INTEGRATED LOT-DELIVERY SUPPLIER-BUYER INVENTORY MODEL WITH DEMAND-DRIVEN PRODUCTION RATE FOR EXPONENTIALLY DETERIORATING ITEMS

WONG WAI HIM

M.Phil

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The Hong Kong Polytechnic University

Department of Applied Mathematics

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WONG Wai Him

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CERTIFICATE OF ORIGINALITY

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____________________________________ (Signed)

____________WONG Wai Him________(Name of student)

Abstract

Business focus has been shifted to the development of management of supply chains. The objective of supply chain management is to optimize the performance of the whole supply chain instead of independent optimization of individual parties. Lot-delivery is a common mode of transferring the concerned product from the supplier(s) to the buyer(s). Since Ghare and Schrader (1963)'s pioneering work, many researchers have presented inventory models on deteriorating items. However, there have been much fewer integrated lot-delivery models than EOQ and EPQ models of deteriorating items in literature. This research investigates integrated lot-delivery models for optimizing the supply chain of exponentially deteriorating items.

In predetermined production rate models, significant proportions of stock are built well before shipments because production rates are usually much higher than demand rates. Therefore, high inventory holding costs and deterioration costs are incurred. In addition, utilization of the production facilities is low. In this research, a demand-driven production rate continuous production model is proposed. In this model, the production rate is related to the delivery interval and is found by optimizing the total cost of the system. By having a lower production rate and a shorter delivery interval, the model can achieve a significant reduction in the total system cost per unit time, under conditions that are found by studying the effect of varying production rate on the total cost. Part of the cost saving can be allocated to finance additional resources for maintenance of the equipment. This model also

addresses the operational issues by minimizing idleness of production facilities and facilitating labour planning. There are two extensions of the proposed model. In some of the literature, researchers have presented EOQ models for deteriorating items with a non-deteriorating period. The proposed model has been extended to include a non-deteriorating period for the item and the effect of a finite production rate and a non-deteriorating period on the system is investigated. In the literature of inventory models, cost parameters have been assumed to be independent of production rate. As the proposed demand-driven production rate model uses a much lower production rate than usual predetermined production rates, the proposed model is extended to investigate a scenario in which some cost parameters increase when production rate decreases.

Chan and Kingman (2005, 2007) presented a synchronized model for a single-vendor multiple-buyer supply chain. The synchronized model performs better than the common order cycle model developed by Banerjee and Banerjee (1994). In this research, the synchronized model is extended for supply chains of exponentially deteriorating items. It has been found that the model has a better performance than the best costs from genetic algorithm and the optimal solutions of the common cycle approach. In view of increasing environmental concerns, a maximum deterioration constraint has been incorporated in the extended model in which the least cost solution that meets the constraint is found.

Utilization of production facilities is low in predetermined production rate models due to high production rates. In this research, a model of producing two products on the same production line for single buyer (or two buyers, one for each product) is presented. Two heuristics have been proposed: one considers time as a continuous variable, and the other is modified from the synchronized model.

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Table of Contents

Appendix

List of Figures

List of Tables

List of Notations

Basic parameters

k: deterioration rate of the item

D: demand rate of the single buyer single-buyer

A_b : ordering and other fixed cost per delivery of the single buyer

C_b : unit deterioration cost of the single buyer

H_b : inventory holding cost per unit per unit time of the single buyer

P: production rate of the vendor

S: production set up cost of the vendor

A_v : vendor's order processing cost and shipment cost per delivery to the single buyer

n_b : number of buyers in the supply chain

D_i : demand rate of buyer i

A_{bi} ordering and other fixed cost per delivery of buyer i

C_{bi} : unit deterioration cost of the buyer i

H_{bi} : inventory holding cost per unit per unit time of buyer i

A_{vi}: vendor's order processing cost and shipment cost per delivery to buyer i

Other parameters

T_T: delivery lead time to the single buyer when deterioration during delivery is considered

T_{Ti}: delivery lead time to buyer i when deterioration during delivery is considered

T_Θ: non-deteriorating period (Section 5.2)

C_{ba}: parameter for C_b (Section 5.3)

C_{bb}: parameter for C_b (Section 5.3)

C_{va}: parameter for C_v (Section 5.3)

C_{vb}: parameter for C_v (Section 5.3)

H_{ba}: parameter for H_b (Section 5.3)

H_{bb}: parameter for H_b (Section 5.3)

H_{va} parameter for H_v (Section 5.3)

H_{vb}: parameter for H_v (Section 5.3)

P_m: maximum permissible production rate (Section 5.4)

MAXDET1: maximum deteriorated quantity to production quantity (Section 7.2)

MAXDET2: maximum deteriorated quantity to annual demand (Section 7.2)

Basic variables and cost functions

Q_0: delivery quantity

T: system cycle time

T_p: production time in a system cycle

T_c: delivery cycle time of the single buyer

T_{ci}: delivery cycle time of buyer i

n: number of deliveries in a system cycle of the single buyer

n_i: number of deliveries in a system cycle of buyer i

TC_b: total cost of the single buyer per unit time

TC_{bi}: total cost of buyer i per unit time

TC_v: total cost of the vendor per unit time

TC_s: total system cost per unit time

Variables for the synchronized model, GA and common cycle approach

N: system cycle time in days

N^{GA}: system cycle time in days from GA

TC_s^{GA}: total system cost per unit time from GA

n^{CC}: optimal number of deliveries in a system cycle for the common cycle approach

TC_s^{CC}: optimal system cost per unit time for the common cycle approach

Cost functions for the 2-product system (Section 7.3)

$TC_{s,1}$: total system cost per unit time for product 1

$TC_{s,2}$: total system cost per unit time for product 2

Chapter 1

Introduction

1.1 Introduction

Optimization of inventory models started with Economic Order Quantity (EOQ) models of non-deteriorating items minimizing the inventory cost per unit time of the buyer. Economic Production Quantity (EPQ) models then followed in which goods are produced during the production stage of the overall cycle and the produced goods are immediately available to satisfy a continuous demand throughout the cycle. In EPQ models, the objective is to minimize the inventory cost per unit time of a supplier. In general, the supplier and the buyer(s) of a product are separate entities in different locations and goods are delivered to the buyer in lots. Business focus then shifted to the development and management of a supply chain which consists of supplier(s) or vendor(s) and buyer(s) of a product. The objective of supply chain management is on the optimization of the whole supply chain or system instead of independent optimization of the individual parties. Coordination between the parties of the system hence plays a vital role in order that the optimization of the whole system can be achieved.

In many inventory models it is assumed that the items are non-perishable or non-deteriorating and can be stored indefinitely to meet the future demands. This assumption, however, is not valid for all inventory items. Whereas some items such as fashioned goods or style goods may become obsolete due to changes in technology and/or changes in customer tastes and preferences, some other commodities may deteriorate in the course of time. Ghare and Schrader (1963) used the term inventory decay for depletion of inventory by 'other-than-demand methods'. Such depletion may take the form of direct spoilage, e.g., fruits and foodstuffs; the form of physical depletion, e.g., highly volatile liquids such as gasoline; or the form of deterioration, e.g., electronic components, radioactive substances, etc. In the same paper, these two authors developed the first EOQ model on exponentially deteriorating items. Raafat (1991) also referred to these three forms in defining decay or deterioration. Goyal and Giri (2001) defined deterioration as 'Deterioration refers to the damage, spoilage, dryness, vaporization, etc. of the products.'. They used the term perishable products for deteriorating items like foodstuffs, human blood, photographic film that are having a certain maximum usable lifetime; and the term decaying products for other deteriorating items like gasoline and radioactive substances that are having no shelf-life. After Ghare and Schrader's (1963) model, various EOQ, EPQ and lot-delivery models have been developed for deteriorating items. When decay during a period is proportional to the inventory level, the time to decay can be assumed by a negative exponential distribution. Exponential distribution has been one of the common distributions for time to deterioration in literature of inventory models for deteriorating items. Exponential deterioration or decay is also referred as having a constant rate of deterioration by many researchers.

The prime objective of supply chain management has been the cost/profit optimization of the system. Nowadays environmental concerns have become critical to the well-being of the society and organizations have to take measures to address the issue. Management, when formulating strategies for optimizing the system, has to consider environmental performance measures in addition to cost/financial performance. For example, management of some supply chains considers controlling the amount of gas emission that is adverse to the environment in their strategy. In addressing environmental issues, management of supply chains of deteriorating items can consider a perspective that is specific for deteriorating items. Deterioration implies raw materials, energy and other items required for the production of the product have been wasted, let alone labour and supervision effort. Wastage of (natural) resources causes impact to the society beyond monetary implications. Controlling the amount of deterioration helps reduce such wastage and enhances environment protection and this perspective could be considered when management of supply chains of deteriorating items formulate their strategy for optimizing the performance of their systems.

1.2 Motivation and Objectives of the Research

In the literature of inventory models of deteriorating items with uniform production and demand rates, the production rates are usually predetermined and usually much larger than the demand rates of the products. For example, the production rate is 3 times the demand rate in Misra's (1975) EPQ model, 3.2 times in Wee et al.'s (2008) lot-delivery model, 4 times in Yan et al.'s (2011) lot-delivery model, and slightly over 2 times in Sarkar's (2013) lot-delivery model, etc. For lot-delivery models, optimal solutions usually entail multiple deliveries within a cycle. This means that production stops before half or more of the deliveries have been made when the production rate is much higher than the demand rate. Although the solution is optimal, inventory holding costs and deterioration costs are still incurred during the "long" non-production stage (in addition to the production stage). On the operational side, the utilization of the production facilities is quite low in such systems, leaving the facilities idle most of the time. The shop floor management also has to consider reallocating the concerned labour to work on other products or production lines in order to minimize labour idleness. In this research, the first major area is the development of a lot-delivery continuous production model for reducing the total system cost. In the proposed model, the relation between the production rate and the delivery cycle time is found and their optimal values, for minimizing the total system cost per unit time, are to be determined from the constant demand rate and the concerned cost parameters. It is found that this lot-for-lot model can give a lower optimal system cost compared with the predetermined production rate model by having lower inventory level, which means that both inventory holding cost and deterioration (and hence wastage) cost are reduced. The continuous production model also addresses the mentioned operational issues by minimizing the idleness of the production facilities and facilitating manpower planning as a group of operators or workers can be allocated "permanently" to a certain production line. There is a concern of the model that there may be more wear and tear of the equipment due to on-going production. While a smaller production rate, compared with the predetermined production rate, may help reduce wear and tear to some extent, the cost saving with the model could allow more financial resources to be allocated for maintenance of the equipment.

By investigating the effect of changing production rate on the system cost of the predetermined production rate model, the conditions under which the proposed lot-delivery continuous production model gives a lower optimal system cost are found. The findings indicate that in many cases, the proposed model can give a lower optimal system cost than the predetermined production rate model.

Initiated by Philip (1974), some researchers developed EOQ models in which the time to deterioration of the items follows a 3-parameter Weibull distribution. In addition to the scale and shape parameters, this distribution includes a location parameter which if positive, means that the item starts deteriorating after a certain period of time. (A discussion of the 3-parameter Weibull distribution will be presented in Section 2.1 of Chapter 2.) Goel and Aggarwal (1980) considered that the exponential distribution is a particular case of the 3-parameter Weibull distribution with the shape parameter being one and the location parameter being zero. Some researchers thought differently. Wu et al. (2006) used the term "non-instantaneous deterioration" for phenomena in which items do not start deterioration immediately when they are received into inventory, and developed an optimal replenishment policy for "non-instantaneous deterioration" for exponentially deteriorating items. (Actually, Philip was the first researcher considering "non-instantaneous deterioration" without creating a term for it.) Inventory models with "non-instantaneous deterioration", whether for the exponential deterioration or for the 3-parameter Weibull distribution, have assumed that all units in a shipment are just "born" and start the non-deteriorating period when they are received by the buyer. Hence, all units in a shipment are at the same state, either all not subject to deterioration or all subject to deterioration, throughout the inventory cycle. In this research, the proposed lot-delivery continuous production model is extended to a model including a non-deteriorating period for exponentially deteriorating items. The non-deteriorating period affects both the vendor's and the buyer's inventory levels due to a finite production rate. At the same instant, some units may still be within the non-deteriorating period while other units have passed that period and start to deteriorate. The objective of this extended model is to find the optimal delivery cycle time and hence the optimal production rate for minimizing the total system cost per unit time.

In the literature of inventory models in which special discounts, time value of money/inflation are not considered, cost parameters are usually assumed to be constant even if the production rates are not constant. For the proposed lot-delivery continuous production model, the optimal production rate found is slightly higher than the demand rate and hence is much smaller than the usual predetermined production rates. Whether cost parameters can be assumed to be constant over a large range of production rates would affect the difference between the proposed continuous production model and the predetermined production rate model. A discussion of the potential variations of cost parameters with production rate will be presented in Chapter 5. The proposed model is extended for optimizing the system cost of a supply chain in which some of the cost parameters increase when the production rate is reduced.

Banerjee and Banerjee (1994) developed a coordinated one-vendor multi-buyer inventory control for minimizing the total system cost by adopting a common delivery cycle for all the buyers of the system. As different buyers may have very different demand rates and dissimilar cost parameters, the common cycle approach may only benefit few partners and also result in a high total system cost. Chan and Kingsman (2005, 2007) proposed a synchronized production-inventory model which enables the delivery cycle times of different buyers and the overall system cycle time to be optimized in a convenient time unit decided by the concerned supply chain. The model provides a lower total system cost than the common cycle approach. The model has another advantage. In the literature of inventory models, time is considered as a continuous variable and the optimal solution obtained needs to be rounded to a convenient time unit before being applied. For this synchronized model, the optimal cycle times and the system cycle time obtained are already integer multiples of the agreed convenient time unit, and therefore, can be adopted directly and practically into the system. The second area of this research is to extend this synchronized one-vendor multi-buyer model to a supply chain of exponentially deteriorating items. This is a predetermined production rate model. For non-deteriorating items, the length of the production period in a system cycle is the product of the system cycle time and the total demand rate divided by the production rate. The main challenge for extending the model to deteriorating items is that the length of the production period also depends on the numbers of deliveries which can be different for different buyers in the supply chain. A two-stage model and algorithm, working on an approximate cost function for getting an initial solution and then on the exact cost function for getting the optimal solution, has been developed in this research to tackle this issue.

There are increasing environmental concerns in the society. As deterioration results in wastage of resources and is adverse to environmental protection, reducing the amount of deterioration helps to address the issue of environmental concerns. If it is required (due to legal regulations or self-awareness) that the amount of deterioration of a supply chain does not exceed a certain level, the management will look for the most cost effective way to achieve this objective. This becomes a goal programming problem. The cost optimization algorithm for the synchronized model is modified to find the minimum cost solution of the supply chain subject to a maximum deterioration constraint.

For predetermined production rate models, the production facilities usually have a low utilization because the production rates are much higher than the demand rates.

The proposed lot-delivery continuous production model minimizes the idleness but does not increase the output of a "production line". To increase the output of the facilities, a model of producing two similar deteriorating items on the same production line for single-buyer is proposed in this research. Two heuristics have been developed. The first heuristic considers time as a continuous variable and the second one is modified from the synchronized model.

1.3 Outline of the Thesis

Chapter 2 presents a literature review of EOQ and EPQ models of deteriorating items. This is followed by a literature review of integrated lot-delivery models of both non-deteriorating and deteriorating items.

Chapter 3 presents an integrated lot-delivery model for a single-vendor single-buyer supply chain of exponentially deteriorating items. It starts with revisiting the EOQ model and the cost function of the vendor for the lot-for-lot policy. A lot-for-lot continuous production model with demand-driven production rate is then developed. Chapter 4 investigates into the effect of changing production rate for the predetermined production rate model. From the results of the analysis, the conditions under which the proposed model gives a lower optimal system cost than the predetermined production rate model are found. Chapter 5 presents some extended models from the lot-delivery continuous production model developed in Chapter 3. One extension is to include a non-deteriorating period in the model.

Another extension of the model is to optimize the total cost of a system in which some cost parameters increase when production rate decreases. Finally, a heuristic for extending the model developed in Chapter 3 to multi-buyer supply chains is proposed.

In Chapter 6, Chan and Kingsman (2005, 2007) synchronized single-vendor multi-buyer production and delivery model is extended for exponentially deteriorating items. The details of the model and algorithmic development are presented. In Chapter 7, the model is modified for solving the goal programming problem of minimizing the system cost subject to a maximum deterioration constraint. In the second part of Chapter 7, a model for producing two similar products on a production line is proposed. Two heuristics have been developed for this model.

Chapter 8 summarizes the thesis and suggests some future research directions extending from the research of the thesis.

Chapter 2

Literature Review

2.1 EOQ and EPQ Models for Deteriorating Items

2.1.1 EOQ models without Price Discounts or Trade Credits

Like non-deteriorating items, development of inventory models for deteriorating items have also started with Economic Order Quantity (EOQ) and Economic Production Quantity (EPQ) models. Ghare and Schrader (1963) were the first authors to study decaying inventory problems. They used the term inventory decay for depletion of inventory by ‘other-than-demand methods’ and elaborated that such depletion may take the forms including direct spoilage, physical depletion and deterioration. They observed that for some items, the decay during a time period is proportional to the inventory level. Hence the continuously declining inventory function can be presented by a negative exponential function. They derived the differential equation $\frac{dI}{dt} + kI = -D$ where I is the inventory level at time t, k is the constant deterioration rate and D is the constant demand rate. Ghare and Schrader then developed an EOQ model for exponentially decaying item with a constant demand rate. By expanding the exponential term and neglecting higher power terms, they obtained a solution for the economic reorder period and the economic order quantity. They introduced a constant parameter “inventory holding cost as a fraction of the maximum inventory” for the inventory holding cost in the cost function. This implies that they have made a further approximation that the inventory level drops linearly in the inventory cycle in the model. Exponential deterioration is also referred as a constant rate of deterioration by many researchers.

Covert and Philip (1973) developed an EOQ model for deteriorating items whose time to deterioration follows a two-parameter Weibull distribution. The instantaneous deterioration rate at time t is given by $\alpha\beta t^{\beta-1}$ where α and β are the scale parameter and the shape parameter respectively of the Weibull distribution. Philip (1974) extended the previous model by considering a 3-parameter Weibull distribution whose instantaneous deterioration rate is $\alpha\beta(t-\gamma)^{\beta-1}$, γ being the location parameter of the distribution. If the location parameter is negative, there is an initial deterioration rate. If it is negative, the item starts to deteriorate after a certain period of time. Setting $\beta=1$, the 3-parameter Weibull distribution is reduced to an exponential distribution and the location parameter disappears from the instantaneous deterioration rate. Both these EOQ models were for constant demand rates without shortages (same as Ghare and Schrader’s model), and assumed that the average inventory level is half of the order quantity (the maximum inventory level) for finding the inventory holding cost, which is only valid for a linear inventory depletion curve. In the solution procedure, terms that cannot be integrated to closed forms are expanded to infinite series. Then term by term integration is applied and the higher power terms are truncated for finding the solution.

Some other researchers also adopted the linearity assumption for inventory depletion curves in their inventory models. Shah and Jaiswal (1977) developed a periodic review inventory model for deteriorating items with stochastic demands but no shortages. In this model, the length of the review period and the reorder level are assumed to be prescribed constants and the objective is to find the optimal order quantity for minimizing the expected total cost. The authors assumed that the average inventory level function is linear in deriving the expected total cost function of the system. In the example provided, the mean and the cumulative distribution function were found from the given probability density function of demand. However, the example was then continued with a symbolic mathematical procedure without any particular numerical values for the length of the review period, the reorder level and the related cost parameters; and hence, no numerical answer could be obtained. Tadikamalla (1978) developed an EOQ model (constant demand rate) without shortages for items following the (2-parameter) Gamma distribution for deterioration. In this model it was also assumed that the average inventory level is half of the maximum inventory level. There is no closed form expression for the instantaneous deterioration rate. The author used an approximate expression followed by numerical methods with a computer program developed for finding the optimal solution.

Shah (1977) developed an order-level lot-size model which is an EOQ model allowing shortages for fulfilling a demand having a constant rate. The exponential distribution and the 2-parameter Weibull distribution were shown as particular cases after the general approach had been presented. (Aggarwal (1979) responded to this model by using the exact expression for the average inventory level. In particular, Aggarwal presented the equations, for exponential deterioration, both in the exact form and in the algebraic form which ignored the higher power terms in the series expansion of the exponential terms.) Jalan and Chaudhuri (1996) extended Covert and Philip (1973)'s model (2-parameter Weibull distribution) by allowing shortages and having a demand rate which is a linearly increasing function of time. Chakrabarty et al. (1998) extended Philip (1974)'s model (3-parameter Weibull distribution) also by allowing shortages and having a demand rate which is a linearly increasing function of time. For these three models, shortages are immediately fulfilled at the beginning of a cycle due to an infinite replenishment rate. The average inventory level for the no-shortage period is assumed to be half of the maximum inventory level (linearity approximation) and the "Weibull terms" are expanded and truncated as in previous models.

Cohen (1977) developed a joint pricing and ordering policy for exponentially deteriorating items. In this model, the demand rate is a function of the unit price. The author approximated exponential terms by truncating higher power terms in the expanded series. Both no shortage and with shortage models were considered with a numerical example for the first case. Goel and Aggarwal (1980) presented a model for determining the pricing and ordering policy for deteriorating items following the 3-parameter Weibull distribution, for both with and without shortage cases. This is a profit maximization model in which the demand rate is a function of the selling price and the revenue function is assumed to be concave. The authors considered the exact expression instead of assuming a linear approximation for finding the inventory holding cost. The solution procedure also involved approximation of exponential terms by neglecting higher power terms in the series expansion. No numerical example has been provided. Goel and Aggarwal considered the exponential distribution as a special case of the 3-parameter Weibull distribution with $\beta = 1$ and $\gamma = 0$. This is arguable as $\beta = 1$ and $\gamma > 0$ means that the item has an instantaneous deterioration rate of α for $t > \gamma$. Items following this distribution deteriorate exponentially after a non-deteriorating period of γ. (As mentioned in Chapter 1, Wu et al. (2006) developed EOQ model with "non-instantaneous deterioration" for exponentially deteriorating items.)

Hollier and Mak (1983) presented mathematical models for optimal inventory replenishment policies, with an infinite replenishment rate, for exponentially deteriorating items (with deterioration rate ν) having a negative exponential demand rate of the form $Ae^{-\alpha t}$. Both constant and variable replenishment period policies were considered. For the second policy the optimal replenishment intervals follow a decreasing sequence. This is expected due to the decreasing demand rate. This is a finite-horizon model. Cheng (1989) developed a model for the infinite-horizon case. Both these models assumed $\alpha > \nu$ (using Hollier and Mak's notations) and Cheng stated that this assumption is to ensure that demand can be met after accounting for decay. However, this is not needed. (Just consider the limiting case of a constant demand rate, $\alpha = 0 < \nu$.) Actually, if $\alpha < \nu$, the inventory level will not drop to a negative value due to "negative negative" cancelled out in the expression for inventory level. Wee (1995) extended the model of Hollier and Mak (1983) for the case of equal replenishment period to allow for shortages. After considering complete backordering, Wee provided a modified equation for partial backordering with a constant fraction of shortages being backordered.

Instead of assuming a constant fraction of shortages for partial backlogging, Chang and Dye (1999) considered the scenario of the backlogging rate being a decreasing function of the waiting time, that is, the longer the waiting time, more shortages will be lost, for exponentially deteriorating items. The authors presented an EOQ model in which the demand rate $f(t)$ is a time continuous monotonic function with $f(t)/f'(t)$ non-decreasing in time *t*. They defined the backlogging rate to be $1/[1+\alpha(t_i - t)]$ where $\alpha > 0$ and t_i is the time at which the *i*th replenishment is made for the non-lost shortages of the (*i*-1)th cycle and part of the demand for the *i*th cycle. This model minimizes the total system cost over a finite planning horizon by finding the optimal starting times for shortages and the optimal replenishment times for cycles of unequal lengths.

Some other researchers also used the backlogging rate of $1/[1+\alpha(t_i - t)]$ for partial backordering in their EOQ models. Wu et al. (2006) developed an EOQ model with this backlogging rate for non-instantaneous deteriorating items that are exponentially deteriorating after a certain period. The demand rate depends on the inventory level and is of the form $D(t)=\alpha+\beta I(t)$ where α and β are positive for positive inventory level, and is α during the shortage period. Each cycle starts with a specified non-deteriorating period during which the inventory is consumed due to demand only. After this period, the inventory is reduced due to demand and deterioration as usual until the inventory drops to zero and the shortage period then starts. The authors presented the mathematical procedure to find the optimal period of the positive inventory stage and the optimal cycle time for minimizing the total cost per unit time. Rajeswari and Vanjikkodi (2012) presented an EOQ model with the same backlogging rate for deteriorating items following the 2-parameter Weibull distribution. This model has an unusual demand rate in the form of $dt^{(1-n)/n} / nT^{1/n}$ where d and n are positive, and T is the cycle time. The cycle time is a parameter. The only decision variable to be found is the length of the positive inventory period in the cycle for minimizing the cost per unit time. The cycle time is one year in the numerical example provided. This means that it has been predetermined that there is one cycle in a year for this example. This model is useful if it has been decided that the cycle time is to be selected from several desirable values, by finding the minimum cost among the optimal costs for all these desirable cycle times. However, the author did not mention this purpose or other rationale for using a predetermined cycle time in this model. Chowdhury et al. (2014) presented an order-level inventory model for a deteriorating item with a quadratic demand rate, same form of backlogging rate for partial backordering, but with a 3-parameter Weibull distribution for time to deterioration. Same as Chang and Dye (1999), this model attempts to minimize the total cost over a finite planning horizon with unequal cycle lengths.

Banerjee and Agrawal (2008) developed a two-warehouse model for items following the 3-parameter Weibull distribution for time to deterioration. In this model, the demand rate is a non-decreasing function of time and shortages are fully backlogged. At the beginning of a replenishment period, a certain order quantity of the items is received instantaneously and the shortages of the previous cycle are immediately fulfilled. The remaining quantity is larger than the storage capacity of the own warehouse (OW). Goods that cannot be kept in OW are immediately sent to the rented warehouse (RW), and are sent back to OW with a transportation cost per unit but in negligible time for satisfying the demand first. After the inventory in RW is consumed, goods in OW will then be consumed for satisfying demand. (Depending on the location parameter, goods in OW may start deterioration before or after the consumption of inventory in RW.) The authors presented a solution procedure for finding the optimal solution to minimize the total cost per unit time. Truncated exponential terms and linear approximation for the inventory curve are taken for simplifying the mathematical expressions.

Begum et al. (2010) developed an EOQ model for deteriorating items following a 3-parameter Weibull distribution and under a price-dependent demand. The demand rate is a linearly decreasing function of selling price and shortages are fully backordered. This is a profit maximization model with the usual method of using truncated series approximations with Weibull distribution in deriving the profit function. A special feature of this model is that the cycle time is an integer of a time unit. The authors considered two cases: (1) finding the optimal period for the positive inventory stage of a cycle and the optimal selling price, with a given cycle time; (2) repeating case (1) for a given value of the cycle time *T*, and "Continue the process unless and until get an optimal solution using other values of *T*.". The authors did not suggest how a range of *T* can be set. In the numerical example given that also does not specify the range of cycle time, the authors showed the net profit for cycle times of 4 to 9 days and concluded that the optimal solution is with a cycle time of 9 days for several decay rates. The authors have proved the convexity for the net profit function in 2 variables for case (1). It seems more reasonable if they have at least indicated the net profit for a cycle time of 10 days in order to justify their conclusion for the optimal solution, by assuming convexity for the profit function with 3 variables for case (2).

2.1.2 EPQ Models without Price Discounts or Trade Credits

Misra (1975) developed the first EPQ model for deteriorating items in which both varying and constant rates of deterioration were considered for constant production and demand rates. The following differential equations were derived:

$$\frac{dI}{dt} + kI = P - D \quad \text{and} \quad \frac{dI}{dt} + kI = -D \,,$$

for the inventory level of the production stage and the non-production stage of a cycle respectively, where I, k, P and D are the inventory level, deterioration rate, production rate and demand rate of the system. The author used the 2-parameter Weibull distribution as an example of varying rate of deterioration and suggested to use a linear approximation for the inventory depletion curve as in EPQ models for non-deteriorating items. For constant deterioration rate (exponential deterioration), the author expanded the exponential terms followed by neglecting higher power terms for approximating the average inventory level and the relation between the production time and the non-production time of a cycle. Closed form expressions for the optimal cycle time and the optimal production lot size were finally obtained by taking further approximations in the mathematical procedure.

Mak (1982) extended Misra (1975)'s model and developed a production lot size model with backlogging for shortages. In this model, there are four stages and the backlog is fulfilled at the first stage of each cycle at a rate of $P-D$ where P and D are the constant production and demand rates respectively. As all the produced units are immediately consumed, there is no deterioration at this stage. The second and third stages are the same as the two stages in Misra's model. The last stage is of negative inventory level (hence, no deterioration) for shortages. After presenting the general mathematical procedure, the author considered exponential deterioration and obtained the cost function in two variables also by approximating the exponential terms by truncating higher power terms in the expanded series. It will require numerical methods to find the optimal solution. By approximating the durations for stage 2 and stage 3 to be linearly related (again a linear approximation for the inventory depletion curve), the author obtained closed form expressions for the optimal solution.

Elsayed and Teresi (1983) developed two EPQ models with constant production rate and allowing shortages. In the first model the demand rate is constant and deterioration is described by a two-parameter Weibull distribution. In the second model the demand rate is the expected value of a normal distribution; and the deterioration rate is the expected value of a two-parameter Weibull distribution. The authors mentioned that they developed computer programs and used search routine to find the optimal solutions. It is unusual that the authors assumed that there is still deterioration during the backorder stage when all the produced units are consumed for fulfilling the current demand and back-orders. The authors derived the total cost function containing integrals for the first model but did not indicate whether they used Philip's method or other methods to handle the integrals which did not have closed form expressions. Also they derived the relation between durations of the different stages of a cycle as if the inventory curves are straight lines for both models. Perhaps the most special feature of these models is that there is no unit cost of deteriorated item in the total cost function. This model has another special feature of having two shortage costs: one shortage cost in dollars/unit short in addition to the usual shortage cost in dollars/unit short/unit time in other models in which shortages are also fully or partially backordered.

Raafat et al. (1991) worked on Mak's (1982) model and derived the exact cost function for exponential deterioration without approximating the exponential terms and without assuming a linear inventory depletion curve. The authors utilized a computerized search technique for finding the optimal solution from the exact cost function. Heng et al. (1991) presented an order-level lot-size inventory model for exponential decay. This model is basically same as Mak (1982) while the solution procedure is "between" that of Mak and Raafat. The authors approximated exponential terms by neglecting higher power terms after expansion, and used a computerized search algorithm to find the optimal solution.

Wee (1993) developed an EPQ model allowing partial backordering for exponentially deteriorating items with constant production and demand rates. The author also approximated the exponential terms by the usual expansion and truncation method for finding the average inventory level and adopted Misra's approximate expression for relating the production time and the non-production time for the two positive inventory level stages. The differential equations for the inventory levels during the shortage stage and the backorder stage are:

$$\frac{dI_3}{dt} = -BD \quad \text{and} \quad \frac{dI_4}{dt} = B(P-D)$$

where B is the fraction of the demand during the shortage stage to be back-ordered, I_3 is the inventory level during the shortage stage, and I_4 is the inventory level during the backorder stage. The second equation seems wrong. All the current demand during the backorder stage should be fulfilled instantaneously to avoid further increase in shortage. Therefore, the backordering should still be made up at a rate of $P-D$ due to constant production and demand rates assumed in this model.

Some researchers considered deteriorating raw materials in production-inventory models. Park (1983) developed an EPQ model for exponentially deteriorating raw materials items and a non-deteriorating product. In this model, raw materials just arrive at the start of a production run of the product. During production of the finished product, raw materials are consumed for making the product and for deterioration. The usual procedure of approximating exponential terms is adopted in the solution procedure. Raafat (1985) extended this model to include exponential deterioration for the finished products. The author applied computerized search techniques on the exact cost function to find the optimal solution for the production system. Both these two models are for single-stage production systems with constant demand and production rates. Goyal and Gunasekaran (1995) developed an integrated production-inventory-marketing model for maximizing the profit of a multi-stage production system. In this model, raw materials, in-process inventory and the finished product are exponentially deteriorating items. The demand rate for the product depends on its unit price and the number of times it is advertised; and is assumed to be uniform over the planning horizon for given values of these parameters. Each order of raw materials can be used for several production batches. The objective of the model is to maximize the profit by finding the optimal order quantity for raw materials and the optimal production batch size for production. The authors used a direct pattern search method for finding the optimal solution. The numerical example shows a 3-stage production system. The reciprocals of the unit processing times for the first and second processes are less than the demand rate. Although each raw material order is split into several production batches, it has not been mentioned that there are several production facilities for these batches to be processed at the same time.

Balkhi and Benkherouf (1996) were probably the first authors to present a production lot size inventory model for exponentially deteriorating items with varying production and demand rates. Based on Raafat et al. (1991), this model is also a 4-stage model allowing shortages to be fully back-ordered. The production rate and the demand rate are functions of time. After presenting the mathematical procedure, the authors suggested that a search numerical procedure can be used for finding the optimal solution. The authors provided a numerical example in which the production rate and the demand rate are increasing exponential functions of time but at a constant ratio at any time.

Bhunia and Maiti (1998) developed EPQ models, with and without shortages, in which both the deterioration rate and the demand rate increase linearly with time and the replenishment rate depends on both the inventory level $Q(t)$ and the demand rate as follows:

deterioration rate: $\theta(t) = \lambda + \mu t \ll 1,\ 0 < \lambda,\ \mu \ll 1$;

demand rate: $D(t) = a + bt,\ a > 0,\ b \ll 0$; and

replenishment rate: $R(t) = \alpha - \beta Q(t) + \gamma D(t),\ \alpha > 0,\ \gamma \geq 0,\ 0 \leq \beta < 1$.

By truncating the Taylor expansion of the exponential terms, the authors simplified the cost function into an algebraic expression and adopted the Newton-Raphson method for finding the solution.

Balkhi (1999) considered a production-inventory model in which production rate, demand rate, and deterioration rates of raw materials and the finished product are functions of time. This is a one-stage production system same as Park (1983) and Raafat (1985) in which raw materials just arrive at the start of the production stage, and shortages of raw materials and the finished product are not allowed. Goyal and Giri (2003) considered a production-inventory problem allowing partial backlogging of a constant fraction of shortages. This model does not consider deterioration of raw materials, while production rate, demand rate and deterioration rate of the product are functions of time. In all these four models, the cost parameters are constant although the production rates are continuously changing. In the numerical examples provided, only the solutions for the first cycle starting at time 0 are shown.

Abad (2003) developed an optimal pricing and production lot-sizing model for exponentially deteriorating items. The production rate is constant and the demand is a decreasing function of the unit selling price. The model allows partial backordering at a rate of $k_0 e^{-k_1 \tau}$ where τ is the waiting time, $k_0 < 1$ and $k_1 \geq 0$. This is a profit maximization model. The author suggested an iterative procedure that can maximize the profit with a certain starting value of the selling price; and suggested to repeat the procedure with different starting values of the selling price.

Teng and Chang (2005) developed an EPQ model for an exponentially deteriorating item whose demand rate depends on both the inventory level $I(t)$ and the unit selling price *p*. The demand rate is $D = \alpha(p) + \beta I(t)$ where $d\alpha / dp < 0$ and $\beta \geq 0$, and shortages are not allowed. Production rate is constant. Instead of the zero inventory level at the beginning and the end of a cycle in usual EPQ models, this model has an initial and ending inventory of Q units ($Q \geq 0$) and the maximum inventory cannot exceed a certain prescribed value. The authors presented the model and an algorithm to find the optimal unit price, production time and ending inventory level for maximizing the profit per unit time.

Rework has been considered as an issue of reverse logistics and green supply chain. Widyadana and Wee (2012) suggested an economic production quantity model for deteriorating items with multiple production setups followed by one rework setup in a system cycle. In the model it is assumed that defective items are generated during production only and all defective items can be reworked to yield good quality items. Production, rework and demand rates are assumed to be constant. Rework starts after the non-defective units from the production setups have been consumed for demand or deterioration. Exponential terms are approximated in deriving the concerned expressions in the model and solution procedure.

2.1.3 EOQ and EPQ Models with Price Discounts, Trade Credits and Time-Value of Money

Wee and Yu (1997) developed an EOQ model for exponentially deteriorating items in which there is an opportunity to place one order with a temporary price discount. For the case of the opportunity temporary price discount occurring at the regular replenishment time, the model, allowing no shortages, is to find out whether the buyer should (i) place one EOQ order based on the reduced price and then resume ordering the EOQ based on the usual price, or (ii) place an order of larger quantity (and the optimal quantity) with the reduced price. If the time to place an order with the price discount does not occur at the regular replenishment time, the model is to find out whether an order should be placed with the reduced price and the optimal order quantity. In this model, the inventory holding cost per unit per unit time is a fraction of the unit purchase price and hence the holding cost for units purchased at the discounted price reduces accordingly. Exponential terms are approximated by truncating the Taylor series in the solution procedure. Chang and Dye (2000) developed an EOQ model similar to Wee and Yu (1997) but for items subject to deterioration following a two-parameter Weibull distribution, also with an opportunity to place a special order with a price discount. The inventory holding cost per unit per time is also reduced for items purchased at the discounted price.

The above price discount models do not allow shortages. Wee (1999) developed a profit maximization inventory model with quantity discount, pricing and partial backordering (a constant fraction of shortages to be backordered) for deteriorating items following the 2-parameter Weibull distribution. The quantity discount is in terms of price breaks based on the order quantity. The unit holding cost is partly constant and partly proportional to the unit cost and the demand rate is a linear decreasing function of the selling price. The model attempts to find the optimal selling price, optimal period of positive inventory stage of a cycle, and optimal order quantity for maximizing the profit, for a given cycle time which is a discrete variable. In the solution procedure, the author assumed that the maximum profit for a cycle time of T^* is the optimal solution if it is larger than the maximum profits for cycle times of T^*-1 and T^*+1. However, the author did not mention how to set a range for the cycle time that is assumed to be a discrete variable. Taleizadeh et al. (2013) developed an EOQ model with a temporary price discount for exponentially deteriorating items allowing shortages to be fully backordered. In this model, the buyer is provided with an opportunity to place an order with a discounted price when an order is to be placed. The holding cost is reduced for items purchased with the reduced price. This is a cost minimization model in which the optimal ordering policy in response to the temporary price discount is determined. Exponential terms are approximated in the solution procedure for both these models.

Hwang and Shinn (1997) developed EOQ model with permissible delay in payments and no shortages for exponentially deteriorating items. This model attempts to determine the optimal unit retail price and the optimal lot size for the items for maximizing the annual profit. The demand rate depends on the retail price and is of the form $D = KP^{-\beta}$ where P is the unit retail price, K and β are positive constants. It is assumed that if the delivery cycle time is larger than the credit period, the purchase cost for the remaining inventory at the retailer (total inventory in the balance portion of the cycle time) has to be financed with an interest rate which is at least as high as that earned by the retailer due to the trade credit. The authors used truncated series for the exponential terms in the price function in the solution procedure.

Chang et al. (2003) developed an EOQ model for deteriorating items for minimizing the total annual cost under the assumption that the purchaser will be offered a trade credit period if the order quantity exceeds a certain value. After presenting the general mathematical procedure, the authors considered the case of constant demand rate and constant deterioration rate. Shah (2010) considered the scenario of the supplier offering a one-time extended credit period to the retailer and formulated an EOQ model for this special ordering for exponentially deteriorating items. Both these models also used the truncated series method for finding the approximate optimal cycle times.

Sarkar (2012) developed an "EOQ" model for deteriorating items with time-varying demand and deterioration rates and permissible delay in payments. This model has a constant finite replenishment rate whose effect on inventory level is included in the model development. It seems more appropriate to refer this as an EPQ model. The demand rate is an increasing quadratic function of time while the deterioration rate at time t is given by $1/(1+R-t)$ where R is the maximum lifetime of the product, and the model does not allows shortages. Due to incorrect mathematical procedure, the production time was found as $t_1 = \frac{1}{\mu}\int_0^T D(t)dt$, as if there is no deterioration during production, where μ and $D(t)$ are the replenishment rate and the demand rate respectively. This model considers several possible credit periods and attempts to find the best option and the associated cycle time, from the optimal solutions for different credit periods, for maximizing the profit per unit time. In the numerical example provided, the product has a maximum lifetime of 3 years but only the solution for the first cycle is shown.

(The correct expressions for the inventory levels should be:

$Q_1(t) = (1+R-t)\int_0^t \frac{\mu - D(u)}{1+R-u}du$ for the production stage and

$Q_2(t) = (1+R-t)\int_t^T \frac{D(u)}{1+R-u}du$ for the non-production stage; and not

$Q_1(t) = (1+R-t)\int_0^t \frac{\mu - D(t)}{1+R-t}dt$ and $Q_2(t) = (1+R-t)\int_t^T \frac{D(t)}{1+R-t}dt$ as indicated in the paper leading to expressions for deterioration being cancelled out.)

Sarkar and Chakrabarti (2013) presented an EPQ model under permissible delay in payments for deteriorating items following the 2-parameter Weibull distribution. The demand rate is exponential and the production rate is a constant multiple of the demand rate. Shortages are fully backordered. The authors found the expressions for inventory level in the positive inventory stage by approximating exponential terms and then derived the cost function for the two cases of the credit period being longer or shorter than the positive inventory period. However, unlike the above models with credit periods, the authors did not provide further details as how the optimal solution is to be found and they only advised the name of the software used.

Wee and Law (1999) developed a profit maximization EPQ model which considers time value of money for deteriorating items following the 2-parameter Weibull distribution. Shortages are allowed and are fully back-ordered. Revenue and costs are discounted at a constant rate to present value with a continuous compounding cash-flow approach. The production rate is constant while the demand rate is a linearly decreasing function of the selling price. The planning horizon is divided into a number of system cycles of equal interval. The model attempts to find the selling price at time 0, the number of system cycles and the duration for one of the four stages of a system cycle for maximizing the net present value of the total profit over the planning horizon. (The durations for the other three stages can be expressed in terms of the chosen stage by approximating the concerned terms by the usual truncating series method.) As mentioned, the usual truncating series method is used in this model for setting up the various components of the objective function.

Wee and Law (2001) presented a replenishment and pricing policy for deteriorating items taking into account the time value of money. This is basically the EOQ version of their EPQ model, Wee and Law (1999), mentioned in the last paragraph. The 4-stage EPQ model becomes a 2-stage EOQ model as there is no production stage and all the shortages are instantaneously back-ordered. Same numerical example is provided except that there is no finite production rate. The optimal solution is, as expected, the same as that for the limiting case of infinite production rate in the numerical example in the EPQ model.

Hou (2006) developed an EOQ model for exponentially deteriorating items with stock-dependent demand rate under inflation and time discounting. The demand rate $D(t)$ is a linear increasing function of the inventory level $I(t)$ and is of the form $D(t) = \alpha + \beta I(t)$ where $\alpha > 0$ and $0 \le \beta \le 1$. During the shortage period, the demand rate is α and shortages are completely back-ordered. The time value of money is represented by a constant net discount rate of inflation which is the difference between the discount rate and inflation rate. The planning horizon is also divided into a number of cycles of the equal cycle time. Unlike the two models of Wee and Law, this is a cost minimization model, with all costs discounted to present values. Hou did not use approximations for exponential terms in the solution procedure. He used the Newton-Raphson method on the exact expression followed by applying an optimal solution procedure suggested by a researcher on quality control to find the optimal solution for his problem.

2.2. Integrated Lot-delivery Vendor and Buyer Models

2.2.1 Integrated Lot-delivery Models for Non-deteriorating Items

Goyal (1977) developed an integrated lot-delivery inventory model for minimizing the total relevant costs of a single supplier-single customer system for non-deteriorating items. At the beginning of a system cycle, a shipment is made to the customer and the supplier's inventory level is $(n-1)Q$ where n is the number of deliveries in a system cycle and Q is the shipment quantity. The effect of rate of production on the supplier's inventory level was not considered and the model is basically one with an infinite production rate. Banerjee (1986) developed a joint economic-lot-size model for non-deteriorating items with a finite production rate and single delivery per production lot. He showed that the optimal system cost, by minimizing the total cost of the two parties per unit time, is lower than the sum of the optimal cost of one of the parties and the corresponding cost of the other party. In other words, system optimization, if agreed, is better than individual optimization. Goyal (1988) showed that allowing several deliveries to the buyer per production batch can result in a lower system cost than a lot-for-lot policy. This model, however, assumed that a production lot is completed before the first delivery. Lu (1995) relaxed this assumption and developed a heuristic for an integrated one-vendor multi-buyer inventory model with the objective of minimizing the vendor's cost while not exceeding the maximum cost(s) that the buyer(s) will accept. He also presented the optimal solution for his model for a one-vendor one-buyer case.

Goyal (1995), in response to Lu (1995), suggested the policy of increasing the size of each successive shipment by a factor of P / D (production rate divided by demand rate). Goyal verified that this policy can reduce the total system cost by using the same numerical example as Lu. Further discussion on shipment sizes continued. Hill (1997) suggested that instead of just predetermining P / D as the factor for increasing the size of successive shipment, a value λ between 1 and P / D should be found for minimizing the total cost. He suggested that the optimal numbers of shipment for $\lambda = 1$ and $\lambda = P / D$ should be found; then a full search should be conducted over λ for each integral value between these two optimal numbers of shipment. However, the author did not outline how to conduct the search over λ which is not discrete. Goyal and Nebebe (2000), and Goyal (2000) suggested different policies for one-vendor one-buyer integrated production-inventory models in which the shipments in a system cycle increase (or increase, then equal) in size.

Banerjee and Banerjee (1994) developed a coordinated one-vendor multi-buyer inventory control model for minimizing the total system cost assuming a common cycle approach. This model assumes that the use of electronic data interchange (EDI) enables the supplier to monitor the consumption pattern of the buyers as they are linked together on a real time basis. There are no buyers' ordering costs as it is not necessary for the buyers to place orders, and the supplier can arrange deliveries based on a prearranged decision system. The model also assumes that all the parties have agreed that shipments are made at fixed intervals common to all buyers. The authors developed a mathematical procedure for minimizing the expected total relevant cost of the system per unit time. They provided a numerical example of three buyers and worked out that the vendor and two buyers can benefit from this coordination policy. The buyer that does not benefit from this policy (even though there is no buyer's ordering cost in this policy) has a mean demand rate of 5000 units per year while the other two buyers have mean demand rates of 1000 and 800 units per year. Probably the two "similar" buyers dominate when a common cycle approach is adopted. Therefore, for a supply chain of more buyers having dissimilar demand rates and probably also dissimilar cost parameters, maybe not many parties can benefit from the common delivery cycle policy, particularly if buyers' ordering costs cannot be eliminated.

The common cycle is a tight constraint for multi-buyer supply chains. Chan and Kingsman (2005, 2007) proposed a model of synchronized delivery and production cycles for a coordinated single-vendor multi-buyer supply chain. The main objective of this model is to reduce the total system cost by allowing different buyers to have different delivery intervals and hence delivery lot sizes. In literature of inventory models, time is usually considered as a continuous variable. The system cycle time and delivery cycle time(s) obtained in the (optimal) solutions are particular values on a continuous basis. In practice, the concerned parties have to round these cycle times to a convenient time unit for implementation into their systems. In this model, the system cycle time and the delivery intervals are integer multiples of a convenient time unit agreed by the parties of the supply chain. An algorithm with day being the convenient time unit has been developed. Assuming the maximum planning horizon to be a year, the range of system cycle times is from 1 unit to 365 units. The optimal solution for each system cycle time is found. Then the least-cost solution is chosen among the 365 optimal solutions. Hence, this model has an additional advantage that the least-cost solution obtained can be directly implemented into the supply chain.

2.2.2 Integrated Lot-delivery Models for Deteriorating Items

Yang and Wee (2000) developed a lot-delivery model for minimizing the cost of a single-vendor single-buyer system for exponentially deteriorating items allowing multiple deliveries per production cycle. The approach of this model has two flaws. (The authors mentioned that "the derived average stock level sometimes turns out to be negative", in Wee et al. (2008) where they provided an improved solution.) Firstly, the vendor starts production and makes the first delivery both at the beginning of the cycle. However, at this instant the vendor's inventory from the previous production lot has already been consumed for meeting the demand of the previous cycle. This implied that the first delivery of a cycle is made without inventory. Secondly, the authors applied Misra's approximate expression, for the relation between the length of the production period and that of the non-production period of an EPQ model, in this lot-delivery model. In EPQ models, some of the produced units are immediately consumed for demand. Therefore, vendor's inventory increases at a higher rate in lot-delivery models than EPQ models, and more units will be deteriorated due to higher inventory level. As a result, a lot-delivery system will require a longer production time in order to produce more units to compensate for the higher loss due to deterioration. In this model, exponential terms are approximated by truncating the series expansion of the terms in deriving the cost function.

Wu and Yee (2001) presented a lot-delivery buyer-seller joint cost model which was basically an extension of the model of Goyal (1977) to exponentially deteriorating as the effect of rate of production was not considered. The seller has an initial inventory level of the total required quantity (for shipments and deterioration in the seller's inventory) minus the shipment quantity due to the first shipment of the cycle. The authors used the approximation of $e^x \approx (2+x)/(2-x)$ in deriving the cost function for the buyer's cost. They used the relation $I_s(t) = I_0(1-k)^t$, where I_0 is the initial inventory level and k is the deterioration, instead of using the relation $I_s(t) = I_0 e^{-kt}$ for the supplier's inventory level in between two deliveries. The authors named this model "…multiple lot-size deliveries" but the lot-size is the same for all the deliveries.

Yang and Wee (2002) developed a single-vendor multi-buyer integrated model for exponentially deteriorating items. The authors derived the cost function with the same flaws as in Yang and Wee (2000). This model assumed same unit price and same holding cost per dollar per unit time for the buyers. In the solution procedure, the total cost function is differentiated with respect to time for a range of numbers of deliveries to find the minimum costs for these numbers of delivery. Then the optimal solution is found by comparing the costs obtained. After providing a numerical example of two buyers, the authors suggested a heuristic for finding the optimal solution assuming a common number of deliveries for all the buyers. A numerical example of three buyers with different demand rates was then provided and the optimal solutions with the solution procedure and the solutions using the heuristic are shown. The authors commented that the percentage errors were small and they suggested that the heuristic solution procedure is low-error and convenient to deal with multiple buyers. Also with the same flaws as in their 2000 model, Yang and Wee (2003) developed an integrated lot-delivery model that includes lot-delivery of raw materials to the producer during the production stage of a system cycle. The solution procedure also involves differentiating the total cost function with respect to time for different numbers of deliveries for raw materials and finished products. This is a multi-buyer model but the numerical example provided has only one buyer.

Jong and Wee (2008) recognized that production has to start before the first delivery of a system cycle in their model for a lot-delivery single-vendor single-buyer system for exponentially deteriorating items. However, the approximate method (truncating terms in series expansion) used for finding the length of the production period only applies for a lot-for-lot delivery system but their model allows multiple deliveries. Wee et al. (2008) provided an improved solution for a single-vendor single buyer integrated model by correcting the two flaws in Yang and Wee (2000). In addition to starting production at an appropriate instant before the first shipment, the authors derived the following expression for production time T_p for the lot-delivery model with a demand rate of D and a production rate of P:

$$T_p = \frac{1}{k}\ln\{1 + \frac{D}{P}(e^{kT} - 1)/[1 - \frac{D}{P}(e^{kT/n} - 1)]\}$$

where k is the deterioration rate, and n is the number of deliveries in a system cycle of cycle time T. As in their previous models, the authors approximated the exponential and logarithm terms in the cost equation in the solution procedure.

Yan et al. (2011) assumed small deterioration rates and used an algebraic method, neglecting square and higher powers for deterioration, in deriving the inventory level and cost functions for their integrated single-vendor single-buyer model. In this model, the vendor and the buyer have different unit holding costs as usual but same unit deterioration cost. The cost function is in terms of two variables: the number of deliveries and the delivery quantity. The authors presented a solution procedure to find the optimal solution. Sarkar (2013) used the same approach as Yan et al. (2011) to derive the cost equation in his models. He considered three probability distributions for deterioration rate: uniform distribution, triangular distribution and beta distribution. As the mean of the distribution is used as the deterioration rate in the cost equation, when the parameters of the distribution are known, Sarkar's model is basically same as Yan el al.'s model with another solution procedure. Chang (2014) considered the same model with a different solution procedure from the other two. In summary, these researchers assumed small deterioration rates and used an algebraic method in their lot-delivery models for deteriorating items with constant deterioration rates with different solution procedure. These models assumed that the unit deterioration cost is the same for both the vendor and the supplier. This assumption probably came from the approach in EPQ models. However, the vendor and the buyer are separate entities in lot-delivery supply chains and hence it is very likely that they have different deterioration costs in addition to having different inventory holding costs. Slight modifications of the cost equation to allow two deterioration costs may be more appropriate.

Wee et al. (2011) developed a replenishment policy for a deteriorating green product with a constant deterioration rate. In this model, the supplier does not manufacture the product but orders the product and receives the whole batch at the beginning of a replenishment period. The goods are delivered to the buyer in one or more shipment(s) with the first shipment delivered at the beginning of the replenishment period. The number of units of the returned product follows a Poisson distribution. The supplier remanufactures the good ones after inspection (proportion of good remanufactured product is assumed to be a constant) and scrap the remaining ones. Remanufacturing rate is not considered in the model. Cost elements of this model include deteriorating cost, remanufacturing cost and scrap processing cost in addition to other usual cost elements; and the authors referred this as a life cycle costing analysis. This model maximizes the profit over the replenishment period by finding the optimal values for the number of shipments in a replenishment and the shipment cycle time. As the replenishment period is not predetermined based on certain criteria but is found from the optimal solution, the use of profit over the replenishment period instead of profit per unit time as the objective function to be optimized does not seem appropriate. In the cost function, the expression for the holding cost for the remanufacturing units over the replenishment period gives a value in dollars per unit time. This is incorrect (should be in dollars only) and is inconsistent with other terms in the cost and profit functions over the period. This problem is due to no consideration of remanufacturing rate in the model. This and all above models assumed that the demand rates and the production rates are constant, and the items deteriorate with constant rates (exponentially deteriorating).

Kim et al. (2014) developed a lot-for-lot delivery model for a supply chain utilizing returnable transport items (RTIs) for shipments, also with constant demand and production rates. In this model, empty RTIs are returned to the supplier with a stochastic return time approximated by an exponential distribution. It is assumed that deterioration only occurs during stockouts of RTIs at the supplier due to late return. Unlike other models, deterioration during production and at the buyer's end is neglected. Deterioration results in a reduction in the selling price of the product which is expressed as an exponential function related to the lot size and the delay. Reduction of the selling price is treated as a cost and added to the other cost elements of the system. The model attempts to minimize the expected total cost of the system by finding the optimal number of RI return lot size, which is proportional to the delivery lot size of the product.

2.3 Conclusion

Ghare and Schrader, Covert and Philip, and Misra were the pioneers in developing EOQ and EPQ models for deteriorating items. Many researchers have then presented models for deteriorating items with more complicated situations: from allowing no shortages to allowing full backordering and partial backordering; from constant demand and production rates to more complicated functions of time or price for demand and/or production rates, etc. In most literature on deteriorating items, the items are subject to deterioration as they are received into inventory. The use of the 3-parameter Weibull distribution for deterioration allows deterioration to start after certain time. More complicated models with the 3-parameter Weibull distribution for deterioration have also been presented. For example, the two-warehouse inventory model of Banerjee and Agrawal (2008) is essentially an EOQ model of a particular scenario. Chowdhury et al. (2014) presented an EOQ model with a quadratic demand rate, partial backordering with a backlogging rate depending on the waiting time, unequal cycle length and time to deterioration following the 3-parameter Weibull distribution. However, the studies of "non-instantaneous" deterioration have been confined to various forms of EOQ models assuming that all units just start the non-deteriorating period when they are received by the buyer. The effect of the "non-instantaneous" deterioration with a finite production rate has not been considered. In literature of EPQ models that have varying production rates, cost parameters are assumed to be constant if price discounts or time value of money is not considered.

Compared with EOQ and EPQ models, there have been much less studies on integrated lot-delivery supply chain models on deteriorating items, and the models are usually for exponentially deteriorating items with constant demand and production rates. In these models, the predetermined production rates are usually much higher than demand rates, resulting in production stops before most of the shipments have been made. This results in inventory being built long before needed and deterioration cost is incurred in addition to inventory holding cost for such inventory. On the operational side, these models result in low utilization of production facilities and give rise to labour planning issues.

In this research, a continuous production lot-delivery model for exponentially deteriorating items is proposed for reducing the total system cost by reducing the average inventory level and deteriorated quantity. The production rate is determined by the demand rate and the cost parameters of the supply chain. The proposed model can help to address the operational issues by minimizing idleness of the production facilities and facilitating labour planning. The proposed model, which has a finite production rate, is extended to allow a non-deteriorating period. The effect of "non-instantaneous" deterioration on both the vendor's inventory and the buyer's inventory is considered. As the production rate for the proposed continuous production model is much less than the usual predetermined production rates with respect to the demand rates, the proposed model is extended to consider a scenario in which some of the cost parameters increase when production rate is reduced. A discussion of the potential reasons for such increase in cost parameters is presented.

In literature of inventory models, time is usually considered as a continuous variable. Some models considered the system cycle time as a discrete variable but did not suggest how to set a range of values for the discrete cycle time. Chan and Kingsman (2005, 2007) proposed a single-vendor multi-buyer model in which both the system cycle and the delivery intervals are multiples of a convenient time unit and presented an algorithm with day as the time unit and a year as the maximum system cycle time. In this research, this model is extended for exponentially deteriorating items. Trying to find the optimal solution just by differentiating the cost function with respect to different numbers of deliveries is impractical for a supply chain with multiple buyers. Instead of starting with one delivery for each buyer, the model and the algorithm presented in this research enables an initial solution to be found followed by an iterative procedure to find the optimal solution.

In view of environmental concerns, green supply chain has become an important area for investigation. Some researchers considered remanufacture or rework in their cost minimization models. Deterioration results in wastage and causes impact to the environment beyond monetary implications. Controlling the amount of deterioration in the first place is perhaps a direction that can help environmental protection. In this research, the cost minimization single-vendor multi-buyer model is modified to a goal programming model. A maximum deterioration constraint is set to control the amount of deterioration within a certain proportion of the produced quantity. The algorithm then finds the minimum cost solution to achieve this goal.

Chapter 3

A Lot-delivery Continuous Production Model for a Single-vendor Single-buyer Supply Chain

3.1 Introduction

In literature of lot-delivery systems for deteriorating items, production rates are predetermined and usually much higher than the demand rates. As the optimal solutions usually entail multiple deliveries for a production batch, production stops at the early part of a system cycle. Inventory has been built well before most of the shipments are made and part of the inventory will be deteriorated during waiting. Substantial inventory holding cost and deterioration cost have been incurred. In addition, intermittent production with the predetermined production rate results in low utilization of the production facilities. In view of this, a lot-delivery continuous production model for a single-vendor single-buyer supply chain is proposed in this research. Instead of using a predetermined production rate, the production rate for this model is determined from the demand rate and the cost parameters of the vendor and the buyer for minimizing the total system cost per unit time. With a shorter cycle time and a smaller production rate, the average inventory level and the total cost would be reduced. In addition to achieving the basic objective of reducing system cost, the continuous production model can minimize the idleness of production facilities and also facilitate labour planning as a group of labour can be "permanently" allocated to the concerned production line. While approximations of exponential and logarithmic terms have been commonly made in inventory models for deteriorating items, the proposed model and its solution procedure are developed using the exact mathematical expressions. The mathematical development of this model is presented in this chapter. This is followed by a numerical example which indicates that the proposed model can reduce the total system cost per unit time. A proof showing why the proposed model, in many cases, can result in a lower cost than the predetermined production rate model will be presented in Chapter 4.

The assumptions of the proposed model are:

1. The item is deteriorating exponentially, that is, the deterioration is a constant rate of the inventory level of the item.
2. The demand rate and the cost parameters are constant.
3. The production rate is constant.
4. Shortages are not allowed.

3.2 Independent Optimization of the Buyer (the EOQ Model)

The buyer has to fulfill the demand of a product which has a constant demand rate of D units per unit time by ordering and receiving the product at a certain fixed ordering/delivery cycle. The product is an exponentially deteriorating item with a constant rate of deterioration of k per unit time. A cost of C_b is incurred for each unit of deteriorated item. There is a fixed ordering cost (and other delivery-related costs, if any, on the buyer side) of A_b per cycle. There is also an inventory holding cost of H_b per unit per unit time for the buyer. The inventory level of the buyer, I_b, is shown in Figure 3.1 where the order quantity and the delivery cycle time are denoted by Q_0 and T_c, respectively.

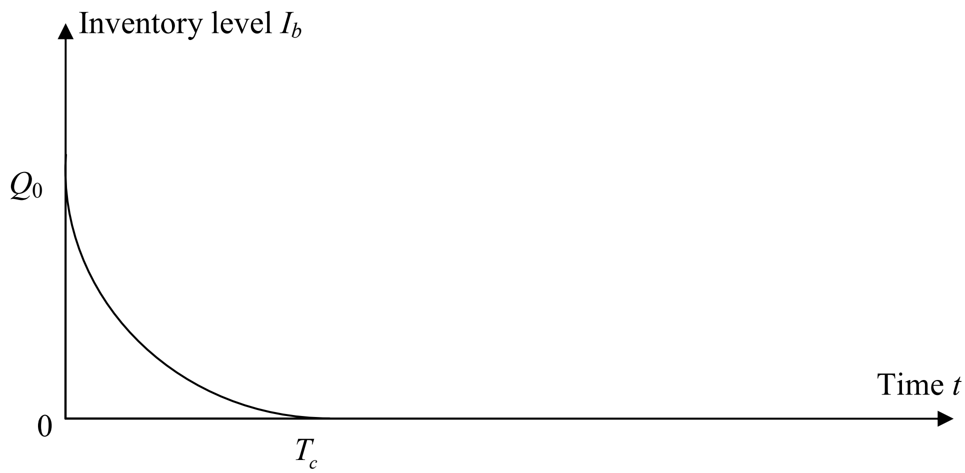


Figure 3.1: Inventory level of the buyer.

The inventory level of the buyer is described by the following differential equation:

$$\frac{dI_b}{dt} = -kI_b - D \text{ or } \frac{dI_b}{dt} + kI_b = -D \tag{3.1}$$

Solving equation (3.1) and with the boundary condition $I_b = 0$ at $t = T_c$, the inventory level of the buyer is given by $I_b = \frac{D}{k}(e^{k(T_c - t)} - 1)$. (3.2)

The order quantity is the inventory level at $t = 0$. Hence, $Q_0 = \frac{D}{k}(e^{kT_c} - 1)$. (3.3)

Average inventory level is given by $\frac{1}{T_c}\frac{D}{k}\int_0^{T_c}(e^{k(T_c - t)} - 1)dt = \frac{D}{kT_c}[\frac{1}{k}(e^{kT_c} - 1) - T_c]$.

The inventory holding cost is given by $\frac{H_b D}{kT_c}[\frac{1}{k}(e^{kT_c}-1)-T_c]$.

Quantity of deteriorated items per cycle is $Q_0 - DT_c = D(\frac{e^{kT_c}-1}{k}-T_c)$.

Cost of deteriorated items per unit time $=\frac{C_b D}{T_c}(\frac{e^{kT_c}-1}{k}-T_c)$.

Total relevant cost per unit time for the buyer, TC_b, is given by

$$TC_b = \frac{A_b}{T_c} + (\frac{H_b}{k}+C_b)(\frac{e^{kT_c}-1}{k})\frac{D}{T_c} - \frac{H_b D}{k} - C_b D . \tag{3.4}$$

The objective is to minimize TC_b, and this can be found by solving:

$$\frac{d}{dT_c}TC_b = -\frac{A_b}{T_c^2} + \frac{D}{k}(\frac{H_b}{k}+C_b)\frac{(kT_c-1)e^{kT_c}+1}{T_c^2} = 0 \quad . \tag{3.5}$$

Some previous researchers expanded the exponential term in equation (3.5) and neglected the higher power terms for obtaining an expression for the optimum ordering interval. Actually, it can be shown as follows that the cost function TC_b is convex. Equation (3.5) can be written as

$(x-1)e^x + 1 = L_1$ where $x = kT_c > 0$ and $L_1 = \frac{A_b k}{D(\frac{H_b}{k}+C_b)} > 0$.

Let $y_1 = (x-1)e^x + 1$.

Then $\frac{dy_1}{dx} = e^x + (x-1)e^x = xe^x > 0$ as $x > 0$.

As $y_1(0)=0$, so for positive *x*, y_1 is increasing and always positive.

Also y_1 has no upper bound as $\lim_{x\to\infty}(x-1)e^x=\infty$. So there exists only one positive solution to the equation $(x-1)e^x+1=L_1$ for any $L_1>0$.

Hence, for any positive values of the parameters, there exists a unique solution to equation (3.5).

Consider the second derivative of TC_b.

$$\frac{d^2}{dT_c^2}TC_b=\frac{d}{dT_c}[-\frac{A_b}{T_c^2}+\frac{D}{k}(\frac{H_b}{k}+C_b)\frac{(kT_c-1)e^{kT_c}+1}{T_c^2}]$$

$$=\frac{2A_b}{T_c^3}+\frac{A_b}{L_1}\frac{T_c^2[ke^{kT_c}+k(kT_c-1)e^{kT_c}]-2T_c[(kT_c-1)e^{kT_c}+1]}{T_c^4}$$

$$=\frac{2A_b}{T_c^3}+\frac{A_b}{L_1}\frac{\{[k^2T_c^2-2(kT_c-1)]e^{kT_c}-2\}}{T_c^3}$$

$$=\frac{1}{T_c^3}\{2A_b+\frac{A_b}{L_1}[k^2T_c^2e^{kT_c}-2(kT_c-1)e^{kT_c}-2]\}$$

Let $y_2=x^2e^x-2(x-1)e^x-2$.

Then $\frac{dy_2}{dx}=x^2e^x+2xe^x-2(x-1)e^x-2e^x=x^2e^x>0$ as $x>0$, and $y_2(0)=0$.

Hence, $y_2>0$ and $\frac{d^2}{dT_c^2}TC_b>0$

The convexity of the cost function of the buyer is therefore proved with the exact mathematical expression.

By applying numerical methods, equation (3.5) can be solved to the required level of accuracy for the optimal delivery interval, T_c^*, for the EOQ problem. The order/delivery quantity for independent optimization of the buyer is therefore given by $Q_0^* = \frac{D}{k}(e^{kT_c^*} - 1)$.

Remark:

Consider the equation $(x-1)e^x + 1 = L_1$.

Expanding the exponential term in the above equation and taking up to second order terms,

$$(x-1)(1 + x + \frac{x^2}{2}) + 1 = L_1$$

$$x + x^2 - 1 - x - \frac{x^2}{2} + 1 = L_1$$

$$\frac{x^2}{2} = L_1$$

$$x = \sqrt{2L_1}$$

$$T_c^* = \frac{1}{k}\sqrt{\frac{2A_b k}{D(\frac{H_b}{k} + C_b)}} = \sqrt{\frac{2A_b}{D(H_b + kC_b)}}$$

This expression is the same as that given by Mak (1982) for optimal ordering cycle time for EOQ model and Mak referred this expression as ‘same as that given by Ghare and Schrader’.

3.3 Independent Optimization of the Vendor (Lot-for-lot Policy)

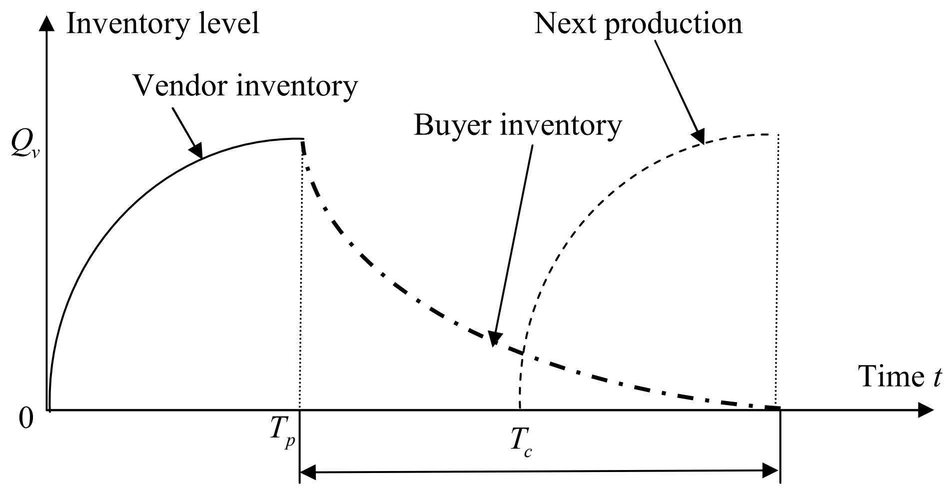


Figure 3.2 Inventory level of "one cycle".

In this section, the independent optimization of the vendor with a lot-for-lot policy and the convexity of the vendor's cost function are investigated with the exact mathematical expressions. Suppose the vendor produces the product at a production rate of P for a period of T_p resulting in an inventory level of Q_v and the lot is then delivered to the buyer instantaneously. Figure 3.2 depicts the inventory level of the vendor during the production period, and that of the buyer for the period $T_p \leq t \leq T_p + T_c$ consuming the lot received at time T_p. Next production will start at time T_c ($T_c \geq T_p$) and the lot will be delivered at time $T_p + T_c$. The system cycle time is equal to the delivery interval of T_c.

The inventory level of the vendor, I_v, is described by the following equation:

$$\frac{dI_v}{dt} = -kI_v + P \tag{3.6}$$

Solving equation (3.6) and with the initial condition $I_v = 0$ at $t = 0$, the inventory level of the vendor is given by $I_v = \frac{P}{k} - \frac{P}{k}e^{-kt}$. (3.7)

At $t = T_p$, $I_v = Q_v$. Hence, $Q_v = \frac{P}{k}(1 - e^{-kT_p})$ (3.8)

Total inventory level $= \frac{P}{k}\int_0^{T_p}(1 - e^{-kt})dt = \frac{P}{k}[T_p + \frac{1}{k}(e^{-kT_p} - 1)]$

Quantity of deteriorated items $= PT_p - Q_v = P(T_p - \frac{1 - e^{-kT_p}}{k})$

The quantity Q_v is delivered at $t = T_p$. The cost elements for the vendor are a production set up cost of S, an order processing and shipment cost of A_v per delivery, a deterioration cost of C_v per unit and a unit holding cost of H_v per unit time. The total relevant cost for the vendor over one cycle is given by

$$S + A_v + \frac{H_v P}{k}[T_p + \frac{1}{k}(e^{-kT_p} - 1)] + C_v P(T_p - \frac{1 - e^{-kT_p}}{k}).$$

The period that is covered by a delivery of Q_v can be found by equating equations (3.3) and (3.8) and after simplification, $T_c = \frac{1}{k}\ln[1 + \frac{P}{D}(1 - e^{-kT_p})]$. (3.9)

Differentiating equation (3.9) with respect to T_c, $\frac{dT_p}{dT_c} = \frac{1 + P(1 - e^{-kT_p})/D}{Pe^{-kT_p}/D} > 0$.

This is also obvious intuitively as a longer production period can cover a longer delivery cycle and vice versa.

The total relevant cost for the vendor per unit time is given by

$$\begin{aligned} TC_v &= \frac{1}{T_c}\{\ S + A_v + \frac{H_v P}{k}[T_p + \frac{1}{k}(e^{-kT_p} - 1)] + C_v P(T_p - \frac{1 - e^{-kT_p}}{k})\} \\ &= \frac{1}{T_c}\{S + A_v + (\frac{H_v}{k} + C_v)(\frac{e^{-kT_p} - 1}{k})P + (\frac{H_v}{k} + C_v)PT_p\} \end{aligned} \tag{3.10}$$

It requires minimizing TC_v for independent optimization of the vendor.

From equation (3.10),

$$\frac{d}{dT_c}(TC_v) = \frac{1}{T_c^2}\{P(\frac{H_v}{k} + C_v)[T_c(1 - e^{-kT_p})\frac{dT_p}{dT_c} + (\frac{1 - e^{-kT_p}}{k}) - T_p] - (S + A_v)\}. \tag{3.11}$$

$$\frac{d}{dT_c}(TC_v) = 0 \Rightarrow P(\frac{H_v}{k} + C_v)[T_c(1 - e^{-kT_p})\frac{dT_p}{dT_c} + (\frac{1 - e^{-kT_p}}{k}) - T_p] = (S + A_v) \tag{3.12}$$

The solution of equation (3.12) gives the optimal production time for optimizing vendor’s cost. The optimal delivery cycle time can then be found by substituting the optimal production time into equation (3.9). The convexity of vendor’s cost function can be shown as follows:

$$\frac{d}{dT_c}([T_c(1-e^{-kT_p})\frac{dT_p}{dT_c}+(\frac{1-e^{-kT_p}}{k})-T_p])$$

$$=(1-e^{-kT_p})\frac{dT_p}{dT_c}+T_c(ke^{-kT_p})\frac{dT_p}{dT_c}+T_c(1-e^{-kT_p})\frac{d}{dT_c}(\frac{dT_p}{dT_c})+e^{-kT_p}\frac{dT_p}{dT_c}-\frac{dT_p}{dT_c}$$

$$=T_c(ke^{-kT_p})\frac{dT_p}{dT_c}+T_c(1-e^{-kT_p})\frac{d}{dT_c}(\frac{dT_p}{dT_c})$$

$$\frac{d}{dT_c}(\frac{dT_p}{dT_c})=[\frac{d}{dT_p}(\frac{dT_p}{dT_c})]\frac{dT_p}{dT_c}$$

$$=[\frac{d}{dT_p}\frac{1+P(1-e^{-kT_p})/D}{Pe^{-kT_p}/D}]\frac{dT_p}{dT_c}$$

$$=\frac{Pe^{-kT_p}(Pke^{-kT_p})/D^2+Pke^{-kT_p}[1+P(1-e^{-kT_p})/D]/D}{(Pe^{-kT_p}/D)^2}\frac{dT_p}{dT_c}>0$$

Hence, $Y=P(\frac{H_v}{k}+C_v)[T_c(1-e^{-kT_p})\frac{dT_p}{dT_c}+(\frac{1-e^{-kT_p}}{k})-T_p]$ is an increasing function of T_c (and T_p) and there is only one solution for equation (3.12) for given values of S and A_v.

Suppose $T_c=T_c^*$ is the solution for equation (3.12).

For $T_c<T_c^*$, $Y<S+A_v$ and hence $\frac{d}{dT_c}(TC_v)=\frac{Y-S-A_v}{T_c^2}<0$.

For $T_c>T_c^*$, $Y>S+A_v$ and hence $\frac{d}{dT_c}(TC_v)>0$ $\frac{d}{dT_c}(TC_v)>0$.

Hence, $T_c=T_c^*$ gives the minimum of TC_v and the convexity of the vendor's cost function is proved due to uniqueness of the solution for equation (3.12).

3.4 Integrated Continuous Production Model

In this section, the development of the proposed continuous production model for a single-vendor single-buyer supply chain is presented. The main objective of this piece of research is to compare the system cost of the proposed model and that of the predetermined production rate model. The proposed model can also help to address the operational issues of minimizing idleness of the production facilities and facilitating labour planning. The continuous production model can be considered as a lot-for-lot policy with the length of the production period equal to the delivery interval. Model development starts with determining the production rate required for continuous production so that the produced quantity just covers the loss due to deterioration before delivery and the required delivery quantity. Total cost function of the model is then derived, followed by proving the convexity of the cost function.

3.4.1 Production Rate for the Continuous Production Model

The inventory levels of the vendor and the buyer for the continuous production model are shown in Figure 3.3.

Let the uniform production rate be *P*. Obviously *P* is larger than *D*.

For continuous production with the lot-for-lot policy, $T_p = T_c$, and $Q_v = Q_0$.

Therefore from equation (3.8), $Q_v = \frac{P}{k}(1 - e^{-kT_c})$. (3.13)

Equating equations (3.3) and (3.13), $\frac{P}{k}(1-e^{-kT_c}) = \frac{D}{k}(e^{kT_c}-1)$.

The required production rate is given by $P = \frac{D(e^{kT_c}-1)}{(1-e^{-kT_c})} = De^{kT_c}$ (3.14)

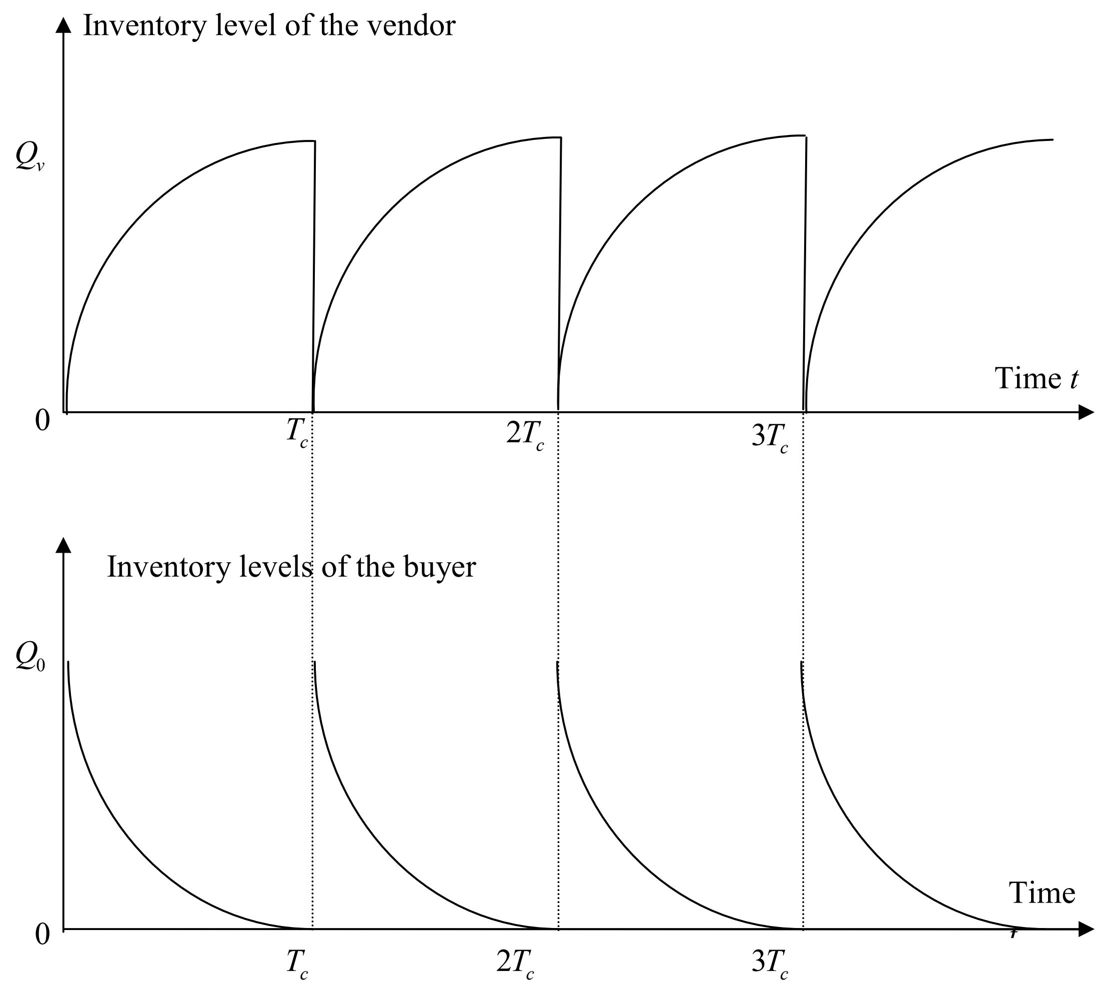


Figure 3.3: The inventory level of the vendor and that of the buyer for a continuous production model.

3.4.2 Total Cost Function of the Model

In the previous sections, the cost functions of the buyer and the vendor for lot-for-lot policy have been derived (equation (3.4) and equation (3.10) respectively):

$$TC_b = \frac{A_b}{T_c} + (\frac{H_b}{k} + C_b)(\frac{e^{kT_c} - 1}{k})\frac{D}{T_c} - \frac{H_b D}{k} - C_b D\,;$$

$$TC_v = \frac{1}{T_c}\{S + A_v + (\frac{H_v}{k} + C_v)(\frac{e^{-kT_p} - 1}{k})P + (\frac{H_v}{k} + C_v)PT_p\}\,.$$

Theoretically the continuous production system can run with one production set up forever. In practice, maintenance of the production facilities is inevitable and manufacturing set up is required after the activity. In this model, the time unit is of "year". It is assumed that there is one manufacturing set up every time unit and hence there is a set up cost of *S* per unit time. With this assumption,

$$TC_v = S + \frac{1}{T_c}\{A_v + (\frac{H_v}{k} + C_v)(\frac{e^{-kT_p} - 1}{k})P + (\frac{H_v}{k} + C_v)PT_p\}\,.$$

Adding up the equations for TC_b and TC_v, substituting $T_p = T_c$ and $P = De^{kT_c}$, the total relevant cost per unit time for the system, TC_s, is given by

$$TC_s = \frac{A_b + A_v}{T_c} + \frac{D}{k}(\frac{H_b}{k} + C_b - \frac{H_v}{k} - C_v)(\frac{e^{kT_c} - 1}{T_c}) + \frac{H_v De^{kT_c}}{k} + C_v De^{kT_c} - \frac{H_b D}{k} - C_b D + S$$

(3.15)

3.4.3 Convexity of the Cost Function

The convexity of the total relevant cost per unit time for the system for the continuous model can be shown as follows:

The derivative of the total cost per unit time is given by

$$\frac{d}{dT_c}TC_s = -\frac{A_b + A_v}{T_c^2} + \frac{1}{T_c^2}\frac{D}{k}[\frac{H_b}{k} + C_b - \frac{H_v}{k} - C_v][(kT_c - 1)e^{kT_c} + 1] + H_v De^{kT_c} + C_v Dke^{kT_c}$$

$$= \frac{1}{T_c^2}\{-(A_b + A_v) + \frac{D}{k}[\frac{H_b}{k} + C_b - \frac{H_v}{k} - C_v][(kT_c - 1)e^{kT_c} + 1] + D(H_v + C_v k)T_c^2 e^{kT_c}\}$$

Setting the derivative to zero,

$$\frac{D}{k}[\frac{H_b}{k} + C_b - \frac{H_v}{k} - C_v][(kT_c - 1)e^{kT_c} + 1] + D(H_v + C_v k)T_c^2 e^{kT_c} - (A_b + A_v) = 0 \qquad (3.16)$$

Case (i): $C_b = C_v$ and $H_b = H_v$

This may happen when both the vendor and the buyer belong to the same company; the buyer gets the produced goods at cost and the same unit holding cost is applicable to both parties.

$$\frac{d}{dT_c}TC_s = 0 \Rightarrow \frac{D(H_v + C_v k)(kT_c)^2 e^{kT_c}}{k^2} = A_b + A_v \qquad (3.17)$$

Since all the quantities are positive, $\frac{D(H_v + C_v k)(kT_c)^2 e^{kT_c}}{k^2} > 0$, is an increasing function and has no finite limit as $T_c \to \infty$, equation (3.17) has a unique solution for any $A_b + A_v > 0$. The unique solution, found by solving the equation by numerical methods, gives the optimum cycle time for production and ordering. Order quantity can be calculated accordingly.

Case (ii): $C_b > C_v$ and $H_b > H_v$

This is the general case when the vendor sells the goods to the buyer with profit and the buyer's unit holding cost is higher than the vendor's unit holding cost.

Setting $m_1 = \frac{D}{k}(\frac{H_b}{k} + C_b - \frac{H_v}{k} - C_v)$ and $m_2 = \frac{D(H_v + C_v k)}{k^2}$ into equation (3.16),

$$\frac{d}{dT_c} TC_s = 0 \Rightarrow m_1[(kT_c - 1)e^{kT_c} + 1] + m_2 (kT_c)^2 e^{kT_c} - (A_b + A_v) = 0 . \qquad (3.18)$$

and $\frac{d}{dT_c} TC_s = \frac{1}{T_c^2} \{m_1[(kT_c - 1)e^{kT_c} + 1] + m_2 (kT_c)^2 e^{kT_c} - (A_b + A_v)\}$.

Let $y_3 = m_1[(kT_c - 1)e^{kT_c} + 1] + m_2 (kT_c)^2 e^{kT_c}$ where $m_1 \geq 0$, $m_2 > 0$, $T_c > 0$.

Then $y_3(0) = 0$, $\frac{dy_3}{dT_c} = m_1 k^2 T_c e^{kT_c} + 2m_2 k^2 T_c e^{kT_c} + m_2 k (kT_c)^2 e^{kT_c} > 0$, and $\lim_{T_c \to \infty} y_3 = \infty$.

Hence, there is a unique solution to equation (3.18) for any $A_b + A_v > 0$.

$$\frac{d^2}{dT_c^2} TC_s = \frac{1}{T_c^4} \left\{m_1 T_c [k^2 T_c^2 e^{kT_c} - 2\left((kT_c - 1)e^{kT_c} + 1\right)] + m_2 k^3 T_c^4 e^{kT_c} + 2(A_b + A_v) T_c\right\}$$

Let $y_4 = k^2 T_c^2 e^{kT_c} - 2\left((kT_c - 1)e^{kT_c} + 1\right)$.

Then $y_4(0) = 0$ and $\frac{dy_4}{dT_c} = k^3 T_c^2 e^{kT_c} > 0$.

Hence, $\frac{d^2}{dT_c^2} TC_s > 0$

The total cost function is therefore convex and there is a unique delivery cycle time that minimizes the total system cost per unit time for the proposed continuous production model.

3.4.4 Solution Procedure

For a single-vendor single-buyer supply chain with known demand rate D and cost parameters S, A_v, H_v, C_v, A_b, H_b, C_b, and deterioration rate k for the exponentially deteriorating product, the optimal solution for minimizing the total system cost per unit time can be found by the following steps.

Step 1: If $C_b = C_v$ and $H_b = H_v$, go to Step 5.

Step 2: Set $m_1 = \frac{D}{k}(\frac{H_b}{k} + C_b - \frac{H_v}{k} - C_v)$, $m_2 = \frac{D(H_v + C_v k)}{k^2}$, and

$$f(x) = m_1[(x-1)e^x + 1] + m_2 x^2 e^x - (A_b + A_v).$$

Step 3: Set $f'(x) = xe^x(m_1 + 2m_2 + m_2 x)$. Solve $f(x) = 0$ by the Newton-Raphson method with the iterative formula: $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$ with a small initial value, say $x_0 = 0.001$.

Step 4: The optimal cycle time is given by $T_c^* = x/k$ where x is the solution obtained in Step 3. Go to Step 8.

Step 5: Set $f(x) = D(H_v + C_v k)x^2 e^{kx} - (A_b + A_v)$ and

$f'(x) = D(H_v + C_v k)(2 + kx)xe^{kx}$.

Step 6: Solve $f(T_c) = 0$ by the Newton-Raphson method with the iterative formula:

$x_{i+1} = x_i - \dfrac{f(x_i)}{f'(x_i)}$ with a small initial value, say $x_0 = 0.01$.

Step 7: The optimal cycle time is given by $T_c^* = x$ where x is the solution obtained in Step 6.

Step 8: Set the production rate at $P^* = kT_c^*$.

Step 9: The delivery quantity is $Q_0^* = \dfrac{D}{k}(e^{kT_c^*} - 1)$ to be shipped at intervals of T_c^*.

The optimal total system cost per unit time can be found by substituting $T_c = T_c^*$ into equation (3.15). Wee et al. (2008) has derived the correct formula for finding the production time required for the predetermined production rate model and has provided a numerical example in the paper. In the next section, the optimal costs in the proposed model will be found using the parameters from that example, and compared with Wee et al.'s optimal costs.

3.4.5 Example 3.1

Using parameters from Wee et al. (2008)'s example,

D = 1000 units per year

P = 3200 units per year (predetermined production rate)

k = 0.1 per year

S = \$400

$A_b + A_v$ = \$25

C_b = \$50 C_v = \$40

H_b = \$5 per unit per year H_v = \$4 per unit per year

$$m_1 = \frac{1000}{0.1}[\frac{5}{0.1} + 50 - \frac{4}{0.1} - 40] = 200000$$

$$m_2 = \frac{1000}{0.1^2}[4 + 40(0.1)] = 800000$$

Let $x = kT_c$.

$$f(x) = 200000[(x-1)e^x + 1] + 800000(x)^2 e^x - 25 = 0$$

$$f'(x) = xe^x[200000 + 2(800000) + 800000x] = xe^x(1800000 + 800000x)$$

Iterations using Newton-Raphson Method:

i	x_i	$f(x_i)$	$f'(x_i)$
0	0.001	-24.09913291	1802.6017
1	0.01436908	163.4129325	26406.248
2	0.00818066	35.70733031	14900.13
3	0.00578422	5.27967951	10498.915
4	0.00528134	0.231311554	9579.1864
5	0.00525719	0.000532819	9535.0562
6	0.00525714	2.86293E-09	9534.9541
7	0.00525714	6.49791E-12	9534.9541

<table>
<tr><th>kT_c</th><th>T_c</th><th>Cost without set up</th><th>Total cost TC_s</th><th><u>Optimal Solution</u></th></tr>
<tr><td>0.003</td><td>0.03</td><td>1103.723716</td><td>1503.7237</td><td rowspan="8">The optimal cycle time is 0.0526 year.

The system cost per year excluding set up cost is $949.89 and the system cost per year with one set up is $1349.89.</td></tr>
<tr><td>0.004</td><td>0.04</td><td>985.6942409</td><td>1385.6942</td></tr>
<tr><td>0.005</td><td>0.05</td><td>951.0851064</td><td>1351.0851</td></tr>
<tr><td>0.0052</td><td>0.052</td><td>949.9429586</td><td>1349.943</td></tr>
<tr><td>0.005257</td><td>0.05257</td><td>949.886024</td><td>1349.886</td></tr>
<tr><td>0.0053</td><td>0.053</td><td>949.9174584</td><td>1349.9175</td></tr>
<tr><td>0.006</td><td>0.06</td><td>958.2297312</td><td>1358.2297</td></tr>
<tr><td>0.008</td><td>0.08</td><td>1035.280601</td><td>1435.2806</td></tr>
</table>

Table 3.1: Finding the optimal solution for Example 3.1.

Remark: It has been verified that some other starting values, e.g., 0.01, 0.05, 0.10, also converge to the same solution. This is expected due to monotonicity of $f(x)$.

In Wee et al. (2008)'s example, the production rate is 3200 units per year. The optimal cost per year for k = 0.1 is \$2695.69 with 5 deliveries in a system cycle. Cycle time is not shown. Applying Newton-Raphson method to the equation in the paper, the cost of \$2695.69 is verified with cycle time 0.38893 year. This means that there are 1/0.38893 or 2.57 set ups per year. Hence the cost per year excluding set up is 2695.69 – 400/0.38893 = \$1667.23.

To compare the costs due to holding inventory and deterioration, the cost excluding set up is reduced by the costs due to deliveries:

(i) For the continuous production with demand-driven production rate model, there are 1/0.05257 or 19.02 deliveries per year and the costs due to holding inventory and deterioration = 949.89 – 25(1/0.05257) =\$474.33 per year.

(ii) For the predetermined production rate of 3200 per year, there are 5/0.33893 or 14.75 deliveries per year. The costs due to holding inventory and deterioration = 1667.23 – 25(5/0.38893) = \$1345.83 per year.

The production rate is $1000e^{0.1(0.05257)} = 1005.27$ units per year for the continuous production model.

The optimal solutions for the demand-driven production rate model are found for the other values of deterioration rate in Wee et al. (2008)'s example and the results are summarized in Table 3.2.

Deterioration rate	Cost excluding manufacturing set up		Cost including manufacturing set up	
	Predetermined production rate	Demand-driven production rate	Predetermined production rate	Demand-driven production rate
0.0001	1178.82	671.16	1904.25	1071.16
0.001	1184.12	674.18	1912.86	1074.18
0.01	1235.91	703.68	1996.92	1103.68
0.1	1667.23	949.89	2695.69	1349.89

Deterioration rate	Costs due to holding inventory and deterioration	
	Predetermined production rate	Demand-driven production rate
0.0001	952.12	335.59
0.001	956.39	337.25
0.01	998.09	351.57
0.1	1345.83	474.33

	k=0.0001		k=0.001		k=0.01	
Production Rate	3200	DDPR	3200	DDPR	3200	DDPR
No. of set ups	1.8136	1	1.8218	1	1.9025	1
No. of deliveries	9.0679	13.4228	9.1092	13.4771	9.5126	14.0845
No. of deliveries in a system cycle	5	-----	5	-----	5	------
Inventory related cost	952.12	335.59	956.39	337.25	998.09	351.57
Total annual cost	1904.25	1071.16	1912.86	1074.18	1996.92	1103.68

(DDPR: demand-driven production rate model)

Table 3.2: Comparison of different rates of deterioration for Example 3.1.

With the same cost parameters and deterioration rates of $k = 0.1$ and $k = 0.2$, the optimal solutions for the predetermined production rates of 3200, 2500 and 4000 units per year and that for the continuous production with demand-driven production rate model are shown in Table 3.3.

Production rate ($k = 0.1$)	2500/year	3200/year	4000/year	Demand-driven production rate: 1005.27/ year
No. of set ups	2.4897	2.5712	2.7484	1
No. of deliveries	12.4483	12.8558	10.9937	19.0223
No. of deliveries in a system cycle	5	5	4	------
Inventory related cost	1304.23	1345.83	1369.32	474.33
Total annual cost	2611.30	2695.69	2743.53	1349.89
Production rate ($k = 0.2$)	2500/year	3200/year	4000/year	Demand-driven production rate: 1008.61/year
No. of set ups	3.0498	3.1498	3.3672	1
No. of deliveries	15.2489	15.7492	13.4687	23.3340
No. of deliveries in a system cycle	5	5	4	------
Inventory related cost	1597.35	1648.30	1677.06	580.95
Total annual cost	3198.48	3301.97	3360.65	1564.30

Table 3.3: Comparison of different predetermined production rates against the continuous production demand-driven production rate model.

3.4.6 Discussion of the Results of Example 3.1

In Example 3.1, the optimal costs of the proposed continuous production model have been compared with the optimal costs of the predetermined production rate model using the parameters of Wee et al.'s example which used a production rate of 3200 units per year. The optimal costs for predetermined production rates of 2500 and 4000 units per year have also been found and compared with the optimal costs of the proposed model. It is found that by having lower inventory related costs and lower set up costs, the proposed model results in a lower total system cost per unit time although the number of deliveries have been increased.

For Wee et al.'s example with a deterioration rate of 0.1, the system cycle time is 0.38893 year and the delivery interval is therefore 0.077786 year as there are 5 deliveries in a system cycle. Substituting $T_c = 0.07786$ into equation (3.15), the system cost is \$1424.10 per year for the proposed model with one set up per year. If the same number of setups as in the predetermined production rate model is used, the system cost will be increased to \$2052.56 per year which is still less than \$2695.69, the optimal cost in Wee et al.'s example. With other costs being the same, the reduction in the total cost is due to a lower average inventory level being obtained with a smaller production rate of the proposed model. The optimal solution of the proposed model, as indicated in the example, results in an even larger cost reduction.

3.4.7 The model including Deterioration during Delivery

In literature of inventory models for deteriorating items, deliveries are assumed to be instantaneous and therefore there is no deterioration during deliveries. This is also an assumption of the proposed method presented in the previous sections. If the buyer's site is not located near the vendor's, say, the buyer is in another city or region and it takes several days for the delivery, it may be necessary to consider deterioration during delivery. In this section, the proposed model is extended to consider deterioration during delivery for situations that this cannot be neglected.

The equation for the inventory change of the goods during delivery is given by $\frac{dI}{dt} = -kI$ whose solution is $I = Ce^{-kt}$ where C is a constant.

Suppose the delivery lead-time is T_T and the delivery occurs between $t = -T_T$ and $t = 0$.

Hence $C = Q_0$ and the delivery quantity required is $Q_T = Q_0 e^{kT_T}$ at $t = -T_T$.

The vendor has to deliver a quantity of $Q_0 e^{kT_T}$ units and the buyer will receive Q_0 units upon receiving the delivery. The production rate, P_2, required to result in an inventory level of $Q_0 e^{kT_T}$ units in a period of T_c can be found from the equation $\frac{P_2}{k}(1 - e^{-kT_c}) = \frac{D}{k}(e^{kT_c} - 1)e^{kT_T}$ which after simplification gives

$$P_2 = De^{kT_T} e^{kT_c} = De^{k(T_c + T_T)} \tag{3.19}$$

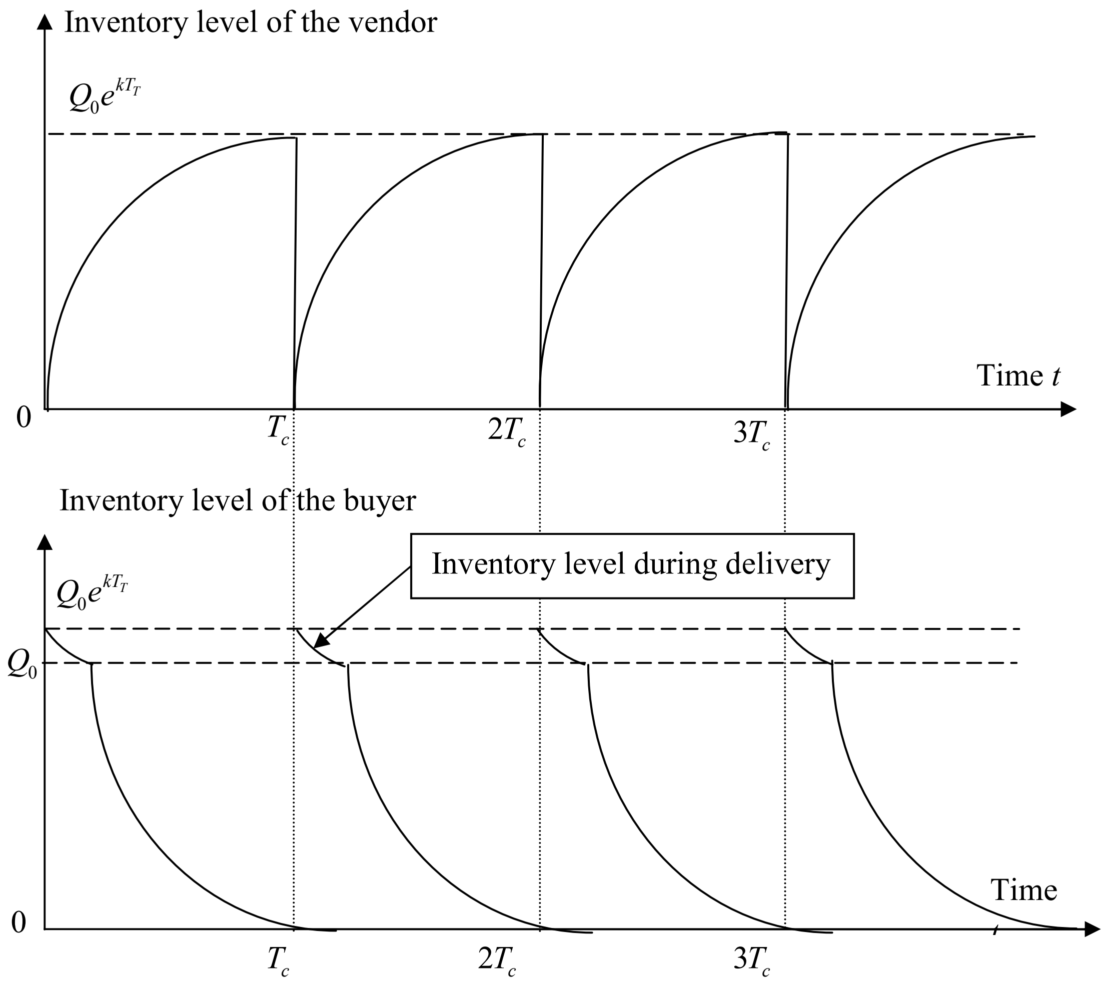


Figure 3.4: The inventory levels of the vendor, that during delivery and that of the buyer.

Consider the delivery to occur between $t = -T_T$ and $t = 0$. The total inventory (area under the inventory curve) is:

$$\int_{-T_T}^{0} Q_0 e^{-kt} dt = \frac{Q_0}{k}(e^{kT_T} - 1) = \frac{D}{k^2}(e^{kT_c} - 1)(e^{kT_T} - 1) \ .$$

Assuming the unit inventory holding cost and unit deterioration cost during delivery are the same as that for the vendor, as this occurs within a cycle of cycle time T_c, the holding cost per unit time for deterioration during delivery is $\frac{H_v D}{k^2 T_c}(e^{kT_c}-1)(e^{kT_T}-1)$.

Quantity of deteriorated items during delivery $= Q_0 e^{kT_T} - Q_0 = \frac{D}{k}(e^{kT_c}-1)(e^{kT_T}-1)$.

Deterioration cost during delivery per unit time = $\frac{C_v D}{k T_c}(e^{kT_c}-1)(e^{kT_T}-1)$.

The total relevant system cost per unit time is found as follows:

$$TC_b = \frac{A_b}{T_c} + (\frac{H_b}{k} + C_b)(\frac{e^{kT_c}-1}{k})\frac{D}{T_c} - \frac{H_b D}{k} - C_b D;$$

$$TC_v = S + \frac{1}{T_c}\{A_v + (\frac{H_v}{k} + C_v)(\frac{e^{-kT_c}-1}{k})De^{k(T_c+T_T)} + (\frac{H_v}{k} + C_v)De^{k(T_c+T_T)}T_c\}$$

Holding cost and deterioration cost during transportation per unit time

$$= \frac{H_v D}{k^2 T_c}(e^{kT_c}-1)(e^{kT_T}-1) + \frac{C_v D}{k T_c}(e^{kT_c}-1)(e^{kT_T}-1).$$

Adding these costs,

$$TC_s = \frac{A_b + A_v}{T_c} + \frac{D}{k}[\frac{H_b}{k} + C_b - \frac{H_v}{k} - C_v](\frac{e^{kT_c}-1}{T_c}) + \frac{H_v D e^{kT_T} e^{kT_c}}{k} + C_v D e^{kT_T} e^{kT_c} - \frac{H_b D}{k} - C_b D + S. \tag{3.20}$$

Set $m_1 = \frac{D}{k}[\frac{H_b}{k} + C_b - \frac{H_v}{k} - C_v]$ unchanged, and $m_2 = \frac{De^{kT_T}(H_v + C_v k)}{k^2}$.

Alternatively, if the unit inventory holding cost and unit deterioration cost during delivery are same as that for the buyer, then

$$TC_s = \frac{A_b + A_v}{T_c} + \frac{D}{k}e^{kT_T}[\frac{H_b}{k} + C_b - \frac{H_v}{k} - C_v](\frac{e^{kT_c} - 1}{T_c}) + \frac{H_v De^{kT_T}e^{kT_c}}{k} + C_v De^{kT_T}e^{kT_c} - \frac{H_b D}{k} - C_b D + S. \tag{3.21}$$

Set $m_1 = \frac{D}{k}e^{kT_T}(\frac{H_b}{k} + C_b - \frac{H_v}{k} - C_v)$, and $m_2 = \frac{De^{kT_T}(H_v + C_v k)}{k^2}$.

Optimal cycle time can be found using Steps 2 to 4 in the solution procedure in Section 3.4.4.

After finding the optimal delivery cycle time T_c^*,

(a) the optimal production rate can be found by using equation (3.19), that is, $P^* = De^{k(T_c^* + T_T)}$;

(b) the shipment quantity is given by $\frac{D}{k}(e^{kT_c^*} - 1)e^{kT_T}$; and

(c) the optimal system cost per unit time can be found by equation (3.20) or (3.21) as appropriate.

3.4.8 Example 3.2

Suppose it takes 0.02 year for a delivery (that is, $T_T = 0.02$) for the supply chain in Example 3.1. Consider a deterioration rate of $k = 0.1$ per year.

If the unit inventory holding cost and unit deterioration cost during delivery are same as that for the vendor, then

$$m_1 = \frac{1000}{0.1}[\frac{5}{0.1} + 50 - \frac{4}{0.1} - 40] = 200000$$

$$m_2 = \frac{1000}{0.1^2}[4 + 40(0.1)]e^{(0.10)(0.02)} = 801601.6$$

Let $x = kT_c$.

$$f(x) = 200000[(x-1)e^x + 1] + 801601.6(x)^2 e^x - 25 = 0$$

$$f'(x) = xe^x[200000 + 2(801601.6) + 801601.6x] = xe^x(1803203.2 + 801601.6x)$$

The optimal cycle time is 0.05253 year, the optimal production rate is 1007.28 units per year, and the optimal cost is $1510.89 per year.

Alternatively, if the unit inventory holding cost and unit deterioration cost during delivery are same as that for the buyer, then

$$m_1 = \frac{1000}{0.1}[\frac{5}{0.1} + 50 - \frac{4}{0.1} - 40]e^{(0.1)(0.02)} = 200400.4$$

$$m_2 = \frac{1000}{0.1^2}[4 + 40(0.1)]e^{(0.10)(0.02)} = 801601.6$$

By the same procedure, the optimal cycle time is 0.05252 year, the optimal production rate is 1007.28 units per year, and the optimal cost is $1551.04 per year.

The same computations are done for a deterioration rate of 0.2 per year, and the results are shown in Table 3.4. The optimal cycle time decreases slightly when deterioration during transportation is considered; and the optimal cost increases as expected.

Deterioration rate $k = 0.1$	No deterioration during delivery	Deterioration during delivery Case 1	Deterioration during delivery Case 2
Optimal cycle time (year)	0.0527	0.05253	0.05252
Optimal production rate (units/year)	1005.27	1007.28	1007.28
Optimal total cost per unit time ($ per year)	1349.89	1510.89	1551.04
$k = 0.2$			
Optimal cycle time (year)	0.04286	0.04278	0.04277
Optimal production rate (units/year)	1008.61	1012.635	1013.633
Optimal total cost per unit time ($ per year)	1564.30	1806.85	1867.23

Case 1: the unit inventory holding cost and unit deterioration cost during delivery are same as that for the vendor

Case 2: the unit inventory holding cost and unit deterioration cost during delivery are same as that for the buyer

Table 3.4: Results for Example 3.2.

3.5 Conclusion

In this chapter, the individual optimization of the buyer and that of the vendor (lot-for-lot policy) have been revisited and the convexity of the buyer's cost function and the vendor's cost function have been shown using the exact mathematical expressions without approximating the exponential terms in the cost functions. A continuous production model, adopting the lot-for-lot policy, is then proposed. Unlike the lot-delivery production-inventory models in the literature, the production rate in this model is not predetermined but is found by optimizing the cost function. After presenting the model development of the proposed model, a numerical example has been provided. The results of the example indicate that the proposed model can reduce the total system cost per unit time by having smaller average inventory levels of the buyer and the vendor. The proposed model is modified to consider deterioration during delivery which is usually ignored in the literature of inventory models.

In addition, the proposed model can minimize the idleness of production facilities and facilitate labour planning. There may be concerns that on-going production may increase the wear and tear of the concerned equipment. It is suggested that part of the cost saving could be allocated as additional resources for maintenance of the equipment.

Chapter 4

Further Comparison of the Demand-driven Production Rate Model and the Predetermined Production Rate Model

4.1 Introduction

In the literature of lot-delivery models, the production rates are predetermined and are usually much higher than the demand rates. In Chapter 3, a lot-for-lot continuous production model has been proposed for a single-vendor single-buyer supply chain. In the proposed model the optimal production rate is found by optimizing the total system cost and is given by $De^{kT_c^*}$, that is, the optimal production rate is a function of the demand rate and the optimal cycle time. The numerical example provided indicated that the proposed model can result in a lower total cost compared with the predetermined production rate model. In this chapter the proposed model, a "demand-driven" production rate model, is further compared with the predetermined production rate model. It will be shown that in many cases, the demand-driven production rate model can give a lower total cost than the other model. To distinguish between the two models, the total relevant cost for the system per unit time for the predetermined production rate model is denoted by $TC_{s(p)}$ in this chapter and that for the other model is denoted by TC_s as in Chapter 3.

Wee et al. (2008) derived the following formula for the total relevant system cost per unit time for a lot-delivery single-vendor single-buyer system:

$$TC_{s(p)} = \frac{S}{T} + \frac{n(A_v + A_b)}{T} + \frac{(H_b - H_v + kC_b - kC_v)nD}{kT}[\frac{1}{k}(e^{\frac{kT}{n}} - 1) - \frac{T}{n}] + \frac{(H_v + kC_v)(PT_p - DT)}{kT} \tag{4.1}$$

where P is the predetermined production rate, n is the number of deliveries in a system cycle of cycle time T, and T_p is the production time within a cycle given by

$$T_p = \frac{1}{k}\ln[1 + \frac{\frac{D}{P}(e^{kT} - 1)}{1 - \frac{D}{P}(e^{\frac{kT}{n}} - 1)}] \ . \tag{4.2}$$

Wee et al. (2008) expanded the exponential and logarithmic terms and truncated the higher order terms to obtain an algebraic expression for $TC_{s(p)}$ and found the optimal number of deliveries and system cycle time for minimizing $TC_{s(p)}$. The Taylor expansion for the exponential function is valid for all real values of the variable. The Taylor expansion for the logarithmic function is given by

$$\ln(1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots$$

It requires $-1 < x \leq 1$ for convergence of this infinite series.

Wee et al. did not discuss the condition for convergence of the expansion for the logarithmic term. The sufficient condition for the convergence can be found as follows:

For a delivery interval of $T_c = \frac{T}{n}$, the minimum production rate is $De^{\frac{kT}{n}}$ in order to satisfy the demand without shortages. For the predetermined production rate model, the production rate must satisfy $P > De^{\frac{kT}{n}}$ due to non-continuous production.

Hence, $\frac{D}{P}(e^{kT} - 1) > 0$, $1 - \frac{D}{P}(e^{\frac{kT}{n}} - 1) > 1 - e^{\frac{-kT}{n}}(e^{\frac{kT}{n}} - 1) = e^{\frac{-kT}{n}} > 0$, and

$$1 - \frac{D}{P}(e^{\frac{kT}{n}} - 1) - \frac{D}{P}(e^{kT} - 1) = 1 - \frac{D}{P}(e^{\frac{kT}{n}} + e^{kT} - 2) > 1 - e^{\frac{-kT}{n}}(e^{\frac{kT}{n}} + e^{kT} - 2) = (2 - e^{kT})e^{\frac{-kT}{n}}$$

If $e^{kT} \le 2$, or $kT \le \ln 2 \approx 0.6931$, then $1 - \frac{D}{P}(e^{\frac{kT}{n}} - 1) - \frac{D}{P}(e^{kT} - 1) \ge 0$, and

$$0 < \frac{\frac{D}{P}(e^{kT} - 1)}{1 - \frac{D}{P}(e^{\frac{kT}{n}} - 1)} < 1. \tag{4.3}$$

The condition $kT \le \ln 2$ is a sufficient one for Eq (4.3) for any production rate larger than the demand-driven production rate and for any number of deliveries. In the literature of inventory models, system cycle time is usually not more than one year; and deterioration rates for exponentially deteriorating items in the numerical examples of most of these models are not more than 0.2 per year. Hence, the condition of $kT \le \ln 2$ is usually satisfied.

In this chapter, expansion for logarithmic terms is not needed as exact mathematical expressions are used. The assumption of $kT \le \ln 2$ is therefore not required. The assumptions are:

1. the system cycle time is within a year, i.e., $T \le 1$,

2. the deterioration rate is not more than 0.863, i.e., $k \le 0.863$.

Therefore, combining assumptions 1 and 2, $kT \le 0.863$. (This will be explained in the next section.)

3. the demand rate and the cost parameters are constant, and

4. the production rate is constant: either a given constant for the predetermined production rate model, or a constant to be determined as in Chapter 3 for the demand-driven production rate model.

4.2 Change of Production Rate for the Predetermined Production Rate Model

In this section, the effect of the change of production rate on the total cost per unit time with a given system cycle time and a given number of deliveries is investigated.

Differentiate equation (4.1) with respect to P, $\frac{\delta}{\delta P} TC_{s(p)} = \frac{(H_v + kC_v)}{kT} \frac{\delta}{\delta P}(PT_p)$.

Hence, $\frac{\delta}{\delta P} TC_{s(p)}$ has the same sign as and is proportional to $\frac{\delta}{\delta P}(PT_p)$.

$$\frac{\delta}{\delta P}(PT_p)=\frac{\delta}{\delta P}\{P\frac{1}{k}\ln[1+\frac{\frac{D}{P}(e^{kT}-1)}{1-\frac{D}{P}(e^{\frac{kT}{n}}-1)}]\}$$

$$=\frac{1}{k}\{\ln[1+\frac{\frac{D}{P}(e^{kT}-1)}{1-\frac{D}{P}(e^{\frac{kT}{n}}-1)}]+P\frac{\delta}{\delta P}\ln[1+\frac{\frac{D}{P}(e^{kT}-1)}{1-\frac{D}{P}(e^{\frac{kT}{n}}-1)}]\}$$

$$=\frac{1}{k}\{\ln[1+\frac{\frac{D}{P}(e^{kT}-1)}{1-\frac{D}{P}(e^{\frac{kT}{n}}-1)}]+P\frac{1}{1+\frac{\frac{D}{P}(e^{kT}-1)}{1-\frac{D}{P}(e^{\frac{kT}{n}}-1)}}\frac{-\frac{D}{P^2}(e^{kT}-1)}{[1-\frac{D}{P}(e^{\frac{kT}{n}}-1)]^2}\}$$

$$=\frac{1}{k}\{\ln[1+\frac{\frac{D}{P}(e^{kT}-1)}{1-\frac{D}{P}(e^{\frac{kT}{n}}-1)}]-\frac{\frac{D}{P}(e^{kT}-1)}{[1+\frac{D}{P}(e^{kT}-e^{\frac{kT}{n}})][1-\frac{D}{P}(e^{\frac{kT}{n}}-1)]}\} \tag{4.4}$$

4.2.1 The Effect of One Delivery in a System Cycle

Consider the case of $n=1$.

$$\frac{\delta}{\delta P}(PT_p)=\frac{1}{k}\{\ln[1+\frac{\frac{D}{P}(e^{kT}-1)}{1-\frac{D}{P}(e^{kT}-1)}]-\frac{\frac{D}{P}(e^{kT}-1)}{1-\frac{D}{P}(e^{kT}-1)}\}<0 \text{ as } \ln(1+u)<u\,.$$

$$\therefore\ \frac{\delta}{\delta P}TC_{s(p)}<0\,.$$

Hence, the total cost per unit time is a decreasing function of production rate for a given value of *T*.

4.2.2 The Effect of Three or More Deliveries in a System Cycle

Numerical experiments show that for $n \geq 3$, $\frac{\delta}{\delta P}(PT_p)$ is positive for any production rate larger than the demand-driven production rate. This observation can be proved with the assumptions mentioned in Section 4.1 as follows:

With given values for T and n, $P > De^{\frac{kT}{n}}$ or $\frac{D}{P} < e^{\frac{-kT}{n}}$.

At the limiting value of $P = De^{\frac{kT}{n}}$,

$$\frac{\delta}{\delta P}(PT_p) = \frac{1}{k}\{\ln[1+\frac{e^{\frac{-kT}{n}}(e^{kT}-1)}{1-e^{\frac{-kT}{n}}(e^{\frac{kT}{n}}-1)}] - \frac{e^{\frac{-kT}{n}}(e^{kT}-1)}{[1+e^{\frac{-kT}{n}}(e^{kT}-e^{\frac{kT}{n}})][1-e^{\frac{-kT}{n}}(e^{\frac{kT}{n}}-1)]}\}$$

$$= \frac{1}{k}[\ln(1+e^{kT}-1) - \frac{e^{kT}-1}{e^{(1-\frac{1}{n})kT}}] = \frac{1}{k}(kT - \frac{e^{kT}-1}{e^{(1-\frac{1}{n})kT}}) = \frac{1}{k}(\frac{kTe^{(1-\frac{1}{n})kT} - e^{kT}+1}{e^{(1-\frac{1}{n})kT}}).$$

Let $y = kT$.

For $n = 2$,

$$kTe^{(1-\frac{1}{n})kT} - e^{kT} + 1 = ye^{\frac{y}{2}} - e^{y} + 1$$

At $y = 0$, $ye^{\frac{y}{2}} - e^{y} + 1 = 0$.

For $y > 0, \dfrac{d}{dy}(ye^{\frac{y}{2}} - e^{y} + 1) = e^{\frac{y}{2}}(1 + \dfrac{y}{2} - e^{\frac{y}{2}}) < 0$.

Hence, $kTe^{(1-\frac{1}{n})kT} - e^{kT} + 1 < 0$.

Therefore, for $n = 2$, at $P = De^{\frac{kT}{2}}, \dfrac{\delta}{\delta P}(PT_p) < 0$.

For $n \geq 3$,

$kTe^{(1-\frac{1}{n})kT} - e^{kT} + 1 \geq ye^{\frac{2y}{3}} - e^{y} + 1$

Solving $ye^{\frac{2y}{3}} - e^{y} + 1 = 0$, $y = 0$ or $y = 4.6223$ (by the Newton-Raphson method) .

It can be easily verified that $ye^{\frac{2y}{3}} - e^{y} + 1 > 0$ for $0 < y < 4.6223$.

So for $0 < y \leq 0.863 < 4.6223$, $kTe^{(1-\frac{1}{n})kT} - e^{kT} + 1 > 0$.

Therefore, for $n \geq 3$, at $P = De^{\frac{kT}{n}}, \dfrac{\delta}{\delta P}(PT_p) > 0$. (4.5a)

The second derivative $\frac{\delta^2}{\delta P^2}(PT_p)$ can be found by differentiating equation (4.4).

$$\frac{\delta^2}{\delta P^2}(PT_p)=\frac{-\frac{D}{P^2}(e^{kT}-1)}{[1+\frac{D}{P}(e^{kT}-e^{\frac{kT}{n}})][1-\frac{D}{P}(e^{\frac{kT}{n}}-1)]}-\frac{[1+\frac{D}{P}(e^{kT}-e^{\frac{kT}{n}})][1-\frac{D}{P}(e^{\frac{kT}{n}}-1)](-\frac{D}{P^2})(e^{kT}-1)\}}{[1+\frac{D}{P}(e^{kT}-e^{\frac{kT}{n}})]^2[1-\frac{D}{P}(e^{\frac{kT}{n}}-1)]^2}$$

$$+\frac{\frac{D}{P}(e^{kT}-1)\{[1+\frac{D}{P}(e^{kT}-e^{\frac{kT}{n}})]\frac{D}{P^2}(e^{\frac{kT}{n}}-1)+[1-\frac{D}{P}(e^{\frac{kT}{n}}-1)](-\frac{D}{P^2})(e^{kT}-e^{\frac{kT}{n}})\}}{[1+\frac{D}{P}(e^{kT}-e^{\frac{kT}{n}})]^2[1-\frac{D}{P}(e^{\frac{kT}{n}}-1)]^2}$$

$$=\frac{D}{P^2}(e^{kT}-1)\{\frac{-1}{[1+\frac{D}{P}(e^{kT}-e^{\frac{kT}{n}})][1-\frac{D}{P}(e^{\frac{kT}{n}}-1)]}+\frac{1+(\frac{D}{P})^2(e^{\frac{kT}{n}}-1)(e^{kT}-e^{\frac{kT}{n}})}{[1+\frac{D}{P}(e^{kT}-e^{\frac{kT}{n}})]^2[1-\frac{D}{P}(e^{\frac{kT}{n}}-1)]^2}\}$$

$$=\frac{D}{P^2}(e^{kT}-1)\{\frac{-[1+\frac{D}{P}(e^{kT}-e^{\frac{kT}{n}})][1-\frac{D}{P}(e^{\frac{kT}{n}}-1)]+1+(\frac{D}{P})^2(e^{\frac{kT}{n}}-1)(e^{kT}-e^{\frac{kT}{n}})}{[1+\frac{D}{P}(e^{kT}-e^{\frac{kT}{n}})]^2[1-\frac{D}{P}(e^{\frac{kT}{n}}-1)]^2}\}$$

$$=\frac{D}{P^2}(e^{kT}-1)\{\frac{-\frac{D}{P}[e^{kT}-2e^{\frac{kT}{n}}+1-2\frac{D}{P}(e^{\frac{kT}{n}}-1)(e^{kT}-e^{\frac{kT}{n}})]}{[1+\frac{D}{P}(e^{kT}-e^{\frac{kT}{n}})]^2[1-\frac{D}{P}(e^{\frac{kT}{n}}-1)]^2}\} \tag{4.6}$$

As $\frac{D}{P}<e^{\frac{-kT}{n}}$,

$$e^{kT}-2e^{\frac{kT}{n}}+1-2\frac{D}{P}(e^{\frac{kT}{n}}-1)(e^{kT}-e^{\frac{kT}{n}})>e^{kT}-2e^{\frac{kT}{n}}+1-2(1-e^{\frac{-kT}{n}})(e^{kT}-e^{\frac{kT}{n}})$$

$$=2e^{(1-\frac{1}{n})kT}-e^{kT}-1$$

Let $f_1(y)=2e^{(1-\frac{1}{n})y}-e^{y}-1$ where $n\geq 3$.

$f_1(0) = 2 - 1 - 1 = 0$

$f_1'(y) = 2(1 - \frac{1}{n})e^{(1-\frac{1}{n})y} - e^y = e^{(1-\frac{1}{n})y}[2(1 - \frac{1}{n}) - e^{\frac{y}{n}})$

$2(1 - \frac{1}{n}) - e^{\frac{y}{n}} \geq 2(1 - \frac{1}{3}) - e^{\frac{y}{3}} > 0 \quad \text{for } y \leq 0.863$

So $f_1'(y) > 0$ and hence, for $y > 0$, $f_1(y) > 0$ and $\frac{\delta^2}{\delta P^2}(PT_p) < 0$. (4.5b)

So $\frac{\delta}{\delta P}(PT_p)$ is a decreasing function of *P*.

For all $n \geq 1$, $\lim_{P \to \infty} \frac{\delta}{\delta P}(PT_p) = \frac{1}{k}\{\ln[1 + \frac{0}{1}] - \frac{0}{[1+0][1-0]}\} = 0$ (4.5c)

From the results of Eqs (4.5a), (4.5b) and (4.5c), $\frac{\delta}{\delta P}(PT_p)$ is positive at the smallest feasible production rate; it decreases when the production rate increases and remains positive and approaches 0 when the production rate is very large. (As $\frac{\delta}{\delta P}(PT_p)$ is always a decreasing function of *P*, it cannot become negative and then increase to approach 0.) Therefore, it is proved that when $n \geq 3$, $k \leq 0.863$ and $T \leq 1$, (i.e., $kT \leq 0.863$, and explains assumption 2.) $\frac{\delta}{\delta P}(PT_p)$ is positive and thus $\frac{\delta}{\delta P}TC_{s(p-s)}$ is positive, for any production rate larger than the demand-driven production rate. The total cost per unit time is an increasing function of production rate for a given value of *T*.

4.2.3 The Effect of Two Deliveries in a System Cycle

When $n = 2$, numerical experiments indicate that $\frac{\delta}{\delta P}(PT_p)$ is positive when the production rate exceeds a certain value. For example, with a demand of 1000 units per year, the experimental results are shown in Table 4.1:

K	0.1	0.1	0.1	0.2	0.3	0.693	0.863
T	0.2	0.5	0.8	0.8	0.8	1	1
Demand-driven production rate	1010.1	1025.3	1040.8	1083.3	1127.5	1415	1540
Minimum production rate for $\frac{\delta}{\delta P}(PT_p) > 0$	1347	1368	1388	1445	1504	1883	2049

Table 4.1: Minimum production rate for $\frac{\delta}{\delta P}(PT_p) > 0$ when $n = 2$ and $D = 1000$.

This experimental observation can be proved as follows:

When $n = 2$, from equation (4.6),

$$\frac{\delta^2}{\delta P^2}(PT_p) = \frac{D}{P^2}(e^{kT}-1)\{\frac{-\frac{D}{P}[e^{kT}-2e^{\frac{kT}{2}}+1-2\frac{D}{P}(e^{\frac{kT}{2}}-1)(e^{kT}-e^{\frac{kT}{2}})]}{[1+\frac{D}{P}(e^{kT}-e^{\frac{kT}{2}})]^2[1-\frac{D}{P}(e^{\frac{kT}{2}}-1)]^2}\}$$

$$= \frac{D}{P^2}(e^{kT}-1)\{\frac{-\frac{D}{P}(e^{\frac{y}{2}}-1)^2(1-2\frac{D}{P}e^{\frac{y}{2}})}{[1+\frac{D}{P}(e^{kT}-e^{\frac{kT}{2}})]^2[1-\frac{D}{P}(e^{\frac{kT}{2}}-1)]^2}\}$$

So $\frac{\delta^2}{\delta P^2}(PT_p)$ and $(e^{\frac{y}{2}}-1)^2(1-2\frac{D}{P}e^{\frac{y}{2}})$ have opposite signs.

Let $\rho = \frac{D}{P}$, $y = kT$ and $h(y) = (e^{\frac{y}{2}} - 1)^2(1 - 2\rho e^{\frac{y}{2}})$.

$h(y) = (e^{\frac{y}{2}} - 1)^2(1 - 2\rho e^{\frac{y}{2}}) = 0 \Rightarrow y = 0$ **or** $2\rho e^{\frac{y}{2}} = 1$, that is, $P = 2De^{\frac{y}{2}}$

When $P < 2De^{\frac{y}{2}}$, $h(y) < 0 \Rightarrow \frac{\delta^2}{\delta P^2}(PT_p) > 0$

When $P > 2De^{\frac{y}{2}}$, $h(y) > 0 \Rightarrow \frac{\delta^2}{\delta P^2}(PT_p) < 0$

Hence, $\frac{\delta}{\delta P}(PT_p)$ is maximum when $P = 2De^{\frac{y}{2}}$.

From Section 4.2.2, when $n = 2$ and $\rho = e^{-\frac{y}{2}}$, $\frac{\delta}{\delta P}(PT_p) < 0$ and $\lim_{P \to \infty} \frac{\delta}{\delta P}(PT_p) = 0$.

So for $n = 2$ and a given value of *T*, as the production rate increases from $P = De^{\frac{kT}{2}}$, $\frac{\delta}{\delta P}(PT_p)$ increases (gets less negative), and becomes positive when the production rate exceeds a particular value which is less than $2De^{\frac{kT}{2}}$. For production rates larger than $2De^{\frac{kT}{2}}$, $\frac{\delta}{\delta P}(PT_p)$ decreases (still being positive) and approaches 0 when the production rate is very large. Therefore, for production rates smaller than that particular value, $\frac{\delta}{\delta P}(PT_p)$ and $\frac{\delta}{\delta P}TC_{s(p)}$ are negative; for production rates larger than that value, $\frac{\delta}{\delta P}(PT_p)$ and therefore $\frac{\delta}{\delta P}TC_{s(p)}$ are positive. This means that for

$n = 2$ and a given value of *T*, the total cost per unit time is not a monotonic function of production rate. There is a particular production rate at which the total cost per unit time is the minimum.

Substituting $n = 2$ and $\rho = D/P$ into equation (4.4),

$$\frac{\delta}{\delta P}(PT_p) = \frac{1}{k}\{\ln[1+\frac{\rho(e^{kT}-1)}{1-\rho(e^{\frac{kT}{2}}-1)}] - \frac{\rho(e^{kT}-1)}{[1+\rho(e^{kT}-e^{\frac{kT}{2}})][1-\rho(e^{\frac{kT}{2}}-1)]}\}. \qquad (4.7)$$

The particular value of production rate, at which $\frac{\delta}{\delta P}(PT_p) = 0$, can be found by solving equation (4.7) for given values of *D*, *k* and *T*. For example, set $y = kT = (0.693)(1)$ and apply the Newton-Raphson Method to solve the equation

$$\frac{\delta}{\delta P}(PT_p) = \frac{1}{0.693}\{\ln[1+\frac{\rho(2-1)}{1-\rho(\sqrt{2}-1)}] - \frac{\rho(2-1)}{[1+\rho(2-\sqrt{2})][1-\rho(\sqrt{2}-1)]}\} = 0$$

with $\frac{\delta}{\delta\rho}[\frac{\delta}{\delta P}(PT_p)] = \rho(e^y-1)\{\frac{e^y-2e^{\frac{y}{2}}+1-2\rho(e^{\frac{y}{2}}-1)(e^y-e^{\frac{y}{2}})}{[1+\rho(e^y-e^{\frac{y}{2}})]^2[1-\rho(e^{\frac{y}{2}}-1)]^2}\}$.

The solution is $\rho = 0$ $(P \to \infty)$ or $\rho = 0.53112$ $(P = 1.883D)$.

For $y = kT = \ln 2$, if $D = 1000$, the minimum production rate for $\frac{\delta}{\delta P}(PT_p) > 0$ is approximately 1883 and agrees with the result in Table 4.1.

The exact production rate at which $\frac{\delta}{\delta P}(PT_p)=0$ depends on the value of *T*. For a given *k*, set $T=1$ and solve the following equation for ρ:

$$\frac{\delta}{\delta P}(PT_p)=\frac{1}{k}\{\ln[1+\frac{\rho(e^k-1)}{1-\rho(e^{\frac{k}{2}}-1)}]-\frac{\rho(e^k-1)}{[1+\rho(e^k-e^{\frac{k}{2}})][1-\rho(e^{\frac{k}{2}}-1)]}\}=0. \tag{4.8}$$

Suppose $\rho=\rho^{\eta}$ is the solution of equation (4.8). For $T<1$, $\frac{\delta}{\delta P}(PT_p)=0$ is reached at a production rate smaller than D/ρ^{η}. Hence, for any $T\leq 1$, if only production rates larger than D/ρ^{η} are considered, the total cost per unit time is an increasing function of production rate. This result will be used in Theorem 1 in Section 4.3.

4.2.4 Summary

For a given value of the system cycle time *T* and a given number of deliveries *n*, the effect of changing production rate on the total system cost per unit time is summarized as follows:

(#1) For $n=1$, increasing production rate reduces the cost.

(#2) For $n=2$, for production rates smaller than a particular value, increasing the producing rate decreases the cost; for production rates larger than that value, increasing production rate increases the cost.

(#3) For $n\geq 3$, increasing production rate increases the cost.

As an illustration, the changes of PT_p with production rate for $D = 1000$ units per year, $k = 0.1$ per year, and $T = 0.5$ year are shown in Figures 4.1, 4.2 and 4.3.

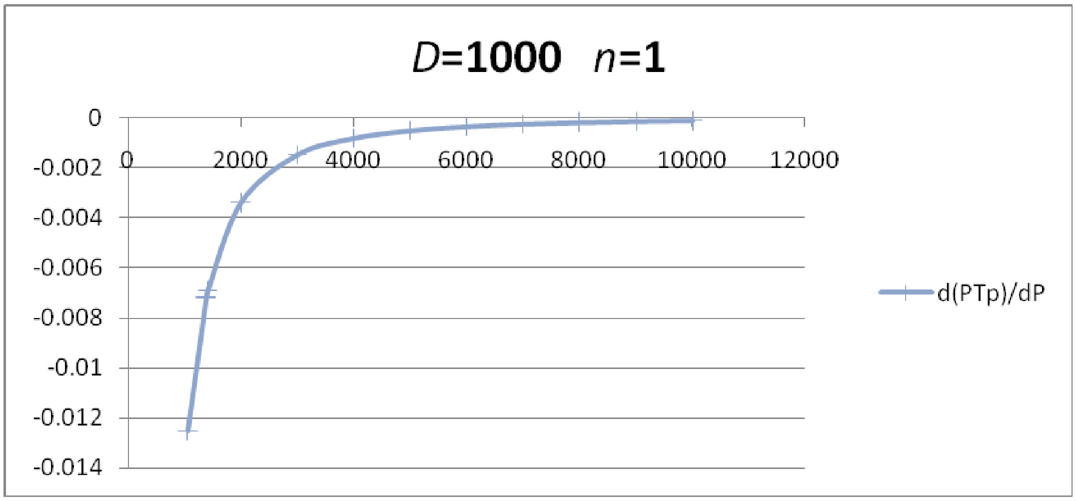


Figure 4.1: $\frac{\delta}{\delta P}(PT_p)$ against production rate for $k = 0.1$, $T = 0.5$ and $n = 1$.

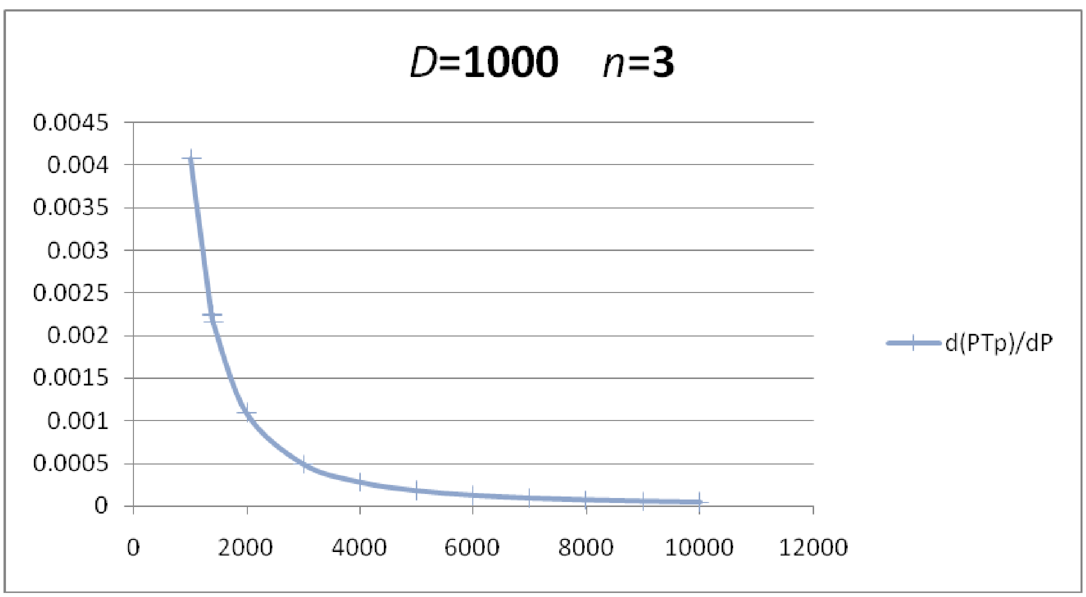


Figure 4.2: $\frac{\delta}{\delta P}(PT_p)$ against production rate for $k = 0.1$, $T = 0.5$ and $n = 3$.

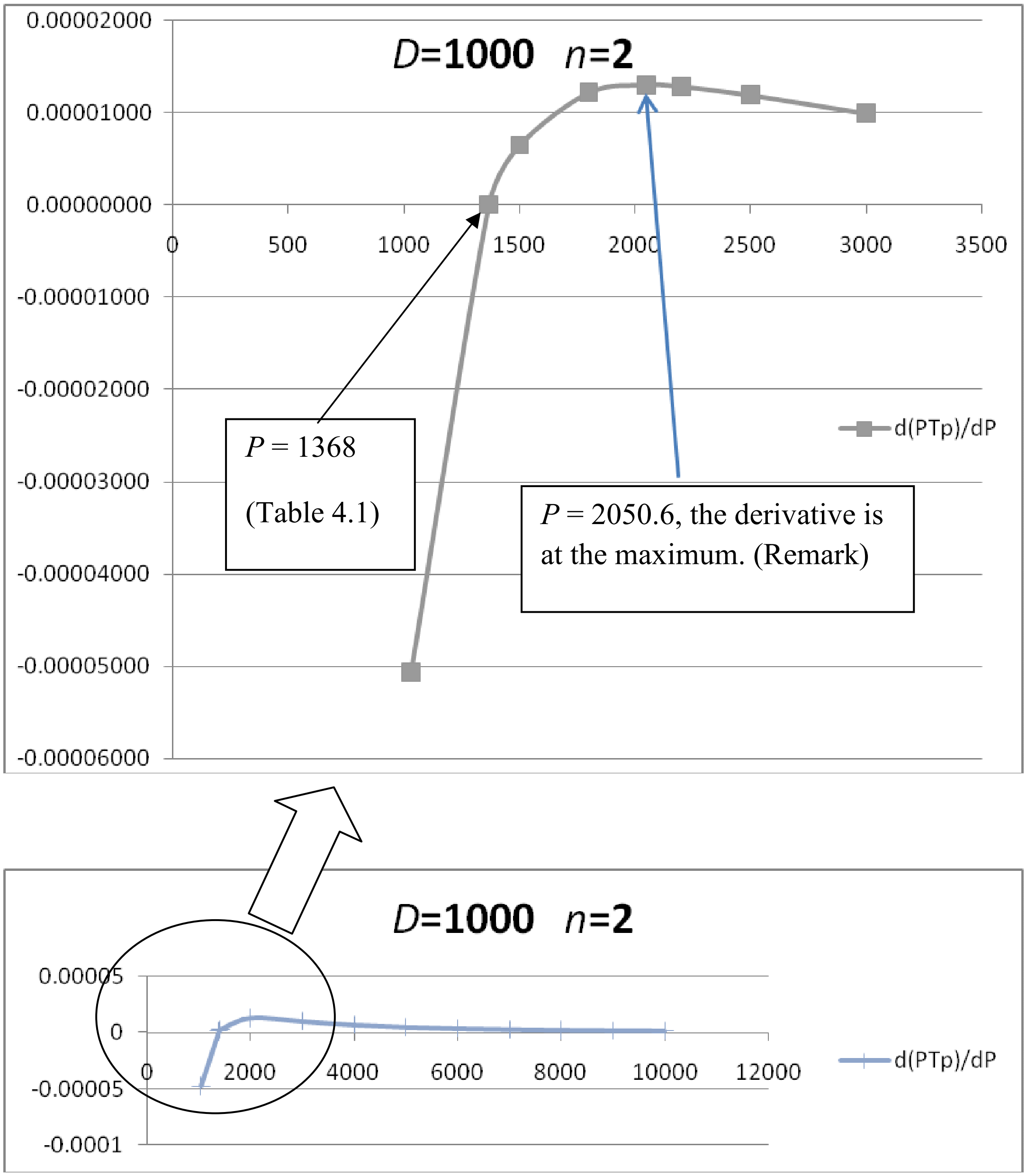


Figure 4.3: $\frac{\delta}{\delta P}(PT_p)$ against production rate for $k = 0.1$, $T = 0.5$ and $n = 2$.

Remark:

At a production rate of $P = 2De^{kT/2} = 2050.6$, $\frac{\delta}{\delta P}(PT_p)$ is at the maximum for $n = 2$.

4.3 Three Theorems for the Two Models

4.3.1 Theorem I

If a single-vendor single-buyer system supplying an exponentially deteriorating item with a demand rate of D satisfies the following conditions:

(a) the rate of deterioration is not more than 0.863, i.e., $k \le 0.863$;

(b) the system cycle time is within one year, i.e., $T \le 1$;

(c) the cost parameters are constant for production rates in the interval $[P_a, P_b]$ where $P_a \ge D/\rho^\eta$, ρ^η being the solution of equation (4.8);

then the optimal total system cost per unit is the smaller value of the optimal costs at the production rates of P_a and P_b.

Proof:

For any production rate P_s in the interval $[P_a, P_b]$,

(i) If the cost is optimal with $n = 1$ and $T = T_s^*$, as $P_s < P_b$, we have

$TC_{s(p)}(P_s, 1, T_s^*) > TC_{s(p)}(P_b, 1, T_s^*) \ge TC_{s(p)}^*(P_b)$ where $TC_{s(p)}^*(P_b)$ is the overall optimal cost at the production rate of P_b for all possible values of n.

The first inequality is due to (#1) in Section 4.2.4.

(ii) If the cost is optimal with $n = 2$ and $T = T_s^*$, as $P_s > P_a \geq D / \rho^{\eta}$, we have

$TC_{s(p)}(P_s, 2, T_s^*) > TC_{s(p)}(P_a, 2, T_s^*) \geq TC_{s(p)}^*(P_a)$ where $TC_{s(p)}^*(P_a)$ is the overall optimal cost at the production rate of P_a for all possible values of *n*.

The first inequality is due to the discussion after equation (4.8).

(iii) If the cost is optimal with $n \geq 3$ and $T = T_s^*$, as $P_s > P_a$, we have

$TC_{s(p)}(P_s, n, T_s^*) > TC_{s(p)}(P_a, n, T_s^*) \geq TC_{s(p)}^*(P_a)$ where $TC_{s(p)}^*(P_a)$ is the overall optimal cost at the production rate of P_a for all possible values of *n*.

The first inequality is due to (#3) in Section 4.2.4.

Hence, for minimizing the total cost per unit time, the optimal system costs at the production rates of P_a and P_b should be found and the production rate that gives the smaller cost should be selected as the predetermined production rate.

4.3.2 Theorem II

If a single-vendor single-buyer system supplying an exponentially deteriorating item with a demand rate of *D* satisfies the following conditions:

(a) the rate of deterioration is not more than 0.863, i.e., $k \leq 0.863$.

(b) the system cycle time is within one year, i.e., $T \le 1$.

(c) the cost parameters are constant for production rates $\ge D$.

then the demand-driven production rate model always gives a better optimal cost than production rate P_s of the predetermined production rate model if the cost for production rate P_s is optimal with $n \ge 3$.

Proof:

Suppose the cost for production rate P_s is optimal with $n \ge 3$ and $T = T_s^*$.

Then $P_s > De^{\frac{kT_s^*}{n}}$ and hence $TC_{s(p)}^*(P_s, n, T_s^*) > TC_{s(p)}(De^{\frac{kT_s^*}{n}}, n, T_s^*)$ due to (#3) in Section 4.2.4.

Consider the two scenarios:

(i) the predetermined production rate model with production rate $De^{\frac{kT_s^*}{n}}$ having n deliveries over a system cycle of cycle time T_s^*, and

(ii) the demand-driven production rate model with delivery cycle time $T_c = \dfrac{T_s^*}{n}$ and production rate $De^{\frac{kT_s^*}{n}}$.

Both scenarios have the same delivery related, inventory holding and deterioration costs due to same production rate and same delivery cycle times. The production set-up cost per unit time of scenario (i) is $\frac{S}{T_s^*}$, and that of scenario (ii) is S.

Since $T_s^* \leq 1$, $S \leq \frac{S}{T_s^*}$. Hence, $TC_{s(p)}(De^{\frac{kT_s^*}{n}}, n, T_s^*) \geq TC_s(T_c = \frac{T_s^*}{n}) \geq TC_s^*$.

$\therefore\ TC_{s(p)}^*(P_s, n, T_s^*) > TC_s^*$,

where TC_s^* is the optimal cost for the demand-driven production rate model.

In Example 3.1 (Chapter 3), there are 4 or 5 deliveries in a system cycle in the optimal solutions for different deterioration rates and predetermined production rates. Therefore, the continuous production with demand-driven production rate model gives a smaller optimal cost for all these cases.

4.3.3 Theorem III

Given that a single-vendor single-buyer system satisfies the conditions in Theorem II. Suppose for the predetermined production rate model of production rate P_s, the cost is optimal with $n = 2$. A sufficient condition for the demand-driven production rate model giving a smaller optimal cost is that $P_s \geq 2D^{\frac{k}{2}}$.

Proof:

Suppose at production rate P_s, the cost is optimal with $n=2$ and $T=T_s^* \le 1$.

$$TC_{s(p)}(P_s,2,T_s^*)=\frac{S}{T_s^*}+\frac{2(A_v+A_b)}{T_s^*}+\frac{(H_b-H_v+kC_b-kC_v)2D}{kT_s^*}[\frac{1}{k}(e^{\frac{kT_s^*}{2}}-1)-\frac{T_s^*}{2}]$$
$$+\frac{(H_v+kC_v)(P_sT_p-DT_s^*)}{kT_s^*}$$

where $T_p=\dfrac{1}{k}\ln[1+\dfrac{\frac{D}{P_s}(e^{kT_s^*}-1)}{1-\frac{D}{P_s}(e^{\frac{kT_s^*}{2}}-1)}]$.

$$TC_s(T_c=\frac{T_s^*}{2})=\frac{2(A_b+A_v)}{T_s^*}+\frac{2D}{k}(\frac{H_b}{k}+C_b-\frac{H_v}{k}-C_v)(\frac{e^{\frac{kT_s^*}{2}}-1}{T_s^*})+\frac{H_vDe^{\frac{kT_s^*}{2}}}{k}+C_vDe^{\frac{kT_s^*}{2}}$$
$$-\frac{H_bD}{k}-C_bD+S$$

$$TC_{s(p)}(P_s,2,T_s^*)-TC_s(T_c=\frac{T_s^*}{2})=S(\frac{1}{T_s^*}-1)+(\frac{H_v}{k}+C_v)(\frac{P_sT_p}{T_s^*}-De^{\frac{kT_s^*}{2}})$$
$$\ge(\frac{H_v}{k}+C_v)(\frac{P_sT_p}{T_s^*}-De^{\frac{kT_s^*}{2}})$$

Let $P_s=aDe^{\frac{kT_s^*}{2}}$.

$$\frac{P_sT_p}{T_s^*}-De^{\frac{kT_s^*}{2}}=De^{\frac{kT_s^*}{2}}\{\frac{a}{kT_s^*}\ln[1+\frac{\frac{D}{aDe^{\frac{kT_s^*}{2}}}(e^{kT_s^*}-1)}{1-\frac{D}{aDe^{\frac{kT_s^*}{2}}}(e^{\frac{kT_s^*}{2}}-1)}]-1\}$$
$$=De^{\frac{kT_s^*}{2}}\{\frac{a}{kT_s^*}\ln[1+\frac{e^{kT_s^*}-1}{(a-1)e^{\frac{kT_s^*}{2}}+1}]-1\}$$

If $a = 2$,

$$\frac{P_s T_p}{T_s^*} - De^{\frac{kT_s^*}{2}} = De^{\frac{kT_s^*}{2}} \{\frac{2}{kT_s^*} \ln[\frac{e^{\frac{kT_s^*}{2}} + e^{kT_s^*}}{e^{\frac{kT_s^*}{2}} + 1}] - 1\}$$

$$= De^{\frac{kT_s^*}{2}} \{\frac{2}{kT_s^*} \ln(e^{\frac{kT_s^*}{2}}) - 1\}$$

$$= De^{\frac{kT_s^*}{2}} (\frac{2}{kT_s^*} \frac{kT_s^*}{2} - 1) = 0$$

that is, $P_s T_p = T_s^* De^{\frac{kT_s^*}{2}}$ when $a = 2$.

It has been proved in Section 4.2.3 that for $n = 2$, $\frac{\delta}{\delta P}(PT_p) > 0$ for $P > 2De^{\frac{kT}{2}}$.

Hence, for $P_s > 2De^{\frac{kT_s^*}{2}}$, $\frac{P_s T_p}{T_s^*} - De^{\frac{kT_s^*}{2}} > \frac{T_s^* De^{\frac{kT_s^*}{2}}}{T_s^*} - De^{\frac{kT_s^*}{2}} = 0$, and therefore

$TC_{s(p)}(P_s, 2, T_s^*) - TC_s(T_c = \frac{T_s^*}{2}) > 0$.

Finally, $TC_s^* \leq TC_s(T_c = \frac{T_s^*}{2}) < TC_{s(p)}(P_s, 2, T_s^*)$

Therefore, the theorem is proved.

Corollary:

Combining Theorem II and Theorem III, the demand-driven production rate model gives a better cost than a predetermined production rate of $P_s \geq 2De^{\frac{k}{2}}$ if the optimal solution for that predetermined production rate is with $n \geq 2$.

Remarks:

(i) With $k = 0.863$, this sufficient condition requires $P_s \geq 2De^{\frac{k}{2}} = 3.0791D$.

(ii) With $k \leq 0.2$ as for most examples in the literature of inventory models, this sufficient condition requires $P_s \geq 2.2103D$.

4.4 Conclusion

Wee et al. (2008) deduced that the average inventory level of the system is given by $\frac{PT_p - DT}{kT}$. For a given system cycle time T and a given number of deliveries n, varying production rate changes the value of PT_p and hence changes the average inventory level of the system due to the change of vendor's inventory level. It has been shown that for 3 or more deliveries in a system cycle, average inventory level increases when production rate is increased. Hence, compared with a predetermined production rate whose optimal number of deliveries is 3 or more, the proposed continuous production model uses a smaller production rate, and reduces the average inventory level. As a result, total system cost per unit time is reduced. In Example 3.1, the optimal numbers of deliveries in a system cycle with the predetermined production rates are more than three. Therefore, even without finding the actual optimal costs with the demand-driven production rate, it can be anticipated that the proposed model gives lower optimal costs for this example.

If one has to choose between a certain predetermined production rate P_s and the proposed demand-driven production rate model, subject to the conditions of $k \le 0.863$ and $T \le 1$, the following indicates how the overall optimal cost can be found with minimum steps:

(a) if $P_s \ge 2De^{\frac{k}{2}}$, the overall optimal cost is the smaller cost of

(i) the optimal cost for the demand-driven production rate model, and

(ii) the optimal cost for the pre-determined production rate model with $n = 1$.

(b) if $P_s < 2De^{\frac{k}{2}}$, the overall optimal cost is the smallest cost of

(i) the optimal cost for the demand-driven production rate model,

(ii) the optimal cost for the predetermined production rate model with $n = 1$,

(iii) the optimal cost for the predetermined production rate model with $n = 2$.

Chapter 5

Extended Models for the Demand-driven Production Rate Model

5.1 Introduction

In Chapter 3 a lot-delivery continuous production model has been presented. In the model the production rate is demand-driven and is found by optimizing the total cost per unit time. The model has been extended to consider deterioration during delivery. In this chapter, further extended models are presented. Deteriorating items are usually assumed to be subject to deterioration once they are produced and received into inventory. A number of researchers have developed inventory models in which the time to deteriorate follows the 3-parameter Weibull distribution, and the items deteriorate some time, prescribed by the location parameter, after they are received. Some researchers have considered "non-instantaneous" deterioration for exponentially deteriorating items (this is a particular case of the 3-parameter Weibull distribution). However, these models assume that the items are just "born" when the buyer receives the shipments. In this chapter, the proposed model is extended to consider the effect of having a non-deteriorating period for an exponentially deteriorating item with a finite production rate. In this extended model, the item starts to deteriorate after a certain period of production. The non-deteriorating period affects both the vendor's and the buyer's inventory systems. The objective of the model is to find the optimal cycle time for minimizing the total system cost per unit time.

Cost parameters are usually assumed to be constant in the literature of production-inventory models, even for non-constant production rate models. In some inventory models, a temporary special price discount is offered for a special order affecting the deterioration cost and the inventory holding cost. This is likely to be a sales decision or an inventory reduction decision. Variation of cost parameters with time is a financial concern of inflation or the time value of money. Cost parameters and production rate have been considered independent. In this chapter a scenario of cost parameters related to production rate is investigated for the proposed demand-driven production rate model.

The proposed demand-driven production model has been presented for a single-vendor single-buyer supply chain. In the last section of this chapter, a heuristic for a single-vendor multi-buyer supply chain is presented.

5.2 A Model with a Non-deteriorating Period

5.2.1 Assumptions

In addition to the assumptions in Section 3.1, this model assumes the following:

1. The "non-deteriorating' period of the item is less than the production time and the delivery cycle time.

2. The production date/time for the items in a shipment is identifiable and the buyer sells the goods on the basis of "first-made-first-sold".

5.2.2 Model Development

Consider an exponentially deteriorating item which has a non-deteriorating period of T_Θ . The item starts to deteriorate after a certain period of production.

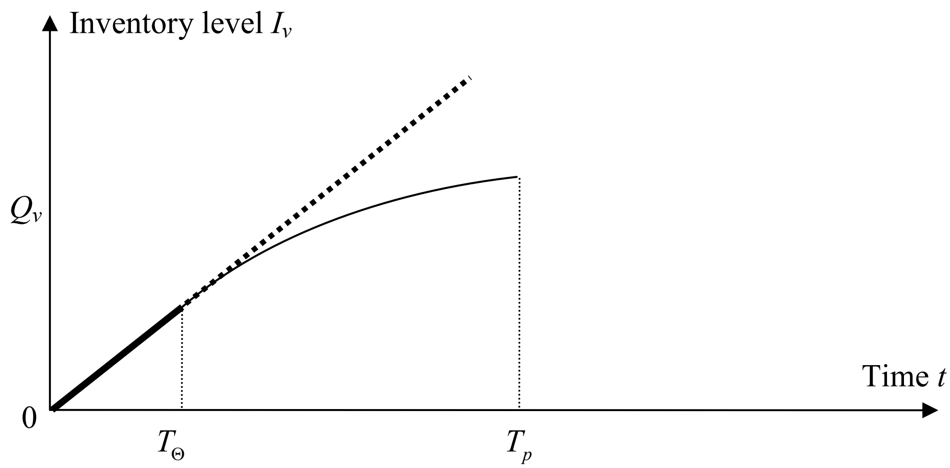


Figure 5.1: Inventory level of the vendor with a non-deteriorating period.

For $0 \le t \le T_\Theta$, $\frac{dI_v}{dt} = P \quad I_v = Pt$.

For $T_\Theta < t \le T_p$,

$$\frac{dI_v}{dt} = -k(I_v - PT_\Theta) + P \;\Rightarrow\; \frac{dI_v}{dt} + kI_v = P(kT_\Theta + 1)$$

Solving the equation with the condition when $t = T_\Theta$, $I_v = PT_\Theta$,

$$I_v = \frac{P(kT_\Theta + 1)}{k} - \frac{P}{k} e^{-k(t - T_\Theta)} \text{ and}$$

$$Q_v = PT_\Theta + \frac{P}{k}[1 - e^{-k(T_p - T_\Theta)}].$$

For $0 < t \le T_p$,

Average inventory level of the vendor

$$= \frac{1}{T_p}\{\frac{PT_\Theta^{\ 2}}{2} + \int_{T_\Theta}^{T_p} [\frac{P(kT_\Theta + 1)}{k} - \frac{P}{k} e^{-k(t-T_\Theta)}]dt\}$$
$$= \frac{1}{T_p}\{\frac{PT_\Theta^{\ 2}}{2} + \frac{P(kT_\Theta + 1)}{k}(T_p - T_\Theta) + \frac{P}{k^2}[e^{-k(T_p-T_\Theta)} - 1]\}$$

Quantity of deteriorated items

$$= P(T_p - T_\Theta) - \frac{P}{k}[1 - e^{-k(T_p-T_\Theta)}]$$

Consider the buyer's inventory.

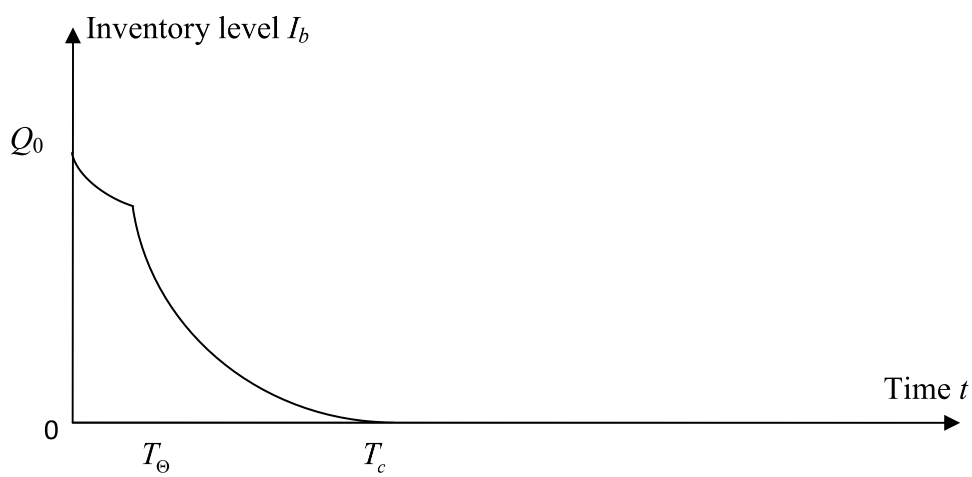


Figure 5.2: Inventory level of the buyer with a non-deteriorating period.

For $0 \le t \le T_\Theta$,

$$\frac{dI_b}{dt} = -k[I_b - P(T_\Theta - t)] - D \;\Rightarrow\; \frac{dI_b}{dt} + kI_b = kP(T_\Theta - t) - D$$

Solving the equation, we have $I_b = P(T_\Theta - t) + \dfrac{P-D}{k} + Ce^{-kt}$ (5.1)

where C is a constant.

For $T_\Theta < t \le T_c$,

$$\frac{dI_b}{dt} = -kI_b - D$$

Solving the equation with the condition when $t = T_c$, $I_b = 0$,

$$I_b = \frac{D}{k}(e^{kT_c}e^{-kt} - 1).$$

Hence, $I_b(t = T_\Theta) = \dfrac{D}{k}(e^{kT_c}e^{-kT_\Theta} - 1) = \dfrac{D}{k}[e^{k(T_c - T_\Theta)} - 1]$ (5.2)

Substitute equation (5.2) into equation (5.1) with $t = T_\Theta$,

$\dfrac{D}{k}[e^{k(T_c - T_\Theta)} - 1] = \dfrac{P-D}{k} + Ce^{-kT_\Theta}$, and therefore, $C = [\dfrac{D}{k}e^{k(T_c - T_\Theta)} - \dfrac{P}{k}]e^{kT_\Theta}$.

So for $0 \le t \le T_\Theta$, $I_b = P(T_\Theta - t) + \dfrac{P-D}{k} + [\dfrac{D}{k}e^{k(T_c - T_\Theta)} - \dfrac{P}{k}]e^{k(T_\Theta - t)}$, and

$$Q_0 = I_b(t = 0) = PT_\Theta + \frac{P-D}{k} + [\frac{D}{k}e^{k(T_c - T_\Theta)} - \frac{P}{k}]e^{kT_\Theta}.$$

For lot-for-lot delivery, $Q_0 = Q_v$ and hence

$$PT_\Theta + \frac{P-D}{k} + [\frac{D}{k}e^{k(T_c - T_\Theta)} - \frac{P}{k}]e^{kT_\Theta} = PT_\Theta + \frac{P}{k}[1 - e^{-k(T_p - T_\Theta)}]$$

$$-\frac{D}{k} + \frac{D}{k}e^{kT_c} - \frac{P}{k}e^{kT_\Theta} = -\frac{P}{k}e^{-kT_p}e^{kT_\Theta} \tag{5.3}$$

The demand-driven production rate for continuous production is found by substituting $T_p = T_c$ into equation (5.3).

$$P = \frac{D(e^{kT_c} - 1)e^{-kT_\Theta}}{1 - e^{-kT_c}} = De^{k(T_c - T_\Theta)} \tag{5.4}$$

At this production rate, the delivery quantity is given by

$$Q_0 = PT_\Theta + \frac{P-D}{k} = D\{T_\Theta e^{k(T_c - T_\Theta)} + \frac{e^{k(T_c - T_\Theta)} - 1}{k}\}.$$

For demand-driven production rate with continuous production, the average inventory level of the buyer is given by

$$\frac{1}{T_c}\{\int_0^{T_\Theta}\{P(T_\Theta - t) + \frac{P-D}{k} + [\frac{D}{k}e^{k(T_c - T_\Theta)} - \frac{P}{k}]e^{k(T_\Theta - t)}\}dt + \int_{T_\Theta}^{T_c}\frac{D}{k}(e^{kT_c}e^{-kt} - 1)dt\}$$

$$= \frac{1}{T_c}\{\frac{PT_\Theta{}^2}{2} + \frac{P-D}{k}t_0 - \frac{D}{k^2}e^{kT_c}(e^{-kT_\Theta} - 1) + \frac{P}{k^2}e^{kT_\Theta}(e^{-kT_\Theta} - 1)$$

$$+ \frac{D}{k^2}(e^{kT_c}e^{-kT_\Theta} - 1) - \frac{D}{k}(T_c - T_\Theta)\}$$

$$= \frac{1}{T_c}\{\frac{PT_\Theta{}^2}{2} + \frac{PT_\Theta}{k} + \frac{D}{k^2}(e^{kT_c} - 1) + \frac{P}{k^2}(1 - e^{kT_\Theta}) - \frac{DT_c}{k}\}$$

Quantity of deteriorated items per buyer cycle

$$= Q_0 - DT_c = PT_\Theta + \frac{P-D}{k} - DT_c$$

The total relevant costs per unit time, for the buyer and the vendor respectively, are

$$TC_b = \frac{A_b}{T_c} + \frac{H_b}{T_c}[\frac{PT_\Theta^{\ 2}}{2} + \frac{PT_\Theta}{k} + \frac{D}{k^2}(e^{kT_c} - 1) + \frac{P}{k^2}(1 - e^{kT_\Theta}) - \frac{DT_c}{k}] + \frac{C_b}{T_c}\{PT_\Theta + \frac{P-D}{k} - DT_c\}$$

$$TC_v = \frac{A_v}{T_c} + \frac{H_v P}{T_c}\{\frac{T_\Theta^{\ 2}}{2} + \frac{(kT_\Theta + 1)}{k}(T_c - T_\Theta) + \frac{1}{k^2}[e^{-k(T_c - T_\Theta)} - 1]\} + \frac{C_v}{T_c}\{P(T_c - T_\Theta) - \frac{P-D}{k}\} + S$$

The total relevant cost for the system per unit time

$$\begin{aligned} TC_s = {} & \frac{A_b + A_v}{T_c} + \frac{H_b D}{k^2 T_c}(e^{kT_c} - 1) + \frac{H_b P}{k^2 T_c}(1 - e^{kT_\Theta}) - \frac{H_v P}{k^2 T_c}[1 - e^{-k(T_c - T_\Theta)}] + \\ & + \frac{PT_\Theta}{T_c}(\frac{H_b - H_v}{2} T_\Theta + \frac{H_b}{k} + C_b - \frac{H_v}{k} - C_v) + P(H_v T_\Theta + \frac{H_v}{k} + C_v) + \\ & \frac{P-D}{kT_c}(C_b - C_v) - \frac{H_b D}{k} - C_b D + S \end{aligned} \tag{5.5}$$

Substitute equation (5.4) into equation (5.5) and differentiate,

$$\begin{aligned} \frac{d}{dT_c} TC_s = {} & \frac{1}{T_c^2}\{-(A_b + A_v) + \frac{H_b D[(kT_c - 1)e^{kT_c} + 1]}{k^2} + D[\frac{H_b}{k^2}(1 - e^{kT_\Theta}) + \\ & T_\Theta(\frac{H_b - H_v}{2} T_\Theta + \frac{H_b}{k} + C_b - \frac{H_v}{k} - C_v)](kT_c - 1)e^{k(T_c - T_\Theta)} + \\ & D(\frac{C_b - C_v}{k} - \frac{H_v}{k^2})[(kT_c - 1)e^{k(T_c - T_\Theta)} + 1] + Dk(H_v T_\Theta + \frac{H_v}{k} + C_v)T_c^{\ 2} e^{k(T_c - T_\Theta)}\} \end{aligned}$$

To find the minimum cost, set $\frac{d}{dT_c} TC_s = 0$ and hence

$$l_1[(kT_c - 1)e^{kT_c} + 1] + l_2(kT_c - 1)e^{k(T_c - T_\Theta)} + l_3[(kT_c - 1)e^{k(T_c - T_\Theta)} + 1] + l_4(kT_c)^2 e^{k(T_c - T_\Theta)}$$
$$-(A_b + A_v) = 0$$

(5.6)

where $l_1 = \frac{H_b D}{k^2}$, $l_2 = D[\frac{H_b}{k^2}(1 - e^{kT_\Theta}) + T_\Theta(\frac{H_b - H_v}{2} T_\Theta + \frac{H_b}{k} + C_b - \frac{H_v}{k} - C_v)]$,

$$l_3 = \frac{D}{k}(C_b - C_v - \frac{H_v}{k}), \quad l_4 = D(\frac{H_v T_\Theta k + H_v + C_v k}{k^2}).$$

It can be shown (Please refer to Appendix A1.) that there is a unique solution for equation (5.6) which gives the minimum of TC_s.

To solve equation (5.6), let $x = kT_c > 0$ and

$$g(x) = l_1[(x-1)e^x + 1] + l_2(x-1)e^{-kT_\Theta}e^x + l_3[(x-1)e^{-kT_\Theta}e^x + 1] + l_4(x)^2 e^{-kT_\Theta}e^x - (A_b + A_v) .$$

By applying the Newton-Raphson Method, $g(x) = 0$ can be solved. The optimal cycle time, T_c^*, can be found by dividing the solution of $g(x) = 0$ by *k*.

Consider the special case of $T_c \le T_\Theta$. Adopting the "first-made-first-sold" policy means that no deterioration takes place. The demand-driven production rate is just the demand rate.

The cost equation is $TC_s(T_c \le T_\Theta) - S = \frac{A_b + A_v}{T_c} + (H_b + H_v)\frac{DT_c}{2}$.

The optimal cycle time T_c^{ND} is given by

$T_c^{ND} = \sqrt{\frac{2(A_b + A_v)}{D(H_b + H_v)}}$ and is only applicable if $T_c^{ND} \le T_\Theta$.

The optimal cycle time for the different cases is shown in Table 5.1.

Conditions	Decision
$T_c^{ND} \le T_\Theta$ and $T_c^* < T_\Theta$	Final optimal cycle time: T_c^{ND}
$T_c^{ND} > T_\Theta$ and $T_c^* < T_\Theta$	Final optimal cycle time: T_Θ
$T_c^{ND} > T_\Theta$ and $T_c^* > T_\Theta$	Find the costs associated with T_c^* and T_Θ, and take the one giving the smaller cost.
$T_c^{ND} \le T_\Theta$ and $T_c^* > T_\Theta$	Find the costs associated with T_c^* and T_c^{ND}, and take the one giving the smaller cost.

Table 5.1: Optimal cycle time

The solution procedure is shown in Section 5.2.3.

5.2.3 Solution Procedure

1. Find T_c^{ND} and T_c^*. If $T_c^{ND} \le T_\Theta$, go to Step 5.

2. If $T_c^* < T_\Theta$, go to Step 4.

3. Find $TC_s^* - S$ and $TC_s(T_c = T_\Theta) - S$. If $TC_s^* - S < TC_s(T_c = T_\Theta) - S$, the optimal delivery cycle time is T_c^*. Otherwise, the final optimal delivery cycle time is T_Θ. Exit.

4. The final optimal delivery cycle time is T_Θ. Exit.

5. If $T_c^* < T_\Theta$, go to Step 7.

6. Find $TC_s^* - S$ and $TC_s(T_c = T_c^{ND}) - S$. If $TC_s^* - S < TC_s(T_c = T_c^{ND}) - S$, the optimal delivery cycle time is T_c^*. Otherwise, the final optimal cycle time is T_c^{ND}. Exit.

7. The final optimal cycle time is T_c^{ND}.

The flowchart for the above solution procedure is shown in Appendix C.

5.2.4 Example 5.1

Take k = 0.01, 0.05, 0.1, and 0.2 per year and use the other parameters from Wee et al. (2008)'s example,

D = 1000 units per year P = 3200 units per year

S = \$400 $A_b + A_v$ = \$25

C_b = \$50 C_v = \$40

H_b = \$5 per unit per year H_v = \$4 per unit per year

The optimal total costs per unit time for this scenario $TC_s^* - S$ and TC_s^* are found with $T_\Theta = 0.01$, 0.02, and so on and stop when $T_c^* < T_\Theta$ as the model is not applicable. These costs are then compared with the costs for the demand-driven production rate model with $T_\Theta = 0$.

Assuming no deterioration, the optimal cycle time T_c^{ND} is given by

$$T_c^{ND} = \sqrt{\frac{2(A_b + A_v)}{D(H_b + H_v)}} = \sqrt{\frac{2(25)}{1000(5+4)}} = 0.0745$$

This is applicable only if $T_\Theta \geq 0.0745$.

The optimal cycle time, T_c^*, and the costs are shown in Table 5.2.

As T_c^*'s are less than 0.0745, the costs of $TC_s(T_c = T_\Theta) - S$ are also calculated.

$k = 0.01$								
T_{Θ}	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07
T_c^*	0.071	0.071	0.071	0.071	0.0709	0.0709	0.0708	0.0707
$TC_s^* - S$	703.7	699.6	695.4	691.0	686.5	681.9	677.1	672.1
TC_s^*	1103.7	1099.6	1095.4	1091.0	1086.5	1081.9	1077.1	1072.1
$TC_s(T_c = T_{\Theta}) - S$		2545	1340	968.33	805	725	686.67	672.14

$k = 0.05$								
T_{Θ}	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07
T_c^*	0.0608	0.0607	0.0607	0.0605	0.0603	0.0600	------	------
$TC_s^* - S$	822.19	801.71	780.39	758.24	735.24	711.38	------	------
TC_s^*	1222.19	1201.71	1180.39	1158.24	1135.24	1111.38	------	------
$TC_s(T_c = T_{\Theta}) - S$		2545	1340	968.33	805	725	686.67	672.14

$k = 0.1$								
T_{Θ}	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07
T_c^*	0.0526	0.0525	0.0524	0.0521	0.0518	0.0513	------	------
$TC_s^* - S$	948.89	908.72	865.64	820.62	773.66	724.69	------	------
TC_s^*	1348.89	1308.72	1265.64	1220.62	1173.66	1124.69	------	------
$TC_s(T_c = T_{\Theta}) - S$		2545	1340	968.33	805	725	686.67	672.14

$k = 0.2$								
T_{Θ}	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07
T_c^*	0.0429	0.0428	0.0426	0.0422	0.0416	------	------	------
$TC_s^* - S$	1164.30	1081.30	993.66	901.32	804.16	------	------	------
TC_s^*	1564.30	1481.30	1393.66	1301.32	1204.16	------	------	------
$TC_s(T_c = T_{\Theta}) - S$		2545	1340	968.33	805	725	686.67	672.14

Table 5.2: Costs for Example 5.1.

In the example, as $TC_s^* - S < TC_s(T_c = T_\Theta) - S$, the optimal cycle time is T_c^*.

The example shows that T_c^* is not quite sensitive to the change of T_Θ, especially for small deterioration rates. However, the change in $TC_s^* - S$ is more significant. For example, with $k = 0.1$, when T_Θ is changed from 0 to 0.05, T_c^* is reduced by 2.5% while $TC_s^* - S$ is reduced by 23.6%.

With $k = 0.1$, the percentage changes with respect to $T_\Theta = 0$ are:

T_Θ	0.01	0.02	0.03	0.04	0.05
% reduction of T_c^*	0.2%	0.4%	1.0%	1.5%	2.5%
% reduction of $TC_s^* - S$	4.2%	8.8%	13.5%	18.5%	23.6%

Table 5.3: Percentage change in Cost with respect to no non-deteriorating period

5.3 A Model of Production Rate Dependent Cost Parameters

5.3.1 Introduction to the Model

In integrated lot-delivery inventory models, cost parameters are usually assumed to be constant. Material cost, one of the components of the production cost, is basically independent of production rate. Depending on the operations of the production system and the accounting methods adopted, other components of production cost may be related to production rate. For example, labour cost per unit is independent of production rate if the amount of labour involved is proportional to production rate, but is inversely proportional to production rate if the same amount of labour is involved regardless of production rates. Whether other cost components such as machine rates and overhead per unit are constant or related to production rate depend on the accounting methods adopted. Machine rates and overheads are independent of production rate if they are set as fixed dollar value per unit produced. However, if they are set as fixed dollar value per unit time, say, machine hourly rates, then these costs per unit produced are inversely proportional to production rate.

In the proposed demand-driven production rate model, the production rate is slightly larger than the demand rate, and is much smaller than the predetermined production rates in the literature of predetermined production rate models in which the production rates are usually more than twice the demand rates. Energy consumption may be reduced for operating the moving parts of the machine with a smaller production rate. This may result in cost reduction and makes the proposed model more favourable. On the other hand, as discussed above some cost components may increase when a smaller production rate is selected.

In this model, production cost is assumed to be partly constant and partly inversely proportional to production rate. Deterioration cost per unit for the vendor is equal to the production cost, and that for the buyer is equal to the purchase price paid by the buyer, if deteriorated items are non-salvageable (or a fraction of the production cost or purchase price if otherwise). In some inventory models, the inventory holding cost is reduced when there is a price discount. This model also assumes that inventory holding cost is proportional to the production cost or purchase price for the vendor and the buyer respectively. With these assumptions, deterioration costs and inventory holding costs are functions of production rate as follows:

$$C_b = C_{ba} + \frac{C_{bb}}{P} \qquad C_v = C_{va} + \frac{C_{vb}}{P}$$

$$H_b = H_{ba} + \frac{H_{bb}}{P} \qquad H_v = H_{va} + \frac{H_{vb}}{P}$$

$$C_{ba} \ge C_{va} \qquad C_{bb} \ge C_{vb}$$

$$H_{ba} \ge H_{va} \qquad H_{bb} \ge H_{vb}$$

These cost functions can also cater the case in which inventory holding costs are independent of production rate, by setting H_{bb} and H_{vb} to zero.

5.3.2 Model Development

Substituting the formulae for the deterioration costs and inventory holding costs in Section 5.3.1 into equation (3.15), for continuous production with demand-driven production rate, the total system cost per unit time is given by

$$\begin{aligned}
TC_s &= \frac{A_b + A_v}{T_c} + \frac{D}{k}\left(\frac{H_{ba}}{k} + C_{ba} - \frac{H_{va}}{k} - C_{va}\right)\left(\frac{e^{kT_c}-1}{T_c}\right) + \frac{1}{ke^{kT_c}}\left(\frac{H_{bb}}{k} + C_{bb} - \frac{H_{vb}}{k} - C_{vb}\right)\left(\frac{e^{kT_c}-1}{T_c}\right) + \\
&\quad \frac{H_{va}De^{kT_c}}{k} + C_{va}De^{kT_c} + \frac{H_{vb}}{k} + C_{vb} - \frac{H_{ba}D}{k} - C_{ba}D - \frac{H_{bb}}{ke^{kT_c}} - \frac{C_{bb}}{e^{kT_c}} + S \\
&= \frac{1}{T_c}\left(A_b + A_v + \frac{H_{bb} - H_{vb}}{k^2} + \frac{C_{bb} - C_{vb}}{k}\right) + \frac{D}{k}\left(\frac{H_{ba}}{k} + C_{ba} - \frac{H_{va}}{k} - C_{va}\right)\left(\frac{e^{kT_c}-1}{T_c}\right) - \\
&\quad \left(\frac{H_{bb}}{k} + C_{bb} - \frac{H_{vb}}{k} - C_{vb}\right)\frac{1}{kT_c e^{kT_c}} + \left(\frac{H_{va}}{k} + C_{va}\right)De^{kT_c} - \left(\frac{H_{bb}}{k} + C_{bb}\right)\frac{1}{e^{kT_c}} + \\
&\quad \frac{H_{vb}}{k} + C_{vb} - \frac{H_{ba}D}{k} - C_{bb}D + S
\end{aligned} \tag{5.7}$$

The objective is to minimize TC_s.

$$\frac{d}{dT_c}TC_s$$

$$= -\frac{1}{T_c^2}(A_b + A_v + \frac{H_{bb} - H_{vb}}{k^2} + \frac{C_{bb} - C_{vb}}{k}) + \frac{D}{k}(\frac{H_{ba}}{k} + C_{ba} - \frac{H_{va}}{k} - C_{va})\frac{(kT_c - 1)e^{kT_c} + 1}{T_c^2} +$$

$$(\frac{H_{bb}}{k} + C_{bb} - \frac{H_{vb}}{k} - C_{vb})\frac{kT_c + 1}{kT_c^2 e^{kT_c}} + (\frac{H_{va}}{k} + C_{va})kDe^{kT_c} + (\frac{H_{bb}}{k} + C_{bb})\frac{k}{e^{kT_c}}$$

$$= \frac{1}{T_c^2 e^{kT_c}}\{-(A_b + A_v + \frac{H_{bb} - H_{vb}}{k^2} + \frac{C_{bb} - C_{vb}}{k})e^{kT_c} + \frac{D}{k}(\frac{H_{ba}}{k} + C_{ba} - \frac{H_{va}}{k} - C_{va})[(kT_c - 1)e^{kT_c} + 1]e^{kT_c} +$$

$$\frac{1}{k}(\frac{H_{bb}}{k} + C_{bb} - \frac{H_{vb}}{k} - C_{vb})(kT_c + 1) + \frac{D(H_{va} + kC_{va})}{k^2}k^2 T_c^2 e^{2kT_c} + (\frac{H_{bb} + kC_{bb}}{k^2})k^2 T_c^2\}$$

Put $\frac{d}{dT_c}TC_s = 0$. Set $x = kT_c > 0$ and

$$p_1 = A_b + A_v + \frac{H_{bb} - H_{vb}}{k^2} + \frac{C_{bb} - C_{vb}}{k} \quad p_2 = \frac{D}{k}(\frac{H_{ba}}{k} + C_{ba} - \frac{H_{va}}{k} - C_{va})$$

$$p_3 = \frac{1}{k}(\frac{H_{bb}}{k} + C_{bb} - \frac{H_{vb}}{k} - C_{vb}) \quad p_4 = \frac{D(H_{va} + kC_{va})}{k^2} \quad p_5 = \frac{H_{bb} + kC_{bb}}{k^2}$$

The quantities p_1, p_2, p_3, p_4, and p_5 are all positive.

$$\frac{d}{dT_c}TC_s = 0 \Rightarrow -p_1 e^x + p_2[(x-1)e^x + 1]e^x + p_3(x+1) + p_4 x^2 e^{2x} + p_5 x^2 = 0$$

Set $h(x) = p_2[(x-1)e^x + 1] + p_3(x+1)e^{-x} + p_4 x^2 e^x + p_5 x^2 e^{-x} - p_1 = 0$ (5.8)

The optimal solution is to be found by solving equation (5.8).

As $(x-1)e^x + 1 > 0$ for $x > 0$, $p_2[(x-1)e^x + 1] + p_3(x+1)e^{-x} + p_4 x^2 e^x + p_5 x^2 e^{-x} > 0$

and it is an increasing function for $x \le 1$ as shown below:

For $x \le 1$, (In general, $k < 1$, and $T_c \le 1$ and hence $x \le 1$ suffices.)

$$\frac{d}{dx}\{p_2[(x-1)e^x+1]+p_3(x+1)e^{-x}+p_4x^2e^x+p_5x^2e^{-x}\}$$
$$= p_2xe^x - p_3xe^{-x} + p_4(x+2)xe^x + p_5(2-x)xe^{-x}$$
$$= p_2xe^x + p_4(x+2)xe^x - \frac{1}{k}(\frac{H_{bb}}{k}+C_{bb}-\frac{H_{vb}}{k}-C_{vb})xe^{-x} + \frac{H_{bb}+kC_{bb}}{k^2}(2-x)xe^{-x}$$
$$= p_2xe^x + p_4(x+2)xe^x + \frac{1}{k}(\frac{H_{vb}}{k}+C_{vb})xe^{-x} + \frac{H_{bb}+kC_{bb}}{k^2}(1-x)xe^{-x} > 0$$

Since $\lim_{x\to\infty} p_2[(x-1)e^x+1]+p_4x^2e^x = \infty$, and

$\lim_{x\to\infty} p_3(x+1)e^{-x}+p_5x^2e^{-x} = \lim_{x\to\infty}\frac{p_3(1)}{e^x}+\lim_{x\to\infty}\frac{p_5(2)}{e^x} = 0$. (by L'Hopital's Rule)

Hence, $p_2[(x-1)e^x+1]+p_3(x+1)e^{-x}+p_4x^2e^x+p_5x^2e^{-x}$ is positive, increasing and has no finite limit. There is a unique solution x^* for equation (5.8) for any $p_1 > 0$.

From $\frac{d}{dT_c}TC_s = \frac{h(x)}{T_c^2e^{kT_c}}$,

$$\frac{d^2}{dT_c^2}TC_s = \frac{T_c^2e^{kT_c}[kh'(x)]-h(x)[2T_ce^{kT_c}+2kT_c^2e^{kT_c}]}{(T_c^2e^{kT_c})^2}.$$

Since $h(x^*)=0$ and $h'(x^*)>0$ as it is an increasing function, $\frac{d^2}{dT_c^2}TC_s > 0$ at $x = x^*$. Therefore, the unique solution of equation (5.8) gives the minimum total cost per unit time.

A discussion for the case of $x > 1$ is shown in Appendix A2.

5.3.3 Example 5.2

Using the following parameters from Wee el al. (2008)'s example:

D = 1000 units per year	P = 3200 units per year
S = \$400	$A_b + A_v$ = \$25
C_b = \$50	C_v = \$40
H_b = \$5 per unit per year	H_v = \$4 per unit per year

These deterioration costs and inventory holding costs are based on a production rate of 3200 units per year. To split these costs into fixed components and variable components, the proportions of the fixed components are taken to be 0.1, 0.2, …, 0.9 of the total costs.

For example, if the fixed component is 0.5 of the cost, then

$$C_b(3200) = 50 = C_{ba} + \frac{C_{bb}}{3200}$$

$$C_{ba} = (0.5)50 = 25, \quad C_{bb} = 80000$$

$$\therefore\ C_b = 25 + \frac{80000}{P}$$

The fixed and variable components of the deterioration costs and the inventory holding costs are shown in Table 5.4.

Proportion of fixed component	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
C_{ba}	5	10	15	20	25	30	35	40	45
C_{va}	4	8	12	16	20	24	28	32	36
H_{ba}	0.5	1	1.5	2	2.5	3	3.5	4	4.5
H_{va}	0.4	0.8	1.2	1.6	2	2.4	2.8	3.2	3.6
C_{bb}	144000	128000	112000	96000	80000	64000	48000	32000	16000
C_{vb}	115200	102400	89600	76800	64000	51200	38400	25600	12800
H_{bb}	14400	12800	11200	9600	8000	6400	4800	3200	1600
H_{vb}	11520	10240	8960	7680	6400	5120	3840	2560	1280

$$C_b = C_{ba} + \frac{C_{bb}}{P} \qquad C_v = C_{va} + \frac{C_{vb}}{P}$$

$$H_b = H_{ba} + \frac{H_{bb}}{P} \qquad H_v = H_{va} + \frac{H_{vb}}{P}$$

Table 5.4: Fixed and variable components of cost parameters for Example 5.2.

For $k = 0.1$, the optimal solutions are as follows:

Proportion of fixed component	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
T_c^*	0.0306	0.0318	0.0331	0.0346	0.0364	0.0385	0.0409	0.0439	0.0477
$De^{kT_c^*}$	1003.1	1003.2	1003.3	1003.5	1003.7	1003.9	1004.1	1004.4	1004.8
C_b	148.56	137.59	126.63	115.67	104.71	93.75	82.80	71.86	60.92
C_v	118.85	110.08	101.30	92.53	83.77	75.00	66.24	57.49	48.74
H_b	14.86	13.76	12.66	11.57	10.47	9.38	8.28	7.19	6.09
H_v	11.89	11.01	10.13	9.25	8.38	7.50	6.62	5.75	4.874
$TC_s^* - S$	1636.5	1575.0	1511.0	1444.1	1374.1	1300.3	1222.1	1138.5	1048.4
TC_s^*	2036.5	1975.0	1911.0	1844.1	1774.1	1700.3	1622.1	1538.5	1448.4

Table 5.5: Optimal solutions for $k = 0.1$ for Example 5.2.

The optimal total system cost per year with a predetermined production rate of 3200 units per year in Wee et al. (2008)'s example is \$2695.69. The demand-driven production rate model with deterioration costs and holding costs related to production rate gives a smaller optimal cost even when the proportion of fixed components of the costs is as low as 0.1. When the proportion of the fixed components is higher, the deterioration costs and holding costs per unit are increased by smaller amounts resulting in a lower optimal cost and hence achieves a higher saving against the predetermined production rate model.

For $k = 0.01$, 0.05 and 0.2, the optimal solutions are as follows:

Proportion of fixed components	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$k = 0.01$ (Wee et. al. (2008) optimal cost: \$1996.92)									
T_c^*	0.0412	0.0428	0.0446	0.0467	0.049	0.0518	0.0552	0.0592	0.0643
$TC_s^* - S$	1214.4	1168.7	1121.2	1071.6	1019.5	964.6	906.5	844.3	777.2
TC_s^*	1614.4	1568.7	1521.2	1071.6	1419.5	1364.6	1306.5	1244.3	1177.2
$k = 0.05$ (Use Wee et. al. (2008)'s method, optimal cost: \$2333.73)									
T_c^*	0.0353	0.0367	0.0382	0.0400	0.0420	0.0444	0.0472	0.0507	0.0551
$TC_s^* - S$	1417.7	1364.4	1308.9	1251.0	1190.2	1126.3	1058.4	986.0	907.7
TC_s^*	1817.7	1764.4	1708.9	1651.0	1590.2	1526.3	1458.4	1386.0	1307.7
$k = 0.2$ (Use Wee et. al. (2008)'s method, optimal cost: \$3301.97)									
T_c^*	0.0250	0.0260	0.0271	0.0283	0.0298	0.0314	0.0334	0.0359	0.0389
$TC_s^* - S$	2003.3	1928.1	1849.8	1768.0	1682.4	1592.1	1496.5	1394.5	1284.5
TC_s^*	2403.3	2328.1	2249.78	2168.0	2082.4	1992.1	1896.5	1794.5	1684.5

Table 5.6: Optimal solutions for $k = 0.01,\ 0.05,\ 0.2$ for Example 5.2.

For these deterioration rates, the demand-driven production rate model with costs related to production rate also give smaller optimal costs than the predetermined production rate model.

5.3.4 Example 5.3

Most of the parameters of Wee et al. (2008)'s example were taken from Goyal (1988)'s example in which the purchasing price and production cost are \$25 and \$20 respectively. Most of the parameters of Goyal (1998)'s example were taken from Banerjee (1986)'s example in which the delivery cycle related fixed cost is \$100.

For k = 0.1 and with the following changes in the parameters, the results are:

(i) $A_b + A_v$ = \$25, C_b = \$25 and C_v = \$20

Wee et al (2008)'s method: optimal cost = \$2336.41

Demand-driven production rate with fixed costs: optimal cost = \$1222.79

Demand-driven production rate with costs related to production rate:

Proportion of fixed components	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
T_c^*	0.0353	0.0367	0.0382	0.0400	0.0421	0.0444	0.0473	0.0507	0.0550
TC_s^*	1817.1	1763.8	1708.4	1650.5	1589.9	1526.0	1458.3	1386.0	1308.0

Table 5.7: Solutions for Example 5.3(i).

The demand-driven production rate model with deterioration costs and holding costs related to production rate gives a smaller optimal cost than the predetermined production rate model even when the proportion of fixed components of the costs is as low as 0.1.

(ii) $A_b + A_v$ = \$50, C_b = \$50 and C_v = \$40

Wee et al (2008)'s method: optimal cost = \$2940.75

Demand-driven production rate with fixed costs: optimal cost = \$1744.05

Demand-driven production rate with costs related to production rate:

Proportion	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
T_c^*	0.0433	0.0450	0.0469	0.0490	0.0515	0.0544	0.0579	0.0621	0.0674
TC_s^*	2713.6	2626.7	2536.2	2441.8	2342.8	2238.6	2128.1	2010.2	1883.0

Table 5.8: Solutions for Example 5.3(ii).

(iii) $A_b + A_v$ = \$50, C_b = \$25 and C_v = \$20

Wee et al (2008)'s method: optimal cost = \$2549.25

Demand-driven production rate with fixed costs: optimal cost = \$1564.30

Demand-driven production rate with costs related to production rate:

Proportion	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
T_c^*	0.0500	0.0519	0.0541	0.0566	0.0595	0.0629	0.0668	0.0717	0.0778
TC_s^*	2403.3	2328.1	2249.8	2168.0	2082.4	1992.1	1896.5	1794.5	1684.5

Table 5.9: Solutions for Example 5.3(iii).

The results of (ii) and (iii) are similar to that of (i). The demand-driven production rate model with deterioration costs and holding costs related to production rate gives a smaller optimal cost than the predetermined production rate model even when the proportion of fixed components of the costs is as low as 0.1.

(iv) $A_b + A_v$ = \$100, C_b = \$50 and C_v = \$40

Wee et al (2008)'s method: optimal cost = \$3294.09

Demand-driven production rate with fixed costs: optimal cost = \$2302.17

Demand-driven production rate with costs related to production rate:

Proportion	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
T_c^*	0.0612	0.0636	0.0663	0.0694	0.0729	0.0770	0.0819	0.0878	0.0952
TC_s^*	3670.5	3547.7	3419.9	3286.5	3146.8	2999.6	2843.6	2677.2	2497.9

Table 5.10: Solutions for Example 5.3(iv).

The demand-driven production rate model with deterioration costs and holding costs related to production rate gives a smaller optimal cost than the predetermined production rate model when the proportion of fixed components of the costs is 0.4 or more.

(v) $A_b + A_v$ = \$100, C_b = \$25 and C_v = \$20

Wee's method: optimal cost = \$2856.37

Demand-driven production rate with fixed costs: optimal cost = \$2047.97

Demand-driven production rate with costs related to production rate:

Proportion	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
T_c^*	0.0708	0.0735	0.0766	0.0801	0.0842	0.0889	0.0946	0.1014	0.1099
TC_s^*	3231.69	3125.37	3014.78	2899.34	2778.38	2651.02	2516.12	2372.2	2217.1

Table 5.11: Solutions for Example 5.3(v).

The demand-driven production rate model with deterioration costs and holding costs related to production rate gives a smaller optimal cost than the predetermined production rate model when the proportion of fixed components of the costs is 0.5 or more.

5.3.5 Interpretation of the Results

When the proportion of fixed components is lower, the unit deterioration costs and the unit inventory holding costs are higher for the same production rate. Hence, the optimal system cost for a lower proportion of fixed components of these costs is higher than that for a higher proportion.

When the delivery related fixed costs are not high, the demand-driven production rate model with deterioration costs and holding costs related to production rate gives a smaller optimal cost than the predetermined production rate model even when the proportion of fixed components of these costs is low. If the delivery related costs are high, the optimal system cost for a low proportion of fixed components of deterioration costs and holding costs is higher than that for the predetermined production rate model. However, if the proportion of fixed components of the costs is medium to high, this extended demand-driven production rate model still gives a lower optimal system cost than the predetermined production rate model.

5.4 A Heuristic for Extending the Model to Multi-buyer Supply Chains

5.4.1 Introduction

The demand-driven production rate continuous production model has been presented for single-vendor single-buyer supply chains. In this section, the model is extended to a single-vendor multi-buyer supply chain. A heuristic method is proposed for optimizing the total system cost per unit time. In this extended model, the vendor produces and supplies the product to n_b buyers ($n_b > 1$). For each delivery to the *i*th buyer, the vendor incurs an ordering processing and shipment cost of A_{vi} while the buyer incurs an ordering and other delivery related cost of A_{bi}. The *i*th buyer has a demand rate of D_i units per unit time, and incurs a cost of C_{bi} per unit of deteriorated item and a unit inventory holding cost of H_{bi} per unit time.

Assumptions:

1. All the cost parameters are constant. In particular, the delivery related fixed costs, A_{vi} and A_{bi}, are also constant regardless of the deliveries to other buyers.

2. The maximum permissible production rate, P_m, is sufficiently large.

3. The production set up cost, S, is apportioned to the buyers in some way such that the set up cost allocated to each buyer, S_i, is still a constant per unit time as in equation (3.15) for each "vendor" and buyer, and $S = \sum_{i=1}^{n_b} S_i$.

With these assumptions, the supply chain can be considered as consisting of several subsystems each having a "vendor" and a buyer. These subsystems are independent of one another and can be optimized individually. Therefore, the whole supply chain can be optimized by independent optimization of all the concerned subsystems.

5.4.2 The Heuristic

If delivery is assumed instantaneous,

(i) Set up equations (3.15) for each subsystem of "vendor" and buyer with the respective demand rate and cost parameters.

(ii) Find the optimum cycle time $T_{ci} = T_{ci}^*$ for each subsystem as illustrated in Example 3.1. The production rate for the *i*th buyer is $D_i e^{kT_{ci}^*}$

(iii) Deliveries are made at cycle time of T_{ci}^* to the *i*th buyer.

(iv) The vendor sets the production rate at $P_1 = \sum_{i=1}^{n_b} D_i e^{kT_{ci}^*}$.

If deterioration during delivery of goods cannot be neglected, and suppose transportation time to site *i* is T_{Ti} .

(i) Set up equation (3.20) or equation (3.21) as appropriate.

(ii) Find the optimum cycle time $T_{ci} = T_{ci}^*$ for each subsystem with the appropriate expressions for m_1 and m_2, as illustrated in Example 3.2. The production rate the *i*th buyer is $D_i e^{k(T_{Ti}+T_{ci}^*)}$.

(iii) Deliveries are made at cycle time of T_{ci}^* to buyer *i*.

(iv) The vendor sets the production rate at $P_2 = \sum_{i=1}^{n_b} D_i e^{k(T_{Ti}+T_{ci}^*)}$.

This heuristic method requires the assumption of a sufficiently large maximum permissible production rate. We consider the production rate of $P_1 = \sum_{i=1}^{n_b} D_i e^{kT_{ci}^*}$. Suppose the delivery cycle is not more than one unit time (year). This gives an upper bound of De^k for the required production rate where *D* is the total demand rate. A production capacity of twice the demand rate is sufficient for a deterioration rate as large as ln2=0.693 and a delivery cycle time of one year. As production rates are usually more than twice the demand rates in the literature of exponentially deteriorating items (while deterioration rates are usually not more than 0.2), production capacity is unlikely to be a constraint even if deterioration during delivery is considered.

Concerning the assumption about apportioning the set up cost, sharing the set up cost equally among all the buyers is not deemed appropriate. It may be reasonable that the set up cost is apportioned to the buyers in proportional to their demand rates. For example, if the demand rates for three buyers are 1000, 2000 and 1000 units per year, a set up cost of \$400 is apportioned as \$100, \$200 and \$100 respectively for the three buyers.

5.4.3 Example 5.4

Consider a supply chain of one vendor and three buyers with the following parameters:

$k = 0.1$ per year	D_i : 1000, 1200, 1800 units per year
$A_{vi} + A_{bi}$: \$25, \$20, \$40	$S =$ \$400
C_{bi} : \$50, \$55, \$45	H_{bi} : \$5, \$5.5, \$4.5 per unit per year
$C_v =$ \$40	$H_v =$ \$4 per unit per year

Set up cost allocated to the three buyers are \$100, \$120, and \$180 respectively.

The first buyer has the same parameters as the buyer in Example 3.1. Hence, the optimal cycle time is 0.0526 year. The annual cost without and with set up are \$949.89 and \$1,049.89, respectively.

For the second buyer,

$$m_1 = \frac{1200}{0.1}[\frac{5.5}{0.1} + 55 - \frac{4}{0.1} - 40] = 360000$$

$$m_2 = \frac{1200}{0.1^2}[4 + 40(0.1)] = 960000$$

$$f(x) = 360000[(x-1)e^x + 1] + 960000(x)^2 e^x - 20 = 0$$

$$f'(x) = xe^x[360000 + 2(960000) + 960000x] = xe^x(2280000 + 960000x)$$

The optimal cycle time is found to be 0.0418 year. The annual cost without and with set up are \$955.93 and \$1,075.93, respectively.

For the third buyer,

$$m_1 = \frac{1800}{0.1}[\frac{4.5}{0.1} + 45 - \frac{4}{0.1} - 40] = 180000$$

$$m_2 = \frac{1800}{0.1^2}[4 + 40(0.1)] = 1440000$$

$$f(x) = 180000[(x-1)e^x + 1] + 1440000(x)^2 e^x - 40 = 0$$

$$f'(x) = xe^x[180000 + 2(1440000) + 1440000x] = xe^x(3060000 + 1440000x)$$

The optimal cycle time is found to be 0.0510 year. The annual cost without and with set up are \$1,566.57 and \$1,746.46, respectively.

The production rate to be set by the vendor is hence:

$$1000\exp[0.1(0.0526)] + 1200\exp[0.1(0.0418)] + 1800\exp[0.1(0.0510)] = 4019.50$$

units per year.

5.5 Conclusion

In this chapter, the proposed demand-driven production rate model has been extended so as to consider a non-deteriorating period for exponentially deteriorating items. The effect of a finite production rate in the system is studied and a cost optimization model has been presented. The proposed continuous production model is a lot-for-lot delivery model. Inventory models with the 3-parameter Weibull distribution or non-instantaneous deteriorating items are EOQ models assuming that all units are just "born" and start their non-deteriorating period when they are received by the buyer. If there are multiple shipments from a production batch, the items in the later shipments have probably passed the non-deteriorating period before they are shipped and received by the buyer. This implies that a lot-for-lot policy is probably assumed in these models. The proposed model is an extension of the EOQ models to an integrated lot-delivery model with a finite production rate.

In the literature of inventory models, cost parameters and production rate are considered independent. The proposed demand-driven production rate model uses a much lower production rate than that in the literature of integrated lot-delivery vendor and buyer systems. A discussion of the potential causes that may increase the deterioration and inventory holding costs has been presented. A scenario of these costs being increased as production rate decreases is investigated and the results are compared with that from the predetermined production rate model. The results of the numerical examples suggest that if the proportion of the fixed components of these costs is medium to high, the proposed model can still give a lower optimal cost than the predetermined production rate model.

In the last section of this chapter, the demand-driven production rate model is extended to a single-vendor multiple buyer supply chain system. A heuristic, extended from the solution procedure of the model for the single buyer case, has been proposed for optimizing the total system cost per unit time.

Chapter 6

A Synchronized Model for a Single-vendor Multi-buyer Supply Chain of Deteriorating Items

6.1 Introduction

Chan and Kingsman (2005, 2007) developed a synchronized delivery and production cycle model for single-vendor multi-buyer supply chains for non-deteriorating items. In the model, the vendor and buyers agree that the delivery cycle times and the system cycle time are integer multiples of a convenient time unit. In addition, the number of deliveries in a system cycle for each buyer is also an integer. This means that all the delivery intervals are integer factors of the system cycle. As the buyers can have different delivery intervals, the optimal system cost will be lower than that obtained by independent optimization as well as common delivery cycle approach.

In this chapter, this synchronized model is extended to a single-vendor multi-buyer supply chain for exponentially deteriorating items with a predetermined production rate.

6.2 Model Development

6.2.1 Introduction to the Model

In this model, the vendor of the supply chain produces a single product, subject to exponential deterioration, with a predetermined production rate of P per year and supplies the product to n_b buyers with demand rates D_i per year for the ith buyer, $i = 1, 2, ..., n_b$. The notations for the cost parameters are the same as those in Section 5.4. It is assumed that the convenient time unit is "day" and the maximum system cycle time is one year, i.e., the feasible system cycle times are $T = N \text{ day(s)} = N / 365 \text{ year}$, where $N = 1, 2, .., 365$. The delivery cycle time for the ith buyer is $T_{ci} = N / n_i$ where n_i is the number of deliveries in a system cycle and n_i is a factor of N. The delivery interval T_{ci} is therefore, also a factor of N.

In a system cycle there is one production batch and production starts before the first shipment. At time $t = 0$, first shipment is made to every buyer of the supply chain. For each system cycle time N, the decision variables are n_i 's, the numbers of deliveries for the n_b buyers. The objective of the model is to determine the optimal system cycle time and the optimal numbers of deliveries for all the buyers for minimizing the total system cost per unit time of the supply chain.

6.2.2 The Exact Cost Function

Suppose production starts at $t = -T_0$ ($T_0 > 0$) such that at $t = 0$, the vendor has just sufficient stock to meet the delivery requirements for the first shipments to all the buyers, and the vendor's inventory level drops to zero instantaneously.

For buyer *i*, delivery cycle time $= \frac{T}{n_i}$, delivery quantity $Q_i = \frac{D_i}{k}(e^{\frac{kT}{n_i}} - 1)$.

Total delivery requirement $= \sum_{i=1}^{n_b} \frac{D_i}{k}(e^{\frac{kT}{n_i}} - 1)$ at $t = 0$.

$$\frac{P}{k}(1 - e^{-kT_0}) = \sum_{i=1}^{n_b} \frac{D_i}{k}(e^{\frac{kT}{n_i}} - 1)$$

Hence, $$T_0 = -\frac{1}{k}\ln[1 - \sum_{i=1}^{n_b} \frac{D_i}{P}(e^{\frac{kT}{n_i}} - 1)]. \tag{6.1}$$

If the buyers do not adopt the same delivery cycle, there may be deliveries in the time interval $(-T_0, 0)$. If so, the stock required for such deliveries has already been produced in the previous system cycle and does not affect the value of T_0. Suppose production stops at time $t = T_1$. Then production time $T_p = T_0 + T_1$. The vendor's inventory level in a system cycle is shown in Figure 6.1.

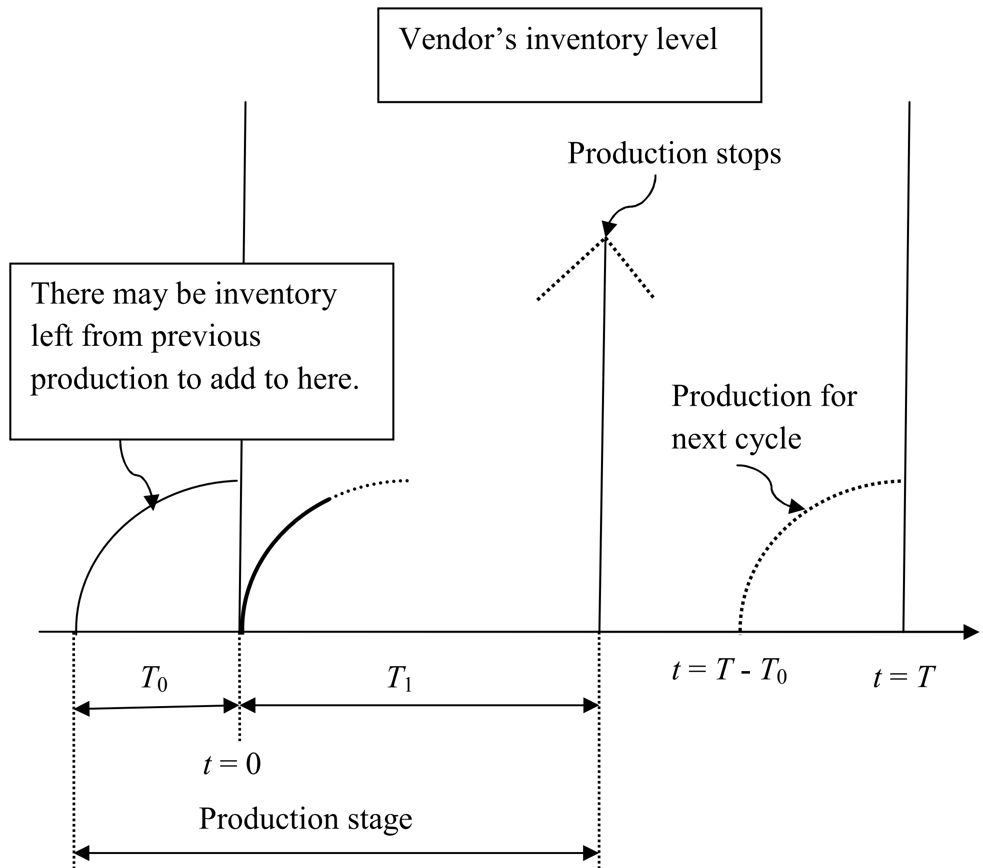


Figure 6.1: Vendor's inventory level in a system cycle.

For the time interval $-T_0 \le t \le T_1$,

The inventory level of the vendor in-between deliveries is described by the equation:

$$\frac{dI_v}{dt} = -kI_v + P$$

The inventory level of the buyers is described by the equation:

$$\sum_{i=1}^{n_b} \frac{dI_{bi}}{dt} = \sum_{i=1}^{n_b} (-kI_{bi} - D).$$

At the instants of deliveries, the delivery quantities are instantaneously transferred from the vendor to the respective buyers, and the total system inventory level remains unchanged.

Since $I_s = I_v + \sum_{i=0}^{n_b} I_{bi} \Rightarrow \frac{dI_s}{dt} = \frac{dI_v}{dt} + \sum_{i=0}^{n_b} \frac{dI_{bi}}{dt}$, for the time interval $-T_0 \le t \le T_1$, the system inventory level is described by the differential equation

$$\frac{dI_s}{dt} = -kI_s + P - \sum_{i=1}^{n_b} D_i \ . \tag{6.2}$$

The solution of equation (6.2) is given by $I_s = \frac{1}{k}(P - \sum_{i=1}^{n_b} D_i) + C_1 e^{-kT}$ where C_1 is a constant to be determined.

The vendor has finished all deliveries for the previous cycle before $t = 0$, and at $t = 0$, the buyers have consumed their inventory received in the previous cycle.

Hence, $I_s(t = 0) = \sum_{i=1}^{n_b} \frac{D_i}{k}(e^{\frac{kT}{n_i}} - 1) + 0 = \sum_{i=1}^{n_b} \frac{D_i}{k}(e^{\frac{kT}{n_i}} - 1)$

With this condition, C_1 can be determined and hence,

$$I_s = \frac{1}{k}(P - \sum_{i=1}^{n_b} D_i) + [\sum_{i=1}^{n_b} \frac{D_i}{k}(e^{\frac{kT}{n_i}} - 1) - \frac{1}{k}(P - \sum_{i=1}^{n_b} D_i)]e^{-kt} \ . \tag{6.3}$$

At $t=-T_0$,

$$I_s(t=-T_0)=\frac{1}{k}(P-\sum_{i=1}^{n_b}D_i)+[\sum_{i=1}^{n_b}\frac{D_i}{k}(e^{\frac{kT}{n_i}}-1)-\frac{1}{k}(P-\sum_{i=1}^{n_b}D_i)]e^{-k(-T_0)}$$
$$=\frac{1}{k}(P-\sum_{i=1}^{n_b}D_i)+[\frac{P}{k}(1-e^{-kT_0})-\frac{1}{k}(P-\sum_{i=1}^{n_b}D_i)]e^{kT_0}$$
$$=\frac{1}{k}(P-\sum_{i=1}^{n_b}D_i)-\frac{P}{k}+\frac{1}{k}\sum_{i=1}^{n_b}D_ie^{kT_0}=\frac{1}{k}\sum_{i=1}^{n_b}D_i(e^{kT_0}-1)$$

$I_s(t=-T_0)$ is composed of inventory at the buyers and possibly inventory at the vendor depending on whether there are still deliveries to be made in the time interval $(-T_0,0)$ for the previous system cycle.

For the time interval $T_1\le t\le T-T_0$, there is no production, and the system inventory is, by removing the term *P* from equation (6.2), described by the differential equation

$$\frac{dI_s}{dt}=-kI_s-\sum_{i=1}^{n_b}D_i \tag{6.4}$$

The solution is $I_s=-\frac{1}{k}\sum_{i=1}^{n_b}D_i+C_2e^{-kt}$ where C_2 is determined as follows:

As $I_s(T-T_0)=I_s(-T_0)=\frac{1}{k}\sum_{i=1}^{n_b}D_i(e^{kT_0}-1)$,

$$\therefore\ C_2=[\frac{1}{k}\sum_{i=1}^{n_b}D_i(e^{kT_0}-1)+\frac{1}{k}\sum_{i=1}^{n_b}D_i]e^{k(T-T_0)}=\frac{1}{k}\sum_{i=1}^{n_b}D_ie^{kT}$$

So $$I_s=-\frac{1}{k}\sum_{i=1}^{n_b}D_i+\frac{1}{k}\sum_{i=1}^{n_b}D_ie^{kT}e^{-kt} \tag{6.5}$$

Equating the expressions for $I_s(t = T_1)$ from equations (6.3) and (6.5),

$$-\frac{1}{k}\sum_{i=1}^{n_n} D_i + \frac{1}{k}\sum_{i=1}^{n_b} D_i e^{kT} e^{-kT_1} = \frac{1}{k}(P - \sum_{i=1}^{n_b} D_i) + [\sum_{i=1}^{n_b}\frac{D_i}{k}(e^{\frac{kT}{n_i}} - 1) - \frac{1}{k}(P - \sum_{i=1}^{n_b} D_i)]e^{-kT_1}$$

$$[\frac{1}{k}\sum_{i=1}^{n_b} D_i e^{kT} - \sum_{i=1}^{n_b}\frac{D_i}{k}(e^{\frac{kT}{n_i}} - 1) + \frac{1}{k}(P - \sum_{i=1}^{n_b} D_i)]e^{-kT_1} = \frac{P}{k}$$

$$e^{kT_1} = \frac{P + \sum_{i=1}^{n_b} D_i (e^{kT} - e^{\frac{kT}{n_i}})}{P} = 1 + \sum_{i=1}^{n_b}\frac{D_i}{P}(e^{kT} - e^{\frac{kT}{n_i}})$$

$$T_p = T_0 + T_1 = \frac{1}{k}\ln\{[\frac{1}{1 - \sum_{i=1}^{n_b}\frac{D_i}{P}(e^{\frac{kT}{n_i}} - 1)}][1 + \sum_{i=1}^{n_b}\frac{D_i}{P}(e^{kT} - e^{\frac{kT}{n_i}})]\}$$

$$= \frac{1}{k}\ln\{1 + \frac{\sum_{i=1}^{n_b}\frac{D_i}{P}(e^{\frac{kT}{n_i}} - 1) + \sum_{i=1}^{n_b}\frac{D_i}{P}(e^{kT} - e^{\frac{kT}{n_i}})}{1 - \sum_{i=1}^{n_b}\frac{D_i}{P}(e^{\frac{kT}{n_i}} - 1)}\}$$

$$\therefore\ T_p = \frac{1}{k}\ln\{1 + \frac{\sum_{i=1}^{n_b}\frac{D_i}{P}(e^{kT} - 1)}{1 - \sum_{i=1}^{n_b}\frac{D_i}{P}(e^{\frac{kT}{n_i}} - 1)}\} \tag{6.6}$$

Hence, equation (4.2) is a particular case of equation (6.6) when there is only one buyer in the supply chain. The expression of T_p indicates that the production time needed for dealing with multiple buyers is not equal to the sum of production times required for dealing with individual buyers.

From equation (3.2), The inventory level for buyer i is described by

$$I_{bi} = \frac{D_i}{k}(-1 + e^{\frac{kT}{n_i}} e^{-kt}) \quad \text{for } 0 \le t \le \frac{T}{n_i}.$$

The average inventory level for buyer i

$$= \frac{n_i D_i}{kT} \int_0^{\frac{T}{n_i}} (-1 + e^{\frac{kT}{n_i}} e^{-kt}) dt = \frac{n_i D_i}{kT} [\frac{1}{k}(e^{\frac{kT}{n_i}} - 1) - \frac{T}{n_i}]$$

Quantity deteriorated per unit time for buyer i

$$= \frac{n_i D_i}{kT} [\frac{1}{k}(e^{\frac{kT}{n_i}} - 1) - \frac{T}{n_i}] k = \frac{n_i D_i}{T} [\frac{1}{k}(e^{\frac{kT}{n_i}} - 1) - \frac{T}{n_i}].$$

Hence, total cost for the buyers per unit time is given by

$$\sum_{i=1}^{n_b} TC_{bi} = \sum_{i=1}^{n_b} \{ \frac{n_i A_{bi}}{T} + (H_{bi} + kC_{bi}) \frac{n_i D_i}{kT} [\frac{1}{k}(e^{\frac{kT}{n_i}} - 1) - \frac{T}{n_i}] \}. \tag{6.7}$$

Total system inventory over one system cycle is given by

$$\int_{-T_0}^{T_1} \{ \frac{1}{k}(P - \sum_{i=1}^{n_b} D_i) + [\sum_{i=1}^{n_b} \frac{D_i}{k}(e^{\frac{kT}{n_i}} - 1) - \frac{1}{k}(P - \sum_{i=1}^{n_b} D_i)] e^{-kt} \} dt$$
$$+ \int_{T_1}^{T-T_0} [-\frac{1}{k} \sum_{i=1}^{n_n} D_i + \frac{1}{k} \sum_{i=1}^{n_b} D_i e^{kT} e^{-kt}] dt$$
$$= (PT_p - \sum_{i=1}^{n_b} D_i T) / k$$

The details are shown in Appendix A3. Hence, the average system inventory level is $(PT_p - \sum_{i=1}^{n_b} D_i T) / kT$ which is reduced to $(PT_p - DT) / kT$ for the case of single buyer as in Wee et al. (2008).

The above expression for the average inventory level can be interpreted as follows: The system has produced a quantity of PT_p units of goods and "shipped out" $\sum_{i=1}^{n_b} D_i T$ units of goods for meeting the demand, over a time period of *T*. Therefore, quantity deteriorated per unit time is $(PT_p - \sum_{i=1}^{n_b} D_i T) / T$. This quantity should be the same as the average system inventory level times the deterioration rate, and hence, the above expression for the average system inventory level can be obtained. The mathematical procedure in Appendix A3 verifies this argument.

The average vendor's inventory level is the average system inventory level minus the average inventory level of the buyers and is given by

$$\frac{1}{kT}(PT_p - \sum_{i=1}^{n_b} D_i T) - \sum_{i=1}^{n_b} \frac{n_i D_i}{kT}[\frac{1}{k}(e^{\frac{kT}{n_i}} - 1) - \frac{T}{n_i}]$$

Total cost for the vendor per unit time is given by

$$TC_v = \frac{S}{T} + \sum_{i=1}^{n_b} \frac{n_i A_{vi}}{T} + (H_v + kC_v)\{\frac{1}{kT}(PT_p - \sum_{i=1}^{n_b} D_i T) - \sum_{i=1}^{n_b} \frac{n_i D_i}{kT}[\frac{1}{k}(e^{\frac{kT}{n_i}} - 1) - \frac{T}{n_i}]\} \quad (6.8)$$

Adding equations (6.7) and (6.8), total system cost per unit time

$$\begin{aligned}
TC_s &= TC_v + \sum_{i=1}^{n_b} TC_{bi} \\
&= \frac{S}{T} + \sum_{i=1}^{n_b} \frac{n_i (A_{vi} + A_{bi})}{T} + \sum_{i=1}^{n_b} \{\frac{(H_{bi} - H_v + kC_{bi} - kC_v)}{kT} n_i D_i [\frac{1}{k}(e^{\frac{kT}{n_i}} - 1) - \frac{T}{n_i}]\} \\
&\quad + (H_v + kC_v)[\frac{1}{kT}(PT_p - \sum_{i=1}^{n_b} D_i T)] \\
&= \frac{S}{T} + \sum_{i=1}^{n_b} \frac{n_i (A_{vi} + A_{bi})}{T} + \sum_{i=1}^{n_b} \{\frac{(H_{bi} - H_v + kC_{bi} - kC_v)}{kT} n_i D_i [\frac{1}{k}(e^{\frac{kT}{n_i}} - 1)]\} \\
&\quad + \frac{(H_v + kC_v) PT_p}{kT} - \sum_{i=1}^{n_b} \frac{(H_{bi} + kC_{bi})}{k} D_i \qquad (6.9)
\end{aligned}$$

where T_p is given by equation (6.6).

The objective is to minimize TC_s subject to the constraints: $N \in \{1, 2, ..., 365\}$ and n_i's are integer factors of N, $i = 1, 2, ..., n_b$.

6.2.3 Analysis of Production Time and Example 6.1

Let T_{pi} be the production time required for meeting the demand of the ith buyer alone, i.e., $T_{pi} = \frac{1}{k}\ln\{1+[\frac{D_i}{P}(e^{kT}-1)]/[1-\frac{D_i}{P}(e^{\frac{kT}{n_i}}-1)]\}$.

It is known that $T_p \neq \sum_i T_{pi}$. However, numerical experiments show that T_p, the actual production time needed for dealing with all the buyers of the supply chain together, is approximately equal to $\sum_i T_{pi}$, i.e., the sum of "individual" production times for the buyers.

It can be shown that:

(i) $T_p > \sum_i T_{pi}$ if at most one of the n_i's is not 1;

(ii) $T_p < \sum_i T_{pi}$ if $n_i \geq 2$ for all i.

(iii) If two or more (but not all) of the n_i's are not 1, then either way can happen.

The proofs for (i) and (ii) are presented in Appendix A4.

Example 6.1

(a) Consider a supply chain of 3 buyers with the following parameters:

$k = 0.1$/year $P = 6000$ units/year $D = (1000, 600, 400)$ units/year

For a system cycle time of $T = 0.5$ year, the findings are:

No. of Deliveries	T_p	$\sum_i T_{pi}$	% error
(1,1,1)	0.172381	0.171461	0.533486
(10,1,1)	0.171051	0.170802	0.145466
(1,5,3)	0.171272	0.171171	0.058829
(1,5,10)	0.171137	0.171144	-0.00425
(2,2,2)	0.17089	0.170897	-0.00458
(3,3,3)	0.170406	0.170713	-0.18024
(10,10,10)	0.169741	0.170459	-0.42315

Table 6.1: Results for Example 6.1(a). [% error = $100(T_p - \sum_i T_{pi}) / T_p$]

(b) Consider a supply chain of 10 buyers with the following parameters:

$k = 0.1$/year $P = 20000$ units/year

$D = (1500, 500, 1200, 800, 1600, 400, 100, 1900, 1000, 1000)$ units/year

For a system cycle time of $T = 0.5$ year, the findings are:

No. of Deliveries	T_p	$\sum_i T_{pi}$	% error
(1,1,1,1,1,1,1,1,1,1)	0.259699	0.256781	1.123445(#)
(1,1,1,1,1,1,1,10,1,1)	0.258548	0.256567	0.766063
(2,2,2,2,2,2,2,2,2,2)	0.256328	0.25635	-0.00858
(10,10,10,10,10,10,10,10,10,10)	0.253753	0.256015	-0.89144

(#) The % error is 2.32056 if $k = 0.2$

Table 6.2: Results for Example 6.1(b).

6.2.4 Solution Procedure

For a given system cycle time, the total system cost per unit time is a function of the numbers of deliveries of all the buyers. Optimization of the total cost function is achieved by finding the optimal numbers of deliveries. However, as indicated in equation (6.6), production time involves the numbers of deliveries of all the buyers and cannot be split among the buyers. If the number of deliveries of one of the buyers is changed, it requires knowing the numbers of deliveries for all other buyers in order to study the impact of this change of delivery on the total cost.

Due to the complexity of the model, it is very difficult, if not impossible, to solve the model analytically. Hence, this research adopts a heuristic approach to find a "good" solution" of the model. The approach has two stages as described below. The first stage is to find a "good" initial solution. The second stage is to find the "optimal" solution.

6.2.4.1 First Stage: Finding a Good Initial Solution

When multiple buyers are involved, it is impractical to start with one delivery for each buyer, or to start with a random set of deliveries. It is desirable that a "good" starting set of deliveries can be found. As discussed in Section 6.2.3, the actual production time is approximately equal to the sum of "individual" production times for the buyers. With this finding, an approximate cost function that can be optimized by independent optimization of the costs of each "vendor-buyer" subsystem is defined. A "good" starting set of deliveries will be found by optimizing this approximate cost function.

Define a function $\varnothing_i$, $i = 1, 2, ..., n_b$, as follows:

$$\varnothing_i = \frac{n_i(A_{vi} + A_{bi})}{T} + \frac{(H_{bi} - H_v + kC_{bi} - kC_v)n_i D_i}{kT}[\frac{1}{k}(e^{\frac{kT}{n_i}} - 1)] + \frac{(H_v + kC_v)(PT_{pi})}{kT} - \frac{(H_{bi} + kC_{bi})}{k} D_i \quad (6.10)$$

There is only one decision variable, n_i , for each $\varnothing_i$.

Let $TC_s^{\varsigma} = \sum_{i=1}^{n_b} \varnothing_i + \frac{S}{T}$. Then

$$TC_s^{\varsigma} = \frac{S}{T} + \sum_{i=1}^{n_b} \frac{n_i(A_{vi} + A_{bi})}{T} + \{\sum_{i=1}^{n_b} \frac{(H_{bi} - H_v + kC_{bi} - kC_v)}{kT} n_i D_i[\frac{1}{k}(e^{\frac{kT}{n_i}} - 1)]\} + \sum_{i=1}^{n_b} \frac{(H_v + kC_v)PT_{pi}}{kT} - \sum_{i=1}^{n_b} \frac{(H_{bi} + kC_{bi})}{k} D_i$$

$$TC_s^{\varsigma} - TC_s = \frac{(H_v + kC_v)P}{kT}(\sum_{i=1}^{n_b} T_{pi} - T_p)$$

TC_s^{ς} is therefore an approximate cost function for the total system cost per unit time. As different $\varnothing_i$'s have different decision variables, each $\varnothing_i$ can be optimized independently. For a given system cycle time *T*, S/T is constant, and TC_s^{ς} is minimized when every $\varnothing_i$ attains its own minimum. Hence, the approximate cost function can be optimized by independent optimization of all the $\varnothing_i$'s. A starting set of deliveries is not required. It can be proved that $\varnothing_i$ is convex as follows:

From equation (6.10),

$$\frac{d\varnothing_i(x)}{dx} = \frac{(A_{vi}+A_{bi})}{T} + \frac{(H_{bi}-H_v+kC_{bi}-kC_v)D_i}{k^2T}(e^{\frac{kT}{x}}-1-\frac{kT}{x}e^{\frac{kT}{x}})$$
$$-\frac{D_i^2(H_v+kC_v)e^{\frac{kT}{x}}}{Pkx^2}\frac{(e^{kT}-1)}{[1+\frac{D_i}{P}(e^{kT}-e^{\frac{kT}{x}})][1-\frac{D_i}{P}(e^{\frac{kT}{x}}-1)]}$$
$$=\frac{1}{T}\{A_{vi}+A_{bi}+\frac{(H_{bi}-H_v+kC_{bi}-kC_v)D_i}{k^2}(e^{\frac{kT}{x}}-1-\frac{kT}{x}e^{\frac{kT}{x}})$$
$$-\frac{D_i^2T(H_v+kC_v)e^{\frac{kT}{x}}}{Pkx^2}\frac{(e^{kT}-1)}{[1+\frac{D_i}{P}(e^{kT}-e^{\frac{kT}{x}})][1-\frac{D_i}{P}(e^{\frac{kT}{x}}-1)]}\}$$

Define $g(x)$ as:

$$g(x)=\frac{(H_{bi}-H_v+kC_{bi}-kC_v)D_i}{k^2}(\frac{kT}{x}e^{\frac{kT}{x}}-e^{\frac{kT}{x}}+1)$$
$$+\frac{D_i^2T(H_v+kC_v)e^{\frac{kT}{x}}}{Pkx^2}\frac{(e^{kT}-1)}{[1+\frac{D_i}{P}(e^{kT}-e^{\frac{kT}{x}})][1-\frac{D_i}{P}(e^{\frac{kT}{x}}-1)]}.$$

Then $\frac{d\varnothing_i(x)}{dx}=\frac{A_{vi}+A_{bi}-g(x)}{T}$ and $\frac{d^2}{dx^2}\varnothing_i(x)=-\frac{g'(x)}{T}$.

To minimize $\varnothing_i$, $\frac{d\varnothing_i(x)}{dx}=0 \Rightarrow g(x)=A_{vi}+A_{bi}$. (6.11)

Let $u=\frac{kT}{x}$ and $g_1(x)=\frac{kT}{x}e^{\frac{kT}{x}}-e^{\frac{kT}{x}}+1$. Then $u>0$ and

$$\frac{d}{dx}[g_1(x)] = \frac{d}{dx}(\frac{kT}{x}e^{\frac{kT}{x}} - e^{\frac{kT}{x}} + 1) = \frac{d}{du}(ue^u - e^u + 1)\frac{d}{dx}(\frac{kT}{x}) = ue^u(-\frac{kT}{x^2}) < 0$$

$$\lim_{x\to 0^+} g_1(x) = \lim_{x\to 0^+}(\frac{kT}{x}e^{\frac{kT}{x}} - e^{\frac{kT}{x}} + 1) = \lim_{x\to 0^+}[e^{\frac{kT}{x}}(\frac{kT}{x} - 1) + 1] = \infty$$

$$\lim_{x\to\infty} g_1(x) = \lim_{x\to\infty}(\frac{kT}{x}e^{\frac{kT}{x}} - e^{\frac{kT}{x}} + 1) = (0)e^0 - e^0 + 1 = 0 .$$

Hence, $g_1(x) = \frac{kT}{x}e^{\frac{kT}{x}} - e^{\frac{kT}{x}} + 1 \in (0,\infty)$, i.e., $g_1(x)$ is positive and a decreasing function of x with no finite upper bound.

Consider $g_2(x) = \frac{D_i^2 T(H_v + kC_v)e^{\frac{kT}{x}}}{Pkx^2} \frac{(e^{kT} - 1)}{[1 + \frac{D_i}{P}(e^{kT} - e^{\frac{kT}{x}})][1 - \frac{D_i}{P}(e^{\frac{kT}{x}} - 1)]}$.

$$\frac{P}{D_i} > e^{\frac{kT}{x}} \Rightarrow 1 - \frac{D_i}{P}(e^{\frac{kT}{x}} - 1) > 1 - \frac{e^{\frac{kT}{x}} - 1}{e^{\frac{kT}{x}}} > 0 \quad \text{and } g_2(x) > 0 .$$

All the factors of $g_2(x)$ are positive and $\frac{d}{dx}(\frac{kT}{x}) < 0$. It is clear that as x increases, the numerator decreases and the denominator increases. $g_2(x)$ is therefore also positive and a decreasing function of x.

$$\lim_{x\to\infty} g_2(x) = \frac{D_i^2 T(H_v + kC_v)e^0}{Pk \lim_{x\to\infty} x^2} \frac{(e^{kT} - 1)}{[1 + \frac{D_i}{P}(e^{kT} - e^0)][1 - \frac{D_i}{P}(e^0 - 1)]} = 0$$

Hence, $g(x) = g_1(x) + g_2(x)$ has the following properties:

(i) it is positive and a decreasing function of x;

(ii) $\lim_{x\to 0^+} g(x) = \infty$ since $\lim_{x\to 0^+}[g_1(x) + g_2(x)] > \lim_{x\to 0^+} g_1(x)$ and $\lim_{x\to 0^+} g_1(x) = \infty$;

(iii) $\lim_{x\to\infty} g(x) = \lim_{x\to\infty}[g_1(x) + g_2(x)] = 0$.

Therefore, for any $A_{vi} + A_{bi} > 0$, there is a unique solution x^* to equation (6.11).

As $g(x)$ is a decreasing function, $g'(x) < 0 \Rightarrow \frac{d^2}{dx^2}\varnothing_i(x) = -\frac{g'(x)}{T} > 0$.

Hence, $\varnothing_i$ is a convex function and its minimum is given by $\varnothing_i(x^*)$. The optimal value, n_i^{ς}, which must be an integral factor of N, for minimizing $\varnothing_i(n_i)$ can be found by the following procedure:

(i) If $x^* \le 1$ (or equivalently, $A_{vi} + A_{bi} \ge g(1)$), set $n_i^{\varsigma} = 1$.

(ii) If $x^* \ge N$ (or $A_{vi} + A_{bi} \le g(N)$), set $n_i^{\varsigma} = N$.

(iii) If x^* is an integer (not expected due to the form of the equation) between 1 and N exclusive and is a factor of N, set $n_i^{\varsigma} = x^*$.

(iv) If $1 < x^* < N$ and (a) is not an integer or (b) is an integer but not a factor of N, find the two consecutive factors of N, α **and** β, such that $\alpha < x^* < \beta$. Calculate $\varnothing_i(\alpha)$ and $\varnothing_i(\beta)$. Set $n_i^{\varsigma} = \alpha$ if $\varnothing_i(\alpha) < \varnothing_i(\beta)$. Otherwise, set $n_i^{\varsigma} = \beta$.

6.2.4.2 Second Stage: Finding the Optimal Solution

The difference between the exact and the approximate cost function is given by

$TC_s - TC_s^{\varsigma} = \frac{P(H_v + kC_v)}{kT}(T_p - \sum_{i=1}^{n_b} T_{pi})$. Hence, from Section 6.2.3:

(i) if at most one of the n_i's is not 1, $T_p > \sum_i T_{pi}$ and $TC_s > TC_s^{\varsigma}$, so the exact cost is larger than the approximate cost;

(ii) if $n_i \geq 2$ for all i, $T_p < \sum_i T_{pi}$ and $TC_s < TC_s^{\varsigma}$, the exact cost is lower than the approximate cost;

(iii) If two or more (but not all) of the n_i's are not 1, then either way can happen.

Lemma 6.1

For a given system cycle time *T*, $T_p - \sum_{i=1}^{n_b} T_{pi}$ decreases, or equivalently $\sum_{i=1}^{n_b} T_{pi} - T_p$ increases, when the number of delivery is increased.

Proof:

Let $\rho_i = \frac{D_i}{P}$, for $i = 1, 2, ..., n_b$. Then $\frac{1}{\rho_i} > e^{\frac{kT}{n_i}}$ and

$$\sum_{i=1}^{n_b} T_{pi} - T_p = \frac{1}{k}\ln\{\prod_{i=1}^{n_b}[1+\frac{\rho_i(e^{kT}-1)}{1-\rho_i(e^{\frac{kT}{n_i}}-1)}]/[1+\frac{\sum_{i=1}^{n_b}\rho_i(e^{kT}-1)}{1-\sum_{i=1}^{n_b}\rho_i(e^{\frac{kT}{n_i}}-1)}]\}.$$

Consider $\Psi = \frac{1}{k}\ln\{\Psi_1\} = \frac{1}{k}\ln\{\prod_{i=1}^{n_b}[1+\frac{\rho_i(e^{kT}-1)}{1-\rho_i(e^{\frac{kT}{x_i}}-1)}]/[1+\frac{\sum_{i=1}^{n_b}\rho_i(e^{kT}-1)}{1-\sum_{i=1}^{n_b}\rho_i(e^{\frac{kT}{x_i}}-1)}]\}.$

Differentiating Ψ with respect to some $x_i = x_p$,

$$\frac{d\Psi}{dx_p} = \frac{1}{k\Psi_1}\frac{d\Psi_1}{dx_p}.$$

The denominator of $\frac{d\Psi_1}{dx_p}$ is a square and its numerator is given by

$$\rho_p e^{\frac{kT}{x_p}}\frac{kT}{(x_p)^2}\prod_{i\neq p}[1+\frac{\rho_i(e^{kT}-1)}{1-\rho_i(e^{\frac{kT}{x_i}}-1)}]\{[1+\frac{\rho_p(e^{kT}-1)}{1-\rho_p(e^{\frac{kT}{x_p}}-1)}]\frac{\sum_{i=1}^{n_b}\rho_i(e^{kT}-1)}{[1-\sum_{i=1}^{n_b}\rho_i(e^{\frac{kT}{x_i}}-1)]^2}$$

$$-[1+\frac{\sum_{i=1}^{n_b}\rho_i(e^{kT}-1)}{1-\sum_{i=1}^{n_b}\rho_i(e^{\frac{kT}{x_i}}-1)}]\frac{\rho_p(e^{kT}-1)}{[1-\rho_p(e^{\frac{kT}{x_p}}-1)]^2}\}$$

$$=\rho_p e^{\frac{kT}{x_p}}\frac{kT}{(x_p)^2}\prod_{i\neq p}[1+\frac{\rho_i(e^{kT}-1)}{1-\rho_i(e^{\frac{kT}{x_i}}-1)}]\frac{(e^{kT}-1)}{[1-\rho_p(e^{\frac{kT}{x_p}}-1)][1-\sum_{i=1}^{n_b}\rho_i(e^{\frac{kT}{x_i}}-1)]}$$

$$\{\frac{[1+\rho_p(e^{kT}-e^{\frac{kT}{x_p}})]\sum_{i=1}^{n_b}\rho_i}{[1-\sum_{i=1}^{n_b}\rho_i(e^{\frac{kT}{x_i}}-1)]}-\frac{\rho_p[1+\sum_{i=1}^{n_b}\rho_i(e^{kT}-e^{\frac{kT}{x_i}})]}{[1-\rho_p(e^{\frac{kT}{x_p}}-1)]}\}$$

Consider the two terms inside {}. The first term has a smaller denominator than the second term. Also the first term has a larger numerator because

$$[1+\rho_p(e^{kT}-e^{\frac{kT}{x_p}})]\sum_{i=1}^{n_b}\rho_i-\rho_p[1+\sum_{i=1}^{n_b}\rho_i(e^{kT}-e^{\frac{kT}{x_i}})]$$
$$=\sum_{i\neq p}^{n_b}\rho_i+\rho_p\sum_{i\neq p}^{n_b}\rho_i(e^{kT}-e^{\frac{kT}{x_p}})-\rho_p\sum_{i\neq p}^{n_b}\rho_i(e^{kT}-e^{\frac{kT}{x_i}})$$
$$=\sum_{i\neq p}^{n_b}\rho_i(1-\rho_p e^{\frac{kT}{x_p}}+\rho_p e^{\frac{kT}{x_i}})>0 \quad (\because\ \rho_p e^{\frac{kT}{x_p}}<1)$$

Hence, both the numerator and the denominator of $\frac{d\Psi_1}{dx_p}$ are positive.

$$\therefore\ \frac{d\Psi_1}{dx_p}>0\Rightarrow\frac{d\Psi}{dx_p}>0$$

This means that $\sum_{i=1}^{n_b}T_{pi}-T_p$ increases with increasing number of delivery or $T_p-\sum_{i=1}^{n_b}T_{pi}$ decreases with increasing number of deliveries. So the lemma is proved.

From the lemma, we have the following theorem:

<u>Theorem</u>

For a given system cycle time *T*, suppose TC_s^{ζ} is minimum with $n_i=n_i^{\zeta}$, then $TC_s(n_i=n_i^{\zeta})<TC_s(n_i<n_i^{\zeta})$.

Proof:

Let $\Delta T_p(1)$ and $\Delta T_p(2)$ be $T_p - \sum_{i=1}^{n_b} T_{pi}$ for $n_i = n_i^{\zeta}$ and $n_i < n_i^{\zeta}$ respectively.

$$TC_s(n_i = n_i^{\zeta}) = TC_s^{\varsigma}(n_i = n_i^{\zeta}) + \frac{P(H_v + kC_v)}{kT} \Delta T_p(1)$$

$$TC_s(n_i < n_i^{\zeta}) = TC_s^{\varsigma}(n_i < n_i^{\zeta}) + \frac{P(H_v + kC_v)}{kT} \Delta T_p(2)$$

$$\begin{aligned} TC_s(n_i = n_i^{\zeta}) - TC_s(n_i < n_i^{\zeta}) &= [TC_s^{\varsigma}(n_i = n_i^{\zeta}) - TC_s^{\varsigma}(n_i < n_i^{\zeta})] \\ &\quad + \frac{P(H_v + kC_v)}{kT}[\Delta T_p(1) - \Delta T_p(2)] \end{aligned}$$

$TC_s^{\varsigma}(n_i = n_i^{\zeta}) - TC_s^{\varsigma}(n_i < n_i^{\zeta}) < 0$ as TC_s^{ς} is minimum with $n_i = n_i^{\zeta}$.

$\Delta T_p(1) - \Delta T_p(2) < 0$ as ΔT_p is smaller when the number of deliveries is higher.

Hence, $TC_s(n_i = n_i^{\zeta}) - TC_s(n_i < n_i^{\zeta}) < 0 + 0 = 0$, i.e., $TC_s(n_i = n_i^{\zeta}) < TC_s(n_i < n_i^{\zeta})$.

This theorem has an important implication to this 2-stage model. After the initial solution is found by minimizing TC_s^{ς} as in Section 6.2.4.1 and the corresponding exact cost is found, reducing the number of deliveries from this initial solution cannot improve the exact cost. Hence, the optimal solution is found by increasing the numbers of deliveries from the initial solution until no improvement can be made.

6.3 The Full Algorithm - Summary

A 2-stage model for minimizing the total system cost per unit time of a single-vendor multi-buyer supply chain has been presented in Section 6.2. The algorithm for finding the optimal solution, consisting of the optimal system cycle time and the optimal numbers of deliveries for the buyers, is summarized as follows:

<u>Algorithm</u>

Step 1: For each N (N=1, 2, 3, …,365), find its integer factors.

Step 2: For the approximate cost function, find the "optimal" solution for numbers of deliveries for different buyers, the "optimal" approximate cost and the corresponding actual cost by using the sub-algorithm. This is the initial solution for this N.

Step 3: Starting with the initial solution, increase the numbers of deliveries by applying the sub-algorithm, until no improvement of the actual cost can be made. This is the minimum cost solution for this N.

Step 4: Find the least cost among the minimum costs of $N = 1,\ 2,\ 3, ...,\ 365$. The corresponding solution is the overall optimal solution.

<u>Sub-algorithm for finding the initial solution for a fixed N</u>

Step 1: Find the integer factors of N and set $T = N / 365$.

Step 2: Start with $i=1$, set $\varnothing_i$ as equation (6.10) and define $f(x)$ as:

$$f(x)=\frac{(H_{bi}-H_v+kC_{bi}-kC_v)D_i}{k^2}\left(\frac{kT}{x}e^{\frac{kT}{x}}-e^{\frac{kT}{x}}+1\right)$$
$$+\frac{D_i^{\,2}T(H_v+kC_v)e^{\frac{kT}{x}}}{Pkx^2}\frac{(e^{kT}-1)}{[1+\frac{D_i}{P}(e^{kT}-e^{\frac{kT}{x}})][1-\frac{D_i}{P}(e^{\frac{kT}{x}}-1)]}-(A_{vi}+A_{bi})$$

Step 3: If $f(1)\le 0$ set $n_i^{\zeta}=1$. Next *i*.

Step 4: If $f(N)\ge 0$, set $n_i^{\zeta}=N$. Next *i*.

Step 5: Find the minimum of $\varnothing_i(x)$. Suppose $\varnothing_i(x)$ is minimum when $x=x^*$. If x^* is an integer and is a factor of *N*, set $n_i^{\zeta}=x^*$. Next *i*.

Step 6: Find α and β, the two consecutive factors of N such that $\alpha<x^*<\beta$.

Step 7: Calculate $\varnothing_i(\alpha)$ and $\varnothing_i(\beta)$. If $\varnothing_i(\alpha)<\varnothing_i(\beta)$, set $n_i^{\zeta}=\alpha$. Otherwise, set $n_i^{\zeta}=\beta$. Next *i*.

Step 8: Find T_p and TC_s with $T=N/365$ and $n_i=n_i^{\zeta}$, $i=1,2,...,n_b$. This is the initial solution.

<u>Sub-algorithm for finding the optimal solution for the same *N*</u>

Step 9: Start with *i* = 1, if $n_i^{\zeta}=N$, next *i*.

Step 10: Let p be the current value of n_i and q be the next higher factor of N. If $TC_s(p) \leq TC_s(q)$, then $n_i^* = p$, where n_i^* denotes the optimal value of n_i. Next i.

Step 11: Otherwise, replace p by q, repeat Step 10 until $TC_s(p) \leq TC_s(q)$ is achieved or N is reached, then $n_i^* = p$ or $n_i^* = N$, respectively. Next i.

Step 12: Record the optimal cost TC_s^* and the optimal solution: $n_i = n_i^*, i = 1, 2, ..., n_b$, for this N.

6.4 Examples

6.4.1 Example 6.2 (5-buyer Example)

(a) In Supply Chain S1, a vendor supplies an exponentially deteriorating product, with a deterioration rate of 0.1 per year to 5 buyers. The production rate is 300,000 units per year while the total demand rate is 150,000 units per year for the 5 buyers. The vendor has a production set up cost S of \$1,000 for each set up. The vendor's order processing & delivery costs A_{vi} for the 5 buyers are: \$100, \$110, \$120, \$130, \$140, respectively. The vendor has a deterioration cost C_v of \$10 per unit and an inventory holding cost H_v of \$1 per unit per year. The demand rates and the cost parameters of the 5 buyers are shown in Table 6.3.

Buyer i	Demand rate D_i (units / year)	Ordering cost A_{bi} (\$)	Deterioration cost C_{bi} (\$/unit)	Inventory holding cost H_{bi} (\$/unit/year)
1	10,000	50	12	1.2
2	20,000	60	13	1.3
3	30,000	70	14	1.4
4	40,000	80	15	1.5
5	50,000	90	16	1.6

Table 6.3: Buyers' parameters for Supply Chain S1.

The following illustrates how to find the initial solution for a system cycle time of $N = 120$ days, i.e., $T = 120/365$ year, by using the algorithm.

For the first buyer, set $T = 120/365$ and

$$f(x) = \frac{[1.2 - 1 + 0.1(12) - 0.1(10)](10000)}{0.1^2}\left(\frac{0.1T}{x}e^{\frac{0.1T}{x}} - e^{\frac{0.1T}{x}} + 1\right) + \frac{10000^2 T[1+0.1(10)]e^{\frac{0.1T}{x}}}{300000(0.1)x^2} \frac{(e^{0.1T} - 1)}{[1 + \frac{10000}{300000}(e^{0.1T} - e^{\frac{0.1T}{x}})][1 - \frac{10000}{300000}(e^{\frac{0.1T}{x}} - 1)]} - 150$$

As $f(1) = 146.76 > 0$ and $f(120) = -149.98 < 0$, the value of x minimizing $f(x)$ is between 1 and 120 and it is found that $x^* \approx 1.40$ by using MATLAB. So $\alpha = 1$ and $\beta = 2$. As $\varnothing_1(1) = 4558.40$ and $\varnothing_1(2) = 4566.17$, $n_1^{\varsigma} = 1$.

By carrying out the same procedure, the values of other n_i^{ς}'s are found. The initial solution is $(1, 2, 3, 4, 5)$ with an actual cost of \$60,229.19 per year.

The optimal solution for $N = 120$ is found as follows:

Number of Deliveries	TC_s ($/year)	Number of Deliveries	TC_s ($/year)
Finding n_1^*			
(1,2,3,4,5)	60,229.19	**(3,2,3,4,5)**	**58,433.62**
(2,2,3,4,5)	58,648.82	(4,2,3,4,5)	58,555,53
Finding n_2^*			
(3,2,3,4,5)	58,433.62	**(3,4,3,4,5)**	**57,291.54**
(3,3,3,4,5)	57,497.83	(3,5,3,4,5)	57,375.58
Finding n_3^*			
(3,4,3,4,5)	57,291.54	**(3,4,5,4,5)**	**56,581.15**
(3,4,4,4,5)	56,702.06	(3,4,6,4,5)	56,693.85
Finding n_4^*			
(3,4,5,4,5)	56,581.15	**(3,4,5,6,5)**	**56,196.97**
(3,4,5,5,5)	56,222.31	(3,4,5,8,5)#	56,645.45
Finding n_5^*			
(3,4,5,6,5)	56,196.97	(3,4,5,6,8)#	56,306.07
(3,4,5,6,6)	**56,011.61**	--------------	--------------

#: 7 is not a factor of 120 so the solutions are "moved up" from 6 to 8.

Table 6.4: Finding the optimal solution for $N = 120$ days for Example 6.2(a).

The optimal solution is (3,4,5,6,6) and the optimal cost is $56,306.07 per year for a system cycle of 120 days. It can be noted that after n_1^* has been found, increasing the value of n_2 only decreases the total cost until n_2^* is found. The same trend is noted for the later iterations with other n_i's.

The optimal solutions for $N = 1, 2, ..., 365$ day(s) have been found and the solutions for the 5 smallest and the 5 largest optimal costs are shown in Table 6.5.

System cycle time N (days)	Optimal numbers of deliveries n_i^*'s	Optimal system cost per unit time TC_s^*
44	**(1,2,2,2,2)**	**45,910.20**
42	(1,2,2,2,2)	45,943.44
46	(1,2,2,2,2)	45,971.73
40	(1,1,2,2,2)	45,976.76
38	(1,1,2,2,2)	46,090.41
349	(1,1,1,1,349)	300,801.90
353	(1,1,1,1,353)	303,356.10
359	(1,1,1,1,359)	307,193.30
2	(1,1,1,1,1)	357,492.00
1	(1,1,1,1,1)	712,558.30

Table 6.5: The 5 smallest and the 5 largest optimal costs for Supply Chain S1.

In this example, the system cycle times of 1, 2, and the 3 largest prime numbers provide the 5 largest optimal costs among the 365 optimal costs. The overall optimal solution is: $N^* = 44$ days, n_i^*'s = (1,2,2,2,2), and $TC_s^* = \$45,910.20$ per year.

(b) In Supply Chain S2, the vendor has the same parameters as the vendor in Supply Chain S1, while the parameters of the 5 buyers are shown in Table 6.6.

Buyer i	Demand rate D_i (units / year)	Ordering cost A_{bi} ($)	Deterioration cost C_{bi} ($/unit)	Inventory holding cost H_{bi} ($/unit/year)
1	10,000	90	16	1.6
2	20,000	80	15	1.5
3	30,000	70	14	1.4
4	40,000	60	13	1.3
5	50,000	50	12	1.2

Table 6.6: Buyers' parameters for Supply Chain S2.

The overall optimal solution for S2 is $N^* = 40$ days, n_i^*'s = (1,1,2,2,2), and $TC_s^* = \$44,224.63$ per year. This cost is lower than that for Supply Chan S1. In Supply Chan S1, the smaller buyers (buyers with lower demand rates) have lower cost parameters. In Supply Chan S2, this is reversed with larger buyers having lower cost parameters, while the buyers' demand rates and the vendor's parameters remain the same. As a result, Supply Chan S2 has a lower optimal cost than S1. In fact, the optimal cost for $N = 44$ days for S2 is $44,281.21 and is lower than the corresponding cost in S1. Also $N = 1, 2, 359, 352, 349$ provide the 5 largest optimal costs for S2 which are smaller than the corresponding costs for S1.

6.4.2 Example 6.3 (10-buyer Example)

In Supply Chain S3, a vendor supplies an exponentially deteriorating product, with a deterioration rate of 0.1 per year to 10 buyers. The production rate is 550,000 units per year while the total demand rate for the 10 buyers is 275,000 units per year. The vendor has a production set up cost S of $6,000 for each set up. The vendor's order processing & delivery costs A_{vi} for the 10 buyers are: $190, $180, $170, $160, $150, $140, $130, $120, $110, and $100, respectively. The vendor has a deterioration cost C_v of $10 per unit and an inventory holding cost H_v of $1 per unit per year. The demand rates and the cost parameters of the 10 buyers are shown in Table 6.7.

Buyer i	Demand rate D_i (units / year)	Ordering cost A_{bi} ($)	Deterioration cost C_{bi} ($/unit)	Inventory holding cost H_{bi} ($/unit/year)
1	5,000	90	21	2.1
2	10,000	85	20	2.0
3	15,000	80	19	1.9
4	20,000	75	18	1.8
5	25,000	70	17	1.7
6	30,000	65	16	1.6
7	35,000	60	15	1.5
8	40,000	55	14	1.4
9	45,000	50	13	1.3
10	50,000	45	12	1.2

Table 6.7: Buyers' parameters for Example 6.3.

The 5 smallest and the 5 largest optimal costs for Supply Chain S3 are shown in Table 6.8.

System cycle time N (days)	Optimal numbers of deliveries n_i^*'s	Optimal system cost per unit time TC_s^*
72	**(1,2,2,3,3,3,3,4,4,4)**	**114,215.70**
84	(2,2,3,3,3,4,4,4,4,4)	114,531.60
76	(1,2,2,2,4,4,4,4,4,4)	114,692.10
80	(1,2,2,2,4,4,4,4,4,5)	114,740.60
66	(1,2,2,2,3,3,3,3,3,3)	114,888.80
5	(1,1,1,1,1,1,1,1,1,1)	600,665.30
4	(1,1,1,1,1,1,1,1,1,1)	747,437.50
3	(1,1,1,1,1,1,1,1,1,1)	993,064.40
2	(1,1,1,1,1,1,1,1,1,1)	1,485,827.00
1	(1,1,1,1,1,1,1,1,1,1)	2,967,132.00

Table 6.8: The 5 smallest and the 5 largest optimal costs for Supply Chain S3.

The overall optimal solution is $N^* = 72$ days, n_i^*'s = (1,2,2,3,3,3,3,4,4,4) and the overall optimal cost is $TC_s^* = \$114,215.70$.

Supply Chain S4 has the same parameters as Supply Chain S3 except that the production set up cost is \$1,000. The overall optimal solution for Supply Chain S4 is found as $N^* = 36$ days, n_i^*'s = (1,1,1,1,1,2,2,2,2,2), and $TC_s^* = \$81,382.24$ per year. This set of deliveries is also the optimal set of deliveries for $N = 36$ days for Supply

Chain S3 because for a given system time, the optimal set of deliveries is independent of the production set up cost. The corresponding cost for Supply Chain S3 is $132,076.70 and ranks 85th among the 365 optimal costs of S3. The figure 132,076.70 can also be obtained from 81382.24 as 81382.24 + (6000-1000)(365)/36 = 132076.70. When other parameters are the same, changing the value of production set up cost can have a large impact on the relative ranking of optimal costs since a larger set up cost favours longer system cycle times. In this case, the system cycle times for the optimal costs ranking 1st to 84th for S3 are all shorter than 36 days.

6.4.3 Example 6.4 (20-buyer Example)

In Supply Chain S5, a vendor supplies an exponentially deteriorating product, with a deterioration rate of 0.1 per year to 20 buyers. The production rate is 1,100,000 units per year while the total demand rate is 536,950 units per year for the 20 buyers. The vendor has a production set up cost S of $10,000 for each set up. The vendor's order processing & delivery costs A_{vi} for the 20 buyers are: $164, $175, $136, $178, $112, $176, $126, $125, $160, $159, $162, $136, $108, $108, $183, $146, $107, $182, $179, and $180, respectively. The vendor has a deterioration cost C_v of $10 per unit and an inventory holding cost H_v of $1 per unit per year. The demand rates and the cost parameters of the 20 buyers are shown in Table 6.9. (These parameters are obtained by randomizing over certain ranges of values.)

Buyer i	Demand rate D_i (units / year)	Ordering cost A_{bi} ($)	Deterioration cost C_{bi} ($/unit)	Inventory holding cost H_{bi} ($/unit/year)
1	11,300	90	19	1.90
2	27,500	69	18	1.98
3	27,950	50	16	2.56
4	10,850	82	18	1.98
5	44,150	60	16	1.76
6	14,450	52	15	1.80
7	29,750	62	14	2.52
8	28,400	90	12	2.04
9	47,750	53	13	2.34
10	8,600	52	15	2.55
11	43,250	90	14	2.24
12	48,650	59	15	1.50
13	12,650	70	17	2.21
14	41,450	76	13	1.82
15	13,100	86	20	2.00
16	26,150	79	14	2.24
17	25,700	77	14	1.96
18	13,550	59	16	1.92
19	37,400	72	15	2.70
20	24,350	77	15	1.80

Table 6.9: Buyers' parameters for Example 6.4.

The overall optimal solution for Supply Chain S5 is $N^* = 72$ days,

n_i^*'s = (2,3,4,2,4,2,4,3,4,2,4,4,2,4,2,3,3,2,4,3), and $TC_s^* = \$230,296.10$ per year.

The 5 smallest and the 5 largest optimal costs for Supply Chain S5 are shown in Table 6.10.

System cycle time N (days)	Optimal numbers of deliveries n_i^*'s	Optimal system cost per unit time TC_s^*
72	**(2,3,4,2,4,2,4,3,4,2,4,4,2,4,2,3,3,2,4,3)**	**230,296.10**
66	(2,3,3,2,3,2,3,3,3,2,3,3,2,3,2,3,3,2,3,2)	230,882.90
76	(2,4,4,2,4,2,4,4,4,2,4,4,2,4,2,4,4,2,4,2)	231,435.80
80	(2,4,4,2,5,2,4,4,4,2,4,4,2,4,2,4,4,2,4,2)	231,862.70
68	(2,2,4,2,4,2,4,2,4,2,4,4,2,4,2,2,4,2,4,2)	231,969.30
359	(1,1,1,1,359,1,1,1,359,1,1,359,1,359,1,1,1,1,1,1,)	1,245,113.00
4	(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,)	1,328,033.00
3	(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,)	1,762,896.00
2	(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,)	2,635,972.00
1	(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,)	5,261,902.00

Table 6.10: The 5 smallest and the 5 largest optimal costs for Supply Chain S5.

Supply Chain S6 has the same parameters as Supply Chain S5 except that the production rate is 1,600,000 units per year, that is, DP ratio changes from approximately 1/2 for S5 to approximately 1/3 for S6. The overall optimal solution for Supply Chain S6 is $N^* = 60$ days, n_i^*'s = (1,2,3,1,3,2,3,2,3,1,3,3,2,3,2,2,2,2,3,2), and the overall optimal cost is $TC_s^* = \$234,682.50$. Hence, Supply Chain S6 has a larger overall optimal cost than Supply Chain S5.

P (units/year)	N (days)	No. of deliveries n_i's	Total cost per unit time TC_s (\$/year)
1,600,000	60	(1,2,3,1,3,2,3,2,3,1,3,3,2,3,2,2,2,2,3,2)	234,682.50 (i)
1,100,000	60	(1,2,3,1,3,2,3,2,3,1,3,3,2,3,2,2,2,2,3,2)	232,728.03 (ii)
1,100,000	60	(2,2,3,2,3,2,3,2,3,2,3,3,2,3,2,2,3,2,3,2)	232,191.40 (iii)
1,100,000	72	(2,3,4,2,4,2,4,3,4,2,4,4,2,4,2,3,3,2,4,3)	230,296.10 (iv)

Table 6.11: Cost Comparison for Supply Chains S5 ($P = 1,100,000$) and S6.

Table 6.11 indicates:

(i) the overall optimal cost for Supply Chain S6,

(ii) the cost for Supply Chain S5 with the same N and n_i's,

(iii) the optimal cost for Supply Chain S5 with $N = 60$ days,

(iv) the overall optimal cost for Supply Chain S5.

The increase in overall optimal cost when the production rate is increased from 1,100,000 units per year for S5 to 1,600,000 units per year for S6 can partly be explained from the findings in Chapter 4. The summary in Section 4.2.4 indicates that, when the system cycle time and the delivery cycle time are unchanged, cost increases when production rate decreases for $n = 1$; cost increases when production rate increases for $n \geq 3$; and cost increases when production rate increases for $n = 2$, if the production rate is sufficiently large.

The overall optimal solution for Supply Chain S6 is that $N^* = 60$ days and the numbers of deliveries are one each for 3 buyers (buyers 1, 4, 10), and are two or more for the other 17 buyers. When the production rate is reduced from 1,600,000 to 1,100,000 units per year keeping the same set of deliveries, the three one's increase the costs while the others reduce the cost. The approximate cost functions $\varnothing_1$, $\varnothing_4$ and $\varnothing_{10}$ increase and the other $\varnothing_i$'s decrease, resulting in an overall decrease in $\sum \varnothing_i$. It is found that there is a larger cost reduction due to the adjustment from approximate cost TC_s^{ς} to actual cost TC_s (Section 6.2.4.2) when the production rate is reduced from 1,600,000 to 1,100,000 units per year time. Hence, the overall optimal cost for S6 (cost (i) in Table 6.11) is already larger than cost (ii), the cost for S5 with the same *N* (60 days) and same n_i's. Cost (ii), the cost for a particular feasible solution for S5 with $N = 60$, is larger than cost (iii) which is the optimal cost for S5 with the same *N*. Finally, cost (iv), the overall optimal cost for S5, is smaller than cost (iii) and this concludes that Supply Chain S5 has a lower overall optimal cost than Supply Chain S6.

6.5 Genetic Algorithm and Comparison of Results

6.5.1 Introduction to Genetic Algorithm

The term genetic algorithm (GA) was first used by Holland (1975) in his book Adaption in Natural and Artificial Systems. A genetic search algorithm is a heuristic search process that resembles natural selection and the "survival of the fitness". Although there are various variations and refinements for GA, any genetic algorithm has the features of reproduction, crossover and mutation. GA has become a popular tool for tackling optimization problems. When it is applied, the algorithm starts with selecting a population which is a set of feasible solutions of the problem under investigation. A member of the population is known as a genotype, a chromosome, or a sting, and is a permutation which corresponds to the order of the decision variables of the objective function. From each string, its fitness value, i.e., the corresponding value of the objective function, can be calculated. Offsprings are produced by means of crossovers among members of the population or mutations of members of the population. The fitness values of new offsprings are found and the population is updated by removing the worst string(s) from the population. When the stopping criterion is met, the process stops and the string that has the highest fitness value in the population is the final solution of the problem. When an optimization problem has been defined (objective function, decision variables, constraints), population size, mutation rate, crossover rate, and stopping criterion are parameters to be defined before applying the genetic algorithm to tackle the problem.

6.5.2 Application of GA to the Model

In this model, the decision variables are the system cycle time N, the numbers of deliveries for the buyers, n_i, $i = 1, 2, ..., n_b$. The length of a string is $n_b + 1$. For example, for a 5-buyer supply chain, the string has a length of 6 and is represented as in Figure 6.2. The fitness value of a string is the total system cost per unit time TC_s.

N	n_1	n_2	n_3	n_4	n_5

Figure 6.2: A string for a 5-buyer supply chain.

The selected values of the parameters for applying GA to the model are as follows:

Population size: 40 Number of Runs: 20

Crossover rate: 0.9 Mutation rate: 0.1

Stopping criterion: 500 iterations without cost improvement

The procedure for the process of GA is as follows:

Step 1: An initial population of size 40 is generated by a random process. The fitness values of the strings in the initial production are found.

Step 2: A pair of parents is randomly selected from the population.

Step 3: (a) If the system cycle times of the two strings chosen in Step 2 are the same, a random number r_1 is generated. If r_1 is less than the crossover rate, two new offsprings are produced after crossover. Otherwise, a new random number r_2 is generated. If r_2 is less than the mutation rate, mutation will be conducted. Two random numbers are generated for producing two new offsprings by mutation. (Even if crossover has been done, a random number will be generated to see if mutation follows.) (b) If the system cycle times of the two strings chosen in Step 2 are not the same, crossover cannot be done as there may be uncommon factors in the two strings. So a random number r_3 is generated. If r_3 is less than the mutation rate, then mutation will be conducted.

Step 4: Repeat Step 2 and Step 3 twenty times for producing 40 offsprings.

Step 5: The new offsprings are combined with the population and the top 40 strings according to their fitness values are kept as the updated population. Check whether there is cost improvement in the updated population against the previous one.

Step 6: Repeat Step 2 to Step 5 until the stopping criterion is met.

Step 7: Repeat the above steps 20 times, i.e., 20 runs. The least-cost solution from the 20 runs is taken as the final solution.

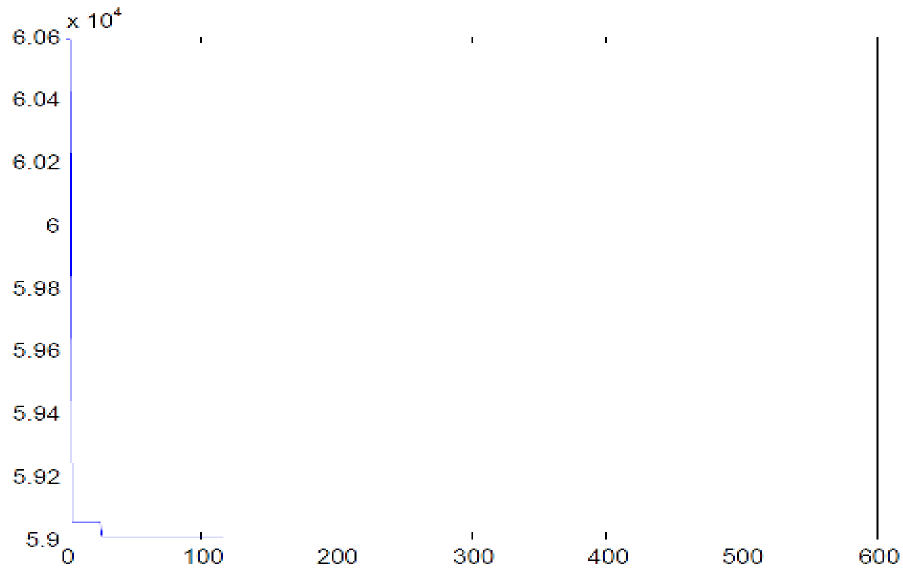


Figure 6.3: GA for 5-buyer.

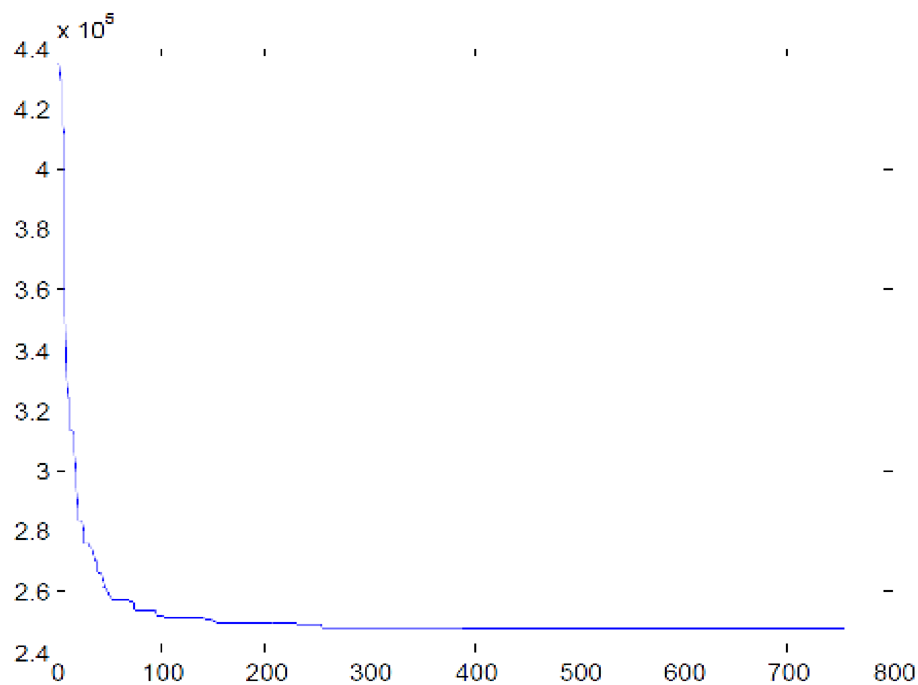


Figure 6.4: GA for 20-buyer.

6.5.3 Comparison of Results

The above GA procedure has been applied to Supply Chains S1 to S6. The best costs from GA, TC_s^{GA}, and the corresponding system cycle times, N^{GA}, are shown in Table 6.12 together with the overall optimal costs and system cycle times found from the algorithm presented in Section 6.3. The percentage difference of the costs is computed as: % difference = $\dfrac{TC_s^{GA} - TC_s^*}{TC_s^*} \times (100)$.

Supply Chain	Optimal solution from the algorithm		Best solution from GA		% difference of the costs
	N^*	TC_s^* (\$/year)	N^{GA}	TC_s^{GA} (\$/year)	
S1	44	45,910.20	42	45,943.44	0.0724
S2	40	44,224.63	42	44,248.69	0.0544
S3	72	114,215.70	72	114,215.70	0
S4	36	81,382.24	32	81,687.06	0.3745
S5	72	230,296.10	72	230,296.10	0
S6	60	234,682.50	66	234,733.49	0.0217

Table 6:12: Comparison of costs from the algorithm and GA.

It is found that:

(i) The best costs from GA are either equal to, or slightly higher than, the overall optimal costs from the algorithm presented in Section 6.3.

(ii) If $N^{GA} = N^*$, GA yields the same set of deliveries and hence the same optimal cost as the algorithm.

(iii) If $N^{GA} \neq N^*$, GA yields the same set of deliveries as the optimal solution for $N = N^{GA}$ from the algorithm, and therefore, TC_s^{GA} is equal to the optimal cost for $N = N^{GA}$ from the algorithm.

The algorithm and the above GA procedure have been applied to other supply chains, and the same findings are noted. The comparison of the optimal solutions and the best solutions from the two algorithms for these supply chains is shown in Appendix B. The experimental results suggest that this GA procedure can at least yield a good solution, if not the optimal solution, whose cost is close to the overall optimal cost.

When the algorithm presented in Section 6.3 is run on computer with MATLAB (R2009a), the results are almost instantaneously shown on the command window. For GA, the average running time for a 20-buyer supply chain is approximately 20 seconds. An experiment has been done for a 40-buyer supply chain. It takes approximately 37 seconds for GA while the results obtained from the algorithm are also almost instantaneously shown on the command window. Both the algorithm and GA experiments are run on a computer with an Intel (R) Core (TM) 2 Quad CPU Q9550 of speed 2.83 GHz.

6.6 Comparison with Common Cycle Model

6.6.1 The Model and Solution Procedure

In this section, the optimal costs obtained from the model presented will be compared with that obtained from the general common cycle model. In the general common cycle model, all the buyers have the same delivery interval and the vendor delivers to every buyer at the same time for all shipments. Time is considered a continuous variable. The system cycle time and the delivery interval are not integer multiples of a convenient time unit.

Let there be *n* common deliveries in a system cycle of time *T*. From equations (6.7), (6.8) and (6.9), the following equations are obtained:

Total cost for the buyers per unit time is given by

$$\sum_{i=1}^{n_b} TC_{bi} = \sum_{i=1}^{n_b} \{\frac{nA_{bi}}{T} + (H_{bi} + kC_{bi})\frac{nD_i}{kT}[\frac{1}{k}(e^{\frac{kT}{n}} - 1) - \frac{T}{n}]\}$$
$$= \sum_{i=1}^{n_b} \{\frac{nA_{bi}}{T} + (H_{bi} + kC_{bi})\frac{nD_i}{kT}[\frac{1}{k}(e^{\frac{kT}{n}} - 1)] - (H_{bi} + kC_{bi})\frac{D_i}{k}]\}$$

Total cost for the vendor per unit time is given by

$$TC_v = \frac{S}{T} + \sum_{i=1}^{n_b} \frac{nA_{vi}}{T} + (H_v + kC_v)\{\frac{1}{kT}(PT_p - \sum_{i=1}^{n_b} D_i T) - \sum_{i=1}^{n_b} \frac{nD_i}{kT}[\frac{1}{k}(e^{\frac{kT}{n}} - 1) - \frac{T}{n}]\}$$
$$= \frac{S}{T} + \sum_{i=1}^{n_b} \frac{nA_{vi}}{T} + (H_v + kC_v)\{\frac{1}{kT}PT_p - \sum_{i=1}^{n_b} \frac{nD_i}{kT}[\frac{1}{k}(e^{\frac{kT}{n}} - 1)]\}$$

Total system cost per unit time

$$TC_s = \frac{S}{T} + \sum_{i=1}^{n_b} \frac{n(A_{vi}+A_{bi})}{T} + \{\sum_{i=1}^{n_b} \frac{(H_{bi}-H_v+kC_{bi}-kC_v)}{kT} nD_i [\frac{1}{k}(e^{\frac{kT}{n}}-1)]\}$$
$$+\frac{(H_v+kC_v)PT_p}{kT} - \sum_{i=1}^{n_b}\frac{(H_{bi}+kC_{bi})}{k}D_i \qquad (6.12)$$

where $T_p = \frac{1}{k}\ln\{1+\frac{\sum_{i=1}^{n_b}\frac{D_i}{P}(e^{kT}-1)}{1-\sum_{i=1}^{n_b}\frac{D_i}{P}(e^{\frac{kT}{n}}-1)}\}$.

The objective is to find minimize TC_s by finding the optimal number of deliveries and the optimal system cycle time. Equation (6.12) can be simplified as follows:

$$TC_s = \frac{S}{T} + \frac{nCP_1}{T} + \frac{nCP_2}{k^2 T}(e^{\frac{kT}{n}}-1) + \frac{(H_v+kC_v)PT_p}{kT} - \frac{CP_3}{k}, \qquad (6.13)$$

where $CP_1 = \sum_{i=1}^{n_b}(A_{vi}+A_{bi})$;

$CP_2 = \sum_{i=1}^{n_b}(H_{bi}-H_v+kC_{bi}-kC_v)D_i$;

$CP_3 = \sum_{i=1}^{n_b}(H_{bi}+kC_{bi})D_i$;

$TDP = \sum_{i=1}^{n_b} D_i / P$; and $T_p = \frac{1}{k}\ln[1+\frac{TDP(e^{kT}-1)}{1-TDP(e^{\frac{kT}{n}}-1)}]$.

Assume the common number of deliveries does not exceed a certain value $NMAX$, say, 365, and the system cycle time does not exceed one year, the optimal solution can be obtained by the following procedure:

Step 1: Find the values for CP_1, CP_2, CP_3, and TDP from the parameters of the supply chain and set an appropriate value for $NMAX$.

Step 2: For $n = 1$ to $NMAX$, find the value of T within the interval (0,1) that minimizes TC_s in equation (6.13) by MATLAB.

Step 3: Find the minimum cost among the $NMAX$ TC_s's obtained in Step 2.

6.6.2 Comparison of Results

The optimal numbers of deliveries, n^{CC} and the optimal costs, TC_s^{CC} for Supply Chains S1 to S6 adopting the common cycle approach are shown in Table 6.13 together with the optimal solutions obtained from the algorithm presented in Section 6.3. The percentage difference of the costs is found as:

$$\% \text{ difference} = \frac{TC_s^{CC} - TC_s^{*}}{TC_s^{*}} \times (100)$$

Supply Chain	Optimal solution from the algorithm	Optimal solution from common cycle		% difference of the costs
	TC_s^* (\$/year)	n^{CC}	TC_s^{CC} (\$/year)	
S1	45,910.20	2	46,392.74	1.05
S2	44,224.63	2	45,124.44	2.03
S3	114,215.70	3	116,831.90	2.29
S4	81,382.24	1	83,097.97	2.11
S5	230,296.10	3	234,476.30	1.82
S6	234,682.50	2	237,732.40	1.30

Table 6:13: Comparison of costs from the algorithm and the common cycle approach.

Although the synchronized model requires that the system cycle time and the delivery intervals must be integer multiples of a convenient time unit, the model allows the buyers to have uncommon delivery intervals that are primarily based on their own demand rates and cost parameters. The results indicate that the model can result in cost saving compared with the common cycle approach.

6.7 Conclusion

Chan and Kingsman (2005, 2007) developed a synchronized delivery and production cycle model for single-vendor multi-buyer supply chains. In the model, the system cycle time and delivery cycle times are integer multiples of a convenient time unit. In this chapter, the model is extended to supply chains of exponentially deteriorating items. Day is selected as the time unit and the maximum system cycle time is one year, i.e., 365 days.

As the production time for multiple buyers is not just the sum of production times for individual production of each of the buyers, the optimal solution for a given system cycle time cannot be obtained just by independent optimization of each "vendor-buyer" subsystem. A two-stage algorithm has been developed for the solution of the extended model. The first stage is to find an initial solution by independent optimization of each "vendor-buyer" subsystem for a given system cycle time N. The second stage is to find the optimal solution for N by increasing the numbers of deliveries until no cost improvement can be achieved. The overall optimal solution is the least–cost solution among the 365 optimal solutions. As an appropriate time unit has been selected beforehand, the overall optimal solution obtained from this model can be directly implemented in practice.

The results of this two-stage algorithm are compared with that obtained from genetic algorithm (GA). It is found that GA provide either the same solutions as the two-stage algorithm; or solutions whose costs are close to, but not better than, that obtained from the two-stage algorithm. The results of this algorithm are also compared with that obtained from the common cycle approach in which time is considered as a continuous variable. It is found that the model can still provide lower optimal costs than the common cycle approach in spite of the constraints on the system and delivery cycle times. In practice, the solution from the common cycle approach has to be rounded to a certain time unit before implementation. As a result, the actual cost will be increased and the cost saving by using the model developed in this chapter will be larger.

Chapter 7

Extended Models for the Synchronized Model

7.1 Introduction

The prime objective of supply chain management has been the cost/profit optimization of the system. There have been increasing environmental concerns in the society and supply chain management has to consider environmental performance measures in addition to the cost/financial performance. Deterioration results in wastage of resources and is adverse to environmental protection. Reducing the amount of deterioration helps to address the issue of environmental concerns. The management of a supply chain of deteriorating items can consider the strategy of controlling the amount of deterioration of the supply chain within a certain level. It will be beneficial to the supply chain if this strategy can be implemented in the most cost effective way. In Chapter 6, a synchronized model for exponentially deterioration items has been presented. The objective of the model is to minimize the total system cost per unit time. In this chapter, the cost optimization algorithm for the synchronized model is modified so as to find the minimum cost solution of the supply chain subject to a predefined maximum deterioration constraint.

For predetermined production rate models, the production facilities have a low utilization when the production rates are much higher than the demand rates. Among the products that a company produces, some may be very similar (for example, just changes in some ingredients) and employ basically the same production process and do not require much time in resetting the process when the production is changed from one product to another. When the production capacity of a production line is large enough, it may be feasible to produce two such products on the same production line. Therefore, the utilization of the production facilities can be increased. In this chapter, a model of producing two similar deteriorating items on the same production line for a single buyer (or for two buyers, one for each product) is proposed. Two heuristics have been developed for this model. The first heuristic considers time as a continuous variable and the second one is modified from the synchronized model.

7.2 Cost Minimization Model subject to a Maximum Deterioration Constraint

7.2.1 Model Development

In this model there is a constraint that the total deteriorated quantity cannot exceed a certain proportion of the production quantity per unit time (a year). The system cycle time is an integer multiple of the convenient time unit (day) and the numbers of deliveries are factors of the system cycle time, as in the synchronized model in Chapter 6.

Suppose the defined maximum deterioration proportion is MAXDET1. Since production quantity is the sum of demand and deteriorated quantity, from $\frac{\textbf{deteriorated quantity}}{\textbf{production quantity}} \le \textbf{MAXDET1}$, the following can be obtained:

$$\frac{\textbf{deteriorated quantity}}{\textbf{demand + deteriorated quantity}} \le \textbf{MAXDET1}$$

$$\frac{\textbf{demand}}{\textbf{deteriorated quantity}} + 1 \ge \frac{1}{\textbf{MAXDET1}}$$

$$\frac{\textbf{deteriorated quantity}}{\textbf{demand}} \le \frac{1}{\frac{1}{\textbf{MAXDET1}} - 1} = \textbf{MAXDET2}\ . \qquad (7.1)$$

Hence, once MAXDET1 is set, MAXDET2 can be found and the feasible solution must satisfy the following constraint:

deteriorated quantity over a year $\le$ (annual demand)(MAXDET2)

Deteriorated quantity is the total production quantity minus total demand per unit time, i.e.,

$$\text{deteriorated quantity} = \frac{PT_p - \sum_{i=1}^{n_b} D_i T}{T} = \frac{PT_p}{T} - \sum_{i=1}^{n_b} D_i \qquad (7.2)$$

So for a given system cycle time *T*, deteriorated quantity is minimized when the production time is the shortest. From equation (6.6), it is obvious that production time is the minimum when the number of delivery is the largest, that is, when daily deliveries are made. For example, if the system cycle time is 10/365 (10 days), deterioration is the minimum when there are 10 deliveries for all buyers. As $T = N / 365$ and $n_i = N$ for all buyers, for minimum deterioration for a given *T*,

$$\frac{PT_p}{T} = \frac{P}{kT}\ln\{1+\frac{\sum_{i=1}^{n_b}\frac{D_i}{P}(e^{kT}-1)}{1-\sum_{i=1}^{n_b}\frac{D_i}{P}(e^{\frac{kT}{n_i}}-1)}\} = \frac{P}{kT}\ln\{1+\frac{\sum_{i=1}^{n_b}\frac{D_i}{P}(e^{kT}-1)}{1-\sum_{i=1}^{n_b}\frac{D_i}{P}(e^{\frac{k}{365}}-1)}\}. \tag{7.3}$$

It can be shown that $\frac{PT_p}{T}$ is an increasing function of T as follows:

$$\frac{T_p}{T} = \frac{1}{kT}\ln\{1+\frac{\sum_{i=1}^{n_b}\frac{D_i}{P}(e^{kT}-1)}{1-\sum_{i=1}^{n_b}\frac{D_i}{P}(e^{\frac{k}{365}}-1)}\} = \frac{1}{u}\ln[1+C(e^u-1)] \text{ where } u = kT > 0 \text{ and}$$

$$C = \frac{\sum_{i=1}^{n_b}\frac{D_i}{P}}{1-\sum_{i=1}^{n_b}\frac{D_i}{P}(e^{\frac{k}{365}}-1)}, \text{ a positive constant}$$

$$\frac{d}{du}(\frac{T_p}{T}) = \frac{u\frac{Ce^u}{1+C(e^u-1)}-\ln[1+C(e^u-1)]}{u^2} = \frac{uCe^u-[1+C(e^u-1)]\ln[1+C(e^u-1)]}{u^2[1+C(e^u-1)]}$$

The denominator of the above derivative is always positive when $u > 0$.

Consider the numerator of the derivative. When $u = 0$, the numerator is 0.

$$\frac{d}{du}\{uCe^u-[1+C(e^u-1)]\ln[1+C(e^u-1)]\}$$

$$= Cue^u + Ce^u - Ce^u\ln[1+C(e^u-1)] - \frac{[1+C(e^u-1)]}{[1+C(e^u-1)]}Ce^u$$

$$= Ce^u\{u-\ln[1+C(e^u-1)]\}$$

$$= Ce^u\{kT-\ln[1+\frac{\sum_{i=1}^{n_b}\frac{D_i}{P}(e^{kT}-1)}{1-\sum_{i=1}^{n_b}\frac{D_i}{P}(e^{\frac{k}{365}}-1)}]\} = Ce^u(kT-kT_p) > 0$$

Hence, $\frac{d}{du}(\frac{T_p}{T}) > 0$ as $u > 0$ and therefore, $\frac{PT_p}{T}$ is an increasing function of *T*.

This implies that the minimum deteriorated quantity is also an increasing function of the system cycle time. Therefore, from the above discussion, we have the following proposition:

<u>Proposition 7.1</u>

(i) For a given system cycle time, if daily deliveries cannot meet the maximum deterioration constraint, there will be no feasible solutions with that system cycle time.

(ii) If a certain system cycle time T_1 is infeasible, then any system cycle time bigger than T_1 is also infeasible.

The objective of the model is to minimize TC_s given by equation (6.9) subject to the constraints: $N \in \{1, 2, ..., 365\}$, n_i's are integer factors of *N*, $i = 1, 2, ..., n_b$, and

$$\frac{PT_p}{T} - \sum_{i=1}^{n_b} D_i \leq \frac{1}{\frac{1}{\text{MAXDET1}} - 1}(\sum_{i=1}^{n_b} D_i) \text{ for a given MAXDET1} < 1.$$

If feasible solution(s) exist(s), the least cost solution that meets the maximum deterioration constraint can be found as follows:

1. Find the range of system cycle time(s) that feasible solution(s) exist(s). Proceed with the following steps for each of the feasible system cycle time(s).

2. Find the minimum cost solution $(n_1^*,\ n_2^*, ...,\ n_{n_b}^*)$ without the maximum deterioration constraint by using the algorithm on pages 151-153 of Chapter 6. Check if this solution meets the maximum deterioration constraint. If the constraint is met, this is the least cost feasible solution for that system cycle time.

3. If the constraint is not met, find the least cost feasible solution $(n^\chi,\ n^\chi, ...,\ n^\chi)$ where n^χ is also a factor of the concerned system cycle time, by adopting the common cycle approach.

4. Set $n_i = \max(n_i^*, n^\chi)$, i.e., increase the number(s) of deliveries for the buyer(s) from n^χ to n_i^* if $n_i^* > n^\chi$ for some buyer(s). This solution, if different from $(n^\chi,\ n^\chi, ...,\ n^\chi)$, has a lower cost without violating the maximum deterioration constraint (actually deterioration is reduced as the number(s) of deliveries is(are) increased). This is the initial feasible solution for later iterations. Consider only reducing number(s) of deliveries for buyers with $n_i = n^\chi > n_i^*$.

5. For buyer(s) whose number(s) of deliveries is larger than n_i^*, reduce the number of delivery to the next smaller factor of N of one buyer at a time, check feasibility and costs to find the minimum cost solution that meets the maximum deterioration constraint. This step may take several iterations to reach the least cost feasible solution for that system cycle time.

6. Among the least cost solutions found for all the feasible system cycle times, the one having the smallest cost is the final solution for the problem.

If $n_i = n^{\chi} > n_i^*$, reducing the number of delivery will reduce the cost (as getting closer to n_i^*) but increase the deteriorated quantity. Hence, feasibility has to be checked before finding the resulting cost so as to determine if a lower cost feasible solution can be obtained. If $n_i = n_i^* > n^{\chi}$, reducing the number of delivery will increase both the cost and the deteriorated quantity, and therefore is not considered. The detailed algorithm is shown in the next section.

7.2.2 The Algorithm

<u>Algorithm 7.2A</u>

Step 1: Set the value for MAXDET1, find MAXDET2 and the maximum deteriorated quantity over a year.

Step 2: Set $T = 1/365$ and find the deteriorated quantity by equation (7.2).

If deteriorated quantity > (annual demand)(MAXDET2) , there is no feasible solution. Stop. Otherwise, go to Step 3.

Step 3: Solve $\dfrac{P}{kT}\ln\{1+\dfrac{\sum_{i=1}^{n_b}\frac{D_i}{P}(e^{kT}-1)}{1-\sum_{i=1}^{n_b}\frac{D_i}{P}(e^{\frac{k}{365}}-1)}\}-\sum_{i=1}^{n_b}D_i = \left(\mathbf{MAXDET2}\right)\sum_{i=1}^{n_b}D_i$.

Suppose the solution is $T = T_m$. Set $m = \min(\lfloor 365T_m \rfloor, 365)$.

Step 4: For N=1 to m, find all the integer factors of N. Set $T = N / 365$ and find the minimum cost solution $(n_1^*, n_2^*, ..., n_{n_b}^*)$ using the minimum cost algorithm in Chapter 6 and find the corresponding deteriorated quantity.

Step 5: If deteriorated quantity <= (annual demand)(MAXDET2), this is the solution for the concerned N. Go back to Step 4 for next N. Otherwise, go to Step 6.

Step 6: Find the minimum cost common cycle solution, where the number of delivery n^{CC} must be a factor of N, by using the sub-algorithm (Algorithm 7.2B), and find the corresponding deteriorated quantity.

Step 7: If deteriorated quantity ≤ (annual demand)(MAXDET2), set $n^{\chi} = n^{CC}$ and go to Step 8. Otherwise, increase the number of delivery, keeping as factors of N, till det. quantity ≤ (annual demand)(MAXDET2) is just obtained. Denote this common number of delivery be n^{χ}.

Step 8: Set $n_i = \max(n^{\chi}, n_i^*)$, for $i = 1, 2, ..., n_b$. This is the initial solution for the following iteration.

Step 9: For $i = 1, 2, ..., n_b$ if $n_i = n_i^*$, next i. Otherwise, go to Step 10.

Step 10: Let p_i be the factor of N that is just smaller than n_i. Check feasibility for the solution with p_i and other n_i 's with respect to the maximum deterioration constraint. (a) Find the corresponding cost if feasible. Go back to Step 9 for next i. (b) If infeasible, it means that the number of delivery cannot be smaller than n_i for that buyer, record that this particular n_i cannot be reduced. Go back to Step 9 for next i.

Step 11: Find the solution with the minimum cost among the feasible solution(s) obtained in Step 10. Update the value of the corresponding n_i by the p_i that gives the minimum cost solution. (For example, if the solution with p_3 is the minimum cost feasible solution among the solutions obtained in Step 10, the updated value of n_3 is p_3, and the values of other n_i 's are unchanged.) Check whether in the minimum cost solution, all the numbers of deliveries are either n_i^* or cannot be reduced. If yes, this is the minimum cost feasible solution for the concerned *N*. Go back to Step 4 for next *N*. Otherwise, go to Step 12.

Step 12: Repeat Step 9 to Step 11 until all the numbers of deliveries are either n_i^* or cannot be reduced. This gives the minimum cost solution that meets the maximum deterioration constraint for the concerned *N*. Go back to Step 4 for next *N*.

Step 13: Find the least cost solution among the *m* solutions obtained in the above steps.

<u>Algorithm 7.2B</u>

The sub-algorithm for finding the minimum cost common cycle solution is shown as follows:

Let $CP_1 = \sum_{i=1}^{n_b}(A_{vi} + A_{bi})$; $CP_2 = \sum_{i=1}^{n_b}(H_{bi} - H_v + kC_{bi} - kC_v)D_i$; $CP_3 = \sum_{i=1}^{n_b}(H_{bi} + kC_{bi})D_i$;

$TDP = \sum_{i=1}^{n_b} D_i / P$; and $T_p = \frac{1}{k}\ln[1 + \frac{TDP(e^{kT} - 1)}{1 - TDP(e^{\frac{kT}{n}} - 1)}]$ and hence

$TC_s = \frac{S}{T} + \frac{nCP_1}{T} + \frac{nCP_2}{k^2T}(e^{\frac{kT}{n}} - 1) + \frac{(H_v + kC_v)PT_p}{kT} - \frac{CP_3}{k}$ as in Section 6.6.1.

Define $\varnothing(x) = \frac{xCP_1}{T} + \frac{CP_2 x}{kT}[\frac{1}{k}(e^{\frac{kT}{x}} - 1)] + \frac{(H_v + kC_v)(PT_p)}{kT} - \frac{CP_3}{k}$, and

$$f(x) = \frac{CP_2}{k^2}(\frac{kT}{x}e^{\frac{kT}{x}} - e^{\frac{kT}{x}} + 1) + \frac{(\sum D_i)^2 T(H_v + kC_v)e^{\frac{kT}{x}}}{Pkx^2}\frac{(e^{kT} - 1)}{[1 + TDP(e^{kT} - e^{\frac{kT}{x}})][1 - TDP(e^{\frac{kT}{x}} - 1)]} - CP_1.$$

Modify the sub-algorithm for finding the initial solution for a fixed *N* in Section 6.3 as follows:

Step 1: Find the integer factors of *N* and set $T = N / 365$.

Step 2: If $f(1) \le 0$ set $n^{CC} = 1$. Go to Step 7.

Step 3: If $f(N) \ge 0$, set $n^{CC} = N$. Go to Step 7.

Step 4: Find the minimum of $\varnothing(x)$. Suppose $\varnothing(x)$ is minimum when $x = x^*$.

If x^* is an integer and is a factor of *N*, set $n^{CC} = x^*$. Go to Step 7.

Step 5: Find α and β, the two consecutive factors of N such that $\alpha < x^* < \beta$.

Step 6: Calculate $\varnothing(\alpha)$ and $\varnothing(\beta)$. If $\varnothing_i(\alpha) < \varnothing_i(\beta)$, set $n^{CC} = \alpha$. Otherwise, set $n^{CC} = \beta$.

Step 7: Find T_p and TC_s with $T = N/365$ and $n = n^{CC}$.

7.2.3 Example 7.1

Consider Supply Chain S1 in Example 6.2 of Chapter 6.

Set MAXDET1 = 0.01. Then MAXDET2 = 0.010101 and $m = 145$ (found by MATLAB). The feasible system cycle times are 1, 2,…, 145 days.

For $T = 120/365$ as an illustration, the minimum cost solution without maximum deterioration constraint is $(n_1^*, n_2^*, ..., n_5^*) = (3,4,5,6,6)$ with a cost of \$56,011.61 per year and a production time of 0.166267 year per cycle.

$$\text{deteriorated quantity} = \frac{PT_p - \sum_{i=1}^{n_b} D_i T}{T} = \frac{300000(0.1662665)}{120/365} - 150000 = 1718.2$$

$$\frac{\text{deteriorated quantity}}{\text{demand}} = \frac{1718.2}{150000} = 0.011455 > 0.010101$$

Hence the minimum cost solution does not meet the maximum deterioration constraint. The minimum cost common cycle delivery solution that meets the maximum deterioration constraint is $(10,10,10,10,10)$ with a cost of \$63845.07 per year and deteriorated quantity / demand = 0.009867.

Checking against the minimum cost solution of $(3,4,5,6,6)$, the initial solution for iteration is $(10,10,10,10,10)$, and all the five numbers of deliveries are considered in the iteration as all the n_i^*'s are less than 10.

Iteration 1:

Numbers of Delivery	Cost per year ($)	Deteriorated quantity / annual demand	Feasible?
(8,10,10,10,10)	63031.81	0.009895	Yes
(10,8,10,10,10)	**63025.87**	**0.009923**	**Yes**
(10,10,8,10,10)	63036.41	0.009951	Yes
(10,10,10,8,10)	63063.44	0.009978	Yes
(10,10,10,10,8)	63106.94	0.010006	Yes

Table 7.1(a)

Iteration 2 [based on **(10,8,10,10,10)**]

Numbers of Delivery	Cost per year ($)	Deteriorated quantity / annual demand	Feasible?
(8,8,10,10,10)	**62212.62**	**0.009950**	**Yes**
(10,6,10,10,10)	62350.38	0.010015	Yes
(10,8,8,10,10)	62217.24	0.010006	Yes
(10,8,10,8,10)	62244.27	0.010033	Yes
(10,8,10,10,8)	62287.79	0.010061	Yes

Table 7.1(b)

Iteration 3 [based on **(8,8,10,10,10)**]

Numbers of Delivery	Cost per year ($)	Deteriorated quantity / annual demand	Feasible?
(6,8,10,10,10)	61465.72	0.009997	Yes
(8,6,10,10,10)	61537.15	0.010043	Yes
(8,8,8,10,10)	**61404.00**	**0.010033**	**Yes**
(8,8,10,8,10)	61431.04	0.010061	Yes
(8,8,10,10,8)	61474.56	0.010089	Yes

Table 7.1(c)

Iteration 4 [based on **(8,8,8,10,10)**]

Numbers of Delivery	Cost per year ($)	Deteriorated quantity / annual demand	Feasible?
(6,8,8,10,10)	**60657.12**	**0.010079**	**Yes**
(8,6,8,10,10)	60728.57	0.010126	No
(8,8,6,10,10)	60827.52	0.010172	No
(8,8,8,8,10)	60622.47	0.010144	No
(8,8,8,10,8)	60666.00	0.010171	No

Table 7.1(d)

Iteration 5: Delivery must be $(n,8,8,10,10)$ where $3 \le n \le 6$

Numbers of Delivery	Cost per year ($)	Deteriorated quantity / annual demand	Feasible?
(5,8,8,10,10)	60333.53	0.010116	No

Table 7.1(e)

Table 7.1(a)-(e) Iterations for Example 7.1.

During the iterations, when the number(s) of delivery is/are reduced, deterioration proportions increase, and costs decrease before n_i^*'s are reached. For the system cycle time of 120 days and the maximum deterioration constraint of MAXDET1 = 0.01, the minimum cost solution is (6,8,8,10,10) and the cost is \$60657.12. The minimum cost feasible solution for $N = 1, 2, ..., 145$ are found. It is found that the optimal solution for this problem is: $N^* = 44$ days, optimal numbers of deliveries: (1,2,2,2,2) and the total cost is \$45,910.20. This is also the overall optimal solution in Example 6.2(a) since this solution meets the maximum deterioration constraint (7.1) as follows:

$$\text{production time} = 0.060651 \text{ year per cycle}$$

$$\text{deteriorated quantity} = \frac{PT_p - \sum_{i=1}^{n_b} D_i T}{T} = \frac{300000(0.060651)}{44/365} - 150000 = 938.28$$

$$\frac{\text{deteriorated quantity}}{\text{demand}} = \frac{938.28}{150000} = 0.006255 < 0.010101 = \text{MAXDET2}$$

7.2.4 Example 7.2

Consider Supply Chain S3 in Example 6.3 of Chapter 6.

The overall optimal solution without the maximum deterioration constraint is given by: $N^* = 72$ days, n_i^*'s = (1,2,2,3,3,3,3,4,4,4) and the overall optimal cost is $TC_s^* = \$114,215.70$. The corresponding production time is 0.099430 year per cycle and it follows that deteriorated quantity / demand = 0.008106. Hence, this solution is also the least cost feasible solution for the maximum deterioration constraint if MAXDET2 is set at a value not less than 0.008106 which is equivalent to MAXDET1 being set at a value not less than 0.008041.

If the maximum deterioration constraint is set as MAXDET1 = 0.005, then MAXDET2 = 0.005025 and $m = 71$, which can be verified by equation (7.3).

Set $T = 71/365$, $\frac{PT_p}{T} = 276374.99$, $\frac{\text{deteriorated quantity}}{\text{demand}}$=0.005000 < 0.005025.

Set $T = 72/365$, $\frac{PT_p}{T} = 276393.82$, $\frac{\text{deteriorated quantity}}{\text{demand}}$=0.005068 > 0.005025.

$\therefore\ m = 71$.

With this maximum deterioration constraint, the least cost feasible solution is found as: $N = 44$, $n_i : (1,2,2,2,2,4,4,4,4,4)$, and $TC_s = \$131,064.90$ per year. The original optimal solution for $N = 44$ without the maximum deterioration constraint is $n_i : (1,1,1,2,2,2,2,2,2,2)$, and $TC_s = \$123,583.40$. The corresponding deteriorated quantity to demand ratio is 0.006379 and is feasible for a MAXDET1 of 0.006339. Comparing the two sets of deliveries, n_1, n_4, and n_5 are unchanged while the other seven n_i's are increased in order to meet the constraint. In Table 7.2, it is verified that all these increases in n_i's are necessary.

Numbers of Delivery	Production time per cycle (year)	Deteriorated quantity / annual demand	Feasible?
$(1,\mathbf{1},2,2,2,4,4,4,4,4)$	0.060582	0.005106	No
$(1,2,\mathbf{1},2,2,4,4,4,4,4)$	0.060585	0.005161	No
$(1,2,2,2,2,\mathbf{2},4,4,4,4)$	0.060585	0.005161	No
$(1,2,2,2,2,4,\mathbf{2},4,4,4)$	0.060587	0.005188	No
$(1,2,2,2,2,4,4,\mathbf{2},4,4)$	0.060588	0.005216	No
$(1,2,2,2,2,4,4,4,\mathbf{2},4)$	0.06059	0.005244	No
$(1,2,2,2,2,4,4,4,4,\mathbf{2})$	0.060592	0.005271	No

(3 is not a factor of 44. Hence, the numbers of deliveries decrease from 4 to 2.)

Table 7.2: Verification of the solution for Example 7.2.

When there is no maximum deterioration constraint or there is such a constraint with MAXDET1 being set at a value not less than 0.008041, the overall optimal cost is \$114,215.70 per year. The original optimal cost for $N = 44$ is \$123,583.40 and is feasible for a MAXDET1 of 0.006339. If the constraint is set as MAXDET1 = 0.005, the optimal total cost per year will become \$131,064.90 per year. An increase of \$16,849.20 or 14.75% of the original overall optimal cost is incurred. These results suggest that the optimal cost is quite sensitive to the value of the maximum deterioration proportion being set.

7.2.5 Example 7.3

Consider Supply Chain S5 in Example 6.4 of Chapter 6.

The overall optimal solution without the maximum deterioration constraint is given by: $N^* = 72$ days, n_i^*'s = (2,3,4,2,4,2,4,3,4,2,4,4,2,4,2,3,3,2,4,3) and the overall optimal cost is $TC_s^* = \$230,296.10$. The corresponding production time is 0.097065 year per cycle and it follows that deteriorated quantity / demand = 0.008052. Hence, this solution is also the least cost feasible solution for the maximum deterioration constraint if MAXDET2 is set at a value not less than 0.008052 which is equivalent to MAXDET1 being set at a value not less than 0.007988. The least cost feasible solutions for MAXDET1 of 0.005 to 0.008 are shown in Table 7.3 below:

MAXDET1	System cycle time N^* (days)	Numbers of deliveries n_i^*'s	System cost per unit time TC_s^*
≥ 0.008	72	(2,3,4,2,4,2,4,3,4,2,4,4,2,4,2,3,3,2,4,3)	230,296.10
0.007	60	(2,3,3,2,3,3,3,3,3,2,3,3,3,3,2,3,3,2,3,3)	233,993.20
0.006	48	(1,2,3,1,3,2,3,2,3,2,3,3,2,3,2,2,3,2,3,2)	241,052.80
0.005	42	(2,3,3,2,3,2,3,3,3,2,3,3,2,3,2,3,3,2,3,2)	257,119.00

Table 7.3: Least cost feasible solutions for Example 7.3.

The results indicate that the optimal costs are sensitive to small values of the maximum deterioration proportion.

7.3 A Model for Two Products on the Same Production Line

7.3.1 The Model

In this model, two similar products are produced by the same production process in which the production rate is much higher than the individual demand rates of the two products. In order to increase the utilization of the production facilities, these two products are arranged to be produced on the same production line. It is assumed that that a common system cycle for the two similar products is agreed between the vendor and the buyer. In each system cycle, there is one production batch for Product 1 and one production batch for Product 2, each entailing a production set up. The number of deliveries for Product 1 is n_1 and that for Product 2 is n_2 in each system cycle. It is assumed that the set up time is small and the production rate is predetermined. Figure 7.1 depicts the situation with $n_1 = 3$ and $n_2 = 4$.

The assumptions of this model are:

1. A common system cycle time is adopted for producing and delivering the two products.
2. The two similar products have the same deterioration rate.
3. The two products are produced at the same production rate.
4. The set up cost is the same for the two products as they are produced by the same production process.

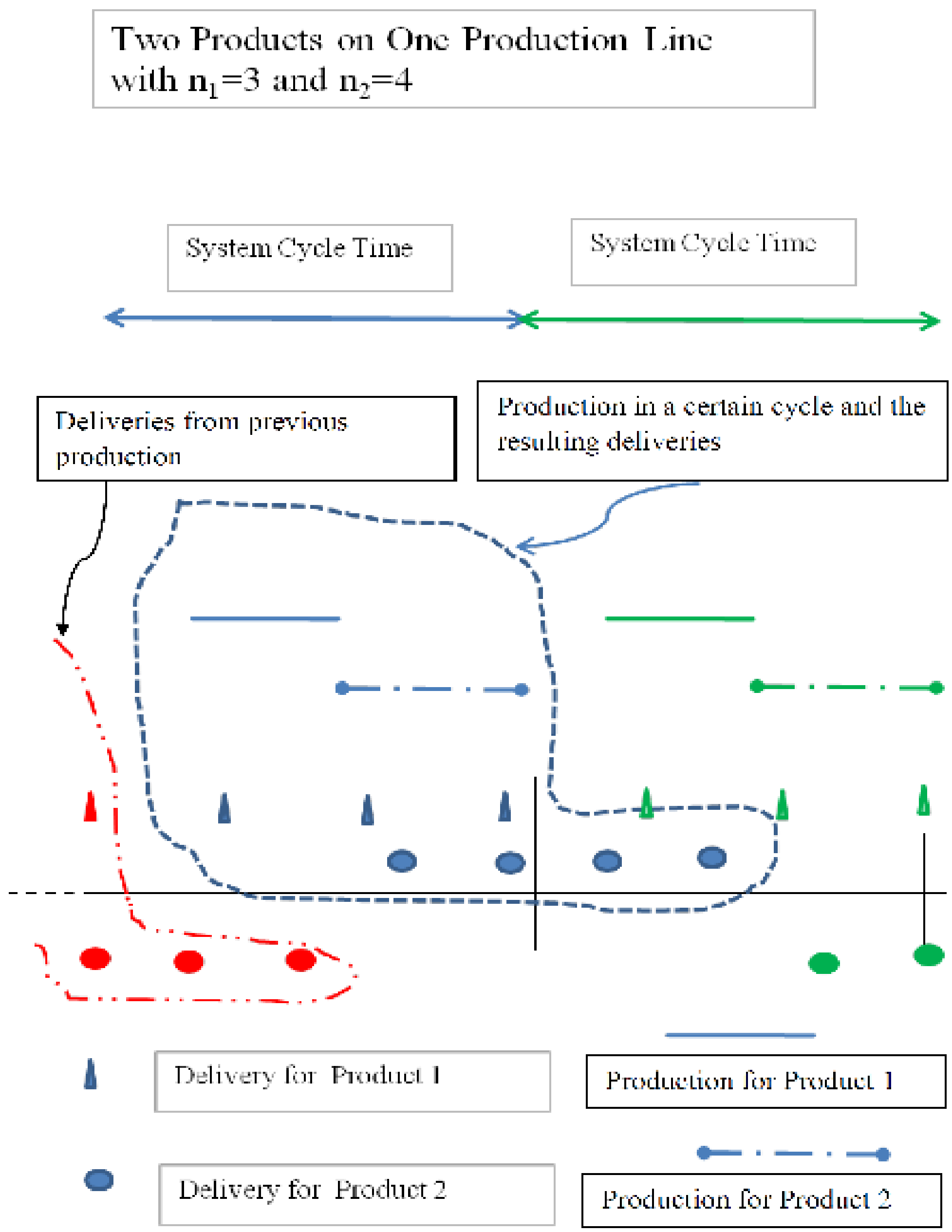


Figure 7.1: Two products on the same production line.

From equation (4.1), the total cost per unit time for the two products are given by

$$TC_{s,i} = \frac{S}{T} + \frac{n_i(A_{vi} + A_{bi})}{T} + \frac{(H_{bi} - H_{vi} + kC_{bi} - kC_{vi})nD_i}{kT}[\frac{1}{k}(e^{\frac{kT}{n_i}} - 1) - \frac{T}{n_i}]$$
$$+ \frac{(H_{vi} + kC_{vi})(PT_{pi} - D_i T)}{kT}, \qquad i = 1,2 \tag{7.4}$$

where $T_{pi} = \frac{1}{k}\ln\{1 + [\frac{D_i}{P}(e^{kT} - 1)] / [1 - \frac{D_i}{P}(e^{\frac{kT}{n_i}} - 1)]\}$.

The total system cost per unit time for the two-product system is given by

$$TC_s = TC_{s,1} + TC_{s,2} \tag{7.5}$$

The objective is to minimize TC_s by finding the optimal system cycle time T and the optimal numbers of deliveries n_1 and n_2 for the two products respectively. Two heuristics are proposed in the coming sections: the first heuristic considers time as a continuous variable, the second heuristic considers cycle times as multiples of a convenient time unit (day) as in the synchronized model in Chapter 6.

7.3.2 Heuristic 1

Consider the following functions:

$$\mathrm{H}_1(x_1, T) = \frac{S}{T} + \frac{x_1(A_{v1} + A_{b1})}{T} + \frac{(H_{b1} - H_{v1} + kC_{b1} - kC_{v1})x_1 D_1}{kT}[\frac{1}{k}(e^{\frac{kT}{x_1}} - 1) - \frac{T}{x_1}]$$
$$+ \frac{(H_{v1} + kC_{v1})(PT_{p1} - D_1 T)}{kT}$$

$$\mathrm{H}_2(x_2,T)=\frac{S}{T}+\frac{x_2(A_{v2}+A_{b2})}{T}+\frac{(H_{b2}-H_{v2}+kC_{b2}-kC_{v2})x_2D_2}{kT}[\frac{1}{k}(e^{\frac{kT}{x_2}}-1)-\frac{T}{x_2}]$$
$$+\frac{(H_{v2}+kC_{v2})(PT_{p2}-D_2T)}{kT}$$

$$\mathrm{Z}(x_1,x_2,T)=\mathrm{H}_1(x_1,T)+\mathrm{H}_2(x_2,T)$$

To optimize the function $\mathrm{Z}(x_1,x_2,T)$, it requires

$$\begin{cases} \dfrac{\delta Z}{\delta x_1}=\dfrac{\delta \mathrm{H}_1}{\delta x_1}=0 \\ \dfrac{\delta Z}{\delta x_2}=\dfrac{\delta \mathrm{H}_2}{\delta x_2}=0 \\ \dfrac{\delta Z}{\delta T}=\dfrac{\delta \mathrm{H}_1}{\delta T}+\dfrac{\delta \mathrm{H}_2}{\delta T}=0 \end{cases}$$

The first and the second equations are analogous to finding the optimal numbers of deliveries for the first product and the second product for a given system cycle time *T*. The third equation is to find an optimal *T* for given numbers of deliveries for the two products. However, as *T* is considered as a continuous variable in this approach, it is difficult to have a logical way to select suitable value(s) for *T* to get the optimal numbers of deliveries. On the other hand, starting with one delivery for each product, followed by progressively increasing the numbers of deliveries, for working on the third equation is one possible way but one delivery for each product could be far from the optimal solution. The following heuristic is proposed. The optimal numbers of deliveries for each product assuming individual production are found. Then this pair of values is used to initiate an iterative procedure for finding the optimal solution. The detailed procedures are shown below:

Step 1: Find the optimal number of deliveries $n_1^{\#}$ for minimizing the cost of the vendor-buyer system for Product 1 as given by equation (7.4) for $i=1$, by using the method in Wee et al. (2008).

Step 2: Find the optimal number of deliveries $n_2^{\#}$ for minimizing the cost of the vendor-buyer system for Product 2 as given equation (7.4) for $i=2$.

Step 3: Set $n_1 = n_1^{\#}$ and $n_2 = n_2^{\#}$. Find the optimal solution for TC_s in equation (7.5) and the optimal cost $TC_s(n_1^{\#}, n_2^{\#})$.

Step 4: Find the optimal costs for, $(n_1^{\#}-1, n_2^{\#}-1)$, $(n_1^{\#}-1, n_2^{\#})$, $(n_1^{\#}-1, n_2^{\#}+1)$, $(n_1^{\#}, n_2^{\#}-1)$, $(n_1^{\#}, n_2^{\#}+1)$, $(n_1^{\#}+1, n_2^{\#}-1)$, $(n_1^{\#}+1, n_2^{\#})$ and $(n_1^{\#}+1, n_2^{\#}+1)$.

Step 5: Find the smallest cost among the 9 costs obtained in Step 3 and Step 4.

Step 6: If $TC_s(n_1^{\#}, n_2^{\#})$ is the smallest, take this as the final solution. Exit.

Step 7: If $TC_s(n_1^{\varsigma}, n_2^{\varsigma})$ is the smallest where $(n_1^{\varsigma}, n_2^{\varsigma}) \neq (n_1^{\#}, n_2^{\#})$, i.e., at least one of the pair is different, update the value(s) and repeat Step 3 to Step 6.

7.3.3 Example 7.4

The demand rates of Product 1 and Product 2 are 1000 units and 1200 units per year respectively. Use the following parameters from Example 3.1 for both products.

P = 3200 units per year $\qquad$ k = 0.1 per year

$S = \$400$ $\quad\quad$ $A_b + A_v = \$25$

$C_b = \$50$ $\quad\quad$ $C_v = \$40$

$H_b = \$5$ per unit per year $\quad\quad$ $H_v = \$4$ per unit per year

Consider individual production of the two products. The optimal solutions are:

$n_1^{\#} = 5$, $TC_{s,1} = \$2695.69$, and $n_2^{\#} = 5$, $TC_{s,2} = \$2886.94$.

Iteration 1: "Centred" at (5, 5), the optimal costs are

n_1	n_2	Optimal cost per unit time ($)
4	4	5602.24
4	**5**	**5583.95**
4	6	5591.63
5	4	5607.91
5	5	5585.88
5	6	5590.53
6	4	5632.40
6	5	5607.25
6	6	5609.39

Iteration 2: "Centred" at (4, 5), the optimal costs are

n_1	n_2	Optimal cost per unit time ($)
3	4	5634.35
3	5	5620.96
3	6	5632.65
4	4	5602.24
4	**5**	**5583.96**
4	6	5591.63
5	4	5607.91
5	5	5585.88
5	6	5590.53

Table 7.4: Iterations for Example 7.4.

The final solution is $(n_1^*, n_2^*) = (4,5)$, $T^* = 0.366642$ year, and $TC_s^* = \$5583.96$ per year. The production times for Product 1 and Product 2 are:

$$T_{p1} = \frac{1}{0.1}\ln[1 + \frac{\frac{1000}{3200}(e^{0.1T^*} - 1)}{1 - \frac{1000}{3200}(e^{\frac{0.1T^*}{4}} - 1)}] = 0.116359 \text{ year, and } T_{p2} = 0.139453 \text{ year.}$$

The sum of production times is less than the system cycle time. The solution is feasible. The optimal solutions and costs for individual production and "integrated" production are shown in Table 7.5. In this case, the total cost for producing two products on the same production line is almost the same as the sum of optimal costs for individual production of the two products.

	Product 1 only	Product 1 and Product 2 on one Production Line : Product 1	Product 2	Product 2 only
System Cycle Time (year)	0.388931	0.366642		0.363307
Production Time (year)	0.123473	0.116359	0.139453	0.138167
No. of deliveries per cycle	5	4	5	5
Set- up cost ($)	1028.4627	1090.9825	1090.9825	1100.9972
Ordering / Delivery related costs ($)	321.3946	272.7456	340.9320	344.0616
Inventory holding & deterioration costs ($)	1345.8329	1333.1689	1455.1425	1441.877
Total annual cost ($)	2695.69	2696.90	2887.06	2886.94

Table 7.5: Comparison of optimal costs for Example 7.4.

7.3.4 Heuristic 2

This heuristic is modified from the synchronized model developed in Chapter 6. The system cycle time and the delivery cycle times are integer multiples of a convenient time unit, day. The maximum system time is 365 days. In this model, production of one product follows the production of the other. Hence, the issue with production for multiple buyers at the same time as in the model of Chapter 6 does not exist here. The heuristic-algorithm is simpler and the procedure is shown below:

Define the following functions:

$$\varnothing_i(n_i) = \frac{n_i(A_{vi} + A_{bi})}{T} + \frac{(H_{bi} - H_{vi} + kC_{bi} - kC_{vi})n_i D_i}{kT}[\frac{1}{k}(e^{\frac{kT}{n_i}} - 1)] + \frac{(H_{vi} + kC_{vi})(PT_{pi})}{kT} - \frac{(H_{bi} + kC_{bi})}{k} D_i \quad i = 1,2 \tag{7.6}$$

Then $TC_{s,i} = \frac{S}{T} + \varnothing_i$, $i = 1,2$. (7.7)

<u>Algorithm</u>

Step 1: For each N (N=1, 2, 3, …,365), find its integer factors.

Step 2: For each of the two products, find the optimal number of deliveries and the optimal cost by using the sub-algorithm. Find the total of these two costs. This is the optimal solution for this N.

Step 3: Find the smallest cost among the total costs of $N = 1, 2, 3, ..., 365$. This is the overall optimal solution for the two-product system.

<u>Sub-algorithm for finding the optimal solution for a fixed *N*</u>

Step 1: Find the integer factors of *N* and set $T = N/365$.

Step 2: Start with $i = 1$, define $f(x)$ as:

$$f(x) = \frac{(H_{bi} - H_v + kC_{bi} - kC_v)D_i}{k^2}\left(\frac{kT}{x}e^{\frac{kT}{x}} - e^{\frac{kT}{x}} + 1\right) + \frac{D_i^2 T(H_{vi} + kC_{vi})e^{\frac{kT}{x}}}{Pkx^2} \frac{(e^{kT} - 1)}{[1 + \frac{D_i}{P}(e^{kT} - e^{\frac{kT}{x}})][1 - \frac{D_i}{P}(e^{\frac{kT}{x}} - 1)]} - (A_{vi} + A_{bi})$$

Step 3: If $f(1) \le 0$ set $n_i^{\zeta} = 1$. Next *i*.

Step 4: If $f(N) \ge 0$, set $n_i^{\zeta} = N$. Next *i*.

Step 5: Find the minimum of $\varnothing_i(x)$. Suppose $\varnothing_i(x)$ is minimum when $x = x^*$. If x^* is an integer and is a factor of *N*, set $n_i^{\zeta} = x^*$. Next *i*.

Step 6: Find α and β, the two consecutive factors of N such that $\alpha < x^* < \beta$.

Step 7: Calculate $\varnothing_i(\alpha)$ and $\varnothing_i(\beta)$. If $\varnothing_i(\alpha) < \varnothing_i(\beta)$, set $n_i^{\zeta} = \alpha$. Otherwise, set $n_i^{\zeta} = \beta$. Next *i*.

Step 8: Find T_{p1} and T_{p2} from the formula of T_{pi} following equation (7.4); and find TC_s (from equations (7.6), (7.7) and (7.5)) with $T = N/365$ and $n_i = n_i^{\zeta}$, $i = 1, 2$. This is the optimal solution for this *N*.

7.3.5 Example 7.5

Apply Heuristic 2 for the 2-product system in Example 7.4. The 5 smallest optimal costs among the 365 optimal costs are given in Table 7.6.

System cycle time N (days)	No. of deliveries n_1	n_2	Product 1 cost ($/year)	Product 2 cost ($/year)	Total cost ($/year)
135	5	5	2702.72	2890.63	5593.35
140	5	5	2699.83	2894.67	5594.50
130	5	5	2709.52	2890.51	5600.03
132	4	6	2701.47	2900.56	5602.03
145	5	5	2700.45	2902.23	5602.68

Table 7.6: 5 smallest optimal costs for Example 7.5.

The overall optimal solution is given by: $N^* = 135$ days, $(n_1^*, n_2^*) = (5,5)$, and $TC_s^* = \$5593.35$ per year. In this case, it is almost the same as the optimal cost in Example 7.4 obtained by applying Heuristic 1 and Wee et al. (2008)'s method which approximates exponential and logarithmic terms in the cost expression for finding the optimal solution. The optimal system cycle time in Example 7.4 is 0.366642 year, that is, 133.8 days, and is not far from 135 days obtained by Heuristic 2.

7.3.6 Example 7.6

In Examples 7.4 and 7.5, the two demand rates are similar. In this example, the demand rates are 1000 units per year and 500 units per year while the other parameters are the same as in the two examples.

Applying Heuristic 1, the optimal numbers of deliveries for the two products are $n_1^{\#} = 5$, and $n_2^{\#} = 3$.

Iteration 1: "Centred" at (5, 3), the optimal costs are

n_1	n_2	Optimal cost per unit time ($)
4	2	4734.97
4	3	4719.35
4	4	4739.88
5	2	4724.84
5	3	4705.27
5	4	4722.98
6	2	4735.86
6	3	4713.06
6	4	4728.49

Table 7.7: Iteration for Heuristic 1 for Example 7.6.

The optimal solution is $(n_1^*, n_2^*) = (5,3)$, $T^* = 0.424200$ year, that is, 154.8 days, and the optimal cost is $TC_s^* = \$4705.27$ per year. Applying Heuristic 2, the overall optimal solution is: $N^* = 150$ days, $(n_1^*, n_2^*) = (5,3)$, and $TC_s^* = \$4716.55$ per year. The overall optimal cost is also very close to the optimal cost obtained by Heuristic 1. The optimal system cycle time is not as close as in the previous example but is not far apart.

7.4 Conclusion

There have been increasing environmental concerns in the society nowadays. Deterioration results in waste. In addition to cost performance, management of supply chains of deteriorating items may need to consider controlling the amount of deterioration generated in their strategy. In this chapter, the model developed in Chapter 6 is extended to include a maximum deterioration constraint. The cost minimization model becomes a goal programming model. The extended model provides the minimum cost solution that limits the amount of deteriorated units within a preset proportion of production volume. The results of the model suggest that the resulting minimum cost is quite sensitive for small values of the deterioration proportion. Supply chain management has to be aware of this phenomenon in formulating their strategy.

In a company, there may be products that are similar and employ the same process and equipment for production. Utilization of production facilities will be increased if such products can be arranged to run on the same production line. In this chapter, two heuristics are proposed to find the optimal solution for producing two similar products on the same production line. One heuristic considers time as a continuous variable and the other is modified from the synchronized model proposed in Chapter 6. The first heuristic can provide a (slightly) lower cost solution. However, as in other "continuous time" models, the cost may be increased for implementing the solution in practice. For example, in Example 7.3, the optimal system cycle time is

133.8 days with 4 and 5 deliveries for the two products. A quick fix may be setting the system cycle time at 133 or 134 days. Then the issue with the delivery cycle time needs to be resolved, and it could mean that the system cycle time needs to be reconsidered. Anyway, the cost will be increased if the solution implemented is different from that obtained from the heuristic. The solution obtained from Heuristic 2 gives a (slightly) higher cost than the solution provided by Heuristic 1 (before implementation) but it can be directly implemented as the cycle times are already integer multiples of an appropriate convenient time unit that has been selected beforehand. Furthermore, Heuristic 2 can be easily extended to more than two products that can be produced on the same production line when production capacity permits. Heuristic 1 is less convenient when applied to several products as there will be many combinations of numbers of deliveries to be considered.

Chapter 8

Conclusion and Future Research Directions

8.1 Introduction

Ghare and Schrader (1963) were the first authors to study decaying inventory problems. After their pioneering work, many researchers have studied and presented inventory models for deteriorating items. While complicated EOQ and EPQ models have been developed, there has been relatively less work on integrated lot-delivery models for deteriorating items. Lot-delivery is a common mode in operations of supply chains. This thesis investigates lot-delivery production-inventory models for optimizing the performance of supply chains of exponentially deteriorating items. The first area of this thesis is the development of a demand-driven production rate continuous production model. It was shown that it would result in a lower system cost than the conventional predetermined production rate models.

In the literature of inventory models, time is usually considered as a continuous variable. Chan and Kingsman (2005, 2007) proposed a synchronized production and delivery cycles model for optimizing the total cost of a single-vendor multi-buyer supply chain. In this model, the system and delivery cycle times are integer multiple of a time unit agreed by the parties of the supply chain. The second area of this thesis is on extending this synchronized model for optimizing the performance of supply chains of deteriorating items.

8.2 The Demand-driven Production Rate model

In the literature of predetermined production rate models, production rates are usually much higher than the demand rates and production stops at the early stage of a system cycle. Although there are multiple deliveries for the goods from a production batch for reducing the total cost, significant amount of goods has still been produced well before shipments. In addition to incurring inventory holding costs, some the items are deteriorated while waiting for deliveries and hence the earlier they are made, the higher the deterioration costs will also be incurred.

In Chapter 3, a continuous production model is developed for a single-vendor dingle-buyer supply chain of exponentially deteriorating items. In this model, production rate is not a preset parameter but is related to the demand rate and the delivery cycle time and is determined by optimizing the total system cost per unit time. It can be shown that for a given demand rate and a given set of cost parameters, a unique optimal cycle time exists that determines the optimal production rate and minimizes the system cost. This optimal production rate is slightly higher than the demand rate, demand-driven, and is much lower than the usual predetermined production rate for the same demand rate. With a lower production rate and a shorter delivery interval, average inventory level is reduced significantly. Hence, smaller inventory holding and deterioration costs are incurred in the proposed model, which result in a lower total system cost than the predetermined production rate model. The proposed model is extended to consider deterioration during transportation which is usually ignored in the literature of inventory models for deteriorating items. A heuristic for extending the model to a single-vendor multi-buyer supply chain is proposed in Chapter 5.

In Chapter 4, the proposed demand-driven production rate model and the predetermined production rate model are compared by studying the effect of changing production rate on the total cost. It was shown that for a given system cycle time and a given delivery cycle time, inventory level increases with production rate if there are three or more deliveries in a system cycle. Under this condition, inventory holding cost and deterioration cost are reduced by selecting a smaller production rate. As a result, a smaller total cost is obtained because set–up cost and delivery-related costs are unchanged. It was shown that if there are two deliveries in a cycle and the production rate is larger than a particular value, inventory level is also an increasing function of production rate. On the other hand, if there is one delivery in a cycle, increasing production rate decreases the cost. These findings support that the model proposed in Chapter 3 can, in many cases, provide a lower cost than the predetermined production rate model because the proposed model utilizes a lower production rate. This also provides a theoretical explanation to the computational results of the example in Chapter 3. The investigation of the effect of changing production rate on the total cost is originated from the objective of validating the experimental observations that the average inventory level of the demand-driven production rate model is lower than that of the predetermined production rate model when the number of deliveries is three or more. The results of this investigation have also an implication for the predetermined production rate model. If a production rate is to be selected from a certain range of production rates for non-continuous production (that is, the conventional predetermined production rate model), and the lowest production rate is higher than a specific value (the value for the two-delivery situation mentioned above), the optimal production rate is either the lowest rate or the highest rate of the range. Therefore, the optimal production rate can be determined by finding and comparing the optimal costs at these two production rates. Finally, if one has to choose between a particular predetermined production rate or the demand-driven production rate based on the objective of cost minimization, the optimal cost(s) for that production rate for one delivery per cycle (and two deliveries if that production rate is smaller than the particular production rate for the two-delivery case) and the optimal cost for the proposed model should be found and compared.

In the literature of inventory models for deteriorating items, some researchers have presented EOQ models in which there is a non-deteriorating period for the items. These models assume that all units just start the non-deteriorating period when they are received by the buyer. In Chapter 5, the proposed demand-driven production rate lot-delivery model is extended to include a non-deteriorating period for exponentially deteriorating items. In this model, some of the units in a shipment have passed the non-deteriorating period (as they are produced "earlier" than that period before shipment) and are already subject to deterioration when they are received by the buyer. The presence of a non-deteriorating period affects both the vendor's and the buyer's inventory levels. It has been shown that a unique optimal solution exists for this extended model. The existence of a non-deteriorating period results in a lower optimal cost than the situation with "immediate' deterioration.

In most of the literature, when time value is not considered, cost parameters have been assumed to be constant including non-constant production rate models. The proposed demand-driven production rate model has also been developed based on this usual assumption. As the proposed model uses a much lower production rate than usual production rates relative to demand rates, cost effectiveness of the proposed model is affected if this "constant cost parameters" assumption is only valid for a limited range of production rates. It will be even more favourable for the proposed model if costs increase with production rate. However, as discussed in Chapter 5, some cost components such as labour cost per unit, machine rate per unit, might increase when production rate is reduced. In view of this potential issue, the model is extended to consider a scenario in which deterioration costs and inventory holding costs are partly constant and partly inversely proportional to production rate. When the constant part is of a lower proportion of the cost, the cost will increase more with a given reduction in production rate. The cost equation has been modified due to the new expressions of these cost parameters. It has been found that for a given set of parameters, there is still a unique optimal solution for this new cost equation. The results of the numerical examples suggest that if the delivery-related cost parameters are small, the demand-driven production rate model gives a lower optimal cost even for low proportions of the constant parts of the deterioration and inventory holding costs. If the delivery-related cost parameters are large, the proposed model can still give a lower optimal cost for medium to high proportions of the constant parts of these costs. As a crude analysis, if material cost (basically independent of production rate), labour cost and overhead each accounts for one-third of the concerned cost parameter, and half of the overhead is independent of production rate, the constant part already accounts for half of the cost parameter even if labour cost is fully production rate dependent.

8.3 A Synchronized Model for a Single-vendor Multi-buyer Supply Chain

Chan and Kingman (2005, 2007) proposed a synchronized model for coordinating production and delivery cycles for a single-vendor multi-buyer supply chain. In Chapter 6, the model is extended to supply chains of exponentially deteriorating items. The model assumes that all the parties have agreed that the basic time unit is day and the maximum system cycle time is 365 days. It was found that the production time for multiple buyers is approximately the same as the sum of production times for individual productions. A two-stage model and algorithm has been developed. An initial solution is found by optimizing individual "vendor-buyer" subsystems independently. The final solution is found by iteratively increasing the numbers of deliveries until there is no further cost improvement. The algorithm finds the optimal cost for each of the feasible system cycle times and the least cost solution is the overall optimal solution. This algorithm has two advantages. It enables a good initial solution for each system cycle time to be found as starting with one delivery for each buyer is impractical when there are multiple buyers in the supply chain. Moreover, it has been shown that once the "optimal" solution based on independent optimization has been found, decreasing the numbers of deliveries cannot reduce the cost. Therefore, the numbers of deliveries should only be increased in the iterative procedures for finding the true optimal solution. Same as Chan and Kingsman's model, the optimal solution from this model can be directly implemented as the system cycle time and delivery intervals are all integers of day, whereas in the "continuous time" models, the results have to be adjusted somehow before implementation due to practical considerations as discussed in the conclusion of Chapter 7.

A genetic search algorithm (GA) is a heuristic search process that resembles natural selection. GA has been a popular meta-heuristic for finding very good, if not optimal, solutions for combinatorial optimization problems. The results from the proposed algorithm are compared with that obtained from GA. It has been found that the best costs from GA are either same as or close to but slightly higher than that from the proposed algorithm, depending on whether GA can reach the optimal system times obtained by the proposed algorithm. Although GA cannot guarantee that the optimal solution can be obtained, the comparative results help to support the use of the proposed model and algorithm in finding optimal solutions for the supply chain. The results from the algorithm are also compared with that obtained by adopting the common cycle approach with time being considered a continuous variable. Although the solutions from the proposed algorithm are subject to the constraint on the system cycle time and delivery cycle times, they still provide lower costs than that from the common cycle "continuous time" approach. The reason is that in the proposed model, the number of deliveries for a buyer is based on the parameters of the concerned "vendor-buyer" subsystem and can be different for different buyers. The other approach, though free of constraints on cycle times, forces every buyer to adopt a common cycle. When the supply chain consists of multiple buyers having dissimilar demand rates and cost parameters, the common cycle approach will result that many of the parties are far from being optimized.

In Chapter 7, there are two extensions of the model proposed in Chapter 6. There have been increasing environmental concerns in the society. This has rendered management's awareness of reducing negative impact on the environment through good practice in the operations of the supply chain. For supply chains of deteriorating items, reducing deterioration helps protect the environment as deterioration results in wastage of raw materials, energy and other resources. The first extension of the model is to address this environmental issue. In this extended model, there is a predefined maximum deterioration proportion which is the ratio of the deteriorated quantity to the production volume, and the algorithm seeks to obtain the least cost solution that meets this constraint. It has been found that the resulting minimum cost can be quite sensitive to the value of the maximum deterioration proportion being set if small values are considered for the proportion. Management may have a challenging task in weighing between different maximum deterioration proportions and their optimal costs in formulating their environmental performance strategy.

In predetermined production rate models, the utilization of the production facilities is usually low due to high production rates compared with demand rates. If products employing the same production process and equipment can be produced on the same production line, utilization of the facilities will be increased. In the second part of Chapter 7, two heuristics have been proposed for producing two similar products on the same production line. The first heuristic considers time as a continuous variable and the second one is modified from the synchronized model presented in Chapter 6. Heuristic 1 will give a smaller cost because there is no constraint on the cycle time and the results of the numerical examples suggest that "convergence" to the final solution is quite fast. However, the results from numerical examples also suggest that the optimal costs from Heuristic 2 are only very slightly higher than that of Heuristic 1. In addition to the practicality issue of implementation of the solution as discussed previously, Heuristic 2 can be more easily extended to cater for more than two products than Heuristic 1. Moreover, Heuristic 1 may reach a local minimum that is not the global minimum, if there are two or more local minima. If the adoption of a practical time unit and a maximum system cycle time is agreed, the procedure of Heuristic 2 ensures that all feasible system cycle times are considered, and hence the overall optimal solution can be obtained even if there are multiple local minima in the concerned system.

8.4 Future Research

One of the potential directions for future research is to extend the two-product model to more than two products as mentioned. Besides, there are several directions for further research arising from this thesis. A characteristic of the two models (the demand-driven production rate model, and the synchronized production and delivery cycles model) is that the delivery related cost parameters, A_v and A_b, are assumed to be constant regardless of the shipment size. It may happen that part of these costs is independent of shipment size, e.g., costs due to order processing and coordination tasks; and part of these costs depends on the shipment size, e.g., handling charges, inspection cost and possibly part of the transportation cost. Under these conditions, these costs can be presented as: $A_{va}+\frac{A_{vb}D}{k}(e^{kT_c}-1)$ and $A_{ba}+\frac{A_{bb}D}{k}(e^{kT_c}-1)$ for each delivery, where A_{va}, A_{vb}, A_{ba} and A_{bb} are the respective parameters.

For the demand-driven production rate model, the total system cost per unit time can therefore be modified from equation (3.15) as follows:

$$
\begin{aligned}
TC_s = {} & \frac{A_{ba}+A_{va}}{T_c}+\frac{D}{k}(A_{bb}+A_{vb}+\frac{H_b}{k}+C_b-\frac{H_v}{k}-C_v)(\frac{e^{kT_c}-1}{T_c})+\frac{H_vDe^{kT_c}}{k} \\
& +C_vDe^{kT_c}-\frac{H_bD}{k}-C_bD+S
\end{aligned}
.
$$

Similarly, for the synchronized model presented in Chapter 6, the total system cost per unit time can be modified from equation (6.9) as follows:

$$TC_s = \frac{S}{T} + \sum_{i=1}^{n_b} \frac{n_i (A_{vai} + A_{bai})}{T}$$
$$+ \{\sum_{i=1}^{n_b} \frac{(kA_{vbi} + kA_{bbi} + H_{bi} - H_v + kC_{bi} - kC_v)}{kT} n_i D_i [\frac{1}{k}(e^{\frac{kT}{n_i}} - 1)]\}$$
$$+ \frac{(H_v + kC_v) PT_p}{kT} - \sum_{i=1}^{n_b} \frac{(H_{bi} + kC_{bi})}{k} D_i$$

These two equations are of the same forms as the original equations. Therefore, after modifying the concerned equations, the solution procedure of both these two models can be applied to investigate supply chains having these characteristics.

The optimal solutions of the demand-driven production rate model usually entail more frequent deliveries than that of the predetermined production rate model. In the synchronized model with the maximum deterioration constraint, numbers of deliveries are increased in order to reduce the deteriorated quantity. In both cases, only costs are concerned for delivery but the environmental issues with more frequent deliveries have not been considered. For the first case, if the optimal solution of a predetermined production rate entails three or more deliveries (or two deliveries if the production rate is high enough) in a cycle, the demand-driven production rate model can give a lower total cost by keeping the same delivery interval as the predetermined production rate. This does not increase delivery frequency but results in less cost saving as the optimal solution is not adopted. As for the second case, a potential direction of research is to extend the maximum deterioration model to consider three factors: cost, deterioration proportion and transportation frequency. The challenge of this direction will be on devising an appropriate way to weigh all these three factors together: whether to use a goal programming model also or by other means for quantifying the impact of transportation frequency on the system.

In the synchronized model of Chapter 6, it has been assumed that a delivery is made to every buyer at the beginning of the cycle. A potential direction is to investigate whether cost can be reduced by relaxing this assumption. It seems possible as production can be started later when some of the first shipments are deferred. The challenge will then be whether an exact expression for production time can be derived without the assumption of simultaneous first shipments for all the buyers.

This thesis is on deteriorating items. Some of the topics, for example, the effect of varying production rate on the total cost (Chapter 4) and the model of cost parameters being production rate dependent (Chapter 5), may also be considered for production-inventory models on non-deteriorating items. A potential research direction is to investigate and develop production rate related models for non-deteriorating items.

Appendix A: Proofs

Appendix A1: Proof for convexity of TC_s in equation (5.5) in Section 5.2.2

In Section 5.2.2, differentiating TC_s in equation (5.5) and setting $\frac{d}{dT_c}TC_s = 0$, we obtain equation (5.6):

$$l_1[(kT_c - 1)e^{kT_c} + 1] + l_2(kT_c - 1)e^{k(T_c - T_\Theta)} + l_3[(kT_c - 1)e^{k(T_c - T_\Theta)} + 1] + l_4(kT_c)^2 e^{k(T_c - T_\Theta)}$$
$$-(A_b + A_v) = 0$$

where $l_1 = \frac{H_b D}{k^2}$, $l_2 = D[\frac{H_b}{k^2}(1 - e^{kT_\Theta}) + T_\Theta(\frac{H_b - H_v}{2}T_\Theta + \frac{H_b}{k} + C_b - \frac{H_v}{k} - C_v)]$,

$$l_3 = \frac{D}{k}(C_b - C_v - \frac{H_v}{k}), \quad l_4 = D(\frac{H_v T_\Theta k + H_v + C_v k}{k^2}).$$

Let $x = kT_c > 0$ and

$$g(x) = l_1[(x-1)e^x + 1] + l_2(x-1)e^{-kT_\Theta}e^x + l_3[(x-1)e^{-kT_\Theta}e^x + 1] + l_4(x)^2 e^{-kT_\Theta}e^x - (A_b + A_v) = 0$$

Differentiating $g(x)$, we obtain

$$\begin{aligned} g'(x) &= l_1 xe^x + l_2 e^{-kT_\Theta} xe^x + l_3 e^{-kT_\Theta} xe^x + 2l_4 e^{-kT_\Theta} xe^x + l_4 e^{-kT_\Theta} x^2 e^x \\ &= xe^x[l_1 + e^{-kT_\Theta}(l_2 + l_3 + 2l_4) + e^{-ktT_\Theta} l_4 x] \end{aligned}$$

It can be shown that

$$l_1[(kT_c - 1)e^{kT_c} + 1] + l_2(kT_c - 1)e^{k(T_c - T_\Theta)} + l_3[(kT_c - 1)e^{k(T_c - T_\Theta)} + 1] + l_4(kT_c)^2 e^{k(T_c - T_\Theta)} > 0$$

and is an increasing function as follows:

As $l_4 > 0$,

$$l_1[(kT_c - 1)e^{kT_c} + 1] + l_2(kT_c - 1)e^{k(T_c - T_\Theta)} + l_3[(kT_c - 1)e^{k(T_c - T_\Theta)} + 1] + l_4(kT_c)^2 e^{k(T_c - T_\Theta)}$$
$$> \frac{H_b D}{k^2}[(kT_c - 1)e^{kT_c} + 1] + D[\frac{H_b}{k^2}(1 - e^{kT_\Theta}) +$$
$$T_\Theta(\frac{H_b - H_v}{2}T_\Theta + \frac{H_b}{k} + C_b - \frac{H_v}{k} - C_v)](kT_c - 1)e^{k(T_c - T_\Theta)} +$$
$$\frac{D}{k}(C_b - C_v - \frac{H_v}{k})[(kT_c - 1)e^{k(T_c - T_\Theta)} + 1]$$
$$= \frac{H_b D}{k^2}[(x-1)e^x + 1] + D[\frac{H_b}{k^2}(1 - e^{kT_\Theta}) + T_\Theta(\frac{H_b - H_v}{2}T_\Theta + \frac{H_b}{k} - \frac{H_v}{k})](x-1)e^{-kT_\Theta}e^x$$
$$- \frac{H_v D}{k^2}[(x-1)e^{-kT_\Theta}e^x + 1] + DT_\Theta(C_b - C_v)(x-1)e^{-kT_\Theta}e^x + \frac{D}{k}(C_b - C_v)[(x-1)e^{-kT_\Theta}e^x + 1]$$

First consider the terms involving H_b and H_v only. In general, we have $x = kT_c < 1$.

$$\frac{H_b D}{k^2}[(x-1)e^x + 1] + D[\frac{H_b}{k^2}(1 - e^{kT_\Theta}) + T_\Theta(\frac{H_b - H_v}{2}T_\Theta + \frac{H_b}{k} - \frac{H_v}{k})](x-1)e^{-kT_\Theta}e^x$$
$$- \frac{H_v D}{k^2}[(x-1)e^{-kT_\Theta}e^x + 1]$$
$$= \frac{H_b D}{k^2}[(x-1)e^x + 1] - \frac{H_v D}{k^2}\{(x-1)[1 - (1 - e^{-kT_\Theta})]e^x + 1\}$$
$$+ D[\frac{H_b}{k^2}(e^{kt_0 T_\Theta} - 1) - T_\Theta(\frac{H_b - H_v}{2}T_\Theta + \frac{H_b}{k} - \frac{H_v}{k})](1-x)e^{-kT_\Theta}e^x$$
$$= \frac{(H_b - H_v)D}{k^2}[(x-1)e^x + 1] + \frac{H_v D}{k^2}(x-1)(1 - e^{-kT_\Theta})e^x$$
$$+ D[\frac{H_b}{k^2}(e^{kT_\Theta} - 1 - kT_\Theta - \frac{k^2 T_\Theta^{\ 2}}{2}) + T_\Theta(\frac{H_v}{2}T_\Theta + \frac{H_v}{k})](1-x)e^{-kT_\Theta}e^x$$
$$\geq \frac{H_v D}{k^2}(x-1)(1 - e^{-kT_\Theta})e^x + D[\frac{H_v}{k^2}(e^{kT_\Theta} - 1 - kT_\Theta - \frac{k^2 T_\Theta^{\ 2}}{2}) + T_\Theta(\frac{H_v}{2}T_\Theta + \frac{H_v}{k})](1-x)e^{-kT_\Theta}e^x$$
$$= \frac{H_v D}{k^2}(x-1)(1 - e^{-kT_\Theta})e^x + H_v D[\frac{1}{k^2}(e^{kT_\Theta} - 1 - kT_\Theta - \frac{k^2 T_\Theta^{\ 2}}{2}) + \frac{T_\Theta}{k} + \frac{T_\Theta^{\ 2}}{2}](1-x)e^{-kT_\Theta}e^x$$
$$= \frac{H_v D}{k^2}(x-1)(1 - e^{-kT_\Theta})e^x + \frac{H_v D}{k^2}(e^{kT_\Theta} - 1)(1-x)e^{-kT_\Theta}e^x$$
$$= \frac{H_v D}{k^2}(x-1)(1 - e^{-kT_\Theta})e^x + \frac{H_v D}{k^2}(1 - e^{-kT_\Theta})(1-x)e^x = 0$$

Now consider the rare case of $x = kT_c \geq 1$.

$$\frac{H_b D}{k^2}[(x-1)e^x + 1] + D[\frac{H_b}{k^2}(1 - e^{kT_\Theta}) + T_\Theta(\frac{H_b - H_v}{2}T_\Theta + \frac{H_b}{k} - \frac{H_v}{k})](x-1)e^{-kT_\Theta}e^x$$
$$-\frac{H_v D}{k^2}[(x-1)e^{-kT_\Theta}e^x + 1]$$
$$= \frac{H_b D}{k^2}[(x-1)e^x + 1] - D[\frac{H_b}{k^2}(e^{kT_\Theta} - 1) - T_\Theta(\frac{H_b - H_v}{2}T_\Theta + \frac{H_b}{k} - \frac{H_v}{k})](x-1)e^{-kT_\Theta}e^x - \frac{H_v D}{k^2}[(x-1)e^{-kT_\Theta}e^x + 1]$$
$$= \frac{H_b D}{k^2}[(x-1)e^x + 1] - D[\frac{H_b}{k^2}(e^{kT_\Theta} - 1 - kT_\Theta - \frac{k^2 T_\Theta^{\ 2}}{2}) + T_\Theta(\frac{H_v}{2}T_\Theta + \frac{H_v}{k})](x-1)e^{-kT_\Theta}e^x$$
$$-\frac{H_v D}{k^2}[(x-1)e^{-kT_\Theta}e^x + 1]$$
$$= \frac{H_b D}{k^2}[(x-1)e^x + 1 - (e^{kT_\Theta} - 1 - kT_\Theta - \frac{k^2 T_\Theta^{\ 2}}{2})(x-1)e^{-kT_\Theta}e^x] - H_v D[(\frac{1}{2}T_\Theta^{\ 2} + \frac{T_\Theta}{k})](x-1)e^{-kT_\Theta}e^x$$
$$-\frac{H_v D}{k^2}[(x-1)e^{-kT_\Theta}e^x + 1]$$
$$= \frac{H_b D}{k^2}\{(x-1)e^x[1 - (e^{kT_\Theta} - 1 - kT_\Theta - \frac{k^2 T_\Theta^{\ 2}}{2})e^{-kT_\Theta}] + 1\} - H_v D[(\frac{1}{2}T_\Theta^{\ 2} + \frac{T_\Theta}{k})](x-1)e^{-kT_\Theta}e^x$$
$$-\frac{H_v D}{k^2}[(x-1)e^{-kT_\Theta}e^x + 1]$$
$$= \frac{H_b D}{k^2}[(x-1)e^x e^{-kT_\Theta}(1 + kT_\Theta + \frac{k^2 T_\Theta^{\ 2}}{2}) + 1] - \frac{H_v D}{k^2}(\frac{k^2 T_\Theta^{\ 2}}{2} + kT_\Theta)(x-1)e^{-kT_\Theta}e^x - \frac{H_v D}{k^2}[(x-1)e^{-kT_\Theta}e^x + 1]$$
$$\geq \frac{H_v D}{k^2}[(x-1)e^{-kT_\Theta}e^x + 1 + (x-1)e^{-kT_\Theta}e^x(kT_\Theta + \frac{k^2 T_\Theta^{\ 2}}{2})] - \frac{H_v D}{k^2}(\frac{k^2 T_\Theta^{\ 2}}{2} + kT_\Theta)(x-1)e^{-kT_\Theta}e^x$$
$$-\frac{H_v D}{k^2}[(x-1)e^{-kT_\Theta}e^x + 1]$$
$$= 0$$

Consider the terms involving C_b and C_v only. Since $(x-1)e^{-T_\Theta}e^x$ is an increasing function in x, we have

$$DT_\Theta(C_b - C_v)(x-1)e^{-kT_\Theta}e^x + \frac{D}{k}(C_b - C_v)[(x-1)e^{-kT_\Theta}e^x + 1]$$
$$> DT_\Theta(C_b - C_v)(-e^{-kT_\Theta}) + \frac{D}{k}(C_b - C_v)(1 - e^{-kT_\Theta})$$
$$= D(C_b - C_v)(-T_\Theta e^{-kT_\Theta} + \frac{1 - e^{-kT_\Theta}}{k})$$
$$= D(C_b - C_v)(\frac{1 - e^{-kT_\Theta} - kT_\Theta e^{-kT_\Theta}}{k}) > 0 \qquad (\#1)$$

Proof of (#1):

Let $f(y)=1-e^{-y}-ye^{-y}$ where $y \ge 0$.

$f'(y)=e^{-y}-e^{-y}+ye^{-y}=ye^{-y}>0$ and $f(0)=1-e^{-0}-(0)e^{-0}=0$

So $f(y)>0$ for positive y and (#1) is proved.

Hence,

$l_1[(kT_c-1)e^{kT_c}+1]+l_2(kT_c-1)e^{k(T_c-T_\Theta)}+l_3[(kT_c-1)e^{k(T_c-T_\Theta)}+1]+l_4(kT_c)^2e^{k(T_c-T_\Theta)}>0$ is proved.

$l_1[(kT_c-1)e^{kT_c}+1]+l_2(kT_c-1)e^{k(T_c-T_\Theta)}+l_3[(kT_c-1)e^{k(T_c-T_\Theta)}+1]+l_4(kT_c)^2e^{k(T_c-T_\Theta)}$ has the same derivative as $g(x)$.

$$
\begin{aligned}
g'(x) &= xe^x[l_1+e^{-kT_\Theta}(l_2+l_3+2l_4)+e^{-kT_\Theta}l_4x] \\
&> xe^x[l_1+e^{-kT_\Theta}(l_2+l_3+2l_4)] \\
&= Dxe^x\{\frac{H_b}{k^2}+e^{-kT_\Theta}[\frac{H_b}{k^2}(1-e^{kT_\Theta})+T_\Theta(\frac{H_b-H_v}{2}T_\Theta+\frac{H_b}{k}+C_b-\frac{H_v}{k}-C_v)+\frac{C_b-C_v}{k}-\frac{H_v}{k^2} \\
&\quad +2(\frac{H_vT_\Theta k+H_v+C_vk}{k^2})]\} \\
&= Dxe^x\{\frac{H_b}{k^2}e^{-kT_\Theta}+e^{-kT_\Theta}[T_\Theta(\frac{H_b-H_v}{2}T_\Theta+\frac{H_b}{k}+C_b-\frac{H_v}{k}-C_v)+\frac{C_b-C_v}{k}+\frac{H_v}{k^2} \\
&\quad +2(\frac{H_vT_\Theta k+C_vk}{k^2})]\}>0
\end{aligned}
$$

So $l_1[(kT_c-1)e^{kT_c}+1]+l_2(kT_c-1)e^{k(T_c-T_\Theta)}+l_3[(kT_c-1)e^{k(T_c-T_\Theta)}+1]+l_4(kT_c)^2e^{k(T_c-T_\Theta)}$

is positive, increasing and has no finite limit. So a unique solution x^* can be found for equation (5.6) for any $A_b+A_v>0$.

To show that the unique solution x^* gives the minimum of TC_s,

from $\frac{d}{dT_c}TC_s = \frac{g(x)}{T_c^2}$, we obtain $\frac{d^2}{dT_c^2}TC_s = \frac{kT_c^2 g'(x) - 2g(x)T_c}{T_c^4}$.

As $g(x^*) = 0$ and $g'(x^*) > 0$ as it is an increasing function, $\frac{d^2}{dT_c^2}TC_s > 0$ at $x = x^*$.

Therefore, the unique solution of equation (5.6) gives the minimum of TC_s. The convexity of TC_s in equation (5.5) is proved.

Appendix 2: Discussion for $x > 1$ in Section 5.3.2

For $x > 1$,

$$\frac{d}{dx}\{p_2[(x-1)e^x+1]+p_3(x+1)e^{-x}+p_4x^2e^x+p_5x^2e^{-x}\}$$
$$= p_2xe^x + p_4(x+2)xe^x + \frac{1}{k}(\frac{H_{vb}}{k}+C_{vb})xe^{-x} + \frac{H_{bb}+kC_{bb}}{k^2}(1-x)xe^{-x}$$
$$= \frac{D}{k}(\frac{H_{ba}}{k}+C_{ba}-\frac{H_{va}}{k}-C_{va})xe^x + \frac{D(H_{va}+kC_{va})}{k^2}(x+2)xe^x + \frac{1}{k}(\frac{H_{vb}}{k}+C_{vb})xe^{-x}$$
$$+\frac{H_{bb}+kC_{bb}}{k^2}(1-x)xe^{-x}$$
$$= \frac{D}{k}(\frac{H_{ba}}{k}+C_{ba})xe^x + \frac{D(H_{va}+kC_{va})}{k^2}(x+1)xe^x + \frac{1}{k}(\frac{H_{vb}}{k}+C_{vb})xe^{-x}$$
$$-\frac{H_{bb}+kC_{bb}}{k^2}(x-1)xe^{-x}$$

As $C_{bb} \geq C_{vb}$ and $H_{bb} \geq H_{vb}$, the derivative could be negative.

Suppose $C_{vb} \geq 0.5C_{bb}$ and $H_{vb} \geq 0.5H_{bb}$. Then for $x \leq 1.5$,

$$\frac{1}{k}(\frac{H_{vb}}{k}+C_{vb})xe^{-x} - \frac{H_{bb}+kC_{bb}}{k^2}(x-1)xe^{-x} \geq \frac{H_{bb}+kC_{bb}}{k^2}(0.5-x+1)xe^{-x} \geq 0 .$$

The first two positive terms in the derivative means that the derivative is still positive up to a certain value of x larger than 1.5.

Consider $y = (x-1)e^{-2x}$.

$$\frac{dy}{dx} = e^{-2x} - 2(x-1)e^{-2x} = e^{-2x}(3-2x)$$

$\frac{dy}{dx} = 0 \Rightarrow x = 1.5$ and the first derivative test shows that this is the maximum.

Hence, $(x-1)e^{-2x} \le 0.5e^{-3} \approx 0.0249$

$$
\begin{aligned}
&\frac{D}{k}(\frac{H_{ba}}{k}+C_{ba})xe^{x}-\frac{H_{bb}+kC_{bb}}{k^2}(x-1)xe^{-x} \\
&=\frac{xe^{x}}{k^2}[D(H_{ba}+kC_{ba})-(H_{bb}+kC_{bb})(x-1)e^{-2x}] \\
&\ge \frac{x}{k^2}[D(H_{ba}+kC_{ba})-0.5e^{-3}(H_{bb}+kC_{bb})]
\end{aligned}
$$

Therefore, if $D(H_{ba}+kC_{ba}) \ge 0.0249(H_{bb}+kC_{bb})$, the derivative will also be positive. (The second and third positive terms in the derivative means that the derivative is still positive when the proportion $D(H_{ba}+kC_{ba})/(H_{bb}+kC_{bb})$ is not less than some positive value which is smaller than 0.0249.)

Hence, it is very probable that the derivative is still positive for $x = kT_c > 1$. When this happens, the discussion for the case $x < 1$ in Section 5.3.2 applies and there is a unique solution that gives the minimum total cost per unit time.

Appendix A3: Derivation of total system inventory in Section 6.2.2

Total system inventory over one system cycle is given by

$$\int_{-T_0}^{T_1}\{\frac{1}{k}(P-\sum_{i=1}^{n_b}D_i)+[\sum_{i=1}^{n_b}\frac{D_i}{k}(e^{\frac{kT}{n_i}}-1)-\frac{1}{k}(P-\sum_{i=1}^{n_b}D_i)]e^{-kt}\}dt$$
$$+\int_{T_1}^{T-T_0}[-\frac{1}{k}\sum_{i=1}^{n_n}D_i+\frac{1}{k}\sum_{i=1}^{n_b}D_ie^{kT}e^{-kt}]dt$$

The equation just above equation (6.1) is given by $\frac{P}{k}(1-e^{-kT_0})=\sum_{i=1}^{n_b}\frac{D_i}{k}(e^{\frac{kT}{n_i}}-1)$.

Hence,

$$\begin{aligned}
&\int_{-T_0}^{T_1}\{\frac{1}{k}(P-\sum_{i=1}^{n_b}D_i)+[\sum_{i=1}^{n_b}\frac{D_i}{k}(e^{\frac{kT}{n_i}}-1)-\frac{1}{k}(P-\sum_{i=1}^{n_b}D_i)]e^{-kt}\}dt\\
&=\frac{1}{k}(P-\sum_{i=1}^{n_b}D_i)(T_1+T_0)-\frac{1}{k}[\sum_{i=1}^{n_b}\frac{D_i}{k}(e^{\frac{kT}{n_i}}-1)-\frac{1}{k}(P-\sum_{i=1}^{n_b}D_i)](e^{-kT_1}-e^{kT_0})\\
&=\frac{1}{k}(P-\sum_{i=1}^{n_b}D_i)(T_1+T_0)-\frac{1}{k}[\frac{P}{k}(1-e^{-kT_0})-\frac{1}{k}(P-\sum_{i=1}^{n_b}D_i)](e^{-kT_1}-e^{kT_0})\\
&=\frac{1}{k}(P-\sum_{i=1}^{n_b}D_i)(T_1+T_0)+\frac{P}{k^2}(e^{-kT_0})(e^{-kT_1}-e^{kT_0})-\frac{1}{k^2}(\sum_{i=1}^{n_b}D_i)(e^{-kT_1}-e^{kT_0}) \qquad (1)
\end{aligned}$$

$$\begin{aligned}
&\int_{T_1}^{T-T_0}[-\frac{1}{k}\sum_{i=1}^{n_b}D_i+\frac{1}{k}\sum_{i=1}^{n_b}D_ie^{kT}e^{-kt}]dt\\
&=-\frac{1}{k}\sum_{i=1}^{n_b}D_i(T-T_0-T_1)-\frac{1}{k^2}(\sum_{i=1}^{n_b}D_ie^{kT})[e^{-k(T-T_0)}-e^{-kT_1}]\\
&=-\frac{1}{k}\sum_{i=1}^{n_b}D_i(T-T_0-T_1)-\frac{1}{k^2}(\sum_{i=1}^{n_b}D_ie^{kT})[e^{kT_0}-e^{k(T-T_1)}] \qquad (2)
\end{aligned}$$

Adding equations (1) and (2),

$$\frac{P(T_1+T_0)}{k}-\frac{T}{k}\sum_{i=1}^{n_b}D_i+\frac{P}{k^2}(e^{-kT_0})(e^{-kT_1}-e^{kT_0})-\frac{1}{k^2}(\sum_{i=1}^{n_b}D_i)(e^{-kT_1}-e^{kT_0})$$

$$-\frac{1}{k^2}(\sum_{i=1}^{n_b}D_ie^{kT})[e^{kT_0}-e^{k(T-T_1)}]$$

$$=\frac{P(T_1+T_0)}{k}-\frac{T}{k}\sum_{i=1}^{n_b}D_i+\frac{P}{k^2}[e^{-k(T_0+T_1)}-1]+\frac{1}{k^2}(\sum_{i=1}^{n_b}D_i)[e^{k(T-T_1)}-e^{-kT_1}]$$

$$=\frac{PT_p}{k}-\frac{T}{k}\sum_{i=1}^{n_b}D_i+\frac{P}{k^2}(e^{-kT_p}-1)+\frac{1}{k^2}(\sum_{i=1}^{n_b}D_i)e^{-kT_1}(e^{kT}-1) \tag{3}$$

From equation (6.6), $e^{kT_p}=1+\dfrac{\sum_{i=1}^{n_b}\frac{D_i}{P}(e^{kT}-1)}{1-\sum_{i=1}^{n_b}\frac{D_i}{P}(e^{\frac{kT}{n_i}}-1)}=\dfrac{1+\sum_{i=1}^{n_b}\frac{D_i}{P}(e^{kT}-e^{\frac{kT}{n_i}})}{1-\sum_{i=1}^{n_b}\frac{D_i}{P}(e^{\frac{kT}{n_i}}-1)}$, therefore,

$$\frac{P}{k^2}(e^{-kT_p}-1)=\frac{P}{k^2}[\frac{1-\sum_{i=1}^{n_b}\frac{D_i}{P}(e^{\frac{kT}{n_i}}-1)}{1+\sum_{i=1}^{n_b}\frac{D_i}{P}(e^{kT}-e^{\frac{kT}{n_i}})}-1]=\frac{P}{k^2}[\frac{\sum_{i=1}^{n_b}\frac{D_i}{P}(1-e^{kT})}{1+\sum_{i=1}^{n_b}\frac{D_i}{P}(e^{kT}-e^{\frac{kT}{n_i}})}]$$

$$=\frac{1}{k^2}[\frac{\sum_{i=1}^{n_b}D_i(1-e^{kT})}{1+\sum_{i=1}^{n_b}\frac{D_i}{P}(e^{kT}-e^{\frac{kT}{n_i}})}]$$

From the equation above equation (6.6),

$e^{kT_1}=1+\sum_{i=1}^{n_b}\frac{D_i}{P}(e^{kT}-e^{\frac{kT}{n_i}})\Rightarrow e^{-kT_1}=\dfrac{1}{1+\sum_{i=1}^{n_b}\frac{D_i}{P}(e^{kT}-e^{\frac{kT}{n_i}})}$ and hence,

$$\frac{P}{k^2}(e^{-kT_p}-1)+\frac{1}{k^2}(\sum_{i=1}^{n_b} D_i)e^{-kT_1}(e^{kT}-1)$$

$$=\frac{1}{k^2}[\frac{\sum_{i=1}^{n_b} D_i(1-e^{kT})}{1+\sum_{i=1}^{n_b}\frac{D_i}{P}(e^{kT}-e^{\frac{kT}{n_i}})}]+\frac{1}{k^2}(\sum_{i=1}^{n_b} D_i)\frac{(e^{kT}-1)}{1+\sum_{i=1}^{n_b}\frac{D_i}{P}(e^{kT}-e^{\frac{kT}{n_i}})}=0$$

Therefore, only $\frac{PT_p}{k}-\frac{T}{k}\sum_{i=1}^{n_b} D_i$ leaves in the last expression of equation (3) and

$$\int_{-T_0}^{T_1}\{\frac{1}{k}(P-\sum_{i=1}^{n_b} D_i)+[\sum_{i=1}^{n_b}\frac{D_i}{k}(e^{\frac{kT}{n_i}}-1)-\frac{1}{k}(P-\sum_{i=1}^{n_b} D_i)]e^{-kt}\}dt$$
$$+\int_{T_1}^{T-T_0}[-\frac{1}{k}\sum_{i=1}^{n_n} D_i+\frac{1}{k}\sum_{i=1}^{n_b} D_i e^{kT}e^{-kt}]dt$$
$$=\frac{PT_p}{k}-\frac{T}{k}\sum_{i=1}^{n_b} D_i$$

The derivation of “Total system inventory over one system cycle is given by

$(PT_p-\sum_{i=1}^{n_b} D_i T)/k$.” is done.

Appendix A4: Proofs for (i) and (ii) in Section 6.2.3

$$T_p = \frac{1}{k}\ln\{1+\frac{\sum_{i=1}^{n_b}\frac{D_i}{P}(e^{kT}-1)}{1-\sum_{i=1}^{n_b}\frac{D_i}{P}(e^{\frac{kT}{n_i}}-1)}\}$$

$$T_{pi} = \frac{1}{k}\ln\{1+[\frac{D_i}{P}(e^{kT}-1)]/[1-\frac{D_i}{P}(e^{\frac{kT}{n_i}}-1)]\}$$

(i) $T_p > \sum_i T_{pi}$ if at most one of the n_i 's is not 1;

(ii) $T_p < \sum_i T_{pi}$ if $n_i \geq 2$ for all i.

First consider 2 buyers. Let $\rho_i = \frac{D_i}{P}$ and T_{pi} be the production time for meeting demand for one buyer alone, $i = 1,2$.

$$T_{p1}+T_{p2} = \frac{1}{k}\ln\{[1+\frac{\rho_1(e^{kT}-1)}{1-\rho_1(e^{\frac{kT}{n_1}}-1)}][1+\frac{\rho_2(e^{kT}-1)}{1-\rho_2(e^{\frac{kT}{n_2}}-1)}]\}$$

$$= \frac{1}{k}\ln\{1+\frac{\rho_1(e^{kT}-1)}{1-\rho_1(e^{\frac{kT}{n_1}}-1)}+\frac{\rho_2(e^{kT}-1)}{1-\rho_1(e^{\frac{kT}{n_2}}-1)}+\frac{\rho_1(e^{kT}-1)}{[1-\rho_1(e^{\frac{kT}{n_1}}-1)]}\frac{\rho_2(e^{kT}-1)}{[1-\rho_2(e^{\frac{kT}{n_2}}-1)]}\}$$

$$= \frac{1}{k}\ln\{1+\frac{\rho_1(e^{kT}-1)[1-\rho_2(e^{\frac{kT}{n_2}}-1)]+\rho_2(e^{kT}-1)[1-\rho_1(e^{\frac{kT}{n_1}}-1)]+\rho_1\rho_2(e^{kT}-1)^2}{[1-\rho_1(e^{\frac{kT}{n_1}}-1)][1-\rho_2(e^{\frac{kT}{n_2}}-1)]}\}$$

$$= \frac{1}{k}\ln\{1+\frac{\rho_1(e^{kT}-1)+\rho_2(e^{kT}-1)-\rho_1\rho_2(e^{kT}-1)(e^{\frac{kT}{n_2}}+e^{\frac{kT}{n_1}}-e^{kT}-1)}{1-\rho_1(e^{\frac{kT}{n_1}}-1)-\rho_2(e^{\frac{kT}{n_2}}-1)+\rho_1\rho_2(e^{\frac{kT}{n_1}}-1)(e^{\frac{kT}{n_2}}-1)}\}$$

The production time for meeting the demands of the two buyers together is

$$T_p = \frac{1}{k}\ln\{1 + \frac{\rho_1(e^{kT}-1)+\rho_2(e^{kT}-1)}{1-\rho_1(e^{\frac{kT}{n_1}}-1)-\rho_2(e^{\frac{kT}{n_1}}-1)}\}$$

(i) If at least one of n_1 and n_2 is one, then $e^{\frac{kT}{n_1}}+e^{\frac{kT}{n_2}}-e^{kT}-1>0$ and the expression inside the logarithm of $T_{p1}+T_{p2}$ has a smaller numerator but a larger denominator than the expression inside the logarithm of T_p.

Hence, $T_p > T_{p1}+T_{p2}$.

Suppose $n_1 = 1$ and $n_2 \geq 1$

$$T_{p1}+T_{p2} = \frac{1}{k}\ln\{1+\frac{\rho_1(e^{kT}-1)+\rho_2(e^{kT}-1)-\rho_1\rho_2(e^{kT}-1)(e^{\frac{kT}{n_2}}+e^{\frac{kT}{n_1}}-e^{kT}-1)}{1-\rho_1(e^{kT}-1)-\rho_2(e^{\frac{kT}{n_2}}-1)+\rho_1\rho_2(e^{kT}-1)(e^{\frac{kT}{n_2}}-1)}\}$$

$$= \frac{1}{k}\ln\{\frac{1+\rho_2(e^{kT}-e^{\frac{kT}{n_2}})}{1-\rho_1(e^{kT}-1)-\rho_2(e^{\frac{kT}{n_2}}-1)+\rho_1\rho_2(e^{kT}-1)(e^{\frac{kT}{n_2}}-1)}\}$$

$$T_p = \frac{1}{k}\ln\{1+\frac{\rho_1(e^{kT}-1)+\rho_2(e^{kT}-1)}{1-\rho_1(e^{kT}-1)-\rho_2(e^{\frac{kT}{n_2}}-1)}\} = \frac{1}{k}\ln\{\frac{1+\rho_2(e^{kT}-e^{\frac{kT}{n_2}})}{1-\rho_1(e^{kT}-1)-\rho_2(e^{\frac{kT}{n_2}}-1)}\}$$

$$T_p - (T_{p1} + T_{p2}) = \frac{1}{k} \ln\{\frac{1 - \rho_1(e^{kT} - 1) - \rho_2(e^{\frac{kT}{n_2}} - 1) + \rho_1\rho_2(e^{kT} - 1)(e^{\frac{kT}{n_2}} - 1)}{1 - \rho_1(e^{kT} - 1) - \rho_2(e^{\frac{kT}{n_2}} - 1)}\}$$

$$= \frac{1}{k} \ln\{1 + \frac{\rho_1\rho_2(e^{kT} - 1)(e^{\frac{kT}{n_2}} - 1)}{1 - \rho_1(e^{kT} - 1) - \rho_2(e^{\frac{kT}{n_2}} - 1)}\}$$

If $\rho_1 + \rho_2 = \rho$ is constant, then $\max(\rho_1\rho_2) = (\frac{\rho}{2})^2$. For $(e^{\frac{kT}{n_2}} - 1)$ to be largest, take $T = 1$ (year) and $n_2 = 1$. If $k = 0.1$ and $\rho_1 = \rho_2 = 0.25$,

$T_p - (T_{p1} + T_{p2}) = 0.007294$, 1.35% of T_p.

If $T = 0.5$, $T_p - (T_{p1} + T_{p2})$ is 0.65% of T_p.

If $n_2 > 1$ and/or $\rho_1 \neq \rho_2$ (keeping $\rho_1 + \rho_2$ constant) and/or reducing ρ, the percentage change will be even smaller. This illustrates that $T_p \approx T_{p1} + T_{p2}$.

From above, in particular, if $n_1 = n_2 = 1$, then $T_p > T_{p1} + T_{p2}$.

Now consider more than 2 buyers.

Suppose $T_p > T_{p1} + T_{p2} + ... + T_{pm}$ with $n_1 = n_2 = ... = n_m = 1$, i.e.,

$$1 + \frac{\sum_{i=1}^{m} \rho_i(e^{kT} - 1)}{1 - \sum_{i=1}^{m} \rho_i(e^{kT} - 1)} > \prod_{i=1}^{m} (1 + \frac{\rho_i(e^{kT} - 1)}{1 - \rho_i(e^{kT} - 1)})$$. Then we have

$$1+\frac{\sum_{i=1}^{m}\rho_i(e^{kT}-1)+\rho_{m+1}(e^{kT}-1)}{1-\sum_{i=1}^{m}\rho_i(e^{kT}-1)-\rho_{m+1}(e^{\frac{kT}{n_{m+1}}}-1)}-\prod_{i=1}^{m}(1+\frac{\rho_i(e^{kT}-1)}{1-\rho_i(e^{kT}-1)})(1+\frac{\rho_{m+1}(e^{kT}-1)}{1-\rho_{m+1}(e^{\frac{kT}{n_{m+1}}}-1)})$$

$$>1+\frac{\sum_{i=1}^{m}\rho_i(e^{kT}-1)+\rho_{m+1}(e^{kT}-1)}{1-\sum_{i=1}^{m}\rho_i(e^{kT}-1)-\rho_{m+1}(e^{\frac{kT}{n_{m+1}}}-1)}-[1+\frac{\sum_{i=1}^{m}\rho_i(e^{kT}-1)}{1-\sum_{i=1}^{m}\rho_i(e^{kT}-1)}](1+\frac{\rho_{m+1}(e^{kT}-1)}{1-\rho_{m+1}(e^{\frac{kT}{n_{m+1}}}-1)})$$

$$=\frac{1+\rho_{m+1}(e^{kT}-e^{\frac{kT}{n_{m+1}}})}{1-\sum_{i=1}^{m}\rho_i(e^{kT}-1)-\rho_{m+1}(e^{\frac{kT}{n_{m+1}}}-1)}-\frac{1}{1-\sum_{i=1}^{m}\rho_i(e^{kT}-1)}\frac{1+\rho_{m+1}(e^{kT}-e^{\frac{kT}{n_{m+1}}})}{1-\rho_{m+1}(e^{\frac{kT}{n_{m+1}}}-1)}$$

$$=\frac{[1+\rho_{m+1}(e^{kT}-e^{\frac{kT}{n_{m+1}}})]\{[1-\sum_{i=1}^{m}\rho_i(e^{kT}-1)][1-\rho_{m+1}(e^{kT}-1)]-[1-\sum_{i=1}^{m}\rho_i(e^{kT}-1)-\rho_{m+1}(e^{kT}-1)]\}}{[1-\sum_{i=1}^{m}\rho_i(e^{kT}-1)-\rho_{m+1}(e^{\frac{kT}{n_{m+1}}}-1)][1-\sum_{i=1}^{m}\rho_i(e^{kT}-1)][1-\rho_{m+1}(e^{\frac{kT}{n_{m+1}}}-1)]}$$

$$=\frac{[1+\rho_{m+1}(e^{kT}-e^{\frac{kT}{n_{m+1}}})]\{\rho_{m+1}(e^{kT}-1)[\sum_{i=1}^{m}\rho_i(e^{kT}-1)]\}}{[1-\sum_{i=1}^{m}\rho_i(e^{kT}-1)-\rho_{m+1}(e^{\frac{kT}{n_{m+1}}}-1)][1-\sum_{i=1}^{m}\rho_i(e^{kT}-1)][1-\rho_{m+1}(e^{\frac{kT}{n_{m+1}}}-1)]}>0$$

So $T_p > T_{p1}+T_{p2}+...+T_{pm}+T_{p(m+1)}$ with $n_1=n_2=...=n_m=1,\ n_{m+1}\geq 1$.

"$T_p > \sum_i T_{pi}$ if at most one of the n_i's is not 1." is proved.

(ii) If $n_1 \geq 2$ and $n_2 \geq 2$,

$$e^{\frac{kT}{n_1}} + e^{\frac{kT}{n_2}} - e^{kT} - 1 \leq e^{\frac{kT}{2}} + e^{\frac{kT}{2}} - e^{kT} - 1 = -(e^{\frac{kT}{2}} - 1)^2 < 0 .$$

$$T_{p1} + T_{p2} = \frac{1}{k}\ln\{1 + \frac{\rho_1(e^{kT}-1) + \rho_2(e^{kT}-1) + \rho_1\rho_2(e^{kT}-1)(1+e^{kT} - e^{\frac{kT}{n_2}} - e^{\frac{kT}{n_1}})}{1-\rho_1(e^{\frac{kT}{n_1}}-1) - \rho_2(e^{\frac{kT}{n_2}}-1) + \rho_1\rho_2(e^{\frac{kT}{n_1}}-1)(e^{\frac{kT}{n_2}}-1)}\} .$$

We make use of "If quantities a, b, c, d are positive and $bc - ad > 0$, then $\frac{a+b}{c+d} - \frac{a}{c} = \frac{bc-ad}{c(c+d)} > 0$." and set the following:

$$b = \rho_1\rho_2(e^{kT}-1)(1+e^{kT} - e^{\frac{kT}{n_2}} - e^{\frac{kT}{n_1}}) \qquad c = [1-\rho_1(e^{\frac{kT}{n_1}}-1) - \rho_2(e^{\frac{kT}{n_2}}-1)]$$

$$a = [\rho_1(e^{kT}-1) + \rho_2(e^{kT}-1)] \qquad d = \rho_1\rho_2(e^{\frac{kT}{n_1}}-1)(e^{\frac{kT}{n_2}}-1)$$

$$bc - ad$$
$$= 1 + e^{kT} - e^{\frac{kT}{n_2}} - e^{\frac{kT}{n_1}} + \rho_1(e^{kT} + e^{\frac{2kT}{n_1}} - e^{kT+\frac{kT}{n_1}} - e^{\frac{kT}{n_1}}) + \rho_2(e^{kT} + e^{\frac{2kT}{n_2}} - e^{kT+\frac{kT}{n_2}} - e^{\frac{kT}{n_2}})$$

Also for non-continuous production, $P > D_1 e^{\frac{kT}{n_1}} + D_2 e^{\frac{kT}{n_2}}$ **or** $1 > \rho_1 e^{\frac{kT}{n_1}} + \rho_2 e^{\frac{kT}{n_2}}$**.**

(Actually, it requires $P > D_1 e^{kT} + D_2 e^{kT}$ allowing $n_1 = n_2 = 1$.)

$$bc - ad$$
$$> (\rho_1 e^{\frac{kT}{n_1}} + \rho_2 e^{\frac{kT}{n_2}})(1+e^{kT} - e^{\frac{kT}{n_2}} - e^{\frac{kT}{n_1}}) + \rho_1(e^{kT} + e^{\frac{2kT}{n_1}} - e^{kT+\frac{kT}{n_1}} - e^{\frac{kT}{n_1}})$$
$$+ \rho_2(e^{kT} + e^{\frac{2kT}{n_2}} - e^{kT+\frac{kT}{n_2}} - e^{\frac{kT}{n_2}})$$
$$= \rho_1(e^{kT} - e^{\frac{kT}{n_1}+\frac{kT}{n_2}}) + \rho_2(e^{kT} - e^{\frac{kT}{n_1}+\frac{kT}{n_2}}) \geq \rho_1(e^{kT} - e^{\frac{kT}{2}+\frac{kT}{2}}) + \rho_2(e^{kT} - e^{\frac{kT}{2}+\frac{kT}{2}}) = 0$$

$$\frac{\rho_1(e^{kT}-1)+\rho_2(e^{kT}-1)+\rho_1\rho_2(e^{kT}-1)(1+e^{kT}-e^{\frac{kT}{n_2}}-e^{\frac{kT}{n_1}})}{1-\rho_1(e^{\frac{kT}{n_1}}-1)-\rho_2(e^{\frac{kT}{n_2}}-1)+\rho_1\rho_2(e^{\frac{kT}{n_1}}-1)(e^{\frac{kT}{n_2}}-1)}$$

$$>\frac{\rho_1(e^{kT}-1)+\rho_2(e^{kT}-1)}{1-\rho_1(e^{\frac{kT}{n_1}}-1)-\rho_2(e^{\frac{kT}{n_1}}-1)}$$

$$\therefore\ T_{p1}+T_{p2}>T_p$$

Revisit the expressions

$$T_{p1}+T_{p2}=\frac{1}{k}\ln\{1+\frac{\rho_1(e^{kT}-1)+\rho_2(e^{kT}-1)+\rho_1\rho_2(e^{kT}-1)(1+e^{kT}-e^{\frac{kT}{n_2}}-e^{\frac{kT}{n_1}})}{1-\rho_1(e^{\frac{kT}{n_1}}-1)-\rho_2(e^{\frac{kT}{n_2}}-1)+\rho_1\rho_2(e^{\frac{kT}{n_1}}-1)(e^{\frac{kT}{n_2}}-1)}\}\ \text{and}$$

$$T_p=\frac{1}{k}\ln\{1+\frac{\rho_1(e^{kT}-1)+\rho_2(e^{kT}-1)}{1-\rho_1(e^{\frac{kT}{n_1}}-1)-\rho_2(e^{\frac{kT}{n_1}}-1)}\}.$$

It can be shown by Mean Value Theorem that

$\ln(1+x_1)-\ln(1+x_2)<x_1-x_2$ **if** $x_1>x_2>0$.

If the difference between the expressions inside the logarithms of $T_{p1}+T_{p2}$ and T_p is $\vartriangle x$, then the difference between $T_{p1}+T_{p2}$ and T_p will be less than $\frac{1}{k}(\vartriangle x)$.

As an illustration, take $k=0.1$, $n_1=n_2=2$, $T=1$, $\rho_1=\rho_2=0.25$.

$$\frac{\rho_1\rho_2(e^{kT}-1)(1+e^{kT}-e^{\frac{kT}{n_2}}-e^{\frac{kT}{n_1}})}{\rho_1(e^{kT}-1)+\rho_2(e^{kT}-1)}=0.000329$$

$$\frac{\rho_1\rho_2(e^{\frac{kT}{n_1}}-1)(e^{\frac{kT}{n_2}}-1)}{1-\rho_1(e^{\frac{kT}{n_1}}-1)-\rho_2(e^{\frac{kT}{n_2}}-1)}=0.000169$$

The difference between the expressions inside the logarithms of $T_{p1}+T_{p2}$ and T_p is very small as the numerator and denominator are almost unchanged.

$$\frac{\rho_1(e^{kT}-1)+\rho_2(e^{kT}-1)+\rho_1\rho_2(e^{kT}-1)(1+e^{kT}-e^{\frac{kT}{n_2}}-e^{\frac{kT}{n_1}})}{1-\rho_1(e^{\frac{kT}{n_1}}-1)-\rho_2(e^{\frac{kT}{n_2}}-1)+\rho_1\rho_2(e^{\frac{kT}{n_1}}-1)(e^{\frac{kT}{n_2}}-1)}-\frac{\rho_1(e^{kT}-1)+\rho_2(e^{kT}-1)}{1-\rho_1(e^{\frac{kT}{n_1}}-1)-\rho_2(e^{\frac{kT}{n_1}}-1)}$$
$$=8.63(10^{-6})$$

So the difference between $T_{p1}+T_{p2}$ and T_p is smaller than $8.63(10^{-5})$. The actual difference is $8.19(10^{-5})$ which is 0.0156% of T_p.

However, if the number of deliveries increase, the percentage increases, e.g. with $n_1=n_2=3$ and other parameters unchanged, it goes up to 0.4481%.

To get the upper bound of this percentage, allowing $n_1,n_2\to\infty$, then

$$T_{p1}+T_{p2}\to\frac{1}{k}\ln\{1+\rho_1(e^{kT}-1)+\rho_2(e^{kT}-1)+\rho_1\rho_2(e^{kT}-1)^2\}\text{ and}$$

$$T_p\to\frac{1}{k}\ln\{1+\rho_1(e^{kT}-1)+\rho_2(e^{kT}-1)\}.$$

Take $k=0.1$, $T=1$, and $\rho_1=\rho_2=0.25$, $\dfrac{T_{p1}+T_{p2}-T_p}{T_p}\to 1.28\%$

Again if $\rho_1\neq\rho_2$, e.g., $\rho_1=0.3$ and $\rho_2=0.2$, it drops to 1.23%.

Now consider more than 2 buyers.

Suppose $T_{p1}+T_{p2}+...+T_{pm}>T_p$ with $n_i\geq 2$ for $i=1,2,...,m$, i.e.,

$\prod_{i=1}^{m}(1+\dfrac{\rho_i(e^{kT}-1)}{1-\rho_i(e^{\frac{kT}{n_i}}-1)})>1+\dfrac{\sum_{i=1}^{m}\rho_i(e^{kT}-1)}{1-\sum_{i=1}^{m}\rho_i(e^{\frac{kT}{n_i}}-1)}$. Then we have

$$\prod_{i=1}^{m}[1+\frac{\rho_i(e^{kT}-1)}{1-\rho_i(e^{\frac{kT}{n_i}}-1)}][1+\frac{\rho_{m+1}(e^{kT}-1)}{1-\rho_{m+1}(e^{\frac{kT}{n_{m+1}}}-1)}]-[1+\frac{\sum_{i=1}^{m}\rho_i(e^{kT}-1)+\rho_{m+1}(e^{kT}-1)}{1-\sum_{i=1}^{m}\rho_i(e^{\frac{kT}{n_i}}-1)-\rho_{m+1}(e^{\frac{kT}{n_{m+1}}}-1)}]$$

$$>[1+\frac{\sum_{i=1}^{m}\rho_i(e^{kT}-1)}{1-\sum_{i=1}^{m}\rho_i(e^{\frac{kT}{n_i}}-1)}][1+\frac{\rho_{m+1}(e^{kT}-1)}{1-\rho_{m+1}(e^{\frac{kT}{n_{m+1}}}-1)}]-[1+\frac{\sum_{i=1}^{m}\rho_i(e^{kT}-1)+\rho_{m+1}(e^{kT}-1)}{1-\sum_{i=1}^{m}\rho_i(e^{\frac{kT}{n_i}}-1)-\rho_{m+1}(e^{\frac{kT}{n_{m+1}}}-1)}]$$

$$=[\frac{1+\sum_{i=1}^{m}\rho_i(e^{kT}-e^{\frac{kT}{n_i}})}{1-\sum_{i=1}^{m}\rho_i(e^{\frac{kT}{n_i}}-1)}][\frac{1+\rho_{m+1}(e^{kT}-e^{\frac{kT}{n_{m+1}}})}{1-\rho_{m+1}(e^{\frac{kT}{n_{m+1}}}-1)}]-[\frac{1+\sum_{i=1}^{m}\rho_i(e^{kT}-e^{\frac{kT}{n_i}})+\rho_{m+1}(e^{kT}-e^{\frac{kT}{n_{m+1}}})}{1-\sum_{i=1}^{m}\rho_i(e^{\frac{kT}{n_i}}-1)-\rho_{m+1}(e^{\frac{kT}{n_{m+1}}}-1)}]$$

The above expression is of the form $\dfrac{a+b}{c+d}-\dfrac{a}{c}$ where

$$a = 1 + \sum_{i=1}^{m} \rho_i (e^{kT} - e^{\frac{kT}{n_i}}) + \rho_{m+1}(e^{kT} - e^{\frac{kT}{n_{m+1}}}) > 0$$

$$b = \rho_{m+1}(e^{kT} - e^{\frac{kT}{n_{m+1}}})[\sum_{i=1}^{m} \rho_i (e^{kT} - e^{\frac{kT}{n_i}})] > 0$$

$$c = 1 - \sum_{i=1}^{m} \rho_i (e^{\frac{kT}{n_i}} - 1) - \rho_{m+1}(e^{\frac{kT}{n_{m+1}}} - 1) > 0$$

$$d = \rho_{m+1}(e^{\frac{kT}{n_{m+1}}} - 1)[\sum_{i=1}^{m} \rho_i (e^{\frac{kT}{n_i}} - 1)] > 0$$

$$\begin{aligned}
& bc - ad \\
&= \rho_{m+1}\{(e^{kT} - e^{\frac{kT}{n_{m+1}}})[\sum_{i=1}^{m} \rho_i (e^{kT} - e^{\frac{kT}{n_i}})][1 - \sum_{j=1}^{m} \rho_j (e^{\frac{kT}{n_j}} - 1) - \rho_{m+1}(e^{\frac{kT}{n_{m+1}}} - 1)] \\
&\quad - (e^{\frac{kT}{n_{m+1}}} - 1)[\sum_{i=1}^{m} \rho_i (e^{\frac{kT}{n_i}} - 1)][1 + \sum_{j=1}^{m} \rho_j (e^{kT} - e^{\frac{kT}{n_j}}) + \rho_{m+1}(e^{kT} - e^{\frac{kT}{n_{m+1}}})]\} \\
&= \rho_{m+1}\left\{\sum_{i=1}^{m} \rho_i \{(e^{kT} - e^{\frac{kT}{n_{m+1}}})(e^{kT} - e^{\frac{kT}{n_i}})[1 - \sum_{j=1}^{m} \rho_j (e^{\frac{kT}{n_j}} - 1) - \rho_{m+1}(e^{\frac{kT}{n_{m+1}}} - 1)]\right. \\
&\quad \left. - (e^{\frac{kT}{n_{m+1}}} - 1)(e^{\frac{kT}{n_i}} - 1)[1 + \sum_{j=1}^{m} \rho_j (e^{kT} - e^{\frac{kT}{n_j}}) + \rho_{m+1}(e^{kT} - e^{\frac{kT}{n_{m+1}}})]\}\right\}
\end{aligned}$$

For each i, we have:

$$\begin{aligned}
& (e^{kT} - e^{\frac{kT}{n_{m+1}}})(e^{kT} - e^{\frac{kT}{n_i}})[1 - \sum_{j=1}^{m+1} \rho_j (e^{\frac{kT}{n_j}} - 1)] - (e^{\frac{kT}{n_{m+1}}} - 1)(e^{\frac{kT}{n_i}} - 1)[1 + \sum_{j=1}^{m+1} \rho_j (e^{kT} - e^{\frac{kT}{n_j}})] \\
&= e^{2kT} - e^{kT + \frac{kT}{n_{m+1}}} - e^{kT + \frac{kT}{n_i}} + e^{\frac{kT}{n_{m+1}} + \frac{kT}{n_i}} - e^{\frac{kT}{n_{m+1}} + \frac{kT}{n_i}} + e^{\frac{kT}{n_{m+1}}} + e^{\frac{kT}{n_i}} - 1 \\
&\quad - (e^{2kT} - e^{kT + \frac{kT}{n_{m+1}}} - e^{kT + \frac{kT}{n_i}} + e^{\frac{kT}{n_{m+1}} + \frac{kT}{n_i}}) \sum_{j=1}^{m+1} \rho_j (e^{\frac{kT}{n_j}} - 1) \\
&\quad - (e^{\frac{kT}{n_{m+1}} + \frac{kT}{n_i}} - e^{\frac{kT}{n_{m+1}}} - e^{\frac{kT}{n_i}} + 1) \sum_{j=1}^{m+1} \rho_j (e^{kT} - e^{\frac{kT}{n_j}}) \qquad \textbf{(\#)}
\end{aligned}$$

Define y as

$$
\begin{aligned}
y &= e^{2kT} - e^{kT+\frac{kT}{n_{m+1}}} - e^{kT+\frac{kT}{n_i}} + e^{\frac{kT}{n_{m+1}}+\frac{kT}{n_i}} - e^{\frac{kT}{n_{m+1}}+\frac{kT}{n_i}} + e^{\frac{kT}{n_{m+1}}} + e^{\frac{kT}{n_i}} - 1 \\
&= e^{2kT} - (e^{kT+\frac{kT}{n_{m+1}}} - e^{\frac{kT}{n_{m+1}}}) - (e^{kT+\frac{kT}{n_i}} - e^{\frac{kT}{n_i}}) - 1 \\
&= e^{2kT} - e^{\frac{kT}{n_{m+1}}}(e^{kT}-1) - e^{\frac{kT}{n_i}}(e^{kT}-1) - 1 \\
&\geq e^{2kT} - e^{\frac{kT}{2}}(e^{kT}-1) - e^{\frac{kT}{2}}(e^{kT}-1) - 1 \\
&= e^{2kT} - 2e^{\frac{3kT}{2}} - 2e^{\frac{kT}{2}} - 1
\end{aligned}
$$

Let $f(z) = e^{2z} - 2e^{\frac{3z}{2}} + 2e^{\frac{z}{2}} - 1$. $f(0) = 0$

$$
\begin{aligned}
f'(z) &= 2e^{2z} - 3e^{\frac{3z}{2}} + e^{\frac{z}{2}} & f'(0) &= 0 \\
f''(z) &= 4e^{2z} - \frac{9}{2}e^{\frac{3z}{2}} + \frac{1}{2}e^{\frac{z}{2}} & f''(0) &= 0 \\
f'''(z) &= 8e^{2z} - \frac{27}{4}e^{\frac{3z}{2}} + \frac{1}{4}e^{\frac{z}{2}} > 0 \ \textbf{ for } z \geq 0
\end{aligned}
$$

Hence, for $z > 0$, $f''(z) > 0$, $f'(z) > 0$, $f(z) > 0$, and so $y > 0$ for $T > 0$.

It requires $P > \sum_{j=1}^{m+1} De^{\frac{kT}{n_j}}$ **or** $1 > \sum_{j=1}^{m+1} \rho_j e^{\frac{kT}{n_j}}$ and we have

$$
\begin{aligned}
y &= (e^{2kT} - e^{kT+\frac{kT}{n_{m+1}}} - e^{kT+\frac{kT}{n_i}} + e^{\frac{kT}{n_{m+1}}+\frac{kT}{n_i}} - e^{\frac{kT}{n_{m+1}}+\frac{kT}{n_i}} + e^{\frac{kT}{n_{m+1}}} + e^{\frac{kT}{n_i}} - 1)(1) \\
&> (e^{2kT} - e^{kT+\frac{kT}{n_{m+1}}} - e^{kT+\frac{kT}{n_i}} + e^{\frac{kT}{n_{m+1}}+\frac{kT}{n_i}} - e^{\frac{kT}{n_{m+1}}+\frac{kT}{n_i}} + e^{\frac{kT}{n_{m+1}}} + e^{\frac{kT}{n_i}} - 1)(\sum_{j=1}^{m+1} \rho_j e^{\frac{kT}{n_j}})
\end{aligned}
$$

(#) becomes

$$
\begin{aligned}
&(e^{kT}-e^{\frac{kT}{n_{m+1}}})(e^{kT}-e^{\frac{kT}{n_i}})[1-\sum_{j=1}^{m+1}\rho_j(e^{\frac{kT}{n_j}}-1)]-(e^{\frac{kT}{n_{m+1}}}-1)(e^{\frac{kT}{n_i}}-1)[1+\sum_{j=1}^{m+1}\rho_j(e^{kT}-e^{\frac{kT}{n_j}})]\\
&>(e^{2kT}-e^{kT+\frac{kT}{n_{m+1}}}-e^{kT+\frac{kT}{n_i}}+e^{\frac{kT}{n_{m+1}}+\frac{kT}{n_i}})[(\sum_{j=1}^{m+1}\rho_j e^{\frac{kT}{n_j}}-\sum_{j=1}^{m+1}\rho_j(e^{\frac{kT}{n_j}}-1)]\\
&-(e^{\frac{kT}{n_{m+1}}+\frac{kT}{n_i}}-e^{\frac{kT}{n_{m+1}}}-e^{\frac{kT}{n_i}}+1)[\sum_{j=1}^{m+1}\rho_j e^{\frac{kT}{n_j}}+\sum_{j=1}^{m+1}\rho_j(e^{kT}-e^{\frac{kT}{n_j}})]\\
&=(e^{2kT}-e^{kT+\frac{kT}{n_{m+1}}}-e^{kT+\frac{kT}{n_i}}+e^{\frac{kT}{n_{m+1}}+\frac{kT}{n_i}})(\sum_{j=1}^{m+1}\rho_j)-(e^{\frac{kT}{n_{m+1}}+\frac{kT}{n_i}}-e^{\frac{kT}{n_{m+1}}}-e^{\frac{kT}{n_i}}+1)[\sum_{j=1}^{m+1}\rho_j(e^{kT})]\\
&=\sum_{j=1}^{m+1}\rho_j(e^{2kT}+e^{\frac{kT}{n_{m+1}}+\frac{kT}{n_i}}-e^{kT+\frac{kT}{n_{m+1}}+\frac{kT}{n_i}}-e^{kT})\geq\sum_{j=1}^{m+1}\rho_j[e^{2kT}-(e^{kT}-1)e^{\frac{kT}{2}+\frac{kT}{2}}-e^{kT}]\\
&=\sum_{j=1}^{m+1}\rho_j[e^{2kT}-e^{2kT}+e^{kT}-e^{kT}]=0
\end{aligned}
$$

$bc-ad>0$ is proved and it can be concluded that $T_p<\sum_i T_{pi}$ if $n_i\geq 2$ for all i.

Appendix B: Data and Results

Appendix B1: Data Sets – Parameters for Supply Chains

(1): Deterioration rate k is common for the vendor and the buyers.

(2): A_{vi} is the vendor's order processing and shipment cost per delivery for buyer i.

vendor	k (1)	P('000)	S	C_v	H_v
	0.1	300	1,000	10	1
Buyer i	A_{vi} (2)	D_i ('000)	A_{bi}	C_{bi}	H_{bi}
1	100	10	50	12	1.2
2	110	20	60	13	1.3
3	120	30	70	14	1.4
4	130	40	80	15	1.5
5	140	50	90	16	1.6

Table B1-1: Parameters for Supply Chain S1

vendor	k (1)	P('000)	S	C_v	H_v
	0.1	300	1,000	10	1
Buyer i	A_{vi} (2)	D_i ('000)	A_{bi}	C_{bi}	H_{bi}
1	140	10	90	16	1.6
2	130	20	80	15	1.5
3	120	30	70	14	1.4
4	110	40	60	13	1.3
5	100	50	50	12	1.2

Table B1-2: Parameters for Supply Chain S2

vendor	k (1)	P('000)	S	C_v	H_v
	0.1	550	6,000 (S3) 1,000 (S4)	10	1
Buyer i	A_{vi} (2)	D_i ('000)	A_{bi}	C_{bi}	H_{bi}
1	190	5	90	21	2.1
2	180	10	85	20	2.0
3	170	15	80	19	1.9
4	160	20	75	18	1.8
5	150	25	70	17	1.7
6	140	30	65	16	1.6
7	130	35	60	15	1.5
8	120	40	55	14	1.4
9	110	45	50	13	1.3
10	100	50	45	12	1.2

Table B1-3: Parameters for Supply Chains S3 and S4.

vendor	k (1)	P('000)	S	C_v	H_v
	0.1	1,100 (S5) 1,600 (S6) 2,100 (S7)	10,000	10	1
Buyer i	A_{vi} (2)	D_i ('000)	A_{bi}	C_{bi}	H_{bi}
1	164	11.3	90	19	1.9
2	175	27.5	69	18	1.98
3	136	27.95	50	16	2.56
4	178	10.85	82	18	1.98
5	112	44.15	60	16	1.76
6	176	14.45	52	15	1.8
7	126	29.75	62	14	2.52
8	125	28.4	90	12	2.04
9	160	47.75	53	13	2.34
10	159	8.6	52	15	2.55
11	162	43.25	90	14	2.24
12	136	48.65	59	15	1.5
13	108	12.65	70	17	2.21
14	108	41.45	76	13	1.82
15	183	13.1	86	20	2
16	146	26.15	79	14	2.24
17	107	25.7	77	14	1.96
18	182	13.55	59	16	1.92
19	179	37.4	72	15	2.7
20	180	24.35	77	15	1.8

Table B1-4: Parameters for Supply Chains S5, S6 and S7.

vendor	k (1)	P('000)	S	C_v	H_v
	0.1	300	1,000	10	1
Buyer i	A_{vi} (2)	D_i ('000)	A_{bi}	C_{bi}	H_{bi}
1	120	10	70	14	1.4
2	120	20	70	14	1.4
3	120	30	70	14	1.4
4	120	40	70	14	1.4
5	120	50	70	14	1.4

Table B1-5: Parameters for Supply Chain S8

vendor	k (1)	P('000)	S	C_v	H_v
	0.1	300	1,000	10	1
Buyer i	A_{vi} (2)	D_i ('000)	A_{bi}	C_{bi}	H_{bi}
1	140	10	90	16	1.6
2	110	20	60	13	1.3
3	120	30	70	14	1.4
4	130	40	80	15	1.5
5	100	50	50	12	1.2

Table B1-6: Parameters for Supply Chain S9

vendor	k (1)	P('000)	S	C_v	H_v
	0.1	300	1,000	10	1
Buyer i	A_{vi} (2)	D_i ('000)	A_{bi}	C_{bi}	H_{bi}
1	100	10	50	12	1.2
2	130	20	80	15	1.5
3	120	30	70	14	1.4
4	110	40	60	13	1.3
5	140	50	90	16	1.6

Table B1-7: Parameters for Supply Chain S10

vendor	k (1)	P('000)	S	C_v	H_v
	0.1	30	1,000	10	1
Buyer i	A_{vi} (2)	D_i ('000)	A_{bi}	C_{bi}	H_{bi}
1	100	1	50	12	1.2.
2	110	2	60	13	1.3
3	120	3	70	14	1.4
4	130	4	80	15	1.5
5	140	5	90	16	1.6

Table B1-8: Parameters for Supply Chain S11

vendor	k (1)	P('000)	S	C_v	H_v
	0.1	300	3,000	10	1
Buyer i	A_{vi} (2)	D_i ('000)	A_{bi}	C_{bi}	H_{bi}
1	100	10	50	12	1.2
2	110	20	60	13	1.3
3	120	30	70	14	1.4
4	130	40	80	15	1.5
5	140	50	90	16	1.6

Table B1-9: Parameters for Supply Chain S12

vendor	k (1)	P('000)	S	C_v	H_v
	0.1	300	3,000	10	1
Buyer i	A_{vi} (2)	D_i ('000)	A_{bi}	C_{bi}	H_{bi}
1	120	10	70	14	1.4
2	120	20	70	14	1.4
3	120	30	70	14	1.4
4	120	40	70	14	1.4
5	120	50	70	14	1.4

Table B1-10: Parameters for Supply Chain S13

vendor	k (1)	P('000)	S	C_v	H_v
	0.1	300	3,000	10	1
Buyer i	A_{vi} (2)	D_i ('000)	A_{bi}	C_{bi}	H_{bi}
1	140	10	90	16	1.6
2	130	20	80	15	1.5
3	120	30	70	14	1.4
4	110	40	60	13	1.3
5	100	50	50	12	1.2

Table B1-11: Parameters for Supply Chain S14

vendor	k (1)	P('000)	S	C_v	H_v
	0.1	300	3,000	10	1
Buyer i	A_{vi} (2)	D_i ('000)	A_{bi}	C_{bi}	H_{bi}
1	140	10	90	16	1.6
2	110	20	60	13	1.3
3	120	30	70	14	1.4
4	130	40	80	15	1.5
5	100	50	50	12	1.2

Table B1-12: Parameters for Supply Chain S15

vendor	k (1)	P('000)	S	C_v	H_v
	0.1	300	3,000	10	1
Buyer i	A_{vi} (2)	D_i ('000)	A_{bi}	C_{bi}	H_{bi}
1	100	10	50	12	1.2
2	130	20	80	15	1.5
3	120	30	70	14	1.4
4	110	40	60	13	1.3
5	140	50	90	16	1.6

Table B1-13: Parameters for Supply Chain S16

vendor	k (1)	P('000)	S	C_v	H_v
	0.1	30	3,000	10	1
Buyer i	A_{vi} (2)	D_i ('000)	A_{bi}	C_{bi}	H_{bi}
1	100	1	50	12	1.2
2	110	2	60	13	1.3
3	120	3	70	14	1.4
4	130	4	80	15	1.5
5	140	5	90	16	1.6

Table B1-14: Parameters for Supply Chain S17

vendor	k (1)	P(‘000)	S	C_v	H_v
	0.1	550	1,000 (S18) 6,000 (S19)	10	1
Buyer i	A_{vi} (2)	D_i (‘000)	A_{bi}	C_{bi}	H_{bi}
1	100	5	45	12	1.2
2	110	10	50	13	1.3
3	120	15	55	14	1.4
4	130	20	60	15	1.5
5	140	25	65	16	1.6
6	150	30	70	17	1.7
7	160	35	75	18	1.8
8	170	40	80	19	1.9
9	180	45	85	20	2.0
10	190	50	90	21	2.1

Table B1-15: Parameters for Supply Chains S18 and S19.

vendor	k (1)	P(‘000)	S	C_v	H_v
	0.1	550	1,000 (S18) 6,000 (S19)	10	1
Buyer i	A_{vi} (2)	D_i (‘000)	A_{bi}	C_{bi}	H_{bi}
1	145	5	67.5	16.5	1.65
2	145	10	67.5	16.5	1.65
3	145	15	67.5	16.5	1.65
4	145	20	67.5	16.5	1.65
5	145	25	67.5	16.5	1.65
6	145	30	67.5	16.5	1.65
7	145	35	67.5	16.5	1.65
8	145	40	67.5	16.5	1.65
9	145	45	67.5	16.5	1.65
10	145	50	67.5	16.5	1.65

Table B1-16: Parameters for Supply Chains S20 and S21.

vendor	k (1)	P(‘000)	S	C_v	H_v
	0.1	1,100 (S22) 2,100 (S23)	10,000	10	1
Buyer i	A_{vi} (2)	D_i (‘000)	A_{bi}	C_{bi}	H_{bi}
1	186	37.85	72	18	3.42
2	161	17.6	69	12	1.68
3	126	7.7	60	19	3.61
4	139	17.15	74	20	2.8
5	196	17.15	56	20	2.2
6	144	20.75	84	14	2.38
7	189	49.1	79	19	2.66
8	105	47.75	68	13	2.08
9	130	5.45	58	19	3.23
10	148	13.55	70	18	2.52
11	159	16.25	68	20	3.4
12	129	25.25	53	20	3.8
13	117	10.85	58	13	1.95
14	121	30.2	88	20	2.4
15	112	35.6	76	15	1.8
16	114	32.9	58	15	3
17	154	32	72	14	1.68
18	194	14.9	65	19	3.23
19	152	38.75	71	14	2.66
20	168	46.85	62	16	2.72

Table B1-17: Parameters for Supply Chains S22, and S23.

vendor	k (1)	P('000)	S	C_v	H_v
	0.1	1,200	10,000	10	1
Buyer i	A_{vi} (2)	D_i ('000)	A_{bi}	C_{bi}	H_{bi}
1	145	8.15	75	14	1.4
2	131	42.8	83	17	2.89
3	104	43.7	66	12	1.68
4	200	38.3	82	19	2.66
5	191	49.55	59	17	2.55
6	200	28.4	80	15	1.8
7	189	26.6	63	19	1.9
8	124	43.7	82	14	1.68
9	149	5.45	60	15	2.55
10	176	26.6	61	20	2.2
11	138	25.25	62	12	1.56
12	173	33.35	85	17	1.7
13	180	16.7	86	18	1.98
14	177	17.6	76	17	1.87
15	100	26.6	66	14	1.4
16	191	13.55	73	16	3.04
17	191	30.2	54	19	1.9
18	192	42.8	72	17	1.87
19	137	13.55	82	15	2.55
20	138	44.6	68	16	1.92

Table B1-18: Parameters for Supply Chain S24.

vendor	k (1)	P('000)	S	C_v	H_v
	0.1	1,000	20,000	20	2
Buyer i	A_{vi} (2)	D_i ('000)	A_{bi}	C_{bi}	H_{bi}
1	226	34.25	110	38	7.6
2	272	49.1	168	32	6.08
3	220	47.75	168	38	4.94
4	342	36.95	161	36	5.76
5	367	8.6	144	36	5.76
6	213	21.65	175	26	5.2
7	325	47.3	147	31	3.72
8	347	12.65	153	25	5
9	321	20.75	100	27	3.78
10	288	6.35	147	31	4.96
11	333	40.1	130	40	6.4
12	363	21.65	168	24	4.56
13	399	48.65	157	32	5.44
14	243	16.25	126	29	4.06
15	230	5.9	136	39	6.63
16	200	37.85	157	40	4.4
17	294	7.25	127	36	6.48
18	326	6.35	176	32	4.16
19	317	8.6	159	26	2.86
20	204	15.8	166	35	5.95

Table B1-19: Parameters for Supply Chain S25.

vendor	k (1)	P('000)	S	C_v	H_v
	0.1	1,100	20,000	20	2
Buyer i	A_{vi} (2)	D_i ('000)	A_{bi}	C_{bi}	H_{bi}
1	277	10.4	120	33	5.28
2	383	41.45	109	39	4.29
3	310	5.45	162	29	2.9
4	244	26.15	150	26	3.38
5	301	5.9	139	36	4.68
6	313	27.5	171	35	5.95
7	397	20.3	168	40	6.4
8	354	17.6	140	30	5.7
9	340	9.5	108	37	5.18
10	205	18.05	109	27	4.59
11	378	11.75	139	40	4.4
12	259	21.65	118	29	4.35
13	375	48.2	163	27	4.59
14	305	42.35	115	37	4.44
15	344	24.35	154	31	5.58
16	398	42.8	142	40	8
17	315	32	138	40	5.2
18	269	29.75	125	36	5.76
19	298	44.6	148	40	5.2
20	294	40.1	130	37	7.03

Table B1-20: Parameters for Supply Chain S26.

vendor	k (1)	P('000)	S	C_v	H_v
	0.1	1,000	20,000	20	2
Buyer i	A_{vi} (2)	D_i ('000)	A_{bi}	C_{bi}	H_{bi}
1	386	42.8	158	32	4.16
2	329	7.25	139	31	3.41
3	300	7.7	115	27	3.51
4	382	9.05	153	38	6.46
5	253	13.1	103	36	4.68
6	302	20.75	120	33	4.95
7	349	36.95	180	35	5.95
8	304	30.65	142	37	6.29
9	248	37.4	120	24	2.88
10	370	8.15	172	36	3.6
11	293	49.1	116	28	3.08
12	343	13.1	163	29	5.8
13	246	41.45	177	34	4.76
14	388	9.05	166	25	4.25
15	273	28.85	173	32	5.76
16	293	32.9	173	25	3.25
17	208	5.9	146	24	4.8
18	290	32.45	137	27	3.51
19	273	17.15	151	26	2.86
20	321	15.8	159	26	5.2

Table B1-21: Parameters for Supply Chain S27.

vendor	k (1)	P(‘000)	S	C_v	H_v
	0.1	1,100	10,000	10	1
Buyer i	A_{vi} (2)	D_i (‘000)	A_{bi}	C_{bi}	H_{bi}
1	65	11.3	67	19	1.9
2	122	27.5	56	18	1.98
3	75	27.95	56	16	2.56
4	68	10.85	81	18	1.98
5	130	44.15	67	16	1.76
6	145	14.45	107	15	1.8
7	183	29.75	66	14	2.52
8	180	28.4	83	12	2.04
9	130	47.75	140	13	2.34
10	100	8.6	150	15	2.55
11	125	43.25	142	14	2.24
12	64	48.65	131	15	1.5
13	96	12.65	112	17	2.21
14	176	41.45	135	13	1.82
15	166	13.1	98	20	2
16	139	26.15	89	14	2.24
17	104	25.7	108	14	1.96
18	84	13.55	151	16	1.92
19	162	37.4	73	15	2.7
20	79	24.35	153	15	1.8

Table B1-22: Parameters for Supply Chain S28.

vendor	k (1)	P(‘000)	S	C_v	H_v
	0.1	1,100	10,000	10	1
Buyer i	A_{vi} (2)	D_i (‘000)	A_{bi}	C_{bi}	H_{bi}
1	143	37.85	78	18	3.42
2	87	17.6	174	12	1.68
3	193	7.7	77	19	3.61
4	118	17.15	100	20	2.8
5	163	17.15	72	20	2.2
6	150	20.75	131	14	2.38
7	196	49.1	111	19	2.66
8	81	47.75	110	13	2.08
9	150	5.45	84	19	3.23
10	74	13.55	197	18	2.52
11	181	16.25	89	20	3.4
12	135	25.25	86	20	3.8
13	96	10.85	55	13	1.95
14	93	30.2	136	20	2.4
15	144	35.6	197	15	1.8
16	52	32.9	177	15	3
17	57	32	157	14	1.68
18	51	14.9	128	19	3.23
19	147	38.75	147	14	2.66
20	98	46.85	195	16	2.72

Table B1-23: Parameters for Supply Chain S29.

vendor	k (1)	P('000)	S	C_v	H_v
	0.1	1,200	10,000	10	1
Buyer i	A_{vi} (2)	D_i ('000)	A_{bi}	C_{bi}	H_{bi}
1	188	8.15	80	14	1.4
2	171	42.8	170	17	2.89
3	179	43.7	112	12	1.68
4	200	38.3	126	19	2.66
5	95	49.55	79	17	2.55
6	83	28.4	73	15	1.8
7	53	26.6	140	19	1.9
8	160	43.7	65	14	1.68
9	158	5.45	149	15	2.55
10	106	26.6	188	20	2.2
11	154	25.25	116	12	1.56
12	188	33.35	53	17	1.7
13	152	16.7	199	18	1.98
14	165	17.6	118	17	1.87
15	170	26.6	118	14	1.4
16	188	13.55	51	16	3.04
17	77	30.2	175	19	1.9
18	81	42.8	105	17	1.87
19	159	13.55	145	15	2.55
20	104	44.6	125	16	1.92

Table B1-24: Parameters for Supply Chain S30.

Appendix B2: Comparison of Results from the Algorithm and GA

Supply Chain		N^* / N^{GA}	n_i^* / n_i^{GA}					TC_s^* / TC_s^{GA} ($/year)	% difference of the costs
S1	Algorithm	44	1	2	2	2	2	45910.20	
	GA	42	1	2	2	2	2	45943.44	0.0724
S2	Algorithm	40	1	1	2	2	2	44224.63	
	GA	42	1	1	2	2	2	44248.69	0.0544
S8	Algorithm	44	1	2	2	2	2	45095.71	
	GA	46	1	2	2	2	2	45149.70	0.1197
S9	Algorithm	44	1	1	1	2	2	44522.79	
	GA	46	1	2	2	2	2	44580.25	0.1291
S10	Algorithm	40	1	1	1	1	2	45612.56	
	GA	44	1	2	2	2	2	45668.62	0.1229
S11	Algorithm	138	1	1	1	2	2	14567.07	
	GA	134	1	2	2	2	2	14570.74	0.0252
S12	Algorithm	72	2	3	3	3	4	58469.43	
	GA	72	2	3	3	3	4	58469.43	0
S13	Algorithm	72	2	2	3	3	4	57604.53	
	GA	72	2	2	3	3	4	57604.53	0
S14	Algorithm	72	2	2	3	4	4	56734.04	
	GA	72	2	2	3	4	4	56734.04	0
S15	Algorithm	72	2	3	3	3	4	57066.21	
	GA	72	2	3	3	3	4	57066.21	0
S16	Algorithm	72	2	2	3	4	4	58137.26	
	GA	66	2	2	3	3	3	58284.60	0.2534
S17	Algorithm	240	2	3	3	4	4	18520.20	
	GA	236	2	2	4	4	4	18613.45	0.5035

Table B2-1: Comparison of results from the algorithm and GA for 5-buyer supply chains.

Supply Chain		N^{*} / N^{GA}	n_i^{*} / n_i^{GA}										TC_s^{*} / TC_s^{GA} ($/year)	% difference of the costs
S3	Algorithm	72	1	2	2	3	3	3	3	4	4	4	114215.70	
	GA	72	1	2	2	3	3	3	3	4	4	4	114215.70	0
S4	Algorithm	36	1	1	1	1	1	2	2	2	2	2	81382.24	
	GA	32	1	1	1	1	1	1	2	2	2	2	81687.06	0.3745
S18	Algorithm	38	1	1	1	1	2	2	2	2	2	2	87853.39	
	GA	36	1	1	1	1	1	2	2	2	2	2	87896.64	0.0492
S19	Algorithm	72	1	2	2	3	3	3	3	3	4	4	120853.90	
	GA	84	2	2	3	3	3	4	4	4	4	4	120985.94	0.1093
S20	Algorithm	38	1	1	1	1	2	2	2	2	2	2	84627.54	
	GA	30	1	1	1	1	1	1	1	2	2	2	85356.35	0.8612
S21	Algorithm	72	1	2	2	3	3	3	3	4	4	4	117538.70	
	GA	76	1	2	2	2	4	4	4	4	4	4	118005.46	0.3971

Table B2-2: Comparison of results from the algorithm and GA for 10-buyer supply chains.

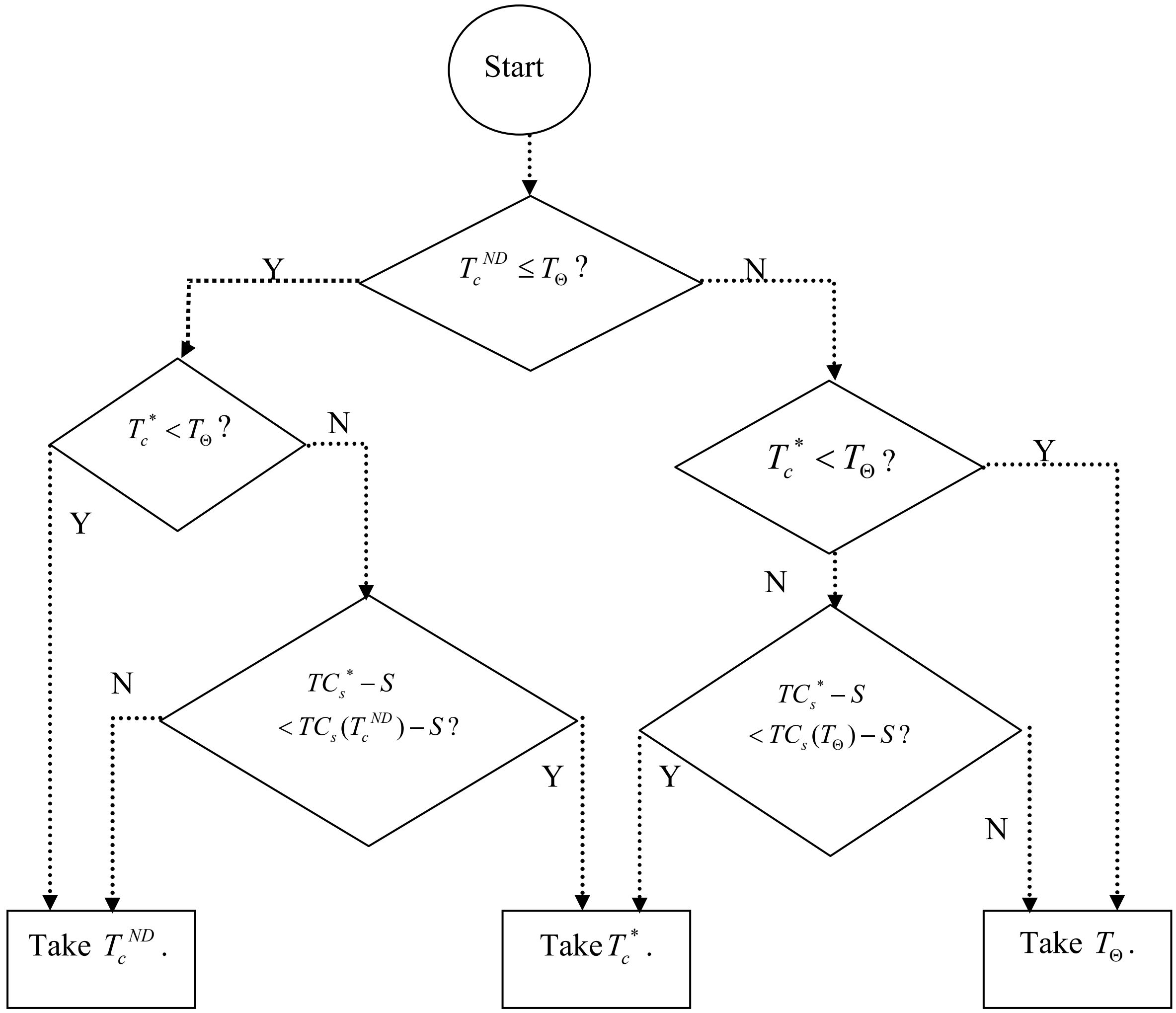
Supply Chain		N^{*} / N^{GA}	n_i^{*} / n_i^{GA}																				TC_s^{*} / TC_s^{GA} ($/year)	% difference of the costs
S5	Algorithm	72	2	3	4	2	4	2	4	3	4	2	4	4	2	4	2	3	3	2	4	3	230296.10	
	GA	72	2	3	4	2	4	2	4	3	4	2	4	4	2	4	2	3	3	2	4	3	230296.10	0
S6	Algorithm	60	1	2	3	1	3	2	3	2	3	1	3	3	2	3	2	2	2	2	3	2	234682.50	
	GA	66	2	2	3	2	3	2	3	2	3	2	3	3	2	3	2	2	3	2	3	2	234733.49	0.0217
S7	Algorithm	54	1	2	2	1	3	1	2	2	3	1	2	2	2	2	1	2	2	1	2	2	235434.80	
	GA	48	1	2	2	1	2	1	2	2	2	1	2	2	1	2	1	2	2	1	2	1	236976.35	0.6548
S22	Algorithm	72	4	2	2	3	2	3	4	4	2	2	3	4	2	4	3	4	3	2	4	4	236671.00	
	GA	78	3	2	2	3	3	3	6	6	2	3	3	6	2	3	3	6	3	3	3	6	240355.83	1.5569
S23	Algorithm	60	3	1	2	2	2	2	3	3	1	2	2	3	1	3	2	3	2	2	3	3	242151.50	
	GA	60	3	1	2	2	2	2	3	3	1	2	2	3	1	3	2	3	2	2	3	3	242151.50	0
S24	Algorithm	72	2	4	4	3	4	3	3	4	2	3	3	3	2	2	3	2	3	3	2	4	245311.80	
	GA	84	2	6	4	4	4	3	3	4	2	4	3	3	3	3	3	3	4	4	3	4	248149.20	1.1567

Table B2-3: Comparison of results from the algorithm and GA for 20-buyer supply chains.

Supply Chain		N^{*} / N^{GA}	n_i^{*} / n_i^{GA}																				TC_s^{*} / TC_s^{GA} ($/year)	% difference of the costs
S25	Algorithm	72	4	4	4	4	2	3	4	2	3	2	4	2	4	2	2	4	2	1	1	3	456849.20	
	GA	75	5	5	5	3	3	3	3	3	3	1	5	3	5	3	3	5	3	1	1	3	462051.55	1.1387
S26	Algorithm	72	2	4	1	3	2	3	3	2	2	3	2	3	4	4	3	4	4	4	4	4	484195.30	
	GA	84	2	4	1	3	2	4	3	3	2	3	2	3	4	4	3	6	4	4	4	6	488373.26	0.8629
S27	Algorithm	72	3	1	2	2	2	3	4	4	3	2	4	2	4	2	3	3	2	3	2	2	433471.40	
	GA	72	3	1	2	2	2	3	4	4	3	2	4	2	4	2	3	3	2	3	2	2	433471.40	0
S28	Algorithm	72	3	3	4	2	4	2	3	3	4	2	3	4	2	3	2	3	3	2	4	3	231993.10	
	GA	60	2	3	3	2	3	2	3	2	3	1	3	3	2	2	2	2	2	2	3	2	233619.44	0.7010
S29	Algorithm	72	4	2	2	3	2	2	4	4	2	2	3	4	2	3	3	4	3	3	3	4	245960.70	
	GA	72	4	2	2	3	2	2	4	4	2	2	3	4	2	3	3	4	3	3	3	4	245960.70	0
S30	Algorithm	72	1	3	3	3	4	3	3	3	1	3	2	3	2	2	2	2	3	4	2	4	251097.10	
	GA	66	1	3	3	3	6	3	3	3	1	3	2	3	2	2	2	2	3	3	2	3	251137.93	0.0163

Table B2-4: Comparison of results from the algorithm and GA for 20-buyer supply chains (continued).

Appendix C: Flowchart for the Solution Procedure in Section 5.2.3



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