi.$$

Applying the same procedure to φ as was done to $v - V$, and using (3.2.73), we have a $K > 0$ such that

$$\begin{aligned}\|\varphi\|_{L^2} &\leqslant Ce^{-Kt} + C\int_0^t e^{-K(t-s)}\|u_x(s) - U_x\|_{L^2}\, ds \\ &\leqslant Ce^{-Kt} + C\int_0^t e^{-K(t-s)}(1+s)^{-\theta/(1-2\theta)} ds \\ &\leqslant C(1+t)^{-\theta/(1-2\theta)}.\end{aligned} \tag{3.2.74}$$

Collecting (3.2.71) and (3.2.74), one has

$$\|v - V\|_{H^1} \leqslant C(1+t)^{-\theta/(1-2\theta)}.$$

Applying the same procedure to w, we have

$$\|w - W\|_{H^1} \leqslant C(1+t)^{-\theta/(1-2\theta)}.$$

Thus the proof of Theorem 3.3 is completed.

3.3 Boundedness of Solutions in Two Dimensions

3.3.1 Case 1: $\tau_1 = \tau_2 = 1$

When $\tau_1 = \tau_2 = 1$, $\beta \neq \delta$ and $n = 2$, the global existence of solution with small initial mass has been proved in [85] if repulsion dominates over attraction. In this subsection, we obtain the global classical solution with uniform-in-time bound for the large initial mass, which substantially improve the results in [85].

Theorem 3.4. *Assume that* $0 \leqslant (u_0, v_0, w_0) \in [W^{1,\infty}(\Omega)]^3$ *with* $u_0 \not\equiv 0$ *and* $\xi\gamma \geqslant \chi\alpha$. *The the ARKS model (3.1.1) with* $\tau_1 = \tau_2 = 1$ *has a unique nonnegative classical solution* $(u, v, w) \in C(\bar{\Omega} \times [0, \infty); \mathbb{R}^3) \cap C^{2,1}(\bar{\Omega} \times (0, \infty); \mathbb{R}^3)$ *such that* $u > 0$ *in* $\Omega \times (0, \infty)$. *Moreover there is a constant* C *independent of time* t *such that* $\|u(\cdot, t)\|_{L^\infty} \leqslant C$.

Next, we are going to prove Theorem 3.4. To estimate the cross-diffusive terms in system (3.1.1), we use the transformation $s = \xi w - \chi v$ such that (3.1.1) can be transformed into the following system

$$\begin{cases} u_t = \Delta u + \nabla \cdot (u \nabla s), & x \in \Omega, t > 0, \\ s_t = \Delta s - \delta s + (\xi\gamma - \chi\alpha)u + \chi(\beta - \delta)v, & x \in \Omega, t > 0, \\ v_t = \Delta v + \alpha u - \beta v, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial s}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x), s(x, 0) = \xi w_0(x) - \chi v_0(x) =: s_0(x), v(x, 0) = v_0(x), & x \in \Omega. \end{cases} \tag{3.3.75}$$

Lemma 3.9. *Assume the conditions in Theorem* 3.4 *are satisfied. Then there exists* $C > 0$ *independent of* T *such that the solution of* (3.3.75) *satisfies*

$$\|\nabla s\|_{L^\infty} \leqslant C \quad \text{if} \quad \xi\gamma = \chi\alpha, \tag{3.3.76}$$

and

$$\|u \ln u\|_{L^1} + \|\nabla s\|_{L^2} \leqslant C \quad \text{if} \quad \xi\gamma > \chi\alpha. \tag{3.3.77}$$

Proof. Using Lemma 2.1 and $u \in L^1(\Omega)$, we have

$$\|v\|_{W^{1,p}} \leqslant c_1, \quad \text{for all } 1 \leqslant p < 2. \tag{3.3.78}$$

Choosing $p = \frac{6}{5}$ in (3.3.78) and using Lemma 2.2, we have

$$\|v\|_{L^3} \leqslant c_2 \|v\|_{W^{1,\frac{6}{5}}} \leqslant c_1 c_2. \tag{3.3.79}$$

If $\xi\gamma = \chi\alpha$, the second equation of (3.3.75) becomes $s_t = \Delta s - \delta s + \chi(\beta - \delta)v$, then using Lemma 2.1 and (3.3.79), we can deduce (3.3.76) by noting that $\|\nabla s\|_{L^\infty} \leqslant \|s\|_{W^{1,\infty}} \leqslant c_3$.

Next, we consider the case $\xi\gamma > \chi\alpha$. Multiplying the first equation of (3.3.75) by $\ln u$ and integrating with respect to x over Ω yields that

$$\frac{d}{dt}\int_\Omega u \ln u dx + \int_\Omega \frac{|\nabla u|^2}{u} dx = -\int_\Omega \nabla u \cdot \nabla s dx. \tag{3.3.80}$$

Multiplying the second equation of (3.3.75) by $-\frac{1}{\xi\gamma - \chi\alpha}\Delta s$ and integrating over Ω, then we have

$$\begin{aligned}
&\frac{1}{2(\xi\gamma - \chi\alpha)}\frac{d}{dt}\int_\Omega |\nabla s|^2 dx + \frac{1}{\xi\gamma - \chi\alpha}\int_\Omega |\Delta s|^2 dx + \frac{\delta}{\xi\gamma - \chi\alpha}\int_\Omega |\nabla s|^2 dx \\
&= \int_\Omega \nabla u \cdot \nabla s dx - \frac{\chi(\beta - \delta)}{\xi\gamma - \chi\alpha}\int_\Omega v \Delta s dx \\
&\leqslant \int_\Omega \nabla u \cdot \nabla s dx + \frac{1}{2(\xi\gamma - \chi\alpha)}\int_\Omega |\Delta s|^2 dx + \frac{\chi^2(\beta - \delta)^2}{2(\xi\gamma - \chi\alpha)}\int_\Omega v^2 dx.
\end{aligned} \tag{3.3.81}$$

Furthermore, using the Hölder's inequality and (3.3.79) we have

$$\int_\Omega v^2 dx \leqslant \left(\int_\Omega v^3 dx\right)^{\frac{2}{3}} |\Omega|^{\frac{1}{3}} \leqslant c_4. \tag{3.3.82}$$

The combination of (3.3.80), (3.3.81) and (3.3.82) entails

$$\begin{aligned}
&\frac{d}{dt}\left\{\int_\Omega u \ln u dx + \frac{1}{2(\xi\gamma - \chi\alpha)}\int_\Omega |\nabla s|^2 dx\right\} + \int_\Omega \frac{|\nabla u|^2}{u} dx \\
&+ \frac{1}{2(\xi\gamma - \chi\alpha)}\int_\Omega |\Delta s|^2 dx + \frac{\delta}{\xi\gamma - \chi\alpha}\int_\Omega |\nabla s|^2 dx \leqslant \frac{c_4\chi^2(\beta - \delta)^2}{2(\xi\gamma - \chi\alpha)}.
\end{aligned} \tag{3.3.83}$$

Using the Gagliardo-Nirenberg inequality and $\|u^{\frac{1}{2}}\|_{L^2} = \|u\|_{L^1}^{\frac{1}{2}} = \|u_0\|_{L^1}^{\frac{1}{2}}$, we have

$$\begin{aligned}\int_\Omega u\ln u dx \leqslant \int_\Omega u^{\frac{3}{2}}dx + c_5 = \|u^{\frac{1}{2}}\|_{L^3}^3 + \|u_0\|_{L^1}\\ \leqslant c_6\left(\|\nabla u^{\frac{1}{2}}\|_{L^2}\|u^{\frac{1}{2}}\|_{L^2}^2 + \|u^{\frac{1}{2}}\|_{L^2}^3\right) + c_5\\ \leqslant \frac{1}{4}\|\nabla u^{\frac{1}{2}}\|_{L^2}^2 + c_6^2\|u_0\|_{L^1}^2 + c_6\|u_0\|_{L^1}^{\frac{3}{2}} + c_5\\ = \int_\Omega \frac{|\nabla u|^2}{u}dx + c_7,\end{aligned} \tag{3.3.84}$$

where $c_7 = c_6^2\|u_0\|_{L^1}^2 + c_6\|u_0\|_{L^1}^{\frac{3}{2}} + c_5$. Substituting (3.3.84) into (3.3.83) and letting $c_8 = c_6 + \frac{c_4\chi^2(\beta-\delta)^2}{2(\xi\gamma-\chi\alpha)}$, we have

$$\frac{d}{dt}\left\{\int_\Omega u\ln u dx + \frac{1}{2(\xi\gamma-\chi\alpha)}\int_\Omega |\nabla s|^2 dx\right\} + \int_\Omega u\ln u dx + \frac{\delta}{\xi\gamma-\chi\alpha}\int_\Omega |\nabla s|^2 dx \leqslant c_8. \tag{3.3.85}$$

Then applying the Gronwall's inequality to (3.3.85), we obtain (3.3.77). □

Lemma 3.10. *Assume the conditions in Theorem* 3.4 *are satisfied. Then there exists a constant* $C > 0$ *such that*

$$\|u\|_{L^2} \leqslant C. \tag{3.3.86}$$

Proof. First, we consider the case $\xi\gamma = \chi\alpha$. We multiply the first equation in (3.3.75) by u and apply the Young's inequality to find a constant c_1 such that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2dx + \int_\Omega |\nabla u|^2dx = -\int_\Omega u\nabla u\cdot\nabla s dx \leqslant \frac{1}{2}\int_\Omega |\nabla u|^2dx + c_1\int_\Omega u^2dx, \tag{3.3.87}$$

where the inequality (3.3.77) in Lemma 3.9 has been used. Using the Gagliardo-Nirenberg inequality, we have

$$\|u\|_{L^2} \leqslant c_2\left(\|\nabla u\|_{L^2}^{\frac{1}{2}}\|u\|_{L^1}^{\frac{1}{2}} + \|u\|_{L^1}\right), \tag{3.3.88}$$

which implies

$$2c_1\|u\|_{L^2}^2 \leqslant 2c_1c_2^2\left(\|\nabla u\|_{L^2}\|u\|_{L^1} + \|u\|_{L^1}^2\right) \leqslant \frac{1}{2}\|\nabla u\|_{L^2}^2 + c_3 \tag{3.3.89}$$

with $c_3 = 2c_1c_2^2(c_1c_2^2+1)\|u_0\|_{L^1}^2$. Adding (3.3.89) and (3.3.87), we have

$$\frac{d}{dt}\|u\|_{L^2}^2 + 2c_1\|u\|_{L^2}^2 \leqslant 2c_3. \tag{3.3.90}$$

Applying the Gronwall's inequality to (3.3.90), we have (3.3.86) for the case $\xi\gamma = \chi\alpha$. Next, we will show that (3.3.86) holds with $\xi\gamma > \chi\alpha$. Multiplying the first equation of (3.3.75) by u, integrating the result with respect to x over Ω, then we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2dx + \int_\Omega |\nabla u|^2dx = \int_\Omega u^2\Delta s dx \leqslant \|u\|_{L^3}^2\|\Delta s\|_{L^3}. \tag{3.3.91}$$

Using (3.3.77), (2.2.6) and the Gagliardo-Nirenberg inequality, we have

$$\begin{aligned}\|u\|_{L^3}^2\|\Delta s\|_{L^3} &\leqslant c_5\|u\|_{L^3}^2\left(\|\nabla\Delta s\|_{L^2}^{\frac{2}{3}}\|\nabla s\|_{L^2}^{\frac{1}{3}} + \|\nabla s\|_{L^2}\right)\\ &\leqslant c_6\left(\varepsilon\|\nabla u\|_{L^2}^2 + c_7\right)^{\frac{2}{3}}(\|\nabla\Delta s\|_{L^2}^{\frac{2}{3}}+1)\\ &\leqslant c_8(\varepsilon\|\nabla u\|_{L^2}^{\frac{4}{3}}+1)(\|\nabla\Delta s\|_{L^2}^{\frac{2}{3}}+1)\\ &\leqslant \frac{1}{2}\|\nabla u\|_{L^2}^2 + \varepsilon\|\nabla\Delta s\|_{L^2}^2 + c_9.\end{aligned} \tag{3.3.92}$$

Substituting (3.3.92) into (3.3.91), we have

$$\frac{d}{dt}\int_\Omega u^2dx + \int_\Omega|\nabla u|^2dx \leqslant 2\varepsilon\|\nabla\Delta s\|_{L^2}^2 + 2c_9. \tag{3.3.93}$$

We differentiate the second equation of (3.3.75) first and then multiply it by $-\nabla(\Delta s)$ and integrate the product in Ω to obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega|\Delta s|^2dx + \int_\Omega|\nabla\Delta s|^2dx + \delta\int_\Omega|\Delta s|^2dx\\ &= (\chi\alpha-\xi\gamma)\int_\Omega\nabla u\cdot\nabla\Delta s dx + \chi(\delta-\beta)\int_\Omega\nabla v\cdot\nabla\Delta s dx\\ &\leqslant \frac{1}{2}\int_\Omega|\nabla\Delta s|^2dx + (\chi\alpha-\xi\gamma)^2\int_\Omega|\nabla u|^2dx + \chi^2(\delta-\beta)^2\int_\Omega|\nabla v|^2dx.\end{aligned} \tag{3.3.94}$$

Multiplying the third equation of (3.3.75) by v and integrating the product in Ω, then we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega v^2dx + \int_\Omega |\nabla v|^2dx + \beta\int_\Omega v^2dx = \alpha\int_\Omega uvdx \leqslant \|u\|_{L^2}^2 + \frac{\alpha^2}{4}\|v\|_{L^2}^2$$

$$\leqslant \varepsilon\|\nabla u\|_{L^2}^2 + \frac{\alpha^2}{4}\|v\|_{L^2}^2 + c_{10}$$

$$= \varepsilon\|\nabla u\|_{L^2}^2 + c_{11}, \tag{3.3.95}$$

where we have used (3.3.82) and

$$\int_\Omega u^2dx \leqslant \varepsilon\|\nabla u\|_{L^2}^2 + c_{12}. \tag{3.3.96}$$

Multiplying (3.3.93) and (3.3.96) by $2(\chi\alpha-\xi\gamma)^2$, (3.3.95) by $\chi^2(\delta-\beta)^2$, and then adding them to (3.3.94) and taking ε sufficiently small, we have

$$\begin{aligned}&\frac{d}{dt}\left(2(\chi\alpha-\xi\gamma)^2\int_\Omega u^2dx + \frac{1}{2}\int_\Omega|\Delta s|^2 + \frac{\chi^2(\delta-\beta)^2}{2}\int_\Omega v^2dx\right)\\ &+2(\chi\alpha-\xi\gamma)^2\int_\Omega u^2dx + \delta\int_\Omega|\Delta s|^2 + \beta\chi^2(\delta-\beta)^2\int_\Omega v^2dx \leqslant c_{13}.\end{aligned} \tag{3.3.97}$$

Then using the Gronwall's inequality, we have (3.3.86). □

Lemma 3.11. *Assume the conditions in Theorem* 3.4 *are satisfied. Then there exists a constant* $C>0$ *such that*

$$\|\nabla v\|_{L^\infty} + \|\nabla w\|_{L^\infty} \leqslant C. \tag{3.3.98}$$

Proof. Multiplying the first equation of (3.1.1) by u^2 to get that

$$\frac{1}{3}\frac{d}{dt}\int_\Omega u^3dx + \frac{8}{9}\int_\Omega|\nabla u^{\frac{3}{2}}|^2dx = 2\chi\int_\Omega u^2\nabla u\cdot\nabla vdx - 2\xi\int_\Omega u^2\nabla u\cdot\nabla wdx$$

$$\leqslant \frac{4\chi}{3}\int_\Omega \left|u^{\frac{3}{2}}\nabla u^{\frac{3}{2}}\cdot\nabla v\right|dx + \frac{4\xi}{3}\int_\Omega\left|u^{\frac{3}{2}}\nabla u^{\frac{3}{2}}\cdot\nabla w\right|dx. \tag{3.3.99}$$

Using (3.3.86) and Lemma 2.1, we have $\|(\nabla v, \nabla w)\|_{L^4} \leqslant c_1$. Then, applying the Cauchy-Schwarz inequality and the Gagliardo-Nirenberg inequality to the right terms of (3.3.99) we have

$$\begin{aligned}
&\frac{4\chi}{3}\int_\Omega \left|u^{\frac{3}{2}}\nabla u^{\frac{3}{2}}\cdot\nabla v\right|dx + \frac{4\xi}{3}\int_\Omega \left|u^{\frac{3}{2}}\nabla u^{\frac{3}{2}}\cdot\nabla w\right|dx\\
&\leqslant \frac{1}{3}\|\nabla u^{\frac{3}{2}}\|_{L^2}^2 + 16\chi^2\|u^{\frac{3}{2}}\|_{L^4}^2\|\nabla v\|_{L^4}^2 + 8\xi^2\|u^{\frac{3}{2}}\|_{L^4}^2\|\nabla w\|_{L^4}^2\\
&= \frac{1}{3}\|\nabla u^{\frac{3}{2}}\|_{L^2}^2 + c_1\left(16\chi^2+8\xi^2\right)\|u^{\frac{3}{2}}\|_{L^4}^2\\
&\leqslant \frac{1}{3}\|\nabla u^{\frac{3}{2}}\|_{L^2}^2 + c_2\left(\|\nabla u^{\frac{3}{2}}\|_{L^2}^{\frac{4}{3}}\|u^{\frac{3}{2}}\|_{L^{\frac{4}{3}}}^{\frac{2}{3}} + \|u^{\frac{3}{2}}\|_{L^{\frac{4}{3}}}^2\right)\\
&\leqslant \frac{1}{3}\|\nabla u^{\frac{3}{2}}\|_{L^2}^2 + c_2c_3^{\frac{3}{2}}\|\nabla u^{\frac{3}{2}}\|_{L^2}^{\frac{4}{3}} + c_2c_3^2\\
&\leqslant \frac{5}{9}\|\nabla u^{\frac{3}{2}}\|_{L^2}^2 + c_4,
\end{aligned} \tag{3.3.100}$$

where we have used $\|u^{\frac{3}{2}}\|_{L^{\frac{4}{3}}} = \|u^2\|_{L^2}^{\frac{3}{4}} \leqslant c_3$. Substituting (3.3.100) into (3.3.99), we can derive that

$$\frac{d}{dt}\int_\Omega u^3dx + \int_\Omega |\nabla u^{\frac{3}{2}}|^2dx \leqslant 3c_4. \tag{3.3.101}$$

By the Gagliardo-Nirenberg inequality we two positive constant c_5 and c_6 which are depended on $\|u\|_{L^2}$ such that

$$\|u^{\frac{3}{2}}\|_{L^2}^6 \leqslant \frac{1}{c_5}\|\nabla u^{\frac{3}{2}}\|_{L^2}^2 + c_6. \tag{3.3.102}$$

Inserting (3.3.102) into (3.3.101) and using $\|u^{\frac{3}{2}}\|_{L^2}^6 = \|u\|_{L^3}^3$, we have

$$\frac{d}{dt}\|u\|_{L^3}^3 + c_5\|u\|_{L^3}^3 \leqslant 3c_4 + c_5c_6 = c_7,$$

which implies

$$\|u(\cdot,t)\|_{L^3} \leqslant e^{-c_6t}\|u_0\|_{L^3}^3 + \frac{c_7}{c_8} = c_8. \tag{3.3.103}$$

Then using Lemma 2.1 and (3.3.103), we obtain (3.3.98). Then the proof of this lemma is completed. □

We are now in a position to prove Theorem 3.4.

Proof of Theorem 3.4. Multiplying the first equation of (3.1.1) by u^{p-1}, and integrating the result equation with respect to x over Ω, using (3.3.98) and applying the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int_\Omega u^p dx &= -\frac{4(p-1)}{p^2}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 dx + (p-1)\int_\Omega u^{p-1}\nabla u\cdot(\chi\nabla v - \xi\nabla w)dx \\
&\leqslant -\frac{4(p-1)}{p^2}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 dx + c_1(p-1)(\chi+\xi)\int_\Omega u^{p-1}\left|\nabla u\right| dx \\
&\leqslant -\frac{4(p-1)}{p^2}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 dx \\
&\quad + c_1(\chi+\xi)\frac{2(p-1)}{p}\left(\frac{1}{pc_1(\chi+\xi)}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 dx + \frac{pc_1(\chi+\xi)}{4}\int_\Omega u^p dx\right) \\
&= -\frac{2(p-1)}{p^2}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 dx + \frac{c_1^2(\chi+\xi)^2}{2}(p-1)\int_\Omega u^p dx,
\end{aligned} \tag{3.3.104}$$

we can obtain that there exists a constant $c_2 = 1 + \frac{c_1^2(\chi+\xi)^2}{2}$ independent of t such that

$$\frac{d}{dt}\int_\Omega u^p dx + p(p-1)\int_\Omega u^p dx \leqslant -\frac{2(p-1)}{p}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 dx + c_2 p(p-1)\int_\Omega u^p dx \tag{3.3.105}$$

for all $t \in (0,T)$ and for all $p \geqslant 2$ (see [85, p.12] for details). Then it follows from (3.3.105) and the well-know Moser-Alikakos iteration procedure (cf. [3], [84, Lemma 3.2], [85, Lemma 4.1]) that there exists a constant $c_3 > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty} \leqslant c_3 \qquad \text{for all} \quad t \in (0,T). \tag{3.3.106}$$

Theorem 3.4 is an immediate consequence of (3.3.106) and Lemma 3.1.

3.3.2 Case 2: $\tau_1 = 1, \tau_2 = 0$

In this subsection, we will consider the case $\tau_1 = 1,\ \tau_2 = 0,\ \beta \neq \delta$. When $\tau_1 = 1$ and $\tau_2 = 0$, then system (3.1.1) is transformed into

$$\begin{cases} u_t = \Delta u - \nabla \cdot (\chi u \nabla v) + \nabla \cdot (\xi u \nabla w), & x \in \Omega, \quad t > 0, \\ v_t = \Delta v + \alpha u - \beta v, & x \in \Omega, \quad t > 0, \\ 0 = \Delta w + \gamma u - \delta w, & x \in \Omega, \quad t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in \Omega. \end{cases} \tag{3.3.107}$$

Then we have the following result of system (3.3.107).

Theorem 3.5. *Let $\tau_1 = 1, \tau_2 = 0$ and Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$. Assume that $0 \leqslant (u_0, v_0) \in [W^{1,\infty}(\Omega)]^2$. Then if $\xi\gamma \geqslant \chi\alpha$, there exists a unique triple (u, v, w) of nonnegative functions belong to $C(\bar{\Omega}\times[0,\infty);\mathbb{R}^3)\cap C^{2,1}(\bar{\Omega}\times(0,\infty);\mathbb{R}^3)$ which solves (3.1.1) classically. Furthermore, there exists a constant C independent of t such that*

$$\|u(\cdot,t)\|_{L^\infty} \leqslant C. \tag{3.3.108}$$

If we can find a constant $c_1 > 0$ independent of t such that the solution of (3.1.1) satisfies

$$\|\nabla v(\cdot,t)\|_{L^\infty} + \|\nabla w(\cdot,t)\|_{L^\infty} \leqslant C, \tag{3.3.109}$$

then we can carry out the Moser-Alikakos iteration procedure same as the proof of Theorem 3.4 to obtain (3.3.108). Then Theorem 3.5 follows immediately from Lemma 3.1 and (3.3.108). Hence to complete the proof of Theorem 3.5, we only need to prove (3.3.109). For system (3.3.107), the transformation $s = \chi v - \xi w$ does not help to estimate the cross-diffusive terms as done for the case $\tau_1 = \tau_2 = 1$.

However, we can find a Lyapunov functional of system (3.1.1) as follows

$$F(u,v,w)=\int_\Omega u\ln u dx+\frac{\chi}{2\alpha}\int_\Omega(\beta v^2+|\nabla v|^2)dx+\frac{\xi}{2\gamma}\int_\Omega(\delta w^2+|\nabla w|^2)dx-\chi\int_\Omega uvdx. \tag{3.3.110}$$

Lemma 3.12. *Let $\tau_1 = 1$ and $\tau_2 = 0$, then the solutions obtained in Lemma* 3.1 *satisfy*

$$\frac{d}{dt}F(u,v,w)+G(u,v,w)=0, \tag{3.3.111}$$

where $F(u,v,w)$ is defined by (3.3.110) *and*

$$G(u,v,w)=\frac{\chi}{\alpha}\int_\Omega v_t^2dx+\int_\Omega u|\nabla(\ln u-\chi v+\xi w)|^2dx. \tag{3.3.112}$$

Proof. Multiplying the first equation of (3.3.107) by $\ln u - \chi v + \xi w$ and integrating with respect to x over Ω, we have

$$\begin{aligned}\int_\Omega u_t(\ln u-\chi v+\xi w)dx &= \int_\Omega \nabla\cdot(\nabla u-\chi u\nabla v+\xi u\nabla w)(\ln u-\chi v+\xi w)dx\\ &= -\int_\Omega u|\nabla(\ln u-\chi v+\xi w)|^2dx.\end{aligned} \tag{3.3.113}$$

Using the fact that $\int_\Omega u_t dx = 0$, we have

$$\int_\Omega u_t(\ln u-\chi v+\xi w)dx=\frac{d}{dt}\int_\Omega u\ln u dx-\chi\frac{d}{dt}\int_\Omega uvdx+\chi\int_\Omega uv_tdx+\xi\int_\Omega u_twdx. \tag{3.3.114}$$

From the second equation of (3.3.107), one has $u=\frac{1}{\alpha}v_t-\frac{1}{\alpha}\Delta v+\frac{\beta}{\alpha}v$, which gives

$$\int_\Omega uv_tdx=\frac{1}{\alpha}\int_\Omega v_t^2dx+\frac{1}{2\alpha}\frac{d}{dt}\int_\Omega|\nabla v|^2dx+\frac{\beta}{2\alpha}\frac{d}{dt}\int_\Omega v^2dx. \tag{3.3.115}$$

Similarly, from the third equation of (3.3.107), we have $u_t=\frac{\delta}{\gamma}w_t-\frac{1}{\gamma}\Delta w_t$, and hence

$$\int_\Omega u_twdx=\frac{\delta}{2\gamma}\frac{d}{dt}\int_\Omega w^2dx+\frac{1}{2\gamma}\frac{d}{dt}\int_\Omega|\nabla w|^2dx. \tag{3.3.116}$$

The combination of (3.3.114), (3.3.115) and (3.3.116) yields

$$\begin{aligned}&\int_\Omega u_t(\ln u-\chi v+\xi w)dx\\&=\frac{d}{dt}\int_\Omega\left\{u\ln u-\chi uv+\frac{\beta\chi}{2\alpha}v^2+\frac{\chi}{2\alpha}|\nabla v|^2+\frac{\xi\delta}{2\gamma}w^2+\frac{\xi}{2\gamma}|\nabla w|^2\right\}dx+\frac{\chi}{\alpha}\int_\Omega v_t^2dx,\end{aligned}$$

which together with (3.3.113) leads to (3.3.111). The proof of this lemma is completed. □

Lemma 3.13. *Let* (u,v,w) *be a smooth solution to* (3.3.107). *Assume that* $\xi\gamma\geqslant\chi\alpha$. *Then there exists a finite constant* C *independent of* t *such that*

$$\int_\Omega u\ln u dx+\frac{\chi}{\alpha}\int_0^t\|v_t(\tau)\|_{L^2}^2\,d\tau\leqslant C,\quad\text{for all } t\in(0,T). \tag{3.3.117}$$

Proof. From the third equation of (3.3.107), we have

$$u=\frac{\delta}{\gamma}w-\frac{1}{\gamma}\Delta w. \tag{3.3.118}$$

Hence, using (3.3.118) and the Cauchy-Schwarz inequality one can derive that

$$\begin{aligned}&\chi\int_\Omega uvdx\\&=\frac{\chi\delta}{\gamma}\int_\Omega vwdx+\frac{\chi}{\gamma}\int_\Omega\nabla w\cdot\nabla vdx\\&\leqslant\frac{\chi\delta}{\gamma}\left(\frac{\xi}{2\chi}\int_\Omega w^2dx+\frac{\chi}{2\xi}\int_\Omega v^2dx\right)+\frac{\chi}{\gamma}\left(\frac{\xi}{2\chi}\int_\Omega|\nabla w|^2dx+\frac{\chi}{2\xi}\int_\Omega|\nabla v|^2dx\right)\\&=\frac{\xi\delta}{2\gamma}\int_\Omega w^2dx+\frac{\chi^2\delta}{2\xi\gamma}\int_\Omega v^2dx+\frac{\xi}{2\gamma}\int_\Omega|\nabla w|^2dx+\frac{\chi^2}{2\xi\gamma}\int_\Omega|\nabla v|^2dx.\end{aligned} \tag{3.3.119}$$

Substituting (3.3.119) into (3.3.110), then we have

$$\begin{aligned}F(u,v,w)&\geqslant\int_\Omega u\ln udx+\left(\frac{\beta\chi}{2\alpha}-\frac{\chi^2\delta}{2\xi\gamma}\right)\int_\Omega v^2dx+\left(\frac{\chi}{2\alpha}-\frac{\chi^2}{2\xi\gamma}\right)\int_\Omega|\nabla v|^2dx\\&=\int_\Omega u\ln udx+\frac{\chi(\xi\gamma\beta-\chi\alpha\delta)}{2\alpha\xi\gamma}\int_\Omega v^2dx+\frac{\chi(\xi\gamma-\chi\alpha)}{2\xi\gamma\alpha}\int_\Omega|\nabla v|^2dx.\end{aligned} \tag{3.3.120}$$

Integrating (3.3.111) with respect to t and using (3.3.120), we have

$$\int_\Omega u\ln u dx + \frac{\chi(\xi\gamma-\chi\alpha)}{2\xi\gamma\alpha}\int_\Omega |\nabla v|^2 dx + \frac{\chi}{\alpha}\int_0^t\int_\Omega v_t^2 dxd\tau + \int_0^t\int_\Omega u|\nabla(\ln u - \chi v + \xi w)|^2 dxd\tau \leqslant F(u_0,v_0) + \frac{\chi|\xi\gamma\beta-\chi\alpha\delta|}{2\alpha\xi\gamma}\int_\Omega v^2 dx. \tag{3.3.121}$$

Using Lemma 2.1 and $u \in L^1(\Omega)$, we have $\|v\|_{W^{1,p}} \leqslant c_1$ for all $1 \leqslant p < 2$. Hence using Lemma 2.2 and choosing $p = 1$, we obtain

$$\|v\|_{L^2} \leqslant c_2\|v\|_{W^{1,1}} \leqslant c_1c_2. \tag{3.3.122}$$

Substituting (3.3.122) into (3.3.121) and using the condition $\xi\gamma - \chi\alpha \geqslant 0$, we have

$$\int_\Omega u\ln u dx + \frac{\chi}{\alpha}\int_0^t\int_\Omega v_t^2 dxd\tau + \int_0^t\int_\Omega u|\nabla(\ln u - \chi v + \xi w)|^2 dxd\tau \leqslant F(u_0,v_0) + c_3,$$

which implies (3.3.117). Then the proof of this lemma is completed. □

Lemma 3.14. *Assume the conditions in Lemma* 3.13 *are satisfied, then there exists a positive constant C such that $\|u(\cdot,t)\|_{L^2} \leqslant C$.*

Proof. Multiplying the first equation of (3.3.107) by u, integrating with respect to x, then we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx + \int_\Omega |\nabla u|^2 dx \\
&= -\chi\int_\Omega u\nabla\cdot(u\nabla v)dx + \xi\int_\Omega u\nabla\cdot(u\nabla w)dx \\
&= \frac{\chi}{2}\int_\Omega \nabla u^2\cdot\nabla v dx - \frac{\xi}{2}\int_\Omega \nabla u^2\cdot\nabla w dx \\
&= -\frac{\chi}{2}\int_\Omega u^2(v_t - \alpha u + \beta v)dx + \frac{\xi}{2}\int_\Omega u^2(\delta w - \gamma u)dx \\
&= \frac{\chi\alpha-\xi\gamma}{2}\int_\Omega u^3 dx + \frac{\xi\delta}{2}\int_\Omega u^2 w dx - \frac{\chi}{2}\int_\Omega u^2 v_t dx - \frac{\chi\beta}{2}\int_\Omega u^2 v dx \\
&\leqslant \frac{\chi\alpha-\xi\gamma}{2}\int_\Omega u^3 dx + \frac{\xi\delta}{2}\int_\Omega u^2 w dx - \frac{\chi}{2}\int_\Omega u^2 v_t dx,
\end{aligned}$$

then using the condition $\xi\gamma - \chi\alpha \geqslant 0$, we have

$$\frac{d}{dt}\int_\Omega u^2 dx + 2\int_\Omega |\nabla u|^2 dx \leqslant \xi\delta \int_\Omega u^2 w dx - \chi \int_\Omega u^2 v_t dx. \tag{3.3.123}$$

Next, we estimate the first term on the right-hand side in (3.3.123). By the Young's inequality (3.2.8), we have that

$$\xi\delta \int_\Omega u^2 w dx \leqslant \frac{1}{2}\int_\Omega u^3 dx + \frac{16}{27}(\xi\delta)^3 \int_\Omega w^3 dx. \tag{3.3.124}$$

The combination of (3.3.123) and (3.3.124) yields that

$$\frac{d}{dt}\int_\Omega u^2 dx + 2\int_\Omega |\nabla u|^2 dx \leqslant \frac{1}{2}\int_\Omega u^3 dx + \frac{16}{27}(\xi\delta)^3 \int_\Omega w^3 dx - \chi \int_\Omega u^2 v_t dx. \tag{3.3.125}$$

To estimate the term $\int_\Omega w^3 dx$, we apply the Agmon-Douglis-Nirenberg L^p estimates [1, 2] to the following linear elliptic equations with the zero Neumann boundary condition:

$$\begin{cases} -\Delta w + \delta w = \gamma u, & x \in \Omega, \quad t \in (0, T_{max}), \\ \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega, \quad t \in (0, T_{max}), \end{cases}$$

where $\delta > 0$, then we can find some constant c_1 to satisfy the following inequality

$$\|w(\cdot, t)\|_{W^{2,p}} \leqslant c_1 \|u(\cdot, t)\|_{L^p}. \tag{3.3.126}$$

Specially, we choose $p = 2$ in (3.3.126) to obtain

$$\|w(\cdot, t)\|_{W^{2,2}} \leqslant c_1 \|u(\cdot, t)\|_{L^2}. \tag{3.3.127}$$

Using (3.3.127), the Gagliardo-Nirenberg inequality and Young's inequality to obtain some constants $c_i > 0, i = 2, 3, 4$ such that

$$\begin{aligned} \frac{16}{27}(\xi\delta)^3 \int_\Omega w^3 dx = \frac{16}{27}(\xi\delta)^3 \|w\|_{L^3}^3 \leqslant c_2 \|w\|_{W^{2,2}}^{\frac{4}{3}} \|w\|_{L^1}^{\frac{5}{3}} &\leqslant c_3 \|u\|_{L^2}^{\frac{4}{3}} \\ &\leqslant c_4 \|u\|_{L^1}^{\frac{1}{3}} \|u\|_{L^3} \\ &\leqslant \frac{1}{2}\|u\|_{L^3}^3 + c_5 \end{aligned} \tag{3.3.128}$$

where $c_5 := \frac{2}{3}^{\frac{3}{2}} c_4^{\frac{3}{2}} \|u_0\|_{L^1}^{\frac{1}{2}}$ and the interpolation inequality has been used to obtain

$$\|u\|_{L^2} \leqslant \|u\|_{L^1}^{\frac{1}{4}} \|u\|_{L^3}^{\frac{3}{4}} = \|u_0\|_{L^1}^{\frac{1}{4}} \|u\|_{L^3}^{\frac{3}{4}}. \tag{3.3.129}$$

Inserting (3.3.128) into (3.3.125), we obtain that

$$\frac{d}{dt}\int_\Omega u^2 dx + 2\int_\Omega |\nabla u|^2 dx \leqslant \int_\Omega u^3 dx - \chi \int_\Omega u^2 v_t dx + c_5. \tag{3.3.130}$$

By the Hölder's inequality and the Gagliardo-Nirenberg inequality we have

$$\begin{aligned} -\chi \int_\Omega u^2 v_t dx \leqslant \chi \|v_t\|_{L^2} \|u\|_{L^4}^2 &\leqslant c_6 \|v_t\|_{L^2} \left(\|\nabla u\|_{L^2}^{\frac{1}{2}} \|u\|_{L^2}^{\frac{1}{2}} + \|u\|_{L^2}\right)^2 \\ &\leqslant 2c_6 \|v_t\|_{L^2} \left(\|\nabla u\|_{L^2} \|u\|_{L^2} + \|u\|_{L^2}^2\right) \\ &\leqslant \|\nabla u\|_{L^2}^2 + \left(c_6^2 \|v_t\|_{L^2}^2 + 2c_6 \|v_t\|_{L^2}\right) \|u\|_{L^2}^2 . \end{aligned} \tag{3.3.131}$$

Collecting (3.3.130) and (3.3.131), we obtain

$$\begin{aligned} &\frac{d}{dt}\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \\ &\leqslant \|u\|_{L^3}^3 + \left(c_6^2 \|v_t\|_{L^2}^2 + 2c_6 \|v_t\|_{L^2}\right) \|u\|_{L^2}^2 + c_5 \\ &\leqslant \varepsilon \|\nabla u\|_{L^2}^2 \|u \ln u\|_{L^1} + c_7(\|u \ln u\|_{L^1}^3 + \|u\|_{L^1}) + \left(c_6^2 \|v_t\|_{L^2}^2 + 2c_6 \|v_t\|_{L^2}\right) \|u\|_{L^2}^2 + c_5. \end{aligned} \tag{3.3.132}$$

Using the facts $\|u \ln u\|_{L^1} \leqslant C$ and $\|u\|_{L^1} \leqslant C$, and letting ε small enough such that $\varepsilon \|u \ln u\|_{L^1} < \frac{1}{2}$, then from (3.3.132) we have

$$\frac{d}{dt}\|u\|_{L^2}^2 + \frac{1}{2}\|\nabla u\|_{L^2}^2 \leqslant \left(c_6^2 \|v_t\|_{L^2}^2 + 2c_6 \|v_t\|_{L^2}\right) \|u\|_{L^2}^2 + c_8. \tag{3.3.133}$$

By the Gagliardo-Nirenberg inequality and Cauchy-Schwarz inequality, we have

$$\|u\|_{L^2}^2 \leqslant c_9(\|\nabla u\|_{L^2}\|u\|_{L^1} + \|u\|_{L^1}^2) \leqslant \frac{1}{2}\|\nabla u\|_{L^2}^2 + c_{10}. \tag{3.3.134}$$

Then adding (3.3.133) and (3.3.135), and using the Young's inequality yields that

$$\begin{aligned}\frac{d}{dt}\|u\|_{L^2}^2+\|u\|_{L^2}^2 &\leqslant \left(c_6^2\|v_t\|_{L^2}^2+2c_6\|v_t\|_{L^2}\right)\|u\|_{L^2}^2+c_8+c_{10}\\ &\leqslant c_{12}\|v_t\|_{L^2}^2\|u\|_{L^2}^2+\frac{1}{2}\|u\|_{L^2}^2+c_{11},\end{aligned} \tag{3.3.135}$$

where $c_{11}:=c_8+c_{10}$ and $c_{12}:=3c_6^2$. From (3.3.135), we have

$$\frac{d}{dt}\left(e^{\int_0^t(\frac{1}{2}-c_{12}\|v_t\|_{L^2}^2)ds}\|u\|_{L^2}^2\right)\leqslant c_{11}e^{\int_0^t(\frac{1}{2}-c_{12}\|v_t\|_{L^2}^2)ds},$$

which yields that

$$\begin{aligned}\|u\|_{L^2}^2\leqslant \|u_0\|_{L^2}^2e^{-\int_0^t(\frac{1}{2}-c_{12}\|v_t\|_{L^2}^2)ds}+c_{11} &= \|u_0\|_{L^2}^2e^{-\frac{t}{2}}e^{c_{12}\int_0^t\|v_t\|_{L^2}^2ds}+c_{11}\\ &\leqslant c_{13}\|u_0\|_{L^2}^2e^{-\frac{t}{2}}+c_{11}\leqslant c_{14},\end{aligned} \tag{3.3.136}$$

where we have used (3.3.117) to obtain $e^{c_{12}\int_0^t\|v_t\|_{L^2}^2ds}\leqslant c_{13}$. Then the proof of this lemma is completed. □

Remark 3.1. Choosing $p=\frac{4}{3}$ in (3.3.126) and using Lemma 3.14, we have

$$\|\nabla w\|_{W^{1,\frac{4}{3}}}\leqslant\|w\|_{W^{2,\frac{4}{3}}}\leqslant c_1\|u\|_{L^{\frac{4}{3}}}\leqslant c_2\|u\|_{L^2}\leqslant c_3. \tag{3.3.137}$$

Using Lemma 2.2 with $n=2$ and the inequality (3.3.137), we can derive that

$$\|\nabla w\|_{L^4}\leqslant c_4\|\nabla w\|_{W^{1,\frac{4}{3}}}\leqslant c_5. \tag{3.3.138}$$

Furthermore, from the Lemma 2.1 and Lemma 3.14, we obtain that there exists a constant c_6 such that

$$\|\nabla v\|_{L^4}\leqslant c_6. \tag{3.3.139}$$

Then as in proof of Lemma 3.11, we can obtain the following lemma.

Lemma 3.15. *There exists a constant C such that for all $t\in(0,T_{max})$ such that* (3.3.109) *holds.*

Proof of Theorem 3.5. Theorem 3.5 can be proved by using Lemma 3.15 and the well-know Moser-Alikakos iteration procedure as in the proof of Theorem 3.4. □

Chapter 4

Cauchy Problem of the MHD-Burgers System

4.1 Introduction

In the chapter, we will investigate the Cauchy problem of the following MHD-Burgers system

$$\begin{cases} u_t + (uv)_x = Du_{xx}, & x \in \mathbb{R}, \quad t > 0, \\ v_t + \left(\frac{1}{2}u^2 + \frac{1}{2}v^2\right)_x = \mu v_{xx}, & x \in \mathbb{R}, \quad t > 0 \end{cases} \tag{4.1.1}$$

with the initial data

$$(u, v)(x, 0) = (u_0, v_0)(x) \to \begin{cases} (u_-, v_-), & \text{as} \quad x \to -\infty, \\ (u_+, v_+), & \text{as} \quad x \to +\infty. \end{cases} \tag{4.1.2}$$

The standard hyperbolic theory (cf.[55, 82]) predicts that the time asymptotic behavior of solutions of the Cauchy problem (4.1.1)-(4.1.2) are closely related to the following Riemann problem:

$$\begin{cases} u_t + (uv)_x = 0, & x \in \mathbb{R}, \quad t > 0, \\ v_t + \left(\frac{1}{2}u^2 + \frac{1}{2}v^2\right)_x = 0, & x \in \mathbb{R}, \quad t > 0, \\ (u, v)(x, 0) = (u_0^r, v_0^r)(x) = \begin{cases} (u_-, v_-), & x < 0, \\ (u_+, v_+), & x > 0. \end{cases} \end{cases} \tag{4.1.3}$$

Writing the equations in (4.1.3) in the vector form

$$\begin{pmatrix} u \\ v \end{pmatrix}_t + \begin{pmatrix} v & u \\ u & v \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_x = 0, \tag{4.1.4}$$

we see that the Jacobian matrix $A := \begin{pmatrix} v & u \\ u & v \end{pmatrix}$ has two real distinct eigenvalues

$$\lambda_1(u, v) = v - u, \quad \lambda_2(u, v) = v + u,$$

with corresponding eigenvectors

$$r_1(u, v) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad r_2(u, v) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Therefore it follows that $\nabla\lambda_1(u, v) \cdot r_1(u, v) = -2 < 0$ and $\nabla\lambda_2(u, v) \cdot r_2(u, v) = 2 > 0$. This shows that the hyperbolic system (4.1.3) is genuinely nonlinear. By the hyperbolic theory [82], the solutions of the Riemann problem (4.1.3) are made up of three types of elementary waves (solutions): constant states, rarefaction waves and shock waves. Moreover, we point out that the rarefaction curves of (4.1.4) are straight lines in the u-v plane (see section 2), and hence the hyperbolic system (4.1.4) is of a Temple class [88].

In this chapter, we shall show that as time goes to infinity, the solution of the Cauchy problem (4.1.1)-(4.1.2) will tend to a constant solution if $(u_+, v_+) = (u_-, v_-)$, or a rarefaction wave if $\lambda_i(u_-, v_-) < \lambda_i(u_+, v_+)$, or a viscous shock wave (i.e. traveling wave) if $\lambda_i(u_-, v_-) > \lambda_i(u_+, v_+)$. Specifically, the following results are proved. If the right state equals the left state, say $(u_+, v_+) = (u_-, v_-) = (\bar{u}, \bar{v})$, then the solution of (4.1.1) with large data (4.1.2) will eventually approach the constant state $(\bar{u}, \bar{v})$. If the right state (u_+, v_+) is connected to the left state (u_-, v_-) by a rarefaction wave, then the Cauchy problem (4.1.1)-(4.1.2) has a unique global solution which tends to the rarefaction wave of the Riemann problem (4.1.3) with large data. Finally if the initial value (4.1.2) is a small perturbation of a viscous shock wave (traveling wave), then the solution of (4.1.1)-(4.1.2) will asymptotically converge to this viscous shock wave with a proper translation, where the wave amplitude can be arbitrarily large. Our results analytically confirm the existence of shock-type waves (and hence turbulence) numerically obtained in both papers [103] and [24].

Mathematical studies on the asymptotics toward rarefaction/shock waves for viscous conservation laws have been undertaken for a long time (e.g. see [41, 65, 66]). For the general 2×2 viscous conservation laws

$$\begin{cases} u_t + [f_1(u,v)]_x = Du_{xx}, & x \in \mathbb{R}, \quad t > 0, \\ v_t + [f_2(u,v)]_x = \mu v_{xx}, & x \in \mathbb{R}, \quad t > 0 \end{cases} \tag{4.1.5}$$

with initial data (4.1.2), Xin [102] and Yang and Zhao [104] established the time asymptotic stability of weak rarefaction waves and strong rarefaction waves with small initial data, respectively. The main hypothesis on the structure of the system (4.1.5) is the strong coupling in the sense that

$$\frac{\partial f_1(u,v)}{\partial v} \cdot \frac{\partial f_2(u,v)}{\partial u} \neq 0, \tag{4.1.6}$$

which is satisfied by the system (4.1.1). To the best of our knowledge, for the strongly coupled system of conservation laws with large initial data, very few results are known. The asymptotic stability of viscous shock waves for the general system of conservation laws have been extensively investigated over many years. Most of results (if not all) require the wave amplitude to be small (e.g. see [28, 62, 64, 83]). The main contributions of this paper have two folds. First, exploiting the peculiar coupling structure of the MHD-Burgers system (4.1.1), the nonlinear stability of strong rarefaction waves of (4.1.1)-(4.1.2) is established with large data. Second, the asymptotic stability of viscous shock waves of (4.1.1)-(4.1.2) is proved for large wave amplitude. Usually these results can not be proved for general hyperbolic systems as mentioned above. Finally, we mention that the asymptotic stability of viscous shock waves to (4.1.1) with $u_+ > 0$ was previously established in [40] based on the idea of [59, 58, 67] by leaving open the case $u_+ = 0$, which causes a singularity in the energy estimates. In this chapter, we will resolve this challenging case (i.e. $u_+ = 0$) by invoking the weighted energy estimates inspired by the ideas of [45, 48, 57, 75].

The rest of the chapter is organized as follows. In section 4.2, we solve the Riemann problem (4.1.3) and then state the main results for the Cauchy problem (4.1.1)-(4.1.2). Then we prove the large-time behavior of solutions with constant states in section 4.3. In section 4.4, we show the stability of rarefaction waves. The proof of nonlinear stability of viscous shock waves is given in section 4.5.

4.2 Riemann Problem

In this section, we first briefly solve the Riemann problem (4.1.3) in the class of functions consisting of constant states, separated by rarefaction waves or shock waves. We begin with the rarefaction waves of (4.1.3) by setting $\xi = x/t$. Then substituting it into the equations of (4.1.4), we find that (u_ξ, v_ξ) is an eigenvector of A for the eigenvalue ξ. Because the matrix A has two real and distinct eigenvalues λ_1 and λ_2, there are two families of rarefaction waves: 1-rarefaction waves and 2-rarefaction waves. The eigenvector (u_ξ, v_ξ) associated with the first eigenvalue λ_1 satisfies

$$\begin{pmatrix} u & u \\ u & u \end{pmatrix} \begin{pmatrix} u_\xi \\ v_\xi \end{pmatrix} = 0, \tag{4.2.7}$$

which gives $u_\xi + v_\xi = 0$ thanks to $u \neq 0$. This gives $\frac{du}{dv} = -1$. Integrating it, we obtain the 1-rarefaction curve $\mathrm{R}_1(u_-, v_-)$ as

$$\mathrm{R}_1(u_-, v_-) = \{(u, v) | u = -v + u_- + v_-, v > v_-\}, \tag{4.2.8}$$

where the entropy condition $\lambda_1(u_-, v_-) < \lambda_1(u, v)$ has been used to guarantee the uniqueness of the 1-rarefaction waves. Similarly, the 2-rarefaction curve $\mathrm{R}_2(u_-, v_-)$

can be represented as

$$R_2(u_-, v_-) = \{(u, v)|u = v + u_- - v_-, v > v_-\}. \tag{4.2.9}$$

Moreover (4.1.3) also has two distinct types of shock waves: 1-shock waves and 2-shock waves. To see this, we use the following jump condition (see [82, 15.11]):

$$\begin{cases} uv - u_-v_- = s(u - u_-), \\ \frac{1}{2}(u^2 + v^2) - \frac{1}{2}(u_-^2 + v_-^2) = s(v - v_-), \end{cases} \tag{4.2.10}$$

where s is the speed of the discontinuity (wave speed). Subtracting the first equation from the second equation of (4.2.10), we have

$$(u - v - u_- + v_-)(u - v + u_- - v_- + 2s) = 0. \tag{4.2.11}$$

For 1-shock waves, the entropy condition $v - u = \lambda_1(u, v) < \lambda_1(u_-, v_-) = v_- - u_-$ implies $u - v - u_- + v_- > 0$. Hence from (4.2.11), we obtain

$$u - v + u_- - v_- + 2s = 0. \tag{4.2.12}$$

Clearly $u \neq u_-$ for otherwise we have $v = v_-$ from the first equation of (4.2.10), and then $(u, v) = (u_-, v_-)$ is not a shock curve. Hence the first equation of (4.2.10) gives

$$s = \frac{uv - u_-v_-}{u - u_-}. \tag{4.2.13}$$

Then substituting (4.2.13) into (4.2.12), one can derive the 1-shock curve $S_1(u_-, v_-)$ as

$$S_1(u_-, v_-) = \{(u, v)|u = -v + u_- + v_-, v < v_-\}. \tag{4.2.14}$$

To deduce 2-shock curves, we add the equations in (4.2.10) to obtain

$$(u + v - u_- - v_-)(u + v + u_- + v_- - 2s) = 0. \tag{4.2.15}$$

Similarly, by using the entropy condition $v + u = \lambda_2(u, v) < \lambda_2(u_-, v_-) = v_- + u_-$ and (4.2.13), we obtain the 2-shock curve $\mathrm{S}_2(u_-, v_-)$ as

$$\mathrm{S}_2(u_-, v_-) = \{(u, v)|u = v + u_- - v_-, v < v_-\}. \tag{4.2.16}$$

Then curves $R_1,\ R_2,\ S_1$ and S_2 divide the u-v plane into four disjoint open regions I, II, III, IV defined as follows

$$\begin{aligned}
\mathrm{I} &= \mathrm{R_1R_2}(u_-, v_-) := \{(u, v)| - v + u_- + v_- < u < v + u_- - v_-\},\\
\mathrm{II} &= \mathrm{R_1S_2}(u_-, v_-) := \{(u, v)|u < -v + u_- + v_-, u < v + u_- - v_-\},\\
\mathrm{III} &= \mathrm{S_1S_2}(u_-, v_-) := \{(u, v)|v + u_- - v_- < u < -v + u_- + v_-\},\\
\mathrm{IV} &= \mathrm{S_1R_2}(u_-, v_-) := \{(u, v)|u > -v + u_- + v_-, u > v + u_- - v_-\}.
\end{aligned} \tag{4.2.17}$$

Hence, depending on the relationship between the end states (u_+, v_+) and (u_-, v_-), the solutions of Riemann problem (4.1.3) are described as

$$\begin{cases}
1-\text{rarefaction waves} & \text{if } u_+ + v_+ = u_- + v_- \text{ and } v_+ > v_-,\\
1-\text{shock waves} & \text{if } u_+ + v_+ = u_- + v_- \text{ and } v_+ < v_-,\\
2-\text{rarefaction waves} & \text{if } u_+ - v_+ = u_- - v_- \text{ and } v_+ > v_-,\\
2-\text{shock waves} & \text{if } u_+ - v_+ = u_- - v_- \text{ and } v_+ < v_-,\\
\text{Composite waves of two rarefaction waves} & \text{if } u_+ + v_+ > u_- + v_-\\
 & \text{and } u_+ - v_+ < u_- - v_-,\\
\text{Composite waves of two viscous shock waves} & \text{if } u_+ + v_+ < u_- + v_-\\
 & \text{and } u_+ - v_+ > u_- - v_-,\\
1-\text{rarefaction waves and } 2-\text{shock waves} & \text{if } u_+ + v_+ < u_- + v_-\\
 & \text{and } u_+ - v_+ < u_- - v_-,\\
1-\text{shock waves and } 2-\text{rarefaction waves} & \text{if } u_+ + v_+ > u_- + v_-\\
 & \text{and } u_+ - v_+ > u_- - v_-.
\end{cases}$$

4.3 Nonlinear Stability of Constant States

If the end states (u_-, v_-) and (u_+, v_+) are connected by a constant, say $(u_+, v_+) = (u_-, v_-) =: (\bar{u}, \bar{v})$, and if the initial value (4.1.2) is a perturbation of the constant state $(\bar{u}, \bar{v})$ in $H^1(\mathbb{R})$, we have the following global asymptotic stability results.

Theorem 4.1. *Let* $(u_0 - \bar{u}, v_0 - \bar{v}) \in H^1(\mathbb{R})$. *Then there exists a unique global solution* $(u,v)(x,t)$ *to the Cauchy problem* (4.1.1)-(4.1.2), *which satisfies*

$$(u - \bar{u}, v - \bar{v}) \in C([0,\infty); H^1) \cap L^2((0,\infty); H^2).$$

Furthermore, the solution has the following asymptotic stability:

$$\sup_{x\in\mathbb{R}} |(u,v)(x,t) - (\bar{u},\bar{v})| \to 0, \quad \text{as} \quad t \to +\infty. \tag{4.3.18}$$

4.3.1 Local Existence and the A Priori Estimates

For the case $(u_+, v_+) = (u_-, v_-) = (\bar{u}, \bar{v})$, we seek the solution of (4.1.1)-(4.1.2) in the following solution space:

$$X_1(0,T) = \{(u,v) : (u - \bar{u}, v - \bar{v}) \in C([0,T]; H^1); (u_x, v_x) \in L^2((0,T); H^1)\}.$$

The construction on the local existence of solutions is standard based on iteration argument and fixed point theorem (cf.[33]). We omit the details for brevity.

Lemma 4.1 (Local existence)**.** *If* $(u_0 - \bar{u}, v_0 - \bar{v}) \in H^1(\mathbb{R})$, *then there exists a positive constant* T_0 *such that the Cauchy problem* (4.1.1)-(4.1.2) *admits a unique smooth solution* $(u,v) \in X_1(0,T_0)$ *satisfying*

$$\|(u(\cdot,t) - \bar{u}, v(\cdot,t) - \bar{v})\|_1 \leqslant 2\|(u_0 - \bar{u}, v_0 - \bar{v})\|_1, \quad for\ all \ \ 0 \leqslant t \leqslant T_0. \tag{4.3.19}$$

Proposition 4.1 (A *priori* estimates)**.** *Suppose the Cauchy problem* (4.1.1)-(4.1.2) *has a solution* $(u,v) \in X_1(0,T)$ *for some* $T > 0$. *Then there exists a constant* C *independent of* T *such that*

$$\begin{aligned} \|(u(\cdot,t) - \bar{u}, v(\cdot,t) - \bar{v})\|_1^2 + D\int_0^t \|u_x(\cdot,\tau)\|_1^2 d\tau + \mu \int_0^t \|v_x(\cdot,\tau)\|_1^2 d\tau \\ \leqslant C\|(u_0 - \bar{u}, v_0 - \bar{v})\|_1^2. \end{aligned} \tag{4.3.20}$$

By continuing a unique local solution with the a *priori* estimates, we have the following proposition.

Proposition 4.2. *There exists a unique global solution* $(u, v) \in X_1(0, \infty)$ *to* (4.1.1)-(4.1.2) *such that*

$$\|(u - \bar{u}, v - \bar{v})\|_1^2 + D\int_0^\infty \|u_x(\cdot, t)\|_1^2\, dt + \mu\int_0^\infty \|v_x(\cdot, t)\|_1^2\, dt \leqslant C\|(u_0 - \bar{u}, v_0 - \bar{v})\|_1^2. \tag{4.3.21}$$

We are now in a position to prove Theorem 4.1.

Proof. The proof of global existence in Theorem 4.1 is a consequence of the Proposition 4.2 . Next we derive (4.3.18). From (4.3.21), one has $\|(u(\cdot, t) - \bar{u}, v(\cdot, t) - \bar{v})\|_{L^2} \leqslant C$ and

$$\|(u_x(\cdot, t), v_x(\cdot, t))\|_{L^2} \to 0 \quad \text{as} \quad t \to \infty.$$

Consequently, for all $x \in \mathbb{R}$, it follows that

$$\begin{aligned}(u(x,t) - \bar{u})^2 &= 2\int_{-\infty}^x (u(y,t) - \bar{u})(u(y,t) - \bar{u})_y dy \\ &\leqslant 2\left(\int (u(y,t) - \bar{u})^2 dy\right)^{\frac{1}{2}} \left(\int u_y^2 dy\right)^{\frac{1}{2}} \\ &\leqslant 2C\|u_x(\cdot, t)\|_{L^2} \to 0 \quad \text{as} \quad t \to \infty.\end{aligned} \tag{4.3.22}$$

This implies $\sup\limits_{x\in\mathbb{R}} |u(x,t) - \bar{u}| \to 0$ as $t \to \infty$. Similarly, we can prove $\sup\limits_{x\in\mathbb{R}} |v(x,t) - \bar{v}| \to 0$ as $t \to \infty$. Hence (4.3.18) is proved and the proof of Theorem 4.1 is completed. □

4.3.2 Proof of the A Priori Estimates

In this subsection, we are devoted to proving Proposition 4.1 based on the energy estimates.

Proof. Letting $\phi = u - \bar{u}$ and $\psi = v - \bar{v}$, and substituting (ϕ, ψ) into (4.1.1), we have

$$\begin{cases}\phi_t + (\bar{u}\psi + \bar{v}\phi + \phi\psi)_x = D\phi_{xx}, \\ \psi_t + \left(\frac{1}{2}\phi^2 + \frac{1}{2}\psi^2 + \bar{u}\phi + \bar{v}\psi\right)_x = \mu\psi_{xx}.\end{cases} \tag{4.3.23}$$

Step 1 (L^2-estimates). Multiplying the first equation of (4.3.23) by ϕ and second equation by ψ, adding them and integrating the resulting equation with respect to x, we end up with

$$\begin{aligned}
&\frac{d}{dt}\int\left(\frac{\phi^2}{2}+\frac{\psi^2}{2}\right)dx + D\int \phi_x^2 dx + \mu\int \psi_x^2 dx \\
&= -\int(\bar{u}\psi+\bar{v}\phi+\phi\psi)_x\phi dx - \int\left(\frac{\phi^2}{2}+\frac{\psi^2}{2}+\bar{u}\phi+\bar{v}\psi\right)_x \psi dx \\
&= -\int\left(\bar{v}\frac{\phi^2}{2}+\bar{v}\frac{\psi^2}{2}+\frac{\psi^3}{3}+\bar{u}\phi\psi+\phi^2\psi\right)_x dx = 0.
\end{aligned} \tag{4.3.24}$$

Hence

$$\int\left(\frac{\phi^2}{2}+\frac{\psi^2}{2}\right)dx + D\int_0^t\int \phi_x^2 dx d\tau + \mu\int_0^t\int \psi_x^2 dx d\tau = \int\left(\frac{\phi_0^2}{2}+\frac{\psi_0^2}{2}\right)dx,$$

which yields

$$\begin{aligned}
\|(u(\cdot,t)-\bar{u}, v(\cdot,t)-\bar{v})\|_{L^2}^2 + 2D\int_0^t \|u_x(\cdot,\tau)\|_{L^2}^2 d\tau + 2\mu\int_0^t \|v_x(\cdot,\tau)\|_{L^2}^2 d\tau \\
= \|(u_0-\bar{u}, v_0-\bar{v})\|_{L^2}^2.
\end{aligned} \tag{4.3.25}$$

Step 2 (H^1-estimates). Multiplying the first equation of (4.3.23) by $-\phi_{xx}$ and second equation by $-\psi_{xx}$, adding them and integrating the results with respect to x yield that

$$\begin{aligned}
&\frac{d}{dt}\int\left(\frac{\phi_x^2}{2}+\frac{\psi_x^2}{2}\right)dx + D\int \phi_{xx}^2 dx + \mu\int \psi_{xx}^2 dx \\
&= \int(\bar{u}\psi+\bar{v}\phi+\phi\psi)_x\phi_{xx} dx + \int\left(\frac{\phi^2}{2}+\frac{\psi^2}{2}+\bar{u}\phi+\bar{v}\psi\right)_x \psi_{xx} dx \\
&\leqslant \int \phi\psi_x\phi_{xx}dx + \int \psi\phi_x\phi_{xx}dx + \int \phi\phi_x\psi_{xx}dx + \int \psi\psi_x\psi_{xx}dx \\
&\quad + \frac{\bar{u}^2}{D}\|\psi_x\|_{L^2}^2 + \frac{D}{4}\|\phi_{xx}\|_{L^2}^2 + \frac{\bar{u}^2}{\mu}\|\phi_x\|_{L^2}^2 + \frac{\mu}{4}\|\psi_{xx}\|_{L^2}^2.
\end{aligned} \tag{4.3.26}$$

Using (4.3.25), we have $\|(\phi,\psi)\|_{L^2} \leqslant C$, and hence

$$\begin{aligned}
&\int \phi\psi_x\phi_{xx}dx + \int \psi\phi_x\phi_{xx}dx \\
&\leqslant \|\psi_x\|_{L^\infty} \|\phi\|_{L^2} \|\phi_{xx}\|_{L^2} + \|\phi_x\|_{L^\infty} \|\psi\|_{L^2} \|\phi_{xx}\|_{L^2} \\
&\leqslant \frac{4}{D}(\|\psi_x\|^2_{L^\infty} \|\phi\|^2_{L^2} + \|\phi_x\|^2_{L^\infty} \|\psi\|^2_{L^2}) + \frac{D}{8} \|\phi_{xx}\|^2_{L^2} \\
&\leqslant C \|\psi_x\|_{L^2} \|\psi_{xx}\|_{L^2} + C \|\phi_x\|_{L^2} \|\phi_{xx}\|_{L^2} + \frac{D}{8} \|\phi_{xx}\|^2_{L^2} \\
&\leqslant C(\|\phi_x\|^2_{L^2} + \|\psi_x\|^2_{L^2}) + \frac{3D}{16} \|\phi_{xx}\|^2_{L^2} + \frac{\mu}{16} \|\psi_{xx}\|^2_{L^2}.
\end{aligned} \tag{4.3.27}$$

Similarly, we have

$$\begin{aligned}
&\int \phi\phi_x\psi_{xx}dx + \int \psi\psi_x\psi_{xx}dx \\
&\leqslant \frac{8}{\mu} \|\phi_x\|^2_{L^\infty} \|\phi\|^2_{L^2} + \frac{\mu}{32}\|\psi_{xx}\|^2_{L^2} + \frac{8}{\mu} \|\psi_x\|^2_{L^\infty} \|\psi\|^2_{L^2} + \frac{\mu}{32}\|\psi_{xx}\|^2_{L^2} \\
&\leqslant C \|\phi_x\|_{L^2} \|\phi_{xx}\|_{L^2} + C \|\psi_x\|_{L^2} \|\psi_{xx}\|_{L^2} + \frac{\mu}{16}\|\psi_{xx}\|^2_{L^2} \\
&\leqslant C(\|\phi_x\|^2_{L^2} + \|\psi_x\|^2_{L^2}) + \frac{D}{16} \|\phi_{xx}\|^2_{L^2} + \frac{3\mu}{16} \|\psi_{xx}\|^2_{L^2}.
\end{aligned} \tag{4.3.28}$$

Substituting (4.3.27) and (4.3.28) into (4.3.26), we have

$$\frac{d}{dt}\int \left(\frac{\phi_x^2}{2} + \frac{\psi_x^2}{2}\right) dx + \frac{D}{2}\int \phi_{xx}^2 dx + \frac{\mu}{2}\int \psi_{xx}^2 dx \leqslant C(\|\phi_x\|^2_{L^2} + \|\psi_x\|^2_{L^2}). \tag{4.3.29}$$

Integrating (4.3.29) over $[0,t]$ and using (4.3.25), we get

$$\|(u_x,v_x)(\cdot,t)\|^2_{L^2} + D\int_0^t \|u_{xx}(\cdot,\tau)\|^2_{L^2}d\tau + \mu\int_0^t \|v_{xx}(\cdot,\tau)\|^2_{L^2}d\tau \leqslant C\|(u_0-\bar{u}, v_0-\bar{v})\|^2_1. \tag{4.3.30}$$

The combination of (4.3.25) and (4.3.30) yields (4.3.20). Then the proof of Proposition 4.1 is completed. □

4.4 Nonlinear Stability of Rarefaction Waves

Without loss of generality, we consider 1-rarefaction wave solutions $(u^r, v^r)(x/t)$ of the Riemann problem (4.1.3) only and analysis can be directly applied to 2-rarefaction wave. Using (4.2.8), we can separate the variables u and v in (4.1.3) such that u satisfies the Riemann problem

$$\begin{cases} u_t + (u_- + v_- - 2u)u_x = 0, \\ u(x,0) = u_0^r(x) = \begin{cases} u_-, & x < 0, \\ u_+, & x > 0 \end{cases} \end{cases} \tag{4.4.31}$$

and v satisfies the Riemann problem

$$\begin{cases} v_t + (2v - u_- - v_-)v_x = 0, \\ v(x,0) = v_0^r(x) = \begin{cases} v_-, & x < 0, \\ v_+, & x > 0. \end{cases} \end{cases} \tag{4.4.32}$$

Employing the method of characteristics, we can solve (4.4.31) and obtain the rarefaction wave $u^r(x/t)$ as follows

$$u^r(x/t) = \begin{cases} u_-, & \frac{x}{t} \leqslant v_- - u_-, \\ \frac{u_- + v_-}{2} - \frac{x}{2t}, & v_- - u_- \leqslant \frac{x}{t} \leqslant v_+ - u_+, \\ u_+, & \frac{x}{t} \geqslant v_+ - u_+. \end{cases} \tag{4.4.33}$$

Similarly, the rarefaction wave $v^r(x/t)$ of (4.4.32) can be obtained as

$$v^r(x/t) = \begin{cases} v_-, & \frac{x}{t} \leqslant v_- - u_-, \\ \frac{u_- + v_-}{2} + \frac{x}{2t}, & v_- - u_- \leqslant \frac{x}{t} \leqslant v_+ - u_+, \\ v_+, & \frac{x}{t} \geqslant v_+ - u_+. \end{cases} \tag{4.4.34}$$

Then the result on asymptotic stability of the 1-rarefaction waves $(u^r, v^r)(x/t)$ is as follows.

Theorem 4.2. *Let* $(u_+, v_+) \in \mathrm{R}_1(u_-, v_-)$ *and* $v_+ > v_-$. *Assume* $(u_0 - u_0^r, v_0 - v_0^r) \in L^2(\mathbb{R})$ *and* $(u_{0x}, v_{0x}) \in L^2(\mathbb{R})$, *then the Cauchy problem* (4.1.1)-(4.1.2) *has a unique global solution* (u, v) *satisfying*

$$\begin{cases} (u - u^r, v - v^r) \in C([0, \infty); L^2) \cap L^\infty((0, \infty); L^2), \\ (u_x, v_x) \in C([0, \infty); L^2) \cap L^\infty((0, \infty); L^2) \cap L^2((0, \infty); H^1), \end{cases}$$

and

$$\sup_{x \in \mathbb{R}} |(u, v)(x, t) - (u^r, v^r)(x/t)| \to 0, \quad \text{as} \quad t \to +\infty. \tag{4.4.35}$$

Remark 4.1. If $(u_+, v_+) \in \mathrm{R}_2(u_-, v_-)$ and $v_+ > v_-$, similar stability result can be obtained.

4.4.1 Smooth Approximate Solutions

To study the nonlinear stability of rarefaction waves, we first construct a smooth approximation of solutions $(u^r, v^r)(x/t)$ of the Riemann problem (4.1.3). It is well-known (e.g. see [65]) that the Riemann problem of the Burgers equation

$$\begin{cases} w_t + w w_x = 0, & x \in \mathbb{R}, \quad t > 0, \\ w(x, 0) = w_0^r(x) = \begin{cases} v_- - u_-, & x < 0, \\ v_+ - u_+, & x > 0, \end{cases} \end{cases} \tag{4.4.36}$$

where $v_- - u_- \leqslant v_+ - u_+$, has a continuous weak solution $w^r(x/t)$ of the form

$$w^r(x/t) = \begin{cases} v_- - u_-, & \frac{x}{t} \leqslant v_- - u_-, \\ \frac{x}{t}, & v_- - u_- \leqslant \frac{x}{t} \leqslant v_+ - u_+, \\ v_+ - u_+, & \frac{x}{t} \geqslant v_+ - u_+. \end{cases} \tag{4.4.37}$$

Then the 1-rarefaction wave solutions $(u^r, v^r)(x/t)$ given by (4.4.33) and (4.4.34) can be written as

$$u^r(x/t) = \frac{u_- + v_- - w^r(x/t)}{2}, \qquad v^r(x/t) = \frac{u_- + v_- + w^r(x/t)}{2}. \tag{4.4.38}$$

We approximate $w^r(x/t)$ by the solution $w(x,t)$ of the following initial value problem:

$$\begin{cases} w_t + ww_x = 0, \qquad x \in \mathbb{R}, \quad t > 0, \\ w(x,0) = w_0(x) := \frac{v_+ - u_+ + v_- - u_-}{2} + \frac{v_+ - u_+ - v_- + u_-}{2} k_q \int_0^{\varepsilon x} (1+y^2)^{-q} dy, \end{cases} \tag{4.4.39}$$

where $\varepsilon > 0$ is a constant to be determined later and k_q is a constant such that $k_q \int_0^\infty (1+y^2)^{-q} dy = 1$ for each $q > \frac{3}{2}$. Then the solution of the Cauchy problem (4.4.39) has the following properties.

Lemma 4.2 ([65]). *If $v_- - u_- < v_+ - u_+$, then the Cauchy problem* (4.4.39) *has a unique smooth global solution $w(x,t)$ satisfying the following:*

(i) $v_- - u_- < w(x,t) < v_+ - u_+, \quad w_x(x,t) > 0, \quad for\ (x,t) \in \mathbb{R} \times \mathbb{R}_+$.

(ii) *For any $p \in [1, \infty]$, there exists a constant $C_{p,q}$ such that for any $t \in \mathbb{R}_+$*

$$\begin{aligned} \|w_x(\cdot,t)\|_{L^p} &\leqslant C_{p,q} \min\left\{\varepsilon^{1-\frac{1}{p}}, \quad (1+t)^{-1+\frac{1}{p}}\right\}, \\ \|w_{xx}(\cdot,t)\|_{L^p} &\leqslant C_{p,q} \min\left\{\varepsilon^{2-\frac{1}{p}}, \quad \varepsilon^{(1-\frac{1}{2q})(1-\frac{1}{p})}(1+t)^{-1-\frac{p-1}{2pq}}\right\}. \end{aligned} \tag{4.4.40}$$

(iii) $\lim_{t\to\infty} \sup_{x\in\mathbb{R}} |w(x,t) - w^r(x/t)| = 0$.

Using (4.4.38) and Lemma 4.2, the smooth approximation of the rarefaction wave profile $(u^r, v^r)(x/t)$ can be constructed via

$$\widetilde{U} = \frac{u_- + v_- - w}{2}, \qquad \widetilde{V} = \frac{u_- + v_- + w}{2}, \tag{4.4.41}$$

which satisfy

$$\begin{cases} \widetilde{U}_t + (\widetilde{U}\widetilde{V})_x = 0, \\ \widetilde{V}_t + (\frac{1}{2}\widetilde{U}^2 + \frac{1}{2}\widetilde{V}^2)_x = 0, \\ (\widetilde{U}, \widetilde{V})(x,0) = (\widetilde{U}_0, \widetilde{V}_0)(x) = \left(\frac{u_- + v_- - w_0(x)}{2}, \frac{u_- + v_- + w_0(x)}{2}\right), \end{cases} \tag{4.4.42}$$

where w_0 is defined in (4.4.39). Moreover the following properties can be readily verified.

Lemma 4.3. *The smooth function* $(\widetilde{U},\widetilde{V})(x,t)$ *given in* (4.4.41) *has the following properties:*

(i) $\widetilde{V}_x = -\widetilde{U}_x > 0$.

(ii) *For any* $p \in [1,+\infty]$, *there exists a positive constant* $C_{p,q}$ *such that*

$$\left\|(\widetilde{U}_x,\widetilde{V}_x)(\cdot,t)\right\|_{L^p} \leqslant C_{p,q}\min\{\varepsilon^{1-\frac{1}{p}},\quad (1+t)^{-1+\frac{1}{p}}\},$$

$$\left\|(\widetilde{U}_{xx},\widetilde{V}_{xx})(\cdot,t)\right\|_{L^p} \leqslant C_{p,q}\min\left\{\varepsilon^{2-\frac{1}{p}},\quad \varepsilon^{(1-\frac{1}{2q})(1-\frac{1}{p})}(1+t)^{-1-\frac{p-1}{2pq}}\right\}.$$

In particular, for $p > 1$, *it holds that*

$$\int_0^\infty \left\|(\widetilde{U}_{xx},\widetilde{V}_{xx})\right\|_{L^p} dt \leqslant C_{p,q}. \tag{4.4.43}$$

(iii) $\lim\limits_{t\to\infty}\sup\limits_{x\in\mathbb{R}}\left|(u^r-\widetilde{U},v^r-\widetilde{V})(x,t)\right| = 0.$

Proof. The properties (ii) and (iii) can de derived from Lemma 4.2 and (4.4.41) directly. We only need to prove (i). Indeed, using (4.4.41) and Lemma 4.2 (i), we have

$$\widetilde{V}_x - \widetilde{U}_x = w_x(x,t) > 0, \qquad \widetilde{U}_x = -\widetilde{V}_x, \tag{4.4.44}$$

which implies $\widetilde{V}_x = -\widetilde{U}_x > 0$. □

4.4.2 Local Existence and the A Priori Estimates

By the approximate smooth solution $(\widetilde{U},\widetilde{V})$ constructed in (4.4.41), we define $(\phi,\psi) = (u-\widetilde{U}, v-\widetilde{V})$ and rewrite the Cauchy problem (4.1.1)-(4.1.2) as

$$\begin{cases}\phi_t + (\widetilde{U}\psi + \widetilde{V}\phi + \phi\psi)_x = D\phi_{xx} + D\widetilde{U}_{xx},\\ \psi_t + \frac{1}{2}(\phi^2+\psi^2)_x + (\widetilde{V}\psi + \widetilde{U}\phi)_x = \mu\psi_{xx} + \mu\widetilde{V}_{xx}\end{cases} \tag{4.4.45}$$

with initial data

$$(\phi,\psi)(x,0) = (\phi_0,\psi_0)(x) = (u_0(x)-\widetilde{U}_0(x), v_0(x)-\widetilde{V}_0(x)), \tag{4.4.46}$$

where (4.4.42) has been used.

We seek the solution of (4.4.45)-(4.4.46) in the space $X_2(0,T)$ defined by

$$X_2(0,T) = \{(\phi,\psi) : (\phi,\psi) \in C([0,T];H^1); (\phi_x,\psi_x) \in L^2((0,T);H^1)\}.$$

Then we can obtain the following proposition on the local existence of solutions of system (4.4.45)-(4.4.46). The proof of the local existence of solutions is standard and is based on an iteration argument and a fixed point theorem (cf. [102]). We state the local existence theorem without proof.

Proposition 4.3 (Local existence)**.** *If* $(\phi_0,\psi_0) \in H^1$*, then there exists a positive constant* T_0 *such that the Cauchy problem* (4.4.45)-(4.4.46) *admits a unique solution* $(\phi,\psi) \in X_2(0,T_0)$ *satisfying*

$$\|(\phi,\psi)(\cdot,t))\|_1 \leqslant 2\,\|(\phi_0,\psi_0)\|_1\,, \quad for\ all \quad 0 \leqslant t \leqslant T_0. \tag{4.4.47}$$

Furthermore, we can show that the solutions of system (4.4.45)-(4.4.46) have the following the a *priori* estimates by using the energy estimates method.

Proposition 4.4 (A *priori* estimates)**.** *Suppose the Cauchy problem* (4.4.45)-(4.4.46) *has a solution* $(\phi,\psi) \in X_2(0,T)$ *for some* $T > 0$*. Then there exists a constant* C *independent of* T *such that*

$$\begin{aligned}\|(\phi,\psi)(\cdot,t)\|_1^2 + D\int_0^t \|\phi_x(\cdot,\tau)\|_1^2\,d\tau + \mu\int_0^t \|\psi_x(\cdot,\tau)\|_1^2\,d\tau + \int_0^t \|\sqrt{\widetilde{V}_x}(\phi-\psi)(\cdot,\tau)\|_{L^2}^2 d\tau \\ + \int_0^t \|\sqrt{\widetilde{V}_x}(\phi_x,\psi_x)(\cdot,\tau)\|_{L^2}^2 d\tau \leqslant C(\|(\phi_0,\psi_0)\|_1^2 + 1), \quad for \quad all \quad t \in [0,T].\end{aligned} \tag{4.4.48}$$

The combination of local existence of solutions and the a *priori* estimates implies the following proposition by using the continuity argument.

Proposition 4.5. *There exists a unique global solution* $(\phi, \psi) \in X_2(0, \infty)$ *to* (4.4.45)-(4.4.46) *such that*

$$\|(\phi,\psi)(\cdot,t)\|_1^2 + D\int_0^\infty \|\phi_x(\cdot,t)\|_1^2\,dt + \mu\int_0^\infty \|\psi_x(\cdot,t)\|_1^2\,dt \leqslant C\,\|(\phi_0,\psi_0)\|_1^2\,. \quad (4.4.49)$$

With the above lemmas in hand, we now prove Theorem 4.2.

Proof of Theorem 4.2 . From Proposition 4.5, one has $\|(\phi, \psi)(\cdot, t)\|_{L^2} \leqslant C$ and

$$\|(\phi_x, \psi_x)(\cdot, t)\|_{L^2} \to 0 \text{ as } t \to \infty.$$

Hence, the same argument as in the proof of Theorem 4.1 leads to

$$\sup_{x\in\mathbb{R}} |u(x,t) - \widetilde{U}(x,t)| \to 0 \quad \text{as} \quad t \to \infty \qquad (4.4.50)$$

and

$$\sup_{x\in\mathbb{R}} |v(x,t) - \widetilde{V}(x,t)| \to 0 \quad \text{as} \quad t \to \infty. \qquad (4.4.51)$$

The combination of (4.4.50) and Lemma 4.3 (iii) gives

$$\sup_{x\in\mathbb{R}} |u(x,t) - u^r(x/t)| \leqslant \sup_{x\in\mathbb{R}} |u(x,t) - \widetilde{U}(x,t)| + \sup_{x\in\mathbb{R}} |u^r(x/t) - \widetilde{U}(x,t)| \to 0 \text{ as } t \to \infty,$$

Similarly, the combination of (4.4.51) and Lemma 4.3 (iii) gives

$$\sup_{x\in\mathbb{R}} |v(x,t) - v^r(x/t)| \to 0 \quad \text{as} \quad t \to \infty.$$

Then the proof of Theorem 4.2 is completed. □

4.4.3 Proof of the A Priori Estimates

To prove Proposition 4.4, we first derive the L^2-estimates of (ϕ, ψ).

Lemma 4.4 (L^2-estimates). *Let $(\phi,\psi)\in X_2(0,T)$ be a solution of* (4.4.45)-(4.4.46) *for some $T>0$. Then it holds that*

$$\begin{aligned}&\|(\phi,\psi)(\cdot,t)\|_{L^2}^2 + D\int_0^t \|\phi_x(\cdot,\tau)\|_{L^2}^2\,d\tau + \mu\int_0^t \|\psi_x(\cdot,\tau)\|_{L^2}^2\,d\tau\\ &\qquad + \int_0^t \|\sqrt{\widetilde{V}_x}(\phi-\psi)(\cdot,\tau)\|_{L^2}^2 d\tau \leqslant C(\|(\phi_0,\psi_0)\|_{L^2}^2+1),\end{aligned} \tag{4.4.52}$$

where C is a constant independent of T.

Proof. We multiply the first equation of (4.4.45) by ϕ and the second by ψ, then integrate the results with respect to x to have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int(\phi^2+\psi^2)dx + D\int\phi_x^2dx + \mu\int\psi_x^2dx\\ &= -\int(\widetilde{U}\psi+\widetilde{V}\phi+\psi\phi)_x\phi dx - \int(\widetilde{V}\psi+\widetilde{U}\phi)_x\psi dx - \int\phi\phi_x\psi dx\\ &\quad + D\int\widetilde{U}_{xx}\phi dx + \mu\int\widetilde{V}_{xx}\psi dx.\end{aligned} \tag{4.4.53}$$

Notice that

$$\begin{cases}(\widetilde{U}\psi+\widetilde{V}\phi+\psi\phi)_x\phi = \left(\widetilde{U}\phi\psi + \frac{\widetilde{V}\phi^2}{2} + \phi^2\psi\right)_x + \frac{1}{2}\widetilde{V}_x\phi^2 - \widetilde{U}\psi\phi_x - \phi\phi_x\psi,\\ (\widetilde{V}\psi+\widetilde{U}\phi)_x\psi = \left(\frac{\widetilde{V}}{2}\psi^2+\widetilde{U}\phi\psi\right)_x + \frac{1}{2}\widetilde{V}_x\psi^2 - \widetilde{U}\phi\psi_x.\end{cases} \tag{4.4.54}$$

Substituting (4.4.54) into (4.4.53) and using Lemma 4.3 (i), we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int(\phi^2+\psi^2)dx + D\int\phi_x^2dx + \mu\int\psi_x^2dx + \frac{1}{2}\int\widetilde{V}_x(\phi^2+\psi^2)dx\\ &= \int\widetilde{U}(\phi\psi)_xdx + D\int\widetilde{U}_{xx}\phi dx + \mu\int\widetilde{V}_{xx}\psi dx\\ &= \int\widetilde{V}_x\phi\psi dx + D\int\widetilde{U}_{xx}\phi dx + \mu\int\widetilde{V}_{xx}\psi dx,\end{aligned}$$

which yields

$$
\begin{aligned}
&\frac{d}{dt}\int(\phi^2+\psi^2)dx+2D\int\phi_x^2dx+2\mu\int\psi_x^2dx+\int\widetilde{V}_x(\phi-\psi)^2dx\\
&=2D\int\widetilde{U}_{xx}\phi dx+2\mu\int\widetilde{V}_{xx}\psi dx\\
&\leqslant 2D\|\widetilde{U}_{xx}\|_{L^2}\,\|\phi\|_{L^2}+2\mu\|\widetilde{V}_{xx}\|_{L^2}\,\|\psi\|_{L^2}\\
&\leqslant D^2\|\widetilde{U}_{xx}\|_{L^2}+\|\widetilde{U}_{xx}\|_{L^2}\,\|\phi\|_{L^2}^2+\mu^2\|\widetilde{V}_{xx}\|_{L^2}+\|\widetilde{V}_{xx}\|_{L^2}\|\psi\|_{L^2}^2,
\end{aligned}
\tag{4.4.55}
$$

where we have used the Hölder and Cauchy-Schwarz inequalities. Applying Gronwall's inequality to (4.4.55), we obtain (4.4.52) by using (4.4.43) and the fact $\widetilde{V}_x > 0$ in Lemma 4.3. □

Lemma 4.5 (H^1-estimates)**.** *Suppose the Cauchy problem* (4.4.45)-(4.4.46) *has a solution* $(\phi,\psi)\in X_2(0,T)$ *for some* $T>0$*. Then there exists a constant* C *independent of* T *such that*

$$
\begin{aligned}
&\|(\phi_x,\psi_x)(\cdot,t)\|_{L^2}^2+D\int_0^t\|\phi_{xx}(\cdot,\tau)\|_{L^2}^2\,d\tau+\mu\int_0^t\|\psi_{xx}(\cdot,\tau)\|_{L^2}^2\,d\tau\\
&\qquad+\int_0^t\|\sqrt{\widetilde{V}_x}(\phi_x,\psi_x)(\cdot,\tau)\|_{L^2}^2d\tau\leqslant C(\|(\phi_0,\psi_0)\|_1^2+1).
\end{aligned}
\tag{4.4.56}
$$

Proof. Multiplying the first equation of (4.4.45) by $-\phi_{xx}$ and the second by $-\psi_{xx}$, and integrating them with respect to x, we end up with

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int(\phi_x^2+\psi_x^2)dx+D\int\phi_{xx}^2dx+\mu\int\psi_{xx}^2dx+\frac{3}{2}\int\widetilde{V}_x(\phi_x^2+\psi_x^2)dx\\
&=\int\phi\phi_x\psi_{xx}dx+\int\psi\psi_x\psi_{xx}dx+\int\phi\psi_x\phi_{xx}dx+\int\phi_x\psi\phi_{xx}dx\\
&-\int\widetilde{V}_{xx}(\phi\phi_x+\psi\psi_x)dx-\int\widetilde{U}_{xx}(\psi\phi_x+\phi\psi_x)dx\\
&-3\int\widetilde{U}_x\phi_x\psi_xdx-D\int\widetilde{U}_{xx}\phi_{xx}dx-\mu\int\widetilde{V}_{xx}\psi_{xx}dx.
\end{aligned}
\tag{4.4.57}
$$

Integrating (4.4.57) with respect to t leads to

$$\frac{1}{2}\int(\phi_x^2+\psi_x^2)dx + D\int_0^t\int\phi_{xx}^2dxd\tau + \mu\int_0^t\int\psi_{xx}^2dxd\tau + \frac{3}{2}\int_0^t\int\widetilde{V}_x(\phi_x^2+\psi_x^2)dxd\tau$$
$$= \frac{1}{2}\left\|(\phi_{0x},\psi_{0x})\right\|_{L^2}^2 + \sum_{j=1}^{9}I_j, \tag{4.4.58}$$

where

$$\sum_{j=1}^{9}I_j = \int_0^t\int\phi\phi_x\psi_{xx}dxd\tau + \int_0^t\int\psi\psi_x\psi_{xx}dxd\tau + \int_0^t\int\phi\psi_x\phi_{xx}dxd\tau$$
$$+ \int_0^t\int\psi\phi_x\phi_{xx}dxd\tau - \int_0^t\int\widetilde{V}_{xx}(\phi\phi_x+\psi\psi_x)dxd\tau - \int_0^t\int\widetilde{U}_{xx}(\psi\phi_x+\phi\psi_x)dxd\tau$$
$$-3\int_0^t\int\widetilde{U}_x\phi_x\psi_xdxd\tau - D\int_0^t\int\widetilde{U}_{xx}\phi_{xx}dxd\tau - \mu\int_0^t\int\widetilde{V}_{xx}\psi_{xx}dxd\tau.$$

Using Lemma 4.4, one has $\left\|(\phi,\psi)(\cdot,t)\right\|_{L^2}^2+\int_0^t\left\|(\phi_x,\psi_x)(\cdot,\tau)\right\|_{L^2}^2d\tau \leqslant C(\left\|(\phi_0,\psi_0)\right\|_{L^2}^2+1)$. Then

$$\begin{aligned}
I_1 &\leqslant \int_0^t\int|\phi\phi_x\psi_{xx}|dxd\tau \\
&\leqslant \frac{2}{\mu}\int_0^t\int\phi^2\phi_x^2dxd\tau + \frac{\mu}{8}\int_0^t\left\|\psi_{xx}\right\|_{L^2}^2d\tau \\
&\leqslant \frac{2}{\mu}\int_0^t\left\|\phi_x\right\|_{L^\infty}^2\left\|\phi\right\|_{L^2}^2d\tau + \frac{\mu}{8}\int_0^t\left\|\psi_{xx}\right\|_{L^2}^2d\tau \\
&\leqslant C\int_0^t\left\|\phi_{xx}\right\|_{L^2}\left\|\phi_x\right\|_{L^2}d\tau + \frac{\mu}{8}\int_0^t\left\|\psi_{xx}\right\|_{L^2}^2d\tau \\
&\leqslant C\int_0^t\left\|\phi_x\right\|_{L^2}^2d\tau + \frac{D}{8}\int_0^t\left\|\phi_{xx}\right\|_{L^2}^2d\tau + \frac{\mu}{8}\int_0^t\left\|\psi_{xx}\right\|_{L^2}^2d\tau \\
&\leqslant C(\left\|(\phi_0,\psi_0)\right\|_{L^2}^2+1) + \frac{D}{8}\int_0^t\left\|\phi_{xx}\right\|_{L^2}^2d\tau + \frac{\mu}{8}\int_0^t\left\|\psi_{xx}\right\|_{L^2}^2d\tau.
\end{aligned} \tag{4.4.59}$$

Applying the same procedure to $I_2,\ I_3$ and I_4, we have

$$\begin{aligned}
&I_2+I_3+I_4\\
&\leqslant \int_0^t\int|\psi\psi_x\psi_{xx}|dxd\tau+\int_0^t\int|\phi\psi_x\phi_{xx}|dxd\tau+\int_0^t\int|\psi\phi_x\phi_{xx}|dxd\tau\\
&\quad+\frac{\mu}{16}\int_0^t\|\psi_{xx}\|_{L^2}^2\,d\tau+\frac{D}{16}\int_0^t\|\phi_{xx}\|_{L^2}^2\,d\tau\\
&\leqslant C(\|(\phi_0,\psi_0)\|_{L^2}^2+1)+\frac{\mu}{8}\int_0^t\|\psi_{xx}\|_{L^2}^2\,d\tau+\frac{D}{8}\int_0^t\|\phi_{xx}\|_{L^2}^2\,d\tau.
\end{aligned}\tag{4.4.60}$$

Using the Hölder and Cauchy-Schwarz inequalities, we can estimate the terms $I_5,\ I_6$ and I_7 as follows:

$$\begin{aligned}
&I_5+I_6+I_7\\
&\leqslant \int_0^t\int|\widetilde{V}_{xx}(\phi\phi_x+\psi\psi_x)|dxd\tau+\int_0^t\int|\widetilde{U}_{xx}(\psi\phi_x+\phi\psi_x)|dxd\tau+3\int_0^t\int|\widetilde{U}_x\phi_x\psi_x|dxd\tau\\
&\leqslant \int_0^t\|\widetilde{V}_{xx}\|_{L^\infty}\left(\|\phi\|_{L^2}\|\phi_x\|_{L^2}+\|\psi\|_{L^2}\|\psi_x\|_{L^2}\right)d\tau+\int_0^t\|\widetilde{U}_{xx}\|_{L^\infty}\left(\|\psi\|_{L^2}\|\phi_x\|_{L^2}+\|\phi\|_{L^2}\|\psi_x\|_{L^2}\right)d\tau\\
&\quad+\frac{3}{2}\int_0^t\|\widetilde{U}_x\|_{L^\infty}(\|\phi_x\|_{L^2}^2+\|\psi_x\|_{L^2}^2)d\tau\\
&\leqslant \frac{1}{2}\int_0^t(\|\widetilde{U}_{xx}\|_{L^\infty}+\|\widetilde{V}_{xx}\|_{L^\infty})\|(\phi,\psi)\|_{L^2}^2\,d\tau+\frac{1}{2}\int_0^t(\|\widetilde{U}_{xx}\|_{L^\infty}+\|\widetilde{V}_{xx}\|_{L^\infty})\|(\phi_x,\psi_x)\|_{L^2}^2\,d\tau\\
&\quad+\frac{3}{2}\int_0^t\|\widetilde{U}_x\|_{L^\infty}(\|\phi_x\|_{L^2}^2+\|\psi_x\|_{L^2}^2)d\tau.
\end{aligned}\tag{4.4.61}$$

From Lemma 4.3 (ii), we have

$$\|(\widetilde{U}_x,\widetilde{U}_{xx},\widetilde{V}_{xx})\|_{L^\infty}\leqslant C\ \text{ and }\ \int_0^t\|(\widetilde{U}_{xx},\widetilde{V}_{xx})\|_{L^\infty}d\tau\leqslant C.\tag{4.4.62}$$

Substituting (4.4.62) into (4.4.61), and using Lemma 4.4, one has

$$\begin{aligned}
I_5+I_6+I_7&\leqslant C\int_0^t(\|\widetilde{U}_{xx}\|_{L^\infty}+\|\widetilde{V}_{xx}\|_{L^\infty})d\tau+C\int_0^t(\|\phi_x\|_{L^2}^2+\|\psi_x\|_{L^2}^2)d\tau\\
&\leqslant C(\|(\phi_0,\psi_0)\|_{L^2}^2+1).
\end{aligned}\tag{4.4.63}$$

Finally, we use the Hölder's inequality, Cauchy-Schwarz inequality and Lemma 4.3 (ii) to estimate the last two terms I_8 and I_9 as follows

$$\begin{aligned} I_8 + I_9 &\leqslant D \int_0^t \int |\widetilde{U}_{xx}\phi_{xx}| dx d\tau + \mu \int_0^t \int |\widetilde{V}_{xx}\psi_{xx}| dx d\tau \\ &\leqslant D \int_0^t \|\widetilde{U}_{xx}\|_{L^2}^2 d\tau + \mu \int_0^t \|\widetilde{V}_{xx}\|_{L^2}^2 d\tau + \frac{D}{4}\int_0^t \|\phi_{xx}\|_{L^2}^2 \, d\tau + \frac{\mu}{4}\int_0^t \|\psi_{xx}\|_{L^2}^2 \, d\tau \\ &\leqslant C + \frac{D}{4}\int_0^t \|\phi_{xx}\|_{L^2}^2 \, d\tau + \frac{\mu}{4}\int_0^t \|\psi_{xx}\|_{L^2}^2 \, d\tau. \end{aligned} \tag{4.4.64}$$

Substituting (4.4.59), (4.4.60), (4.4.63) and (4.4.64) into (4.4.58), we obtain

$$\begin{aligned} &\frac{1}{2}\int (\phi_x^2 + \psi_x^2) dx + \frac{D}{2}\int_0^t \int \phi_{xx}^2 dx d\tau + \frac{\mu}{2}\int_0^t \int \psi_{xx}^2 dx d\tau + \frac{3}{2}\int_0^t \int \widetilde{V}_x(\phi_x^2 + \psi_x^2) dx d\tau \\ &\leqslant \frac{1}{2}\|(\phi_{0x}, \psi_{0x})\|_{L^2}^2 + C(\|(\phi_0, \psi_0)\|_{L^2}^2 + 1) \\ &\leqslant C(\|(\phi_0, \psi_0)\|_1^2 + 1), \end{aligned}$$

which implies (4.4.56). Then we complete the proof of Lemma 4.5. □

Then, the combination of Lemma 4.4 and Lemma 4.5 yields Proposition 4.4.

4.5 Nonlinear Stability of Viscous Shock Waves

The existence of traveling wave solutions of (4.1.1) with $0 \leqslant u_+ < u_-$ and $0 \leqslant v_+ < v_-$ was established in [40] by the phase plane analysis and the nonlinear stability of traveling wave solutions was prove only for $u_+ > 0$ by the method of energy estimates, whereas the stability for $u_+ = 0$ remains open. In this section, we shall solve this open question by using the weighted energy estimates. Toward this end, we identify the decay rates of traveling wave solutions as $z \to \pm\infty$ and choose appropriate exponential weight functions. For completeness, we shall briefly recall the existence of traveling wave solutions for $u_+ = 0$ and derive the asymptotic decay rates of traveling wave solutions.

The traveling wave solution of (4.1.1) with (4.1.2) is a special solution in the form

$$(u, v)(x, t) = (U, V)(z), \quad z = x - st,$$

where $(U, V) \in C^\infty(\mathbb{R})$ satisfies

$$\begin{cases} -sU' + (UV)' = DU'', \\ -sV' + \frac{1}{2}(U^2 + V^2)' = \mu V'' \end{cases} \tag{4.5.65}$$

with boundary condition

$$U(\pm\infty) = u_\pm, \quad V(\pm\infty) = v_\pm, \quad U'(\pm\infty) = V'(\pm\infty) = 0, \tag{4.5.66}$$

where $' = \frac{d}{dz}$. Integrating (4.5.65) once yields that

$$\begin{cases} DU' = -sU + UV + \varrho_1, \\ \mu V' = -sV + \frac{1}{2}(U^2 + V^2) + \varrho_2, \end{cases} \tag{4.5.67}$$

where ϱ_1 and ϱ_2 are constants satisfying

$$\begin{cases} \varrho_1 = su_+ - u_+v_+ = su_- - u_-v_-, \\ \varrho_2 = sv_+ - \frac{1}{2}(u_+^2 + v_+^2) = sv_- - \frac{1}{2}(u_-^2 + v_-^2) \end{cases}$$

which gives

$$\begin{cases} s(u_+ - u_-) = u_+v_+ - u_-v_-, \\ s(v_+ - v_-) = \frac{1}{2}(u_+^2 + v_+^2) - \frac{1}{2}(u_-^2 + v_-^2). \end{cases} \tag{4.5.68}$$

Then (4.5.68) with $u_+ = 0$ yields

$$s^2 - v_-s = 0, \tag{4.5.69}$$

and hence $s = 0$ or $s = v_-$, which corresponds to the wave speed of the 1st and 2nd characteristic family of shock waves of (4.1.1). If $(u_+, v_+) \in \mathrm{S}_1(u_-, v_-)$, using

(4.2.14), we have $u_+ = -v_+ + u_- + v_-$ and $v_+ < v_-$, which yield $0 = u_+ > u_-$ and $v_+ < v_-$. Similarly, when $(u_+, v_+) \in S_2(u_-, v_-)$, from (4.2.16), we obtain $u_+ - u_- = v_+ - v_-$ and $v_+ < v_-$, which imply that

$$0 = u_+ < u_- \quad \text{and} \quad v_+ < v_-, \tag{4.5.70}$$

In this chapter, we consider the case $s = v_-$ only and the analysis for $s = 0$ is similar. We first have the following existence results for the 2-shock profile $(U, V)(x - st)$.

Lemma 4.6. *Let $u_\pm$ and $v_\pm$ satisfy* (4.5.70). *Then there exists a monotone shock profile $(U, V)(x - st)$ to the system* (4.5.65)-(4.5.66) *with wave speed $s = v_-$, which is unique up to a translation and satisfies $U_z < 0$, $V_z < 0$. Furthermore, the solution profile $(U, V)(x - st)$ decays exponentially at $\pm\infty$ with rates*

$$\begin{aligned} U - u_\pm &\sim e^{\sigma_\pm z}, \text{ as } z \to \pm\infty; \\ V - v_\pm &\sim e^{\sigma_\pm z}, \text{ as } z \to \pm\infty \end{aligned} \tag{4.5.71}$$

where

$$\sigma_- = \frac{u_-}{\sqrt{D\mu}}, \quad \sigma_+ = \begin{cases} \frac{v_+ - s}{D}, & D > \mu, \\ \frac{v_+ - s}{\mu}, & D < \mu. \end{cases} \tag{4.5.72}$$

Proof. The existence of monotone shock profiles $(U, V)(x - st)$ to system (4.5.65)-(4.5.66) has been proved in [40] by the phase plane analysis. It remains only to derive the asymptotic decay rates which are eigenvalues of the linearized system at equilibria $(u_\pm, v_\pm)$. To see this, we linearize the system (4.5.67) at $(u_\pm, v_\pm)$ and obtain the corresponding Jacobian matrix

$$J(u_\pm, v_\pm) = \begin{bmatrix} \frac{v_\pm - s}{D} & \frac{u_\pm}{D} \\ \frac{u_\pm}{\mu} & \frac{v_\pm - s}{\mu} \end{bmatrix} \tag{4.5.73}$$

whose eigenvalue σ satisfies

$$\sigma^2 + \frac{D + \mu}{D\mu}(s - v_\pm)\sigma + \frac{(s - v_\pm)^2 - u_\pm^2}{D\mu} = 0. \tag{4.5.74}$$

By (4.5.70) and $s = v_-$, we can readily check that the equilibrium (u_-, v_-) is a saddle and (u_+, v_+) is a stable node. Then solving the equation (4.5.74), we obtained the decay rates as announced. □

Then we proceed to consider the asymptotic stability of traveling wave solutions obtained in Lemma 4.6 under the small initial perturbation of the form

$$\int_{-\infty}^{+\infty} \begin{pmatrix} u_0(x) - U(x) \\ v_0(x) - V(x) \end{pmatrix} dx = x_0 \begin{pmatrix} u_+ - u_- \\ v_+ - v_- \end{pmatrix} + \beta r_1(u_-, v_-). \tag{4.5.75}$$

The coefficients x_0 and β are uniquely determined by the initial data $(u_0(x), v_0(x))$. When $\beta \neq 0$, the diffusion wave will appear. The stability of viscous shock waves with diffusion wave for small wave strength have been investigated previously (e.g. see [62, 83]). The stability of shock waves with diffusion wave and large wave strength still remains open up to present. In this paper we do not consider the diffusion wave (i.e. assuming $\beta = 0$) but consider large wave strength. Then by conservation law (4.1.1), we can derive that

$$\begin{aligned} &\int_{-\infty}^{+\infty} \begin{pmatrix} u(x,t) - U(x + x_0 - st) \\ v(x,t) - V(x + x_0 - st) \end{pmatrix} dx = \int_{-\infty}^{+\infty} \begin{pmatrix} u_0(x) - U(x + x_0) \\ v_0(x) - V(x + x_0) \end{pmatrix} dx \\ &= \int_{-\infty}^{+\infty} \begin{pmatrix} u_0(x) - U(x) \\ v_0(x) - V(x) \end{pmatrix} dx + \int_{-\infty}^{+\infty} \begin{pmatrix} U(x) - U(x + x_0) \\ V(x) - V(x + x_0) \end{pmatrix} dx \\ &= \int_{-\infty}^{+\infty} \begin{pmatrix} u_0(x) - U(x) \\ v_0(x) - V(x) \end{pmatrix} dx - x_0 \begin{pmatrix} u_+ - u_- \\ v_+ - v_- \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned} \tag{4.5.76}$$

Thus we decompose the solution of (4.1.1) into the form

$$(u, v)(x, t) = (U, V)(x + x_0 - st) + (\phi_z, \psi_z)(z, t), \tag{4.5.77}$$

where

$$(\phi(z,t), \psi(z,t)) = \int_{-\infty}^{z} (u(y,t) - U(y + x_0 - st), v(y,t) - V(y + x_0 - st))\, dy.$$

Clearly, for all $t > 0$, if follows from (4.5.76) that

$$\phi(\pm\infty, t) = \psi(\pm\infty, t) = 0.$$

Without loss of generality, we assume $x_0 = 0$, otherwise we make a translation for the traveling wave solutions. Hence, the initial value of the perturbation (ϕ, ψ) is given by

$$(\phi_0, \psi_0)(z) = \int_{-\infty}^{z} (u_0 - U, v_0 - V)(y)dy. \tag{4.5.78}$$

Then we have the following stability results on the traveling wave solutions.

Theorem 4.3. *Let* (4.5.70) *hold, and let* $(U, V)(x - st)$ *be a traveling wave solution obtained in Lemma* 4.6. *If* $D \geqslant \mu$, *there exists a constant* $\varepsilon_0 > 0$ *such that if* $\|u_0 - U\|_{1,w} + \|v_0 - V\|_{1,w} + \|(\phi_0, \psi_0)\|_w \leqslant \varepsilon_0$, *then the Cauchy problem* (4.1.1)-(4.1.2) *has a unique global solution* $(u, v)(x, t)$ *satisfying*

$$(u - U, v - V) \in C([0, \infty); H_w^1) \cap L^2((0, \infty); H_w^2), \tag{4.5.79}$$

where the weight function w *is defined by as*

$$w(z) := 1 + e^{\eta z},\ \ \eta = \frac{s - v_+}{D} > 0. \tag{4.5.80}$$

Furthermore, the solution has the following asymptotic stability:

$$\sup_{x\in\mathbb{R}} |(u, v)(x, t) - (U, V)(x - st)| \to 0, \quad \text{as} \quad t \to +\infty. \tag{4.5.81}$$

Remark 4.2. To establish the L^2-energy estimates, the conditions $D \geqslant \mu$ is needed, see (4.5.94). The nonlinear stability result for the case $D < \mu$ still remains unknown.

Remark 4.3. When $u_+ = 0$ and $D \geqslant \mu$, it can be easily verified that there exist two constants $C_2 > C_1 > 0$ such that the traveling wave solution (U, V) obtained in Lemma 4.6 satisfies

$$C_1 w(z) \leqslant \frac{1}{U(z)} \leqslant C_2 w(z) \ \ \text{for all } z \in \mathbb{R}. \tag{4.5.82}$$

4.5.1 Reformulation of the Problem

Substituting (4.5.77) into (4.1.1), using (4.5.65) and integrating the system with respect to z, we obtain the equations for the perturbation (ϕ, ψ)

$$\begin{cases} \phi_t = D\phi_{zz} + (s - V)\phi_z - U\psi_z - \phi_z\psi_z, \\ \psi_t = \mu\psi_{zz} + (s - V)\psi_z - U\phi_z - \frac{1}{2}(\phi_z^2 + \psi_z^2) \end{cases} \tag{4.5.83}$$

with initial data

$$(\phi, \psi)(z, 0) = (\phi_0, \psi_0)(z), \quad z \in \mathbb{R}, \tag{4.5.84}$$

where (ϕ_0, ψ_0) is defined in (4.5.78). We look for solutions of the reformulated system (4.5.83) in the following solution space:

$$X_3(0, T) = \left\{(\phi, \psi) : (\phi, \psi) \in C([0, T]; H_w^2), (\phi_z, \psi_z) \in L^2((0, T); H_w^2)\right\},$$

where the weight function w is defined by (4.5.80).

Clearly, if $\phi \in H_w^2$, then $\phi \in H^2$ because $w \geqslant 1$. Define

$$N(t) := \sup_{\tau \in [0,t]} (\|\phi(\cdot, \tau)\|_{2,w} + \|\psi(\cdot, \tau)\|_{2,w}).$$

By the Sobolev embedding inequality, one has

$$\sup_{\tau \in [0,t]} \{\|\phi(\cdot, \tau)\|_{L^\infty}, \|\phi_z(\cdot, \tau)\|_{L^\infty}, \|\psi(\cdot, \tau)\|_{L^\infty}, \|\psi_z(\cdot, \tau)\|_{L^\infty}\} \leqslant N(t). \tag{4.5.85}$$

Then we have the following local existence theorem on the reformulated problem (4.5.83).

Proposition 4.6 (Local existence)**.** *For any $\varepsilon_2 > 0$, there exists a positive constant T_0 depending on ε_2 such that if $(\phi_0, \psi_0) \in H_w^2$ with $N(0) \leqslant \varepsilon_2$, then the problem* (4.5.83)-(4.5.84) *has a unique solution $(\phi, \psi) \in X_3(0, T_0)$ satisfying $N(t) \leqslant 2N(0)$ for any $0 \leqslant t \leqslant T_0$.*

The local existence in Proposition 4.6 can be proved by the standard argument (cf. [74]). So we omit the details for brevity. Then Theorem 4.3 is a consequence of the following theorem.

Theorem 4.4. *Let* (4.5.70) *hold, and let* $D \geqslant \mu$. *Then there exists a positive constant* ε_1*, such that if* $N(0) \leqslant \varepsilon_1$*, then the Cauchy problem* (4.5.83)-(4.5.84) *has a unique global solution* $(\phi, \psi) \in X_3(0, \infty)$ *satisfying*

$$\begin{aligned} \|\phi(\cdot,t)\|_{2,w}^2 + \|\psi(\cdot,t)\|_{2,w}^2 + \int_0^t (\|\phi_z(\cdot,\tau)\|_{2,w}^2 + \|\psi_z(\cdot,\tau)\|_{2,w}^2) d\tau \\ \leqslant C\Big(\|\phi_0\|_{2,w}^2 + \|\psi_0\|_{2,w}^2\Big) \leqslant CN^2(0) \end{aligned} \tag{4.5.86}$$

for any $t \in [0, +\infty)$. *Moreover, it follows that*

$$\sup_{z\in\mathbb{R}} |(\phi_z, \psi_z)(z,t)| \to 0 \quad as \quad t \to \infty. \tag{4.5.87}$$

The global existence of (ϕ, ψ) announced in Theorem 4.4 follows from the local existence of solutions in Proposition 4.6 and the following *a priori* estimates.

Proposition 4.7 (A *priori* estimates)**.** *Assume that* $(\phi, \psi) \in X_3(0, T)$ *is a solution obtained in Proposition* 4.6 *for a positive constant* T. *Then there is a positive constant* $\varepsilon_3 > 0$*, independent of* T*, such that if*

$$N(t) \leqslant \varepsilon_3$$

for any $0 \leqslant t \leqslant T$*, then the solution* (ϕ, ψ) *of* (4.5.83)-(4.5.84) *satisfies* (4.5.86) *for any* $0 \leqslant t \leqslant T$.

Now we are in a position to prove Theorem 4.4. In fact we only need to prove (4.5.87). From the global estimate (4.5.86) we have

$$\|(\phi_z(\cdot,t), \psi_z(\cdot,t))\|_{1,w} \to 0 \quad \text{as} \quad t \to +\infty. \tag{4.5.88}$$

Hence, for all $z \in \mathbb{R}$, we have

$$\begin{aligned}\phi_z^2(z,t) &= 2\int_{-\infty}^{z} \phi_z\phi_{zz}(y,t)dy \\ &\leqslant 2\left(\int_{-\infty}^{\infty} \phi_z^2 dy\right)^{1/2}\left(\int_{-\infty}^{\infty} \phi_{zz}^2 dy\right)^{1/2} \to 0 \quad \text{as} \quad t \to +\infty.\end{aligned} \tag{4.5.89}$$

Applying the same procedure to ψ_z leads to

$$\psi_z(z,t) \to 0 \quad \text{as} \quad t \to +\infty \quad \text{for all} \quad z \in \mathbb{R}. \tag{4.5.90}$$

Thus (4.5.87) is proved.

4.5.2 Weighted Energy Estimates

In this subsection, we shall prove Proposition 4.7 by using the weighted energy estimates. In the following, we assume $N(t) < \min\{\mu, D\}$ without loss of generality.

Lemma 4.7 (L^2-estimates)**.** *Let the assumptions of Theorem* 4.4 *hold and* $(\phi,\psi) \in X_3(0,T)$ *is a solution obtained in Proposition* 4.6*. Then there exists a constant* $C > 0$ *such that*

$$\|\phi(\cdot,t)\|_w^2+\|\psi(\cdot,t)\|_w^2+D\int_0^t \|\phi_z(\cdot,\tau)\|_w^2\,d\tau+\mu\int_0^t \|\psi_z(\cdot,\tau)\|_w^2\,d\tau \leqslant C(\|\phi_0\|_w^2+\|\psi_0\|_w^2). \tag{4.5.91}$$

Proof. Multiplying the first equation of (4.5.83) by ϕ/U and the second by ψ/U, integrating the resultant equations with respect to z and adding them, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int \frac{\phi^2+\psi^2}{U}dz + D\int \frac{\phi_z^2}{U}dz + \mu\int \frac{\psi_z^2}{U}dz \\ &= \frac{1}{2}\int\left[\left(\frac{D}{U}\right)_{zz} - \left(\frac{s-V}{U}\right)_z\right]\phi^2 dz + \frac{1}{2}\int\left[\left(\frac{\mu}{U}\right)_{zz} - \left(\frac{s-V}{U}\right)_z\right]\psi^2 dz \\ &\quad - \int \frac{\phi\phi_z\psi_z}{U}dz - \frac{1}{2}\int \frac{\psi(\phi_z^2+\psi_z^2)}{U}dz.\end{aligned} \tag{4.5.92}$$

Using (4.5.65) and the fact that $u_+ = 0$, it can be checked that

$$\left(\frac{D}{U}\right)_{zz} - \left(\frac{s-V}{U}\right)_z = \frac{2U_z}{U^3}(s - v_+)u_+ = 0. \tag{4.5.93}$$

The combination of (4.5.65) and the facts $U_z < 0, V_z < 0, s - V > 0, U > 0$ and $D \geqslant \mu$ gives

$$\left(\frac{\mu}{U}\right)_{zz} - \left(\frac{s-V}{U}\right)_z = \frac{D-\mu}{DU}\left(V_z + \frac{(s-V)U_z}{U}\right) \leqslant 0. \tag{4.5.94}$$

Substituting (4.5.93) and (4.5.94) into (4.5.92) and integrating the equation with respect to t, with the fact $\|(\phi,\psi)(\cdot,t)\|_{L^\infty} \leqslant N(t)$, we derive

$$\begin{aligned}
&\frac{1}{2}\int \frac{\phi^2+\psi^2}{U}dz + D\int_0^t\int \frac{\phi_z^2}{U}dzd\tau + \mu\int_0^t\int \frac{\psi_z^2}{U}dzd\tau \\
&\qquad + \frac{D-\mu}{2D}\int_0^t\int \left[-V_z + \frac{s-V}{U}(-U_z)\right]\frac{\psi^2}{U}dzd\tau \\
&= \frac{1}{2}\int \frac{\phi_0^2+\psi_0^2}{U}dz - \int_0^t\int \frac{\phi\phi_z\psi_z}{U}dzd\tau - \frac{1}{2}\int_0^t\int \frac{\psi(\phi_z^2+\psi_z^2)}{U}dzd\tau \\
&\leqslant \frac{1}{2}\int \frac{\phi_0^2+\psi_0^2}{U}dz + \frac{N(t)}{2}\int_0^t\int\left(\frac{\phi_z^2}{U}+\frac{\psi_z^2}{U}\right)dzd\tau + \frac{N(t)}{2}\int_0^t\int \frac{\phi_z^2+\psi_z^2}{U}dzd\tau \\
&\leqslant \frac{1}{2}\int \frac{\phi_0^2+\psi_0^2}{U}dz + N(t)\int_0^t\int \frac{\phi_z^2}{U}dzd\tau + N(t)\int_0^t\int \frac{\psi_z^2}{U}dzd\tau,
\end{aligned}$$

which yields that

$$\int \frac{\phi^2+\psi^2}{U}dz + 2(D-N(t))\int_0^t\int \frac{\phi_z^2}{U}dzd\tau + 2(\mu-N(t))\int_0^t\int \frac{\psi_z^2}{U}dzd\tau \leqslant \int \frac{\phi_0^2+\psi_0^2}{U}dz. \tag{4.5.95}$$

Then using the assumption $N(t) < \min\{\mu, D\}$ and Remark 4.3, we obtain (4.5.91) from (4.5.95). □

Lemma 4.8 (H^1-estimates). *Let the assumptions of Lemma 4.7 hold. Then it follows that*

$$\begin{aligned} \|\phi(\cdot,t)\|_{1,w}^2 + \|\psi(\cdot,t)\|_{1,w}^2 + D\int_0^t \|\phi_z(\cdot,\tau)\|_{1,w}^2\,d\tau + \mu\int_0^t \|\psi_z(\cdot,\tau)\|_{1,w}^2\,d\tau \\ \leqslant C\left(\|\phi_0\|_{1,w}^2 + \|\psi_0\|_{1,w}^2\right), \end{aligned} \tag{4.5.96}$$

where $C > 0$ is a constant.

Proof. We differentiate (4.5.83) with respect to z to get

$$\begin{cases} \phi_{zt} = D\phi_{zzz} - V_z\phi_z + (s-V)\phi_{zz} - U_z\psi_z - U\psi_{zz} - (\phi_z\psi_z)_z, \\ \psi_{zt} = \mu\psi_{zzz} - V_z\psi_z + (s-V)\psi_{zz} - U_z\phi_z - U\phi_{zz} - \frac{1}{2}(\phi_z^2+\psi_z^2)_z. \end{cases} \tag{4.5.97}$$

Multiplying the first equation of (4.5.97) by ϕ_z/U and the second by ψ_z/U, after some algebra, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int \frac{\phi_z^2+\phi_z^2}{U}dz + D\int\frac{\phi_{zz}^2}{U}dz + \mu\int\frac{\psi_{zz}^2}{U}dz \\ &= \frac{1}{2}\int\left[\left(\frac{D}{U}\right)_{zz} - \left(\frac{s-V}{U}\right)_z\right]\phi_z^2dz + \frac{1}{2}\int\left[\left(\frac{\mu}{U}\right)_{zz} - \left(\frac{s-V}{U}\right)_z\right]\psi_z^2dz \\ &\quad - 2\int\frac{U_z}{U}\phi_z\psi_zdz - \int\frac{V_z}{U}(\phi_z^2+\psi_z^2)dz - \int\frac{\psi_z^2\psi_{zz} + 2\psi_z\phi_z\phi_{zz} + \psi_{zz}\phi_z^2}{U}dz \\ &\leqslant -2\int\frac{U_z}{U}\phi_z\psi_zdz - \int\frac{V_z}{U}(\phi_z^2+\psi_z^2)dz - \int\frac{\psi_z^2\psi_{zz} + 2\psi_z\phi_z\phi_{zz} + \psi_{zz}\phi_z^2}{U}dz, \end{aligned} \tag{4.5.98}$$

where (4.5.93) and (4.5.94) have been used. Using (4.5.67) and the facts $s = v_-$, $0 = u_+ < U < u_-$ and $v_+ < V < v_-$, it is easy to check that

$$\left|\frac{U_z}{U}\right| = \left|\frac{V-s}{D}\right| \leqslant \frac{v_- - v_+}{D}, \quad |V_z| \leqslant \left|-\frac{s}{\mu}V + \frac{1}{2\mu}(U^2+V^2) + \frac{\rho_2}{\mu}\right| \leqslant C. \tag{4.5.99}$$

Integrating (4.5.98) in t and using the fact $\psi_z^2 \leqslant \frac{C\psi_z^2}{U}$ and (4.5.85), we obtain from

(4.5.98) and (4.5.99) that

$$
\begin{aligned}
&\int \frac{\phi_z^2+\phi_z^2}{U}dz + 2D\int_0^t\int \frac{\phi_{zz}^2}{U}dzd\tau + 2\mu\int_0^t\int \frac{\psi_{zz}^2}{U}dzd\tau \\
&\leqslant \int \frac{\phi_{0z}^2+\phi_{0z}^2}{U}dz + \frac{v_- - v_+}{D}\int_0^t\int\left(U\psi_z^2+\frac{\phi_z^2}{U}\right)dzd\tau + C\int_0^t\int \frac{\phi_z^2+\psi_z^2}{U}dzd\tau \\
&\quad + 2N(t)\left(\int_0^t\int \frac{\psi_z^2+\psi_{zz}^2}{2U}dzd\tau + \int_0^t\int \frac{\phi_z^2+\phi_{zz}^2}{U}dzd\tau + \int_0^t\int \frac{\phi_z^2+\psi_{zz}^2}{2U}dzd\tau\right) \\
&\leqslant \int \frac{\phi_{0z}^2+\phi_{0z}^2}{U}dz + C\left(1+N(t)\right)\int_0^t\int \frac{\phi_z^2+\psi_z^2}{U}dzd\tau + 2N(t)\int_0^t\int \frac{\phi_{zz}^2+\psi_{zz}^2}{U}dzd\tau,
\end{aligned}
$$

which entails that

$$
\begin{aligned}
&\int \frac{\phi_z^2+\phi_z^2}{U}dz + 2(D-N(t))\int_0^t\int \frac{\phi_{zz}^2}{U}dzd\tau + 2(\mu-N(t))\int_0^t\int \frac{\psi_{zz}^2}{U}dzd\tau \\
&\leqslant \int \frac{\phi_{0z}^2+\phi_{0z}^2}{U}dz + C\left(1+N(t)\right)\int_0^t\int \frac{\phi_z^2+\psi_z^2}{U}dzd\tau \qquad (4.5.100)\\
&\leqslant C(\|\phi_{0z}\|_w^2+\|\psi_{0z}\|_w^2) + C\left(1+N(t)\right)\int_0^t(\|\phi_z(\cdot,\tau)\|_w^2+\|\psi_z(\cdot,\tau)\|_w^2)d\tau,
\end{aligned}
$$

where we have used the fact $\frac{1}{U(z)} \leqslant C_2 w(z)$ for all $z \in \mathbb{R}$ (see Remark 4.3). The combination (4.5.91) and (4.5.100) gives that

$$
\begin{aligned}
&\int \frac{\phi_z^2+\phi_z^2}{U}dz + 2(D-N(t))\int\int_0^t \frac{\phi_{zz}^2}{U}dzd\tau + 2(\mu-N(t))\int\int_0^t \frac{\psi_{zz}^2}{U}dzd\tau \\
&\leqslant C\left(\|\phi_0\|_{1,w}^2+\|\psi_0\|_{1,w}^2\right).
\end{aligned} \qquad (4.5.101)
$$

Using the fact $C_1 w(z) \leqslant \frac{1}{U(z)}$ for all $z \in \mathbb{R}$ (see Remark 4.3) and the assumption $N(t) < \min\{\mu, D\}$, we obtain (4.5.96) from (4.5.101). □

Next, we give the estimates of the second order derivative of (ϕ, ψ).

Lemma 4.9 (H^2-estimates)**.** *Let the assumptions of Lemma* 4.7 *hold. Then there* *exists a constant $C > 0$ such that*

$$\|\phi(\cdot,t)\|_{2,w}^2 + \|\psi(\cdot,t)\|_{2,w}^2 + D\int_0^t \|\phi_z(\cdot,\tau)\|_{2,w}^2\,d\tau + \mu\int_0^t \|\psi_z(\cdot,\tau)\|_{2,w}^2\,d\tau \leqslant C\left(\|\phi_0\|_{2,w}^2 + \|\psi_0\|_{2,w}^2\right). \tag{4.5.102}$$

Proof. We differentiate (4.5.83) with respect to z twice to get

$$\begin{cases} \phi_{zzt} = D\phi_{zzzz} - V_{zz}\phi_z - 2V_z\phi_{zz} + (s-V)\phi_{zzz} - U_{zz}\psi_z - 2U_z\psi_{zz} \\ \qquad - U\psi_{zzz} - (\phi_z\psi_z)_{zz}, \\ \psi_{zzt} = \mu\psi_{zzzz} - V_{zz}\psi_z - 2V_z\psi_{zz} + (s-V)\psi_{zzz} - U_{zz}\phi_z - 2U_z\phi_{zz} \\ \qquad - U\phi_{zzz} - \frac{1}{2}(\phi_z^2 + \psi_z^2)_{zz}. \end{cases} \tag{4.5.103}$$

Multiplying the first equation of (4.5.103) by ϕ_{zz}/U and the second equation by ψ_{zz}/U, using the facts

$$\begin{cases} \phi_{zzzz}\cdot\frac{\phi_{zz}}{U} = \left(\phi_{zzz}\cdot\frac{\phi_{zz}}{U}\right)_z - \frac{\phi_{zzz}^2}{U} - \frac{1}{2}\left(\phi_{zz}^2\left(\frac{1}{U}\right)_z\right)_z + \frac{1}{2}\left(\frac{1}{U}\right)_{zz}\phi_{zz}^2, \\ (s-V)\phi_{zzz}\cdot\frac{\phi_{zz}}{U} = \frac{1}{2}\left(\phi_{zz}^2\frac{(s-V)}{U}\right)_z - \frac{1}{2}\left(\frac{s-V}{U}\right)_z\phi_{zz}^2, \\ \psi_{zzzz}\cdot\frac{\psi_{zz}}{U} = \left(\psi_{zzz}\cdot\frac{\psi_{zz}}{U}\right)_z - \frac{\psi_{zzz}^2}{U} - \frac{1}{2}\left(\psi_{zz}^2\left(\frac{1}{U}\right)_z\right)_z + \frac{1}{2}\left(\frac{1}{U}\right)_{zz}\psi_{zz}^2, \\ (s-V)\psi_{zzz}\cdot\frac{\psi_{zz}}{U} = \frac{1}{2}\left(\psi_{zz}^2\frac{(s-V)}{U}\right)_z - \frac{1}{2}\left(\frac{s-V}{U}\right)_z\psi_{zz}^2, \end{cases}$$

we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int\frac{\phi_{zz}^2+\psi_{zz}^2}{U}dz + D\int\frac{\phi_{zzz}^2}{U}dz + \mu\int\frac{\psi_{zzz}^2}{U}dz \\ &= \frac{1}{2}\int\left[\left(\frac{D}{U}\right)_{zz} - \left(\frac{s-V}{U}\right)_z\right]\phi_{zz}^2dz + \frac{1}{2}\int\left[\left(\frac{\mu}{U}\right)_{zz} - \left(\frac{s-V}{U}\right)_z\right]\psi_{zz}^2dz \\ &\quad - \int\frac{V_{zz}}{U}(\phi_z\phi_{zz} + \psi_z\psi_{zz})dz - 2\int\frac{V_z}{U}(\phi_{zz}^2 + \psi_{zz}^2)dz - \int\frac{U_{zz}}{U}(\psi_z\phi_{zz} + \phi_z\psi_{zz})dz \\ &\quad - 4\int\frac{U_z}{U}\phi_{zz}\psi_{zz}dz - \int\frac{(\phi_z\psi_z)_{zz}\phi_{zz}}{U}dz - \frac{1}{2}\int\frac{(\phi_z^2+\psi_z^2)_{zz}\psi_{zz}}{U}dz. \end{aligned} \tag{4.5.104}$$

Using (4.5.93) and (4.5.94), one has

$$\frac{1}{2}\int\left[\left(\frac{D}{U}\right)_{zz}-\left(\frac{s-V}{U}\right)_{z}\right]\phi_{zz}^2dz+\frac{1}{2}\int\left[\left(\frac{\mu}{U}\right)_{zz}-\left(\frac{s-V}{U}\right)_{z}\right]\psi_{zz}^2dz\leqslant 0. \tag{4.5.105}$$

The combination of (4.5.104) and (4.5.105) yields that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\frac{\phi_{zz}^2+\psi_{zz}^2}{U}dz+D\int\frac{\phi_{zzz}^2}{U}dz+\mu\int\frac{\psi_{zzz}^2}{U}dz\\&\leqslant-\int\frac{V_{zz}}{U}(\phi_z\phi_{zz}+\psi_z\psi_{zz})dz-2\int\frac{V_z}{U}(\phi_{zz}^2+\psi_{zz}^2)dz-\int\frac{U_{zz}}{U}(\psi_z\phi_{zz}+\phi_z\psi_{zz})dz\\&\quad-4\int\frac{U_z}{U}\phi_{zz}\psi_{zz}dz-\int\frac{(\phi_z\psi_z)_{zz}\phi_{zz}}{U}dz-\frac{1}{2}\int\frac{(\phi_z^2+\psi_z^2)_{zz}\psi_{zz}}{U}dz.\end{aligned} \tag{4.5.106}$$

Using (4.5.65), (4.5.99) and the facts $0=u_+<U<u_-$ and $v_+<V<v_-$, one can derive that

$$\begin{aligned}|U_z|&=\left|\frac{(V-s)U}{D}\right|\leqslant\frac{(v_--v_+)u_-}{D},\\|U_{zz}|&=\left|\frac{(V-s)U_z+UV_z}{D}\right|\leqslant\frac{(v_--v_+)^2u_-}{D^2}+C\cdot\frac{u_-}{D}\leqslant C,\\|V_{zz}|&=\left|\frac{(V-s)V_z+UU_z}{\mu}\right|\leqslant C.\end{aligned} \tag{4.5.107}$$

Then we have the following estimates by using (4.5.107) and the Cauchy-Schwarz inequality

$$\begin{aligned}-\int\frac{V_{zz}}{U}(\phi_z\phi_{zz}+\psi_z\psi_{zz})dz&\leqslant C\int\frac{|\phi_z\phi_{zz}+\psi_z\psi_{zz}|}{U}dz\\&\leqslant C\int\frac{\phi_z^2+\psi_z^2+\phi_{zz}^2+\psi_{zz}^2}{U}dz,\end{aligned} \tag{4.5.108}$$

$$-2\int\frac{V_z}{U}(\phi_{zz}^2+\psi_{zz}^2)dz\leqslant C\int\frac{\phi_{zz}^2+\psi_{zz}^2}{U}dz,$$

$$
\begin{aligned}
-\int \frac{U_{zz}}{U}(\psi_z\phi_{zz}+\phi_z\psi_{zz})dz &\leqslant C\int \frac{\phi_z^2+\psi_z^2+\phi_{zz}^2+\psi_{zz}^2}{U}dz,\\
-4\int \frac{U_z}{U}\phi_{zz}\psi_{zz}dz &\leqslant C\int \frac{\phi_{zz}^2+\psi_{zz}^2}{U}dz.
\end{aligned} \tag{4.5.109}
$$

Using (4.5.85), (4.5.99) and the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
&-\int \frac{(\phi_z\psi_z)_{zz}\phi_{zz}}{U}dz\\
&=\int \frac{(\phi_z\psi_z)_z\phi_{zzz}}{U}dz-\int \frac{(\phi_z\psi_z)_z\phi_{zz}U_z}{U^2}dz\\
&\leqslant N(t)\int \frac{|\psi_{zz}\phi_{zzz}+\phi_{zz}\phi_{zzz}|}{U}+\frac{v_- - v_+}{D}N(t)\int \frac{\phi_{zz}^2+|\psi_{zz}\phi_{zz}|}{U}dz\\
&\leqslant \frac{3v_- - 3v_+ + D}{2D}N(t)\int \frac{\phi_{zz}^2}{U}dz+\frac{v_- - v_+ + D}{2D}N(t)\int \frac{\psi_{zz}^2}{U}dz+N(t)\int \frac{\phi_{zzz}^2}{U}dz,
\end{aligned} \tag{4.5.110}
$$

and

$$
\begin{aligned}
&-\frac{1}{2}\int \frac{(\phi_z^2+\psi_z^2)_{zz}\psi_{zz}}{U}dz\\
&\leqslant \frac{v_- - v_+ + D}{2D}N(t)\int \frac{\phi_{zz}^2}{U}dz+\frac{3v_- - 3v_+ + D}{2D}N(t)\int \frac{\psi_{zz}^2}{U}dz+N(t)\int \frac{\psi_{zzz}^2}{U}dz.
\end{aligned} \tag{4.5.111}
$$

Inserting (4.5.108), (4.5.110), (4.5.110) and (4.5.111) into (4.5.106), one has

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \frac{\phi_{zz}^2+\psi_{zz}^2}{U}dz+(D-N(t))\int \frac{\phi_{zzz}^2}{U}dz+(\mu-N(t))\int \frac{\psi_{zzz}^2}{U}dz\\
&\leqslant \frac{2v_- - 2v_+ + D}{D}N(t)\int \left(\frac{\phi_{zz}^2}{U}+\frac{\psi_{zz}^2}{U}\right)dz+C\int \frac{\phi_z^2+\psi_z^2+\phi_{zz}^2+\psi_{zz}^2}{U}dz.
\end{aligned} \tag{4.5.112}
$$

Integrating (4.5.112) with respect to t, then using (4.5.82), Lemma 4.8 and the assumption $N(t) < \min\{\mu, D\}$, we obtain (4.5.102). Then the proof of Lemma 4.9 is completed. □

Chapter 5

Conclusions and Future Work

In chapter 2, based on the existence of Lyapunov functional, we obtain the critical mass $\frac{4\pi(1+\lambda)}{\alpha\chi}$ for the volume filling chemotaxis model (2.1.1) with the squeezing probability function $q(u) = \frac{1}{(1+u)^\lambda}(\lambda > 0)$ in two dimensions, which means that the solution exists globally with uniform-in-time bound if $\int_\Omega u_0 dx < \frac{4\pi(1+\lambda)}{\alpha\chi}$ and blows up in finite or infinite time if $\int_\Omega u_0 dx > \frac{4\pi(1+\lambda)}{\alpha\chi}$. Furthermore, when $\lambda > 1$, we proved that if there exist some initial data such that the corresponding solutions of (2.1.1) blow up, then it has to blow up at infinite time.

In chapter 3, we study the initial-boundary value problem of the ARKS chemotaxis model. The asymptotic behavior of solutions to the ARKS chemotaxis model was studied in one dimension. In two dimensional spaces, we show that if the repulsion dominates over attraction, then the globally bounded classical solutions exist for large initial data. Moreover we present a Lyapunov function at the first time for the irreducible three-component ARKS chemotaxis model which plays a central role to obtain our results.

In chapter 4, the asymptotic nonlinear stability of solutions to the Cauchy problem of a strongly coupled Burgers system arising in MHD turbulence was established. Our results confirm the existence of shock waves (or turbulence) numerically found in the literature [24, 103].

Based on the results of this thesis, a few interesting problems are proposed below to pursue in the future.

(i). In chapter 2, we have proved that if the initial mass $\int_\Omega u_0 dx > \frac{4\pi(1+\lambda)}{\alpha\chi}$, there exist some initial data such that the corresponding solutions of the volume-filling chemotaxis model blow up at infinite time if $\lambda > 1$. For the classical chemotaxis model, it has been proved in [29] that there exists a radially symmetric solution blows up in finite time by using the asymptotic expansion method. Hence we suspect that the solutions of volume-filling chemotaxis model (2.1.1) with $0 < \lambda \leqslant 1$ may blow up in finite time for $\int_\Omega u_0 dx > \frac{4\pi(1+\lambda)}{\alpha\chi}$. However, this needs to be verified.

(ii). The combination of volume-filling and cell kinetic model has been studied for different cases in [16, 91, 92, 93, 100]. More precisely, the global existence and pattern formation as well as the existence of a compact global attractor of solution have been studied in the literature [93, 100]. However in both papers [93, 100], they assumed that $q(U_{max}) = 0$, which implies the chemotactic force is switched off at $u = U_{max}$. For the case that there is no value of u at which chemotaxis is switched off (i.e., chemotaxis vanishes as $u \to \infty$), it was proved in [16] that the solution globally exists, however with the assumption that the signal production has a priori bound. If there exist some constants $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$, $M_1 > 0$ and $M_2 > 0$ such that $D(u) \geqslant M_1(u+1)^{-\alpha}$ and $\phi(u) \leqslant M_2(u+1)^{\beta-1}$ for all $u \geqslant 0$, and the cell kinetic function satisfies $f(u) \leqslant a - bu^r$ with $r > 1, a \geqslant 0, b > 0$, then the existence of global solution with uniform-in-time bound for $0 < \alpha + \beta < \frac{n}{2}$ has been established in [91]. For the chemotaxis model (2.1.1)-(2.1.3), we can check that $\alpha + \beta = 1$ in two dimensions, and hence it would be interesting to study the global existence of volume-filling chemotaxis model (2.1.1)-(2.1.3) in two dimensional spaces with the same kinetic function as in [91].

(iii). For the ARKS chemotaxis model discussed in chapter 3, we show that if the repulsion dominates over attraction, then the globally bounded classical solutions exist for large initial data. However if attraction prevails over repulsion, few results are known. There are still a few remaining open question to explore as follows.

Case 1. $\tau_1 = 1,\ \tau_2 = 0,\ \chi\alpha > \xi\gamma,\ \beta \neq \delta$ (Ongoing work). We have obtained a Lyapunov functional for this case. With the Lyapunov functional, we can study the critical mass problem to the system (3.1.1) in two dimensional spaces by using the constructed method as in chapter 2.

Case 2. $\tau_1 = \tau_2 = 1,\ \chi\alpha > \xi\gamma,\ \beta \neq \delta$. For this case, the system (3.1.1) can not be transformed into the classical chemotaxis model. However, we may use the asymptotic expansion method as in [29] to show that there exists a radially symmetric solution blows up in finite time.

(iv). In chapter 4, using the energy estimates method, we obtain the asymptotic stability of viscous waves for the MHD-Burgers system by assuming the condition that $D \geqslant \mu$. The energy estimates method is invalid for the case $D < \mu$. The nonlinear stability of viscous waves with $D < \mu$ may be explored by the spectral analysis. Moreover, the nonlinear stability of rarefaction wave with large initial data has been obtained in chapter 4. However there is not any result for the convergence rate. It is deserved to study convergence rate of the rarefaction wave with large initial data in the future.

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