MULTI-PERIOD MEAN-VARIANCE ASSET-LIABILITY PORTFOLIO SELECTION

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Certificate of Originality

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______________________________(Signed)

WU Xianping (Name of student)

Dedicate to my parents.

Abstract

The thesis is concerned with multi-period mean-variance asset-liability portfolio selection. It is a nonseparable problem in the sense of dynamic programming as it cannot be decomposed by a stage-wise backward recursion. In this thesis, we resort to tackling the nonseparability of the problem and seeking analytical optimal solutions and efficient frontiers.

On the one hand, we formulate the mean-variance model by fixing the terminal mean and deal with it using the parameterized method. By a variable substitution and Lagrange multiplier method, we can turn the nonseparable problem to a solvable stochastic linear quadratic optimal control problem. One prominent feature of the dynamic mean-variance formulations is that the optimal portfolio policy is always linear with respect to the current wealth and liability. According to this feature, we derive the analytical optimal policies and efficient frontiers. The analytical form of the Lagrange multiplier is also given in expression of the expectation of the final surplus. The results are much more explicit and accurate compared with the similar model solved by the embedding technique. It is worth mentioning that the relationship of returns between the assets and liability plays an important role in the whole derivation. We consider different cases such as the returns of assets of liability are stochastically correlated at the same period and in different periods as well as uncorrelated, compare their differences and illustrate their effects on optimal strategy and efficient frontier theoretically and numerically.

On the other hand, by putting weights on the two criteria, we transform the mean-variance problem into a single-objective optimization problem. Instead of the parameterized method, we employ the mean-field formulation to solve different asset-liability mean-variance model with various constraints such as uncertain exit time, and bankruptcy control, respectively. In fact, when uncertain exit time or bankruptcy are considered in the model, the parameterized method and the embedding technique will not work smoothly. We shed light on the efficiency and accuracy of mean-field formulation when dealing with the issue of dynamic nonseparability in those models. By taking "mean" of the constraints and some simple calculation, the state space and the control space are enlarged in the language of optimal control. The objective function then becomes separable in the expanded space which enables us to solve the problem by dynamic programming. The analytical form of optimal policy and efficient frontier are derived. It is showed that when the uncertain exit time reduces to terminal exit time or the control over bankruptcy is left out and deterministic expected return is taken, the results of the parameterized method and mean-field formulation are proved to be the same. This further suggests that the two approaches to solve multi-period mean-variance model are accurate.

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Contents

Notation

A'	the transpose of matrix (or vector) A.
A^{-1}	the inverse of matrix A.
A^{+}	the Moore-Penrose pseudoinverse of matrix A
$\|M\|$	the determinant of square matrix M
I_n	identity matrix of dimension n
s_t	the return rate of the riskless asset at time period t
e_t^i	the return rate for asset i at time period t
p_t^i	the excess return rate for asset i at time period t
q_t	the return rate of the liability at time period t
α_t	the probability mass function at time period t
x_t	the wealth of the investor at the beginning of the t-th time period
l_t	the liability of the investor at the beginning of the t-th time period
π_t^i	the amount invested in the i-th risky asset at the beginning of the t-th time period

Chapter 1

Introduction

1.1 Background

Most of us own a portfolio of assets, which may include real assets, such as a house, a car, or a laptop, and financial assets, such as stocks and bonds. Portfolio selection, which is concerned with finding the most desirable group of funds to hold, plays an important role in the process of gathering wealth. Rational investors prefer a higher expected return as well as a lower risk. However, the portfolio with maximum expected return is not always the one with lowest risk. Mean-variance portfolio selection refers to the design of optimal portfolios balancing the gain with the risk, which are in expression of expectation and variance of the final return, respectively. In order to trace out the efficient frontier for this bi-objective optimization problem, a typically method is to put weights on the two criteria and transform the problem into a single-objective optimization problem.

The mean-variance framework of portfolio selection originated by Markowitz, the 1990 Nobel Laureate in Economics. The principles introduced in Markowitz (1952) are still at the core of many modern approaches for asset allocation, investment analysis and risk management. In recent years, research on mean-variance portfolio selection problems have been well developed. Li and Ng (2000) extended Markowitz's model in single period to dynamic version and derived analytical solution by the embedding technique. Costa and Nabholz (2007) generalized the results of Li and Ng (2000) for the case in which the intermediate variances and expected values of the portfolio are also considered in the performance criterion and/or constraints. Zhou and Li (2000) introduced the stochastic linear quadratic control as a general framework to study the continuous-time mean-variance portfolio selection problem and obtained analytical optimal policy and explicit expression of efficient frontier. Li et al. (2002) developed it to a constrained one where short-selling is not allowed. Yin and Zhou (2004) studied a discrete-time mean-variance portfolio selection problem where the market parameters depend on the market mode (regime) that jumps among a finite number of states and revealed their relationship with the continuous-time counterparts. Czichowsky (2013) developed a time-consistent formulation of mean-variance portfolio selection problem based on a local notion of optimality called local mean-variance efficiency in a general semimartingale setting for both discrete and continuous time cases. Cui et al. (2014) presented a mean-field formulation to tackle the multi-period mean-variance portfolio selection problem and derived analytical optimal strategies and efficient frontiers. Pang et al. (2014) considered continuous mean-variance portfolio selection under partial information by dynamic programming approach through exploiting the properties of the filtering process and the wealth process.

Asset-liability management is a financial tool for an investor that sets out to maximize their wealth. The aim of asset-liability management is to reduce risk as well as increase returns and it has been used successfully for banks, pension funds, insurance companies and wise individuals. A judicious investment considers assets and liabilities simultaneously. A financial institution taking liabilities into account can operate more soundly and lucratively. Krouse (1970) noticed that many mean-variance models concentrated only upon to assets with little or no effort being directed to the liabilities. The mean-variance framework of asset-liability management was first investigated by Sharpe and Tint (1990) in single-period setting. Leippold et al. (2004) derived the closed form optimal policies and mean-variance frontiers under exogenous and endogenous liabilities using a geometric approach; Chiu and Li (2006) employed the stochastic optimal control theory to analytically solve the asset-liability management in a continuous time setting; Xie et al. (2008) considered the situation in an incomplete market by using the general stochastic linear-quadratic control technique. Chen and Yang (2011) studied the case with regime switching; Zeng and Li (2011) investigated the model under benchmark and mean-variance criteria in a jump diffusion market. Wu and Li (2012) considered the regime switching and cash flow together in the model.

An important assumption of the simple portfolio selection models is that the investment time horizon is deterministic, which means that the investor determines the exit time at the beginning of the investment. In the real world, however, the investor might be forced to abandon his or her original investment plan for some unexpected events or accidents, such as sudden huge consumption, serious illness, retirement and etc. Therefore, it seems more realistic to relax the restrictive assumption that the investment horizon is pre-determined with certainty. Yaari (1965) formulated an optimal consumption problem for an individual with an uncertain date of death, under a pure deterministic investment environment. Hakansson (1969) extended Yaari's work to a multi-period setting with a risky asset and an uncertain exit time. Merton (1971) introduced an uncertain retiring time into a dynamic optimal investment and consumption problem, where the uncertain time was defined as the first jump of an independent Poisson process. Li and Xie (2010) incorporated a market-related exogenous uncertain time horizon into a continuous-time mean-variance portfolio selection problem. Yi et al. (2008) investigated a multi-period mean-variance portfolio selection problem with an uncertain exit time. Wu and Li (2011) studied a multi-period mean-variance portfolio selection problem with regime switching and an uncertain exit time. Yao et al. (2013) considered an asset-liability management problem under a multi-period mean-variance model with uncontrolled cash flow and uncertain time-horizon.

Due to the volatility of the financial market, it is impossible to eliminate the possibility of bankruptcy in multi-period investment setting. We assume in this thesis bankruptcy occurs when the surplus (total wealth minus liability) falls below a preset level. Once an investor goes bankruptcy, he/she will suffer a great loss such as retrieve part of his/her wealth (even take nothing back), high liability and low credit. It is crucial for a successful investment to take bankruptcy into account. Zhu et al. (2004) generalized the multi-period mean-variance model by considering a good risk control over bankruptcy. Bielecki et al. (2005) studied the continuous-time mean-variance problem with bankruptcy prohibition. Wei and Ye (2007) studied the multi-period optimization portfolio with bankruptcy control when the random returns of risky assets depend on the state of the stochastic market. Wu and Zeng (2013) investigated the case in a regime-switching market.

Most studies above are under a circumstance that the time and returns are independent. In fact, the returns of risky assets or liability always exhibit certain degree of dependency among different time periods. Correlated returns are necessary and meaningful to be considered in the mean-variance portfolio selection. Since the model becomes difficult to solve, there are a few works about it in the literature. Balvers and Mitchell (1997) was the first to derive an explicit analytical solution to the dynamic portfolio problem when the returns are autocorrelated by a normal ARMA(1,1) process. Xu and Li (2008) investigated a dynamic portfolio selection in a market with only one risky asset and one risk-free asset and Zhang and Li (2012) extended it to the case with uncertain exit time. Gao and Li (2014) considered the capital market consisting of all risky assets. By embedding technique, all the last three derived analytical optimal strategies.

We now get an overview of the analytical solution of the mean-variance portfolio selection problem. For the single period model proposed by Markowitz (1952), Merton (1972) gave the analytic solution in the case where the covariance matrix is positive definite and short-selling is allowed. However, a multi-period or continuous-time treatment is considerably more delicate. In order to solve this dynamic problem in dynamic programming approach, it must satisfy the principle of optimality: An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision(See Bellman (2010)). In other words, the problem should be separable so that the objective function can be decomposed by a stage-wise backward recursion. Since the variance term in multi-period mean-variance model is nonlinear with respect to the expected wealth, it does not satisfy the smoothing property, i.e.,

$$\mathrm{Var}\big(\mathrm{Var}(\cdot|\mathcal{F}_i)|\mathcal{F}_j\big) \neq \mathrm{Var}(\cdot|\mathcal{F}_j), \quad \forall\ i > j,$$

where $\mathcal{F}_j$ is the information set available at time j and $\mathcal{F}_{j-1} \subset \mathcal{F}_j$. So the multi-period mean-variance problem is nonseparable in the sense of dynamic programming. All traditional dynamic programming-based optimal stochastic control solution methods are then invalid. The main approaches to tackle it include martingale theory, embedding technique, parameterized methoed and mean-field formulation. Let us first review the embedding technique by Li and Ng (2000)) in detail which is widely used to solve the noseparability (See Leippold et al. (2004), Chiu and Li (2006), Yi et al. (2008), Li and Xie (2010), Zhang and Li (2012), Yao et al. (2013) etc.). Suppose that the capital market consists of one risk-free asset and n risky assets with given return s_t and random return $\mathbf{e}_t = [e_t^1, \cdots, e_t^n]'$. An investor joining the market at the beginning of period 0 with initial wealth x_0 and plans to invest his/her wealth within a time horizon T. Then the multi-period mean-variance portfolio selection problem can be formulated as (see Li and Ng (2000)):

$$
\begin{aligned}
MV(\omega) \quad & \max_{\pi} \mathbb{E}(x_T) - \omega \mathrm{Var}(x_T), \\
& \text{s.t.} x_{t+1} = s_t\left(x_t - \sum_{i=1}^{n} \pi_t^i\right) + \sum_{i=1}^{n} e_t^i \pi_t^i \\
& \qquad = s_t x_t + \mathbf{P}_t' \pi_t, \quad t = 0, 1, 2, \cdots, T-1.
\end{aligned}
$$

where x_t denotes the wealth of the investor at the beginning of period t, π_t^i denotes the amount invested in the ith risky asset at the beginning of period t, $\omega > 0$ is the trade-off parameter between the mean and variance representing the degree of the investor's risk aversion. This is hard to solve directly by dynamic programming approach. Adopting an embedding scheme, they considered instead a family of auxiliary problems, $\mathcal{A}(\omega, \lambda)$, parameterized in λ,

$$
\begin{aligned}
\mathcal{A}(\omega, \lambda) \quad & \min_{\pi} \ \mathbb{E}(\omega x_T^2 - \lambda x_T), \\
& \text{s.t.} \ \ x_{t+1} = s_t x_t + \mathbf{P}_t' \pi_t \quad t = 0, 1, 2, \cdots, T-1.
\end{aligned}
$$

Note that problem $\mathcal{A}(\omega, \lambda)$ is a separable linear-quadratic stochastic control formulation and can be thus solved analytically. The optimal solution to the original problem can the be located via the solution to the auxiliary problem.

The second method is the parameterized method. By introducing an auxiliary variable d and an equality constraint $\mathbb{E}(x_T) = d$ for the expected terminal wealth, Li et al. (2002) studied the following slightly modified and equivalent version of $(MV(\omega))$ (the no-shorting constraint is omitted here),

$$
\begin{aligned}
(MV(d)) \quad & \min_{\pi} \mathrm{Var}(x_T) = \mathbb{E}(x_T - d)^2, \\
& \text{s.t.} \ \mathbb{E}(x_T) = d, \\
& \qquad x_{t+1} = s_t x_t + \mathbf{P}_t' \pi_t \quad t = 0, 1, 2, ..., T-1.
\end{aligned}
$$

Introducing a Lagrangian multiplier λ and applying Lagrangian relaxation to $(MV(d))$

give rise to the following linear quadratic stochastic control (LQSC) problem,

$$
\begin{aligned}
(L(\lambda)) \quad & \min \ \mathbb{E}(x_T - d)^2 - \lambda \mathbb{E}(x_T - d), \\
& \text{s.t.} \ \ x_{t+1} = s_t x_t + \mathbf{P}_t' \pi_t \quad t = 0, 1, 2, \cdots, T-1.
\end{aligned}
$$

By maximizing the dual function $L(\lambda)$ over all Lagrangian multiplier $\lambda \in \mathbb{R}$, we can derive the optimal policy of $(MV(d))$. Set $\gamma = d + \lambda/2$, the Lagrangian problem $(L(\lambda))$ can be further written as the following LQSC problem,

$$
\begin{aligned}
(MVH(\gamma)) \quad & \min \ \mathbb{E}(x_T - \gamma)^2, \\
& \text{s.t.} \ x_{t+1} = s_t x_t + \mathbf{P}_t' \pi_t \quad t = 0, 1, 2, \cdots, T-1,
\end{aligned}
$$

which is a special mean-variance hedging problem. Under a quadratic objective function, the investor can hedge the target γ by his/her portfolio. $(MVH(\gamma))$ has been well studied and can be solved by LQSC theory (see Li et al. (2002)), martingale/convex duality theory (see Schweizer et al. (1996), Xia and Yan (2006)) and sequential regression method (see Černỳ and Kallsen (2009)).

The third method is the mean-filed formulation approach developed by Cui et al. (2014). The so-called *mean-field* type stochastic control problem refers to the problem where either the objective functional or the dynamic system involves state processes and their expectations. Note that the multi-period or continuous-time Markowitz-type mean-variance portfolio selection problems are typical mean-field type stochastic control problems, where the variance term appears as a quadratic function of the expected terminal state. In this line of literature, the theory of the mean-field optimal controls for forward systems has been well established and extensively applied, especially to mean-field LQ control problems proposed by Yong (2013) and some financial applications such as those studied in Li et al. (2002), Li and Zhou (2006), Fu et al. (2010). Despite the active research efforts in recent years (see Meyer2012, Nourian et al. (2013)), the related topic of mean-field formulations for multi-period mean-variance models remains a relatively new and largely unexplored area. In Cui et al. (2014), they developed a unified framework of mean-field formulations to investigate three multi-period mean-variance models in the literature: the classical multi-period mean-variance model in Li and Ng (2000), the multi-period mean-variance model with intertemporal restrictions in Costa and Nabholz (2007), and the generalized mean-variance model with risk control over bankruptcy in Zhu et al. (2004). They demonstrated that the mean-field approach represents a new promising way in dealing with nonseparable stochastic control problems related to the mean-variance formulations and even improves solution quality of some existing results in the literature.

1.2 Contributions and organization of the Thesis

In this thesis, we study asset-liability management under a multi-period mean-variance portfolio selection framework. The main difficulty to solve the problem is the nonseparability. As mentioned above, most multi-period mean-variance models derive the analytical optimal policies based on the embedding technique. One of the prominent features of the embedding technique is that it builds a bridge between multi-period portfolio selection problems and standard stochastic control models. Embedding scheme is indeed an efficient way to deal with problems with the nonseparable property. However, it is prone to involve inefficient and complicated calculation during the derivation of the optimal strategies and efficient frontiers by embedding since an auxiliary problem should be built and a long list of notation should be established, especially when adding some constraints such as asset-liability management, uncertain exit time and risk control over bankruptcy and/or serial correlated returns. We resort to exploring new method to solve the multi-period asset-liability mean-variance portfolio selection problem efficiently.

In Chapter 2 we present a brief introduction of multi-period mean-variance asset-liability portfolio selection problem. Some lemmas which will be used in the following chapters are also given.

Chapter 3 tackles the multi-period mean-variance portfolio of asset-liability management problem using the parameterized method addressed in Li et al. (2002). By a variable substitution and Lagrange multiplier method, we can turn the nonseparable problem in the sense of dynamic programming to a solvable stochastic linear quadratic optimal control problem. One prominent feature of the dynamic mean-variance formulations is that the optimal portfolio policy is always linear with respect to the current wealth and liability. According to this feature, we derive the analytical optimal policies and efficient frontiers. The analytical form of the Lagrange multiplier is also given in expression of the expectation of the final surplus. The results are much more explicit and accurate compared with the similar model solved by the embedding technique. It is worth mentioning that the relationship of returns plays an important role in the whole derivation. We first deduce the case when assets and liability are correlated just in the same time period, then it is reduced to the uncorrelated setting. Numerical examples are presented to shed light on the results established in this work.

When uncertain exit time or bankruptcy are considered in the model, neither the parameterized method nor the embedding technique will work smoothly. Chapter 4 is devoted on the mean-field formulation for the multi-period asset-liability mean-variance portfolio selection with an uncertain exit time. Note that the multi-period or continuous-time Markowitz-type mean-variance portfolio selection problems are typical mean-field type stochastic control problems, where the variance term appears as a quadratic function of the expected terminal state. We shed light on the efficiency and accuracy of mean-field formulation when dealing with the issue of dynamic non-separability in those models. By taking "mean" of the constraints and some simple calculation, the state space and the control space are enlarged in the language of optimal control. The objective function then becomes separable in the expanded space which enables us to solve the problem by dynamic programming. In the first section we introduce mean-filed formulation and use it to deal with the noseparability of a simple multi-period mean-variance problem without liability. Then we employed the mean-field formulation to solve the asset-liability management. We derive strictly the optimal strategies and efficient frontiers of the mean-variance model with correlation of assets and liability and the results with uncorrelation of assets and liability respectively. Numerical examples are presented to illustrate the efficiency and accuracy of the mean-field formulation to solve the multi-period mean-variance model. It is showed that compared to the embedding technique (see Yi et al. (2008)), the mean-field approach makes the whole process to derive the optimal strategy simpler and more direct. When the uncertain exit time reduces to terminal exit time and take deterministic expected return, the results of the parameterized method and mean-field formulation are proved to be the same. This in turn suggests that the two approaches to solve multi-period mean-variance model are accurate.

Chapter 5 deals with the multi-period mean-variance portfolio selection problem with risk control over bankruptcy. Mean-field formulation is proved to be also efficient when we take bankruptcy into account. The effect of control over bankruptcy is showed theoretically and numerically. When the bankruptcy control is left out and the terminal expected expectation is deterministic as the model in Chapter 3, the results are also the same as it.

Chapter 6 resolves the problem of Chapter 3 when the returns of assets and liability are correlated among different time periods, which is much more complex but is always the case in real financial market. We prove that the similar results hold when the expectation, the variance, the covariance are extended to conditional expectation, conditional variance, conditional covariance, respectively. In other words, the results in this Chapter can be reduced to that of Chapter 3 when the assets and liability are independent in different periods. In fact, it is not an easy thing. On the one hand, there are not enough references about the mean-variance model when the returns are serially correlated. On the other hand, since we do not have the deterministic distribution of the correlated returns but just adopt a formulation with a general form, how to calculate the expectation of it is crucial. We deal with this by using a approximate formulation. The differences of different cases are illustrated by numerical examples.

The whole thesis deals with the multi-period mean-variance asset-liability portfolio selection problem with different constraints, such as uncertain exit time, bankruptcy control and correlated returns in parameterized method or mean-field approach. We can also consider other situations such as regime switching or time consistent problem. Chapter 7 concludes the whole thesis and plans for the future work.

Chapter 2

Preliminary

The purpose of this chapter is to review the basic concepts of multi-period asset-liability mean-variance model and present some lemmas which will be used in the following chapters.

2.1 Multi-period Asset-Liability Mean-Variance Model

Assume that an investor joining the market at the beginning of period 0 with an initial wealth x_0 and initial liability l_0, plans to invest his/her wealth within a time horizon T. He/she can reallocates his/her portfolio at the beginning of each following $T-1$ consecutive periods. The capital market consists of one risk-free asset, n risky assets and one liability. At time period t, the given deterministic return of the risk-free asset, the random returns of the n risky assets, and the random return of the liability are denoted by s_t (> 1), vector $\mathbf{e}_t = [e_t^1, \cdots, e_t^n]'$ and q_t, respectively. The random vector $\mathbf{e}_t = [e_t^1, \cdots, e_t^n]'$ and the random variable q_t are defined over the probability space $(\Omega, \mathcal{F}, P)$ and are supposed to be statistically independent at different time periods.

We assume that the only information known about $\mathbf{e}_t$ and q_t are their first two unconditional moments, $\mathbb{E}[\mathbf{e}_t] = \big(\mathbb{E}[e_t^1], \cdots, \mathbb{E}[e_t^n]\big)'$, $\mathbb{E}[q_t]$ and $(n+1) \times (n+1)$

positive definite covariance

$$\mathrm{Cov}\left(\begin{pmatrix}\mathbf{e}_t\\ q_t\end{pmatrix}\right)=\mathbb{E}\left[\begin{pmatrix}\mathbf{e}_t\\ q_t\end{pmatrix}\begin{pmatrix}\mathbf{e}_t' & q_t\end{pmatrix}\right]-\mathbb{E}\left[\begin{pmatrix}\mathbf{e}_t\\ q_t\end{pmatrix}\right]\mathbb{E}\left[\begin{pmatrix}\mathbf{e}_t' & q_t\end{pmatrix}\right].$$

From the above assumptions, we have

$$\begin{pmatrix} s_t^2 & s_t\mathbb{E}[\mathbf{e}_t'] & s_t\mathbb{E}[q_t]\\ s_t\mathbb{E}[\mathbf{e}_t] & \mathbb{E}[\mathbf{e}_t\mathbf{e}_t'] & \mathbb{E}[\mathbf{e}_t q_t]\\ s_t\mathbb{E}[q_t] & \mathbb{E}[q_t\mathbf{e}_t'] & \mathbb{E}[q_t^2]\end{pmatrix}\succ 0.$$

We further define the excess return vector of risky assets $\mathbf{P}_t=(P_t^1,\cdots,P_t^n)'$ as $(e_t^1-s_t,\cdots,e_t^n-s_t)'$. The following is then true for $t=0,1,\cdots,T-1$:

$$\begin{aligned}&\begin{pmatrix} s_t^2 & s_t\mathbb{E}[\mathbf{P}_t'] & s_t\mathbb{E}[q_t]\\ s_t\mathbb{E}[\mathbf{P}_t] & \mathbb{E}[\mathbf{P}_t\mathbf{P}_t'] & \mathbb{E}[\mathbf{P}_t q_t]\\ s_t\mathbb{E}[q_t] & \mathbb{E}[q_t\mathbf{P}_t'] & \mathbb{E}[q_t^2]\end{pmatrix}\\ =&\begin{pmatrix}1 & \mathbf{0}' & 0\\ -\mathbf{1} & I & \mathbf{0}\\ 0 & \mathbf{0}' & 1\end{pmatrix}\begin{pmatrix} s_t^2 & s_t\mathbb{E}[\mathbf{e}_t'] & s_t\mathbb{E}[q_t]\\ s_t\mathbb{E}[\mathbf{e}_t] & \mathbb{E}[\mathbf{e}_t\mathbf{e}_t'] & \mathbb{E}[\mathbf{e}_t q_t]\\ s_t\mathbb{E}[q_t] & \mathbb{E}[q_t\mathbf{e}_t'] & \mathbb{E}[q_t^2]\end{pmatrix}\begin{pmatrix}1 & -\mathbf{1}' & 0\\ \mathbf{0} & I & \mathbf{0}\\ 0 & \mathbf{0}' & 1\end{pmatrix}\\ \succ&0,\end{aligned}$$

where $\mathbf{1}$ and $\mathbf{0}$ are the n-dimensional all-one and all-zero vectors, respectively, and I is the $n\times n$ identity matrix, which further implies, for $t=0,1,\cdots,T-1$,

$$\begin{aligned}\mathbb{E}[\mathbf{P}_t\mathbf{P}_t']&\succ 0,\\ s_t^2(1-\mathbb{E}[\mathbf{P}_t']\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t])&>0,\\ \mathbb{E}[q_t^2]-\mathbb{E}[q_t\mathbf{P}_t']\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t q_t]&>0.\end{aligned}$$

We further denote

$$\begin{aligned}B_t&\triangleq\mathbb{E}[\mathbf{P}_t']\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t],\\ \widehat{B}_t&\triangleq\mathbb{E}[q_t\mathbf{P}_t']\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t],\\ \widetilde{B}_t&\triangleq\mathbb{E}[q_t\mathbf{P}_t']\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t q_t].\end{aligned}$$

Thus, $0<B_t<1,\forall\ t=0,1,\cdots,T-1$. If the returns of asset and liability are uncorrelated at every period, then

$$\widehat{B}_t=\mathbb{E}[q_t]B_t\quad\text{and}\quad\widetilde{B}_t=(\mathbb{E}[q_t])^2B_t.$$

Let x_t and l_t be the wealth and liability of the investor at the beginning of period t respectively, then $x_t - l_t$ is the surplus. At period t, if π_t^i, $i = 1, 2, \cdots, n$ is the amount invested in the i-th risky asset, then, $x_t - \sum_{i=1}^n \pi_t^i$ is the amount invested in the risk-free asset. We assume in this paper that the liability is exogenous, which means it is uncontrollable and cannot be affected by the investor's strategies. Denote the information set at the beginning of period t, $t = 1, 2, \cdots, T-1$, as $\mathcal{F}_t = \sigma(\mathbf{P}_0, \mathbf{P}_1, \cdots, \mathbf{P}_{t-1}, q_0, q_1, \cdots, q_{t-1})$ and the trivial σ-algebra over Ω as $\mathcal{F}_0$. Therefore, $\mathbb{E}[\cdot|\mathcal{F}_0]$ is just the unconditional expectation $\mathbb{E}[\cdot]$. We confine all admissible investment strategies to be $\mathcal{F}_t$-adapted Markov controls, i.e., $\pi_t = (\pi_t^1,\ \pi_t^2,\ \cdots,\ \pi_t^n)' \in \mathcal{F}_t$. Then, $\mathbf{P}_t$ and π_t are independent, $\{x_t, l_t\}$ is an adapted Markovian process and $\mathcal{F}_t = \sigma(x_t, l_t)$.

If we consider the multi-period mean-variance portfolio selection problem without liability ($q_t = l_t = 0$), which is to say, the capital market consists of n risky assets and one risk-free asset, then the information set at the beginning of period t is $\mathcal{F}_t = \sigma(\mathbf{P}_0, \mathbf{P}_1, \cdots, \mathbf{P}_{t-1})$, and the positive definite covariance matrix of $\mathbf{e}_t$ is

$$\mathrm{Cov}(\mathbf{e}_t) = \mathbb{E}[\mathbf{e}_t\mathbf{e}_t'] - \mathbb{E}[\mathbf{e}_t]\mathbb{E}[\mathbf{e}_t'] = \begin{bmatrix} \sigma_{t,11} & \cdots & \sigma_{t,1n} \\ \vdots & \ddots & \vdots \\ \sigma_{t,1n} & \cdots & \sigma_{t,nn} \end{bmatrix} \succ 0,$$

where $\sigma_{t,ij}$ is the covariance between assets i and j. We also have $\mathbb{E}[\mathbf{P}_t\mathbf{P}_t'] \succ 0$ and $0 < B_t < 1, \forall\ t = 0, 1, \cdots, T-1$.

2.2 Some Lemmas

Lemma 2.1 (Sherman-Morrison formula)**.** *Suppose that A is an invertible square matrix and μ and ν are two given vectors. If*

$$1 + \nu' A^{-1}\mu \neq 0,$$

then the following holds,

$$(A + \mu\nu')^{-1} = A^{-1} - \frac{A^{-1}\mu\nu' A^{-1}}{1 + \nu' A^{-1}\mu}.$$

For any matrix A, we denote by A^+ the Moore-Penrose pseudoinverse of A satisfying

$$AA^+A = A, A^+AA^+ = A^+, (AA^+)' = AA^+, (A^+A)' = A^+A.$$

It can be proved that A^+ is unique for any matrix A and if the inverse A^{-1} of A exists, then $A^+ = A^{-1}$.

Suppose that M and N are symmetric matrices with the same order. We denote $M \succ N$ ($M \succcurlyeq N$) if and only if (iff) $M - N$ is positive definite (semidefinite). Let M be a symmetrical square matrix partitioned as

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{12}' & M_{22} \end{pmatrix},$$

where M_{11} and M_{22} are also symmetrical square matrices. Denoted by $|M|$ the determinant of a square matrix M. Then the following lemmas hold.

Lemma 2.2. *If* $M_{22} \succ 0$, *then* $|M| = |M_{22}| \left|M_{11} - M_{12}M_{22}^{-1}M_{12}'\right|$.

Lemma 2.3. *If* $M \succeq N \succeq 0$, *then* $|M| \geqslant |N|$.

The proofs of Lemma 2.2 and Lemma 2.3 can be found in Zhang (2011).

Lemma 2.4. *A symmetrical square matrix* $M \succeq 0$ *is equivalent to* $M_{22} \succeq 0$, $M_{22}M_{22}^+M_{12}' = M_{12}'$ *and* $M_{11} - M_{12}M_{22}^+M_{12}' \succeq 0$.

The proof of Lemma 2.4 can be found in Albert (1969).

Chapter 3

A Parameterized Method for Optimal Multi-Period Mean-Variance Asset-Liability Portfolio Selection

In this chapter, we study asset-liability management under a multi-period mean-variance portfolio selection framework using the parameterized method. The model is formulated in minimizing the variance with deterministic expected return. By the Lagrange multiplier method and a variable substitution we turn the problem to a much simpler one which has the same optimal strategy with the original problem and can be solved by dynamic programming. We first deduce the case when the returns of assets and liability are correlated, then we reduce it to the uncorrelated setting. The analytical optimal policies and efficient frontiers are derived. The analytical form of the Lagrange multiplier is also given in expression of the expectation of the final surplus. Numerical examples of different cases are presented to shed light on the results established in this work.

3.1 Formulation

The multi-period mean-variance asset-liability model is to seek the best strategy, π_t^* $= [(\pi_t^1)^*, (\pi_t^2)^*, \cdots, (\pi_t^n)^*]'$, $t = 0, 1, \cdots, T-1$, which is the solution of the following dynamic stochastic optimization problem,

$$\begin{cases} \min & \mathrm{Var}(x_T - l_T) \equiv \mathbb{E}[(x_T - l_T - d)^2], \\ \text{s.t.} & \mathbb{E}[x_T - l_T] = d, \\ & x_{t+1} = s_t\left(x_t - \sum_{i=1}^n \pi_t^i\right) + \sum_{i=1}^n e_t^i \pi_t^i \\ & \qquad = s_t x_t + \mathbf{P}_t' \pi_t, \\ & l_{t+1} = q_t l_t, \qquad\qquad t = 0, 1, \cdots, T-1. \end{cases} \tag{3.1}$$

Introducing a Lagrange multiplier $2\omega > 0$ yields

$$\begin{cases} \min & \mathbb{E}[(x_T - l_T - d)^2] - 2\omega(\mathbb{E}[x_T - l_T] - d), \\ \text{s.t.} & \{x_t, l_t, \pi_t\} \text{ satisfies the dynamic system of problem (3.1)}, \end{cases} \tag{3.2}$$

which is equivalent to the following problem,

$$\begin{cases} \min & \mathbb{E}[(x_T - l_T - d - \omega)^2], \\ \text{s.t.} & \{x_t, l_t, \pi_t\} \text{ satisfies the dynamic system of problem (3.1)}, \end{cases} \tag{3.3}$$

in the sense that the two problems have the same optimal strategy. It can be rewritten as

$$\begin{cases} \min & \mathbb{E}[(x_T - \gamma - l_T)^2], \\ \text{s.t.} & \{x_t, l_t, \pi_t\} \text{ satisfies the dynamic system of problem (3.1)}, \end{cases} \tag{3.4}$$

where $\gamma = d + \omega$. Set

$$y_t := x_t - \gamma \prod_{k=t}^{T-1} s_k^{-1}, \tag{3.5}$$

and denote $\prod_{k=T}^{T-1} s_k^{-1} := 1$. Then the dynamic system of problem (3.1) turns to

$$\begin{cases} y_{t+1} &= s_t y_t + \mathbf{P}_t' \pi_t, \\ l_{t+1} &= q_t l_t, \qquad\qquad t = 0, 1, \cdots, T-1, \end{cases} \tag{3.6}$$

where $y_0 = x_0 - \gamma \prod_{k=0}^{T-1} s_k^{-1}$. The problem (3.4) can be reformulated into

$$\begin{cases} \min & \mathbb{E}[(y_T - l_T)^2], \\ \text{s.t.} & \{y_t, l_t, \pi_t\} \text{ satisfies equation (3.6)}, \end{cases} \tag{3.7}$$

and it is the 'same' with the following problem:

$$\begin{cases} \min & \mathbb{E}[y_T^2 - 2 l_T y_T], \\ \text{s.t.} & \{y_t, l_t, \pi_t\} \text{ satisfies equation (3.6)}, \end{cases} \tag{3.8}$$

The 'same' here means they have the same optimal strategy. By studying the problem (3.8), we can obtain the optimal strategy of the original problem (3.1).

3.2 Optimal Strategy

3.2.1 The Optimal Strategy with Correlation of Assets and Liability

In this subsection, assume that the returns of assets and liability are correlated at every period, i.e., $\mathbf{P}_t$ and q_t are dependent each other at period $t = 0, 1, \cdots, T-1$.

Theorem 3.1. *Assume that the returns of assets and liability are correlated at every period. Then the optimal strategy of problem (3.1) is given by*

$$\pi_t^* = -\mathbb{E}^{-1}[\mathbf{P}_t \mathbf{P}_t'] \mathbb{E}[\mathbf{P}_t] s_t \left(x_t - \gamma^* \prod_{k=t}^{T-1} s_k^{-1} \right) + \left(\prod_{k=t+1}^{T-1} \frac{\mathbb{E}[q_k] - \widehat{B}_k}{(1 - B_k) s_k} \right) \mathbb{E}^{-1}[\mathbf{P}_t \mathbf{P}_t'] \mathbb{E}[q_t \mathbf{P}_t] l_t, \tag{3.9}$$

where

$$\gamma^* = \frac{x_0 \prod_{k=0}^{T-1} (1 - B_k) s_k - d - l_0 \prod_{k=0}^{T-1} \left(\mathbb{E}[q_k] - \widehat{B}_k \right)}{\prod_{k=0}^{T-1} (1 - B_k) - 1}. \tag{3.10}$$

Proof. We prove it by making use of the dynamic programming approach. For the information set $\mathcal{F}_t$, the cost-to-go functional of problem (3.8) at period t is

$$J_t(y_t, l_t) = \min_{\pi_t} \mathbb{E}\big[J_{t+1}(y_{t+1}, l_{t+1})\big|\mathcal{F}_t\big],$$

where the terminal condition $J_T(y_T, l_T) = y_T^2 - 2l_T y_T$.

We start from the last stage $T-1$. While $t = T-1$, we have

$$\begin{aligned}
&\mathbb{E}\big[J_T(y_T, l_T)\big|\mathcal{F}_{T-1}\big] \\
&= \mathbb{E}\big[y_T^2 - 2l_T y_T\big|\mathcal{F}_{T-1}\big] \\
&= s_{T-1}^2 y_{T-1}^2 + 2s_{T-1}y_{T-1}\mathbb{E}[\mathbf{P}'_{T-1}]\pi_{T-1} + \pi'_{T-1}\mathbb{E}[\mathbf{P}_{T-1}\mathbf{P}'_{T-1}]\pi_{T-1} \\
&\quad - 2\mathbb{E}[q_{T-1}]s_{T-1}l_{T-1}y_{T-1} - 2\mathbb{E}[q_{T-1}\mathbf{P}'_{T-1}]l_{T-1}\pi_{T-1}.
\end{aligned}$$

Minimizing it with respect to π_{T-1} yields the optimal decision at period $T-1$ as below

$$\pi_{T-1}^* = -\,\mathbb{E}^{-1}[\mathbf{P}_{T-1}\mathbf{P}'_{T-1}]\mathbb{E}[\mathbf{P}_{T-1}]s_{T-1}y_{T-1} + \mathbb{E}^{-1}[\mathbf{P}_{T-1}\mathbf{P}'_{T-1}]\mathbb{E}[q_{T-1}\mathbf{P}_{T-1}]l_{T-1}.$$

Substituting π_{T-1}^* to $\mathbb{E}\big[J_T(y_T, l_T)\big|\mathcal{F}_{T-1}\big]$, we obtain

$$\begin{aligned}
&J_{T-1}(y_{T-1}, l_{T-1}) = \min_{\pi_{T-1}} \mathbb{E}\big[J_T(y_T, l_T)\big|\mathcal{F}_{T-1}\big] \\
&= (1 - B_{T-1})s_{T-1}^2 y_{T-1}^2 - 2\big(\mathbb{E}[q_{T-1}] - \widehat{B}_{T-1}\big)s_{T-1}l_{T-1}y_{T-1} - \widetilde{B}_{T-1}l_{T-1}^2.
\end{aligned}$$

In order to derive the cost-to-go functional and the optimal decision at period t clearly, we patiently repeat the procedure at time $T-2$. While $t = T-2$, we have

$$\begin{aligned}
&\mathbb{E}\big[J_{T-1}(y_{T-1}, l_{T-1})\big|\mathcal{F}_{T-2}\big] \\
&= \mathbb{E}\big[(1 - B_{T-1})s_{T-1}^2 y_{T-1}^2 - 2\big(\mathbb{E}[q_{T-1}] - \widehat{B}_{T-1}\big)s_{T-1}l_{T-1}y_{T-1} - \widetilde{B}_{T-1}l_{T-1}^2\big|\mathcal{F}_{T-2}\big] \\
&= (1 - B_{T-1})s_{T-1}^2\Big(s_{T-2}^2 y_{T-2}^2 + 2s_{T-2}y_{T-2}\mathbb{E}[\mathbf{P}'_{T-2}]\pi_{T-2} + \pi'_{T-2}\mathbb{E}[\mathbf{P}_{T-2}\mathbf{P}'_{T-2}]\pi_{T-2}\Big) \\
&\quad - 2\big(\mathbb{E}[q_{T-1}] - \widehat{B}_{T-1}\big)\mathbb{E}\big[q_{T-2}]s_{T-1}s_{T-2}l_{T-2}y_{T-2} \\
&\quad - 2\big(\mathbb{E}[q_{T-1}] - \widehat{B}_{T-1}\big)\mathbb{E}[q_{T-2}\mathbf{P}'_{T-2}]s_{T-1}l_{T-2}\pi_{T-2} \\
&\quad - \widetilde{B}_{T-1}\mathbb{E}[q_{T-2}^2]l_{T-2}^2.
\end{aligned}$$

We derive the following optimal decision at period $T-2$ by minimizing the above functional with respect to π_{T-2}

$$\begin{aligned}\pi^*_{T-2} = &-\mathbb{E}^{-1}[\mathbf{P}_{T-2}\mathbf{P}'_{T-2}]\mathbb{E}[\mathbf{P}_{T-2}]s_{T-2}y_{T-2}\\ &+\frac{\mathbb{E}[q_{T-1}]-\widehat{B}_{T-1}}{(1-B_{T-1})s_{T-1}}\mathbb{E}^{-1}[\mathbf{P}_{T-2}\mathbf{P}'_{T-2}]\mathbb{E}[q_{T-2}\mathbf{P}_{T-2}]l_{T-2}.\end{aligned}$$

Then the cost-to-go functional at period $T-2$ is

$$\begin{aligned}&J_{T-2}(y_{T-2},l_{T-2}) = \min_{\pi_{T-2}}\mathbb{E}\big[J_{T-1}(y_{T-1},l_{T-1})\big|\mathcal{F}_{T-2}\big]\\ &= (1-B_{T-1})(1-B_{T-2})s^2_{T-1}s^2_{T-2}y^2_{T-2}\\ &\quad -2\big(\mathbb{E}[q_{T-1}]-\widehat{B}_{T-1}\big)\big(\mathbb{E}\big[q_{T-2}\big]-\widehat{B}_{T-2}\big)s_{T-1}s_{T-2}l_{T-2}y_{T-2}\\ &\quad -\left(\frac{\big(\mathbb{E}[q_{T-1}]-\widehat{B}_{T-1}\big)^2}{1-B_{T-1}}\widetilde{B}_{T-2}+\widetilde{B}_{T-1}\mathbb{E}[q^2_{T-2}]\right)l^2_{T-2}.\end{aligned}$$

While $t=T-3$, we can similarly get

$$\begin{aligned}&\mathbb{E}\big[J_{T-2}(y_{T-2},l_{T-2})\big|\mathcal{F}_{T-3}\big]\\ &=\mathbb{E}\Big[(1-B_{T-1})(1-B_{T-2})s^2_{T-1}s^2_{T-2}y^2_{T-2}\\ &\quad -2\big(\mathbb{E}[q_{T-1}]-\widehat{B}_{T-1}\big)\big(\mathbb{E}\big[q_{T-2}\big]-\widehat{B}_{T-2}\big)s_{T-1}s_{T-2}l_{T-2}y_{T-2}\\ &\quad -\left(\frac{\big(\mathbb{E}[q_{T-1}]-\widehat{B}_{T-1}\big)^2}{1-B_{T-1}}\widetilde{B}_{T-2}+\widetilde{B}_{T-1}\mathbb{E}[q^2_{T-2}]\right)l^2_{T-2}\Big|\mathcal{F}_{T-3}\Big]\\ &=(1-B_{T-1})(1-B_{T-2})s^2_{T-1}s^2_{T-2}\\ &\quad\Big(s^2_{T-3}y^2_{T-3}+2s_{T-3}y_{T-3}\mathbb{E}[\mathbf{P}'_{T-3}]\pi_{T-3}+\pi'_{T-3}\mathbb{E}[\mathbf{P}_{T-3}\mathbf{P}'_{T-3}]\pi_{T-3}\Big)\\ &\quad -2\big(\mathbb{E}[q_{T-1}]-\widehat{B}_{T-1}\big)\big(\mathbb{E}\big[q_{T-2}\big]-\widehat{B}_{T-2}\big)\mathbb{E}[q_{T-3}]s_{T-1}s_{T-2}s_{T-3}l_{T-3}y_{T-3}\\ &\quad -2\big(\mathbb{E}[q_{T-1}]-\widehat{B}_{T-1}\big)\big(\mathbb{E}\big[q_{T-2}\big]-\widehat{B}_{T-2}\big)\mathbb{E}[q_{T-3}\mathbf{P}'_{T-3}]s_{T-1}s_{T-2}l_{T-3}\pi_{T-3}\\ &\quad -\left(\frac{\big(\mathbb{E}[q_{T-1}]-\widehat{B}_{T-1}\big)^2}{1-B_{T-1}}\widetilde{B}_{T-2}+\widetilde{B}_{T-1}\mathbb{E}[q^2_{T-2}]\right)\mathbb{E}[q^2_{T-3}]l^2_{T-3}.\end{aligned}$$

Thus the optimal decision at period $T-3$ is

$$\pi^*_{T-3} = -\mathbb{E}^{-1}[\mathbf{P}_{T-3}\mathbf{P}'_{T-3}]\mathbb{E}[\mathbf{P}_{T-3}]s_{T-3}y_{T-3} + \frac{\mathbb{E}[q_{T-1}] - \widehat{B}_{T-1}}{(1-B_{T-1})s_{T-1}}\frac{\mathbb{E}[q_{T-2}] - \widehat{B}_{T-2}}{(1-B_{T-2})s_{T-2}}\mathbb{E}^{-1}[\mathbf{P}_{T-3}\mathbf{P}'_{T-3}]\mathbb{E}[q_{T-3}\mathbf{P}_{T-3}]l_{T-3},$$

and the cost-to-go functional at period $T-3$ is

$$\begin{aligned}
J_{T-3}(y_{T-3}, l_{T-3}) &= \min_{\pi_{T-3}} \mathbb{E}\big[J_{T-2}(y_{T-2}, l_{T-2})\big|\mathcal{F}_{T-3}\big] \\
&= (1-B_{T-1})(1-B_{T-2})(1-B_{T-3})s^2_{T-1}s^2_{T-2}s^2_{T-3}y^2_{T-3} \\
&\quad - 2\big(\mathbb{E}[q_{T-1}] - \widehat{B}_{T-1}\big)\big(\mathbb{E}[q_{T-2}] - \widehat{B}_{T-2}\big)\big(\mathbb{E}[q_{T-3}] - \widehat{B}_{T-3}\big)s_{T-1}s_{T-2}s_{T-3}l_{T-3}y_{T-3} \\
&\quad - \Bigg[\frac{\big(\mathbb{E}[q_{T-1}] - \widehat{B}_{T-1}\big)^2}{1-B_{T-1}}\frac{\big(\mathbb{E}[q_{T-2}] - \widehat{B}_{T-2}\big)^2}{1-B_{T-2}}\widetilde{B}_{T-3} \\
&\quad + \Bigg(\frac{\big(\mathbb{E}[q_{T-1}] - \widehat{B}_{T-1}\big)^2}{1-B_{T-1}}\widetilde{B}_{T-2} + \widetilde{B}_{T-1}\mathbb{E}[q^2_{T-2}]\Bigg)\mathbb{E}[q^2_{T-3}]\Bigg]l^2_{T-3}.
\end{aligned}$$

Inspired by the above three stages, we conjecture that the cost-to-go functional at period t can be expressed by the following form

$$\begin{aligned}
J_t(y_t, l_t) = & \Bigg(\prod_{k=t}^{T-1}(1-B_k)s^2_k\Bigg)y^2_t - 2\Bigg(\prod_{k=t}^{T-1}(\mathbb{E}[q_k] - \widehat{B}_k)s_k\Bigg)l_t y_t \\
& - \sum_{j=t}^{T-1}\Bigg(\prod_{k=j+1}^{T-1}\frac{(\mathbb{E}[q_k] - \widehat{B}_k)^2}{1-B_k}\Bigg)\widetilde{B}_j\Bigg(\prod_{m=t}^{j-1}\mathbb{E}[q^2_m]\Bigg)l^2_t.
\end{aligned} \tag{3.11}$$

Next, we prove it in mathematical induction. Assume that the cost-to-go functional (3.11) holds at period $t+1$. Then we shall prove that it still holds at time t. For the given information set $\mathcal{F}_t$, we have

$$\begin{aligned}
&\mathbb{E}\big[J_{t+1}(y_{t+1}, l_{t+1})\big|\mathcal{F}_t\big] \\
&= \mathbb{E}\Bigg[\Bigg(\prod_{k=t+1}^{T-1}(1-B_k)s^2_k\Bigg)y^2_{t+1} - 2\Bigg(\prod_{k=t+1}^{T-1}\big(\mathbb{E}[q_k] - \widehat{B}_k\big)s_k\Bigg)l_{t+1}y_{t+1}
\end{aligned}$$

$$- \sum_{j=t+1}^{T-1} \left(\prod_{k=j+1}^{T-1} \frac{\big(\mathbb{E}[q_k] - \widehat{B}_k\big)^2}{1 - B_k} \right) \widetilde{B}_j \left(\prod_{m=t+1}^{j-1} \mathbb{E}[q_m^2] \right) l_{t+1}^2 \Big| \mathcal{F}_t \Big]$$

$$= \left(\prod_{k=t+1}^{T-1} (1 - B_k) s_k^2 \right) \big(s_t^2 y_t^2 + 2 s_t y_t \mathbb{E}[\mathbf{P}_t'] \pi_t + \pi_t' \mathbb{E}[\mathbf{P}_t \mathbf{P}_t'] \pi_t \big)$$

$$- 2 \left(\prod_{k=t+1}^{T-1} \big(\mathbb{E}[q_k] - \widehat{B}_k\big) s_k \right) \big(\mathbb{E}[q_t] s_t l_t y_t + \mathbb{E}[q_t \mathbf{P}_t'] l_t \pi_t \big)$$

$$- \sum_{j=t+1}^{T-1} \left(\prod_{k=j+1}^{T-1} \frac{\big(\mathbb{E}[q_k] - \widehat{B}_k\big)^2}{1 - B_k} \right) \widetilde{B}_j \left(\prod_{m=t+1}^{j-1} \mathbb{E}[q_m^2] \right) \mathbb{E}[q_t^2] l_t^2.$$

Minimizing the above functional with respect to π_t, we get the optimal strategy decision at time t as follows

$$\pi_t^* = -\mathbb{E}^{-1}[\mathbf{P}_t \mathbf{P}_t'] \mathbb{E}[\mathbf{P}_t] s_t y_t + \left(\prod_{k=t+1}^{T-1} \frac{\mathbb{E}[q_k] - \widehat{B}_k}{(1 - B_k) s_k} \right) \mathbb{E}^{-1}[\mathbf{P}_t \mathbf{P}_t'] \mathbb{E}[q_t \mathbf{P}_t] l_t.$$

Substituting it to $\mathbb{E}\big[J_{t+1}(y_{t+1}, l_{t+1}) \big| \mathcal{F}_t\big]$ yields

$$J_t(y_t, l_t) = \min_{\pi_t} \mathbb{E}\big[J_{t+1}(y_{t+1}, l_{t+1}) \big| \mathcal{F}_t\big]$$

$$= \left(\prod_{k=t+1}^{T-1} (1 - B_k) s_k^2 \right) s_t^2 y_t^2 - 2 \left(\prod_{k=t+1}^{T-1} \big(\mathbb{E}[q_k] - \widehat{B}_k\big) s_k \right) \mathbb{E}[q_t] s_t l_t y_t$$

$$- \left(\prod_{k=t+1}^{T-1} (1 - B_k) s_k^2 \right) \mathbb{E}[\mathbf{P}_t'] \mathbb{E}^{-1}[\mathbf{P}_t \mathbf{P}_t'] \mathbb{E}[\mathbf{P}_t] s_t^2 y_t^2$$

$$+ 2 \left(\prod_{k=t+1}^{T-1} \big(\mathbb{E}[q_k] - \widehat{B}_k\big) s_k \right) \mathbb{E}[q_t \mathbf{P}_t'] \mathbb{E}^{-1}[\mathbf{P}_t \mathbf{P}_t'] \mathbb{E}[\mathbf{P}_t] s_t l_t y_t$$

$$- \left(\prod_{k=t+1}^{T-1} \frac{\big(\mathbb{E}[q_k] - \widehat{B}_k\big)^2}{1 - B_k} \right) \mathbb{E}[q_t \mathbf{P}_t'] \mathbb{E}^{-1}[\mathbf{P}_t \mathbf{P}_t'] \mathbb{E}[q_t \mathbf{P}_t] l_t^2$$

$$- \sum_{j=t+1}^{T-1} \left(\prod_{k=j+1}^{T-1} \frac{\big(\mathbb{E}[q_k] - \widehat{B}_k\big)^2}{1 - B_k} \right) \widetilde{B}_j \left(\prod_{m=t+1}^{j-1} \mathbb{E}[q_m^2] \right) \mathbb{E}[q_t^2] l_t^2$$

$$= \left(\prod_{k=t}^{T-1}(1-B_k)s_k^2\right)y_t^2 - 2\left(\prod_{k=t}^{T-1}\left(\mathbb{E}[q_k]-\widehat{B}_k\right)s_k\right)l_t y_t$$

$$- \sum_{j=t}^{T-1}\left(\prod_{k=j+1}^{T-1}\frac{\left(\mathbb{E}[q_k]-\widehat{B}_k\right)^2}{1-B_k}\right)\widetilde{B}_j\left(\prod_{m=t}^{j-1}\mathbb{E}[q_m^2]\right)l_t^2,$$

which proves (3.11).

To derive the expression (3.10) of γ, we first consider the value of the optimal objective function in (3.8). In fact,

$$\mathbb{E}\big[y_T^2 - 2l_T y_T\big] = \mathbb{E}\big[y_T^2 - 2l_T y_T\big|\mathcal{F}_0\big] = J_0(y_0, l_0)$$

$$= y_0^2\prod_{k=0}^{T-1}(1-B_k)s_k^2 - 2l_0 y_0\prod_{k=0}^{T-1}\left(\mathbb{E}[q_k]-\widehat{B}_k\right)s_k$$

$$- l_0^2\sum_{j=0}^{T-1}\left(\prod_{k=j+1}^{T-1}\frac{\left(\mathbb{E}[q_k]-\widehat{B}_k\right)^2}{1-B_k}\right)\widetilde{B}_j\left(\prod_{m=0}^{j-1}\mathbb{E}[q_m^2]\right).$$

Then

$$\mathrm{Var}(x_T - l_T) = \mathbb{E}[(x_T - l_T - d)^2]$$

$$= \mathbb{E}[(x_T - l_T - d)^2] - 2\omega(\mathbb{E}[x_T - l_T] - d) + \omega^2 - \omega^2$$

$$= \mathbb{E}[(x_T - l_T - d)^2 - 2\omega(x_T - l_T - d) + \omega^2] - \omega^2$$

$$= \mathbb{E}[(x_T - l_T - d - \omega)^2] - \omega^2$$

$$= \mathbb{E}[(y_T - l_T)^2] - \omega^2$$

$$= \mathbb{E}[y_T^2 - 2l_T y_T] + \mathbb{E}[l_T^2] - \omega^2$$

$$= y_0^2\prod_{k=0}^{T-1}(1-B_k)s_k^2 - 2l_0 y_0\prod_{k=0}^{T-1}\left(\mathbb{E}[q_k]-\widehat{B}_k\right)s_k$$

$$- l_0^2\sum_{j=0}^{T-1}\left(\prod_{k=j+1}^{T-1}\frac{\left(\mathbb{E}[q_k]-\widehat{B}_k\right)^2}{1-B_k}\right)\widetilde{B}_j\left(\prod_{m=0}^{j-1}\mathbb{E}[q_m^2]\right) + l_0^2\prod_{k=0}^{T-1}\mathbb{E}[q_k^2] - \omega^2.$$

Since

$$y_0 = x_0 - \gamma \prod_{k=0}^{T-1} s_k^{-1} = x_0 - (d+\omega) \prod_{k=0}^{T-1} s_k^{-1},$$

we have

$$\begin{aligned} y_0^2 \prod_{k=0}^{T-1}(1-B_k)s_k^2 &= \left(x_0 - (d+\omega)\prod_{k=0}^{T-1} s_k^{-1}\right)^2 \prod_{k=0}^{T-1}(1-B_k)s_k^2 \\ &= \left(x_0 \prod_{k=0}^{T-1} s_k - (d+\omega)\right)^2 \prod_{k=0}^{T-1}(1-B_k), \end{aligned}$$

and

$$\begin{aligned} y_0 \prod_{k=0}^{T-1}(\mathbb{E}[q_k] - \widehat{B}_k)s_k &= \left(x_0 - (d+\omega)\prod_{k=0}^{T-1} s_k^{-1}\right) \prod_{k=0}^{T-1} \big(\mathbb{E}[q_k] - \widehat{B}_k\big)s_k \\ &= \left(x_0 \prod_{k=0}^{T-1} s_k - (d+\omega)\right) \prod_{k=0}^{T-1} \big(\mathbb{E}[q_k] - \widehat{B}_k\big). \end{aligned}$$

Hence,

$$\begin{aligned} &\mathrm{Var}(x_T - l_T) \\ &= \left(x_0 \prod_{k=0}^{T-1} s_k - (d+\omega)\right)^2 \prod_{k=0}^{T-1}(1-B_k) - 2l_0\left(x_0 \prod_{k=0}^{T-1} s_k - (d+\omega)\right)\prod_{k=0}^{T-1}\big(\mathbb{E}[q_k] - \widehat{B}_k\big) \\ &\quad -l_0^2 \sum_{j=0}^{T-1}\left(\prod_{k=j+1}^{T-1} \frac{\big(\mathbb{E}[q_k]-\widehat{B}_k\big)^2}{1-B_k}\right)\widetilde{B}_j\left(\prod_{m=0}^{j-1}\mathbb{E}[q_m^2]\right) + l_0^2 \prod_{k=0}^{T-1}\mathbb{E}[q_k^2] - \omega^2 \\ &= \left[\prod_{k=0}^{T-1}(1-B_k) - 1\right]\left(\omega - \frac{\big(x_0\prod_{k=0}^{T-1} s_k - d\big)\prod_{k=0}^{T-1}(1-B_k) - l_0\prod_{k=0}^{T-1}\big(\mathbb{E}[q_k]-\widehat{B}_k\big)}{\prod_{k=0}^{T-1}(1-B_k)-1}\right)^2 \\ &\quad + \frac{\prod_{k=0}^{T-1}(1-B_k)}{1-\prod_{k=0}^{T-1}(1-B_k)}\left(d - x_0\prod_{k=0}^{T-1} s_k + l_0 \prod_{k=0}^{T-1}\frac{\mathbb{E}[q_k]-\widehat{B}_k}{1-B_k}\right)^2 + l_0^2 C_0, \end{aligned} \tag{3.12}$$

where

$$C_0 = -\prod_{k=0}^{T-1}\frac{\big(\mathbb{E}[q_k]-\widehat{B}_k\big)^2}{1-B_k} - \sum_{j=0}^{T-1}\left(\prod_{k=j+1}^{T-1}\frac{(\mathbb{E}[q_k]-\widehat{B}_k)^2}{1-B_k}\right)\widetilde{B}_j\left(\prod_{m=0}^{j-1}\mathbb{E}[q_m^2]\right) + \prod_{k=0}^{T-1}\mathbb{E}[q_k^2]. \tag{3.13}$$

Since $0 < B_t < 1$ for $t = 0, 1, \cdots, T-1$,

$$0 < \prod_{k=0}^{T-1}(1 - B_k) < 1.$$

This implies that the variance $\mathrm{Var}(x_T - l_T)$ in (3.12) is concave in ω. To obtain the minimum variance $\mathrm{Var}(x_T - l_T)$ and optimal strategy for the original portfolio selection problem (3.1), one needs to maximize the value in (3.12) over $\omega \in \mathbb{R}$ according to the Lagrange duality theorem in Luenberger (1968). Taking the first order for (3.12) with respect to ω yields

$$\omega^* = \frac{\left(x_0 \prod_{k=0}^{T-1} s_k - d\right) \prod_{k=0}^{T-1}(1 - B_k) - l_0 \prod_{k=0}^{T-1} \left(\mathbb{E}[q_k] - \widehat{B}_k\right)}{\prod_{k=0}^{T-1}(1 - B_k) - 1}.$$

A simple calculation of $\gamma^* = d + \omega^*$ implies the desired result (3.10). □

3.2.2 Efficient Frontier

The *efficient frontier* consists of the envelope curve of all portfolios that lie between the global minimum variance portfolio and the maximum return portfolio (Elton et al. (2009)). It is the subset of portfolios that will be taken by the investors who prefer less risk to more and prefer more return to less. Before analyzing the efficient frontier, we prove the following important result.

Lemma 3.1. *If* $\mathbb{E}\left[\begin{pmatrix} \mathbf{P}_k \\ q_k \end{pmatrix} \begin{pmatrix} \mathbf{P}'_k & q_k \end{pmatrix}\right]$ *is positive definite for* $k = 0, 1, \cdots, T-1$, *then*

$$C_0 \geqslant 0,$$

where C_0 *is defined as (3.13).*

Proof. Let $L_k = \begin{pmatrix} \mathbf{P}_k \\ 1 \end{pmatrix}$ and $Q_k = \begin{pmatrix} \mathbf{P}_k \\ q_k \end{pmatrix}$, then

$$\begin{pmatrix} \mathbb{E}[\mathbf{P}_k\mathbf{P}'_k] & \mathbb{E}[\mathbf{P}_k] \\ \mathbb{E}[\mathbf{P}'_k] & 1 \end{pmatrix} = \mathbb{E}\left[\begin{pmatrix} \mathbf{P}_k \\ 1 \end{pmatrix} \begin{pmatrix} \mathbf{P}'_k & 1 \end{pmatrix}\right] = \mathbb{E}[L_k L'_k], \tag{3.14}$$

$$\begin{pmatrix} \mathbb{E}[\mathbf{P}_k\mathbf{P}'_k] & \mathbb{E}[q_k\mathbf{P}_k] \\ \mathbb{E}[q_k\mathbf{P}'_k] & \mathbb{E}[q_k^2] \end{pmatrix} = \mathbb{E}\left[\begin{pmatrix} \mathbf{P}_k \\ q_k \end{pmatrix} \begin{pmatrix} \mathbf{P}'_k & q_k \end{pmatrix}\right] = \mathbb{E}[Q_k Q'_k], \tag{3.15}$$

$$\begin{pmatrix} \mathbb{E}[\mathbf{P}_k\mathbf{P}'_k] & \mathbb{E}[\mathbf{P}_k] \\ \mathbb{E}[q_k\mathbf{P}'_k] & \mathbb{E}[q_k] \end{pmatrix} = \mathbb{E}\left[\begin{pmatrix} \mathbf{P}_k \\ q_k \end{pmatrix} \begin{pmatrix} \mathbf{P}'_k & 1 \end{pmatrix}\right] = \mathbb{E}[Q_k L'_k]. \tag{3.16}$$

Taking determinant on both sides for (3.14)-(3.16) and according to Lemma 2.2, we get

$$\begin{vmatrix} \mathbb{E}[\mathbf{P}_k\mathbf{P}'_k] & \mathbb{E}[\mathbf{P}_k] \\ \mathbb{E}[\mathbf{P}'_k] & 1 \end{vmatrix} = \left(1 - \mathbb{E}[\mathbf{P}'_k]\mathbb{E}^{-1}[\mathbf{P}_k\mathbf{P}'_k]\mathbb{E}[\mathbf{P}_k]\right) |\mathbb{E}[\mathbf{P}_k\mathbf{P}'_k]| = |\mathbb{E}[L_k L'_k]|\,, \tag{3.17}$$

$$\begin{vmatrix} \mathbb{E}[\mathbf{P}_k\mathbf{P}'_k] & \mathbb{E}[q_k\mathbf{P}_k] \\ \mathbb{E}[q_k\mathbf{P}'_k] & \mathbb{E}[q_k^2] \end{vmatrix} = \left(\mathbb{E}[q_k^2] - \mathbb{E}[q_k\mathbf{P}'_k]\mathbb{E}^{-1}[\mathbf{P}_k\mathbf{P}'_k]\mathbb{E}[q_k\mathbf{P}_k]\right) |\mathbb{E}[\mathbf{P}_k\mathbf{P}'_k]| = |\mathbb{E}[Q_k Q'_k]|\,, \tag{3.18}$$

$$\begin{vmatrix} \mathbb{E}[\mathbf{P}_k\mathbf{P}'_k] & \mathbb{E}[\mathbf{P}_k] \\ \mathbb{E}[q_k\mathbf{P}'_k] & \mathbb{E}[q_k] \end{vmatrix} = \left(\mathbb{E}[q_k] - \mathbb{E}[q_k\mathbf{P}'_k]\mathbb{E}^{-1}[\mathbf{P}_k\mathbf{P}'_k]\mathbb{E}[\mathbf{P}_k]\right) |\mathbb{E}[\mathbf{P}_k\mathbf{P}'_k]| = |\mathbb{E}[Q_k L'_k]|\,. \tag{3.19}$$

By the assumption of $\mathbb{E}[Q_k Q'_k] \succ 0$, the inverse $\mathbb{E}^{-1}[Q_k Q'_k]$ of $\mathbb{E}[Q_k Q'_k]$ exists. Then $\mathbb{E}^{+}[Q_k Q'_k] = \mathbb{E}^{-1}[Q_k Q'_k]$. Since

$$\mathbb{E}\left[\begin{pmatrix} L_k \\ Q_k \end{pmatrix} \begin{pmatrix} L'_k & Q'_k \end{pmatrix}\right] = \begin{pmatrix} \mathbb{E}[L_k L'_k] & \mathbb{E}[L_k Q'_k] \\ \mathbb{E}[Q_k L'_k] & \mathbb{E}[Q_k Q'_k] \end{pmatrix} \succcurlyeq 0, \tag{3.20}$$

it follows from Lemma 2.4 that

$$\mathbb{E}[L_k L'_k] - \mathbb{E}[L_k Q'_k]\mathbb{E}^{-1}[Q_k Q'_k]\mathbb{E}[Q_k L'_k] \succcurlyeq 0.$$

Obviously,

$$\mathbb{E}[L_k Q'_k]\mathbb{E}[Q_k Q'_k]^{-1}\mathbb{E}[Q_k L'_k] = \mathbb{E}[L_k Q'_k]\mathbb{E}^{-1}[Q_k Q'_k]\left(\mathbb{E}[L_k Q'_k]\right)' \succcurlyeq 0.$$

Consequently,

$$\mathbb{E}[L_kL_k'] \geqslant \mathbb{E}[L_kQ_k']\mathbb{E}^{-1}[Q_kQ_k']\mathbb{E}[Q_kL_k']. \tag{3.21}$$

Then according to (3.21) and Lemma 2.3, it follows that

$$|\mathbb{E}[L_kL_k']| \geqslant \left|\mathbb{E}[L_kQ_k']\mathbb{E}^{-1}[Q_kQ_k']\mathbb{E}[Q_kL_k']\right| = |\mathbb{E}[L_kQ_k']| \left|\mathbb{E}^{-1}[Q_kQ_k']\right| |\mathbb{E}[Q_kL_k']| . \tag{3.22}$$

Notice that $|\mathbb{E}[Q_kL_k']| = |\mathbb{E}[L_kQ_k']|$ and $|\mathbb{E}^{-1}[Q_kQ_k']| = |\mathbb{E}[Q_kQ_k']|^{-1}$, then (3.22) implies

$$|\mathbb{E}[Q_kL_k']|^2 \leqslant |\mathbb{E}[Q_kQ_k']| \, |\mathbb{E}[L_kL_k']| . \tag{3.23}$$

By (3.17)-(3.19) and (3.23), we obtain

$$\begin{aligned}
&\left(1-\mathbb{E}[\mathbf{P}_k']\mathbb{E}^{-1}[\mathbf{P}_k\mathbf{P}_k']\mathbb{E}[\mathbf{P}_k]\right)\left(\mathbb{E}[q_k^2]-\mathbb{E}[q_k\mathbf{P}_k']\mathbb{E}^{-1}[\mathbf{P}_k\mathbf{P}_k']\mathbb{E}[q_k\mathbf{P}_k]\right)\\
&\geqslant \left(\mathbb{E}[q_k]-\mathbb{E}[q_k\mathbf{P}_k']\mathbb{E}^{-1}[\mathbf{P}_k\mathbf{P}_k']\mathbb{E}[\mathbf{P}_k]\right)^2 .
\end{aligned}$$

Namely,

$$\left(\mathbb{E}[q_k]-\widehat{B}_k\right)^2 \leqslant \left(\mathbb{E}[q_k^2]-\widetilde{B}_k\right)(1-B_k).$$

Then

$$\widetilde{B}_k \leqslant \mathbb{E}[q_k^2] - \frac{\left(\mathbb{E}[q_k]-\widehat{B}_k\right)^2}{1-B_k}.$$

Therefore,

$$\begin{aligned}
&\sum_{j=0}^{T-1}\left(\prod_{k=j+1}^{T-1}\frac{\left(\mathbb{E}[q_k]-\widehat{B}_k\right)^2}{1-B_k}\right)\widetilde{B}_j\left(\prod_{m=0}^{j-1}\mathbb{E}[q_m^2]\right)\\
&\leqslant \sum_{j=0}^{T-1}\left(\prod_{k=j+1}^{T-1}\frac{\left(\mathbb{E}[q_k]-\widehat{B}_k\right)^2}{1-B_k}\right)\left(\mathbb{E}[q_j^2]-\frac{(\mathbb{E}[q_j]-\widehat{B}_j)^2}{1-B_j}\right)\left(\prod_{m=0}^{j-1}\mathbb{E}[q_m^2]\right)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=0}^{T-1} \Bigg(\prod_{k=j+1}^{T-1} \frac{\big(\mathbb{E}[q_k] - \widehat{B}_k\big)^2}{1 - B_k} \Bigg) \mathbb{E}[q_j^2] \Bigg(\prod_{m=0}^{j-1} \mathbb{E}[q_m^2] \Bigg) \\
&\quad - \sum_{j=0}^{T-1} \Bigg(\prod_{k=j+1}^{T-1} \frac{\big(\mathbb{E}[q_k] - \widehat{B}_k\big)^2}{1 - B_k} \Bigg) \frac{(\mathbb{E}[q_j] - \widehat{B}_j)^2}{1 - B_j} \Bigg(\prod_{m=0}^{j-1} \mathbb{E}[q_m^2] \Bigg) \\
&= \sum_{j=0}^{T-1} \Bigg(\prod_{k=j+1}^{T-1} \frac{\big(\mathbb{E}[q_k] - \widehat{B}_k\big)^2}{1 - B_k} \Bigg) \Bigg(\prod_{m=0}^{j} \mathbb{E}[q_m^2] \Bigg) - \sum_{j=0}^{T-1} \Bigg(\prod_{k=j}^{T-1} \frac{\big(\mathbb{E}[q_k] - \widehat{B}_k\big)^2}{1 - B_k} \Bigg) \Bigg(\prod_{m=0}^{j-1} \mathbb{E}[q_m^2] \Bigg) \\
&= \Bigg(\prod_{k=T}^{T-1} \frac{\big(\mathbb{E}[q_k] - \widehat{B}_k\big)^2}{1 - B_k} \Bigg) \Bigg(\prod_{m=0}^{T-1} \mathbb{E}[q_m^2] \Bigg) - \Bigg(\prod_{k=0}^{T-1} \frac{\big(\mathbb{E}[q_k] - \widehat{B}_k\big)^2}{1 - B_k} \Bigg) \Bigg(\prod_{m=0}^{-1} \mathbb{E}[q_m^2] \Bigg) \\
&= \Bigg(\prod_{m=0}^{T-1} \mathbb{E}[q_m^2] \Bigg) - \Bigg(\prod_{k=0}^{T-1} \frac{\big(\mathbb{E}[q_k] - \widehat{B}_k\big)^2}{1 - B_k} \Bigg) \\
&= \Bigg(\prod_{k=0}^{T-1} \mathbb{E}[q_k^2] \Bigg) - \Bigg(\prod_{k=0}^{T-1} \frac{\big(\mathbb{E}[q_k] - \widehat{B}_k\big)^2}{1 - B_k} \Bigg).
\end{aligned}
$$

As a result, it follows from the above inequality that

$$
-\prod_{k=0}^{T-1} \frac{\big(\mathbb{E}[q_k] - \widehat{B}_k\big)^2}{1 - B_k} - \sum_{j=0}^{T-1} \Bigg(\prod_{k=j+1}^{T-1} \frac{\big(\mathbb{E}[q_k] - \widehat{B}_k\big)^2}{1 - B_k} \Bigg) \widetilde{B}_j \Bigg(\prod_{m=0}^{j-1} \mathbb{E}[q_m^2] \Bigg) + \prod_{k=0}^{T-1} \mathbb{E}[q_k^2] \geqslant 0,
$$

that is, $C_0 \geqslant 0$. This completes the proof of the lemma. $\square$

It follows equation (3.12) with ω^* that we have the following minimum variance theorem.

Theorem 3.2. *Assume that the returns of assets and liability are correlated at every period. Then the efficient frontier is given by*

$$
\mathrm{Var}(x_T - l_T) = \frac{\prod_{k=0}^{T-1}(1 - B_k)}{1 - \prod_{k=0}^{T-1}(1 - B_k)} \Bigg(d - x_0 \prod_{k=0}^{T-1} s_k + l_0 \prod_{k=0}^{T-1} \frac{\mathbb{E}[q_k] - \widehat{B}_k}{1 - B_k} \Bigg)^2 + l_0^2 C_0,
$$

where C_0 is defined as (3.13).

3.2.3 The Optimal Strategy with Uncorrelation of Assets and Liability

Assume that the returns of asset and liability are uncorrelated at every period. Then

$$\widehat{B}_t = \mathbb{E}[q_t]B_t \quad \text{and} \quad \widetilde{B}_t = (\mathbb{E}[q_t])^2 B_t.$$

Hence, we have the following results

$$\begin{aligned}
\prod_{k=t}^{T-1} \frac{\mathbb{E}[q_k] - \widehat{B}_k}{(1-B_k)s_k} &= \prod_{k=t}^{T-1} \mathbb{E}[q_k] s_k^{-1}, \\
\prod_{k=t}^{T-1} \big(\mathbb{E}[q_k] - \widehat{B}_k\big) &= \prod_{k=t}^{T-1} \mathbb{E}[q_k]\big(1-B_k\big), \\
\prod_{k=t}^{T-1} \frac{\mathbb{E}[q_k] - \widehat{B}_k}{1-B_k} &= \prod_{k=t}^{T-1} \mathbb{E}[q_k], \\
\prod_{k=t}^{T-1} \frac{\big(\mathbb{E}[q_k] - \widehat{B}_k\big)^2}{1-B_k} &= \prod_{k=t}^{T-1} \big(\mathbb{E}[q_k]\big)^2 (1-B_k)
\end{aligned}$$

and

$$\begin{aligned}
C_0 = &- \prod_{k=0}^{T-1} \big(\mathbb{E}[q_k]\big)^2 \big(1-B_k\big) - \sum_{j=0}^{T-1} \left(\prod_{k=j+1}^{T-1} \big(\mathbb{E}[q_k]\big)^2 \big(1-B_k\big)\right) \big(\mathbb{E}[q_j]\big)^2 B_j \left(\prod_{m=0}^{j-1} \mathbb{E}[q_m^2] \right) \\
&+ \prod_{k=0}^{T-1} \mathbb{E}[q_k^2].
\end{aligned} \tag{3.24}$$

Therefore, we have the following two theorems.

Theorem 3.3. *Assume that the returns of assets and liability are uncorrelated at every period. Then the optimal strategy of problem (3.1) is given by*

$$\pi_t^* = -\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t] s_t \Big(x_t - \gamma^* \prod_{k=t}^{T-1} s_k^{-1} - l_t \prod_{k=t}^{T-1} \mathbb{E}[q_k] s_k^{-1} \Big), \tag{3.25}$$

where

$$\gamma^* = \frac{x_0 \prod_{k=0}^{T-1}(1-B_k)s_k - d - l_0 \prod_{k=0}^{T-1}\mathbb{E}[q_k]\big(1-B_k\big)}{\prod_{k=0}^{T-1}(1-B_k) - 1}. \tag{3.26}$$

Theorem 3.4. *Assume that the returns of assets and liability are uncorrelated at every period. Then the efficient frontier is given by*

$$\mathrm{Var}(x_T - l_T) = \frac{\prod_{k=0}^{T-1}(1-B_k)}{1-\prod_{k=0}^{T-1}(1-B_k)}\left(d - x_0 \prod_{k=0}^{T-1} s_k + l_0 \prod_{k=0}^{T-1}\mathbb{E}[q_k]\right)^2 + l_0^2 C_0,$$

where C_0 is defined as (3.24).

3.3 Numerical Examples

We consider an example of constructing a pension fund consisting of S&P 500 (SP), the index of Emerging Market (EM), Small Stock (MS) of U.S market and a bank account. Based on the data provided in Elton et al. (2009), Table 3.1 presents the expected values, variances and correlation coefficients of the annual return rates of these three indices. And the annual risk free rate is supposed to be 5% ($s_t = 1.05$).

Table 3.1: Data for assets and liability example

	SP	EM	MS	liability
Expected return	14%	16%	17%	10%
Standard deviation	18.5%	30%	24%	20%
Correlation coefficient				
SP	1	0.64	0.79	ρ_1
EM	0.64	1	0.75	ρ_2
MS	0.79	0.75	1	ρ_3
liability	ρ_1	ρ_2	ρ_3	1

Thus, for any time t, we have

$$\mathbb{E}[\mathbf{P}_t] = \begin{pmatrix} 0.09 \\ 0.11 \\ 0.12 \end{pmatrix}, \ \mathrm{Cov}(\mathbf{P}_t) = \begin{pmatrix} 0.0342 & 0.0355 & 0.0351 \\ 0.0355 & 0.0900 & 0.0540 \\ 0.0351 & 0.0540 & 0.0576 \end{pmatrix},$$

$$\mathbb{E}[\mathbf{P}_t\mathbf{P}_t'] = \begin{pmatrix} 0.0423 & 0.0454 & 0.0459 \\ 0.0454 & 0.1021 & 0.0672 \\ 0.0459 & 0.0672 & 0.0720 \end{pmatrix}.$$

The correlation of assets and the liability is $\rho = (\rho_1, \rho_2, \rho_3)$, where

$$\rho_i = \frac{\mathrm{Cov}(q_t, P_t^i)}{\sqrt{\mathrm{Var}(q_t)}\sqrt{\mathrm{Var}(P_t^i)}}$$

is the correlation coefficient of the i-th asset and the liability. This means

$$\mathbb{E}[q_t P_t^i] = \mathbb{E}[q_t]\mathbb{E}[P_t^i] + \rho_i\sqrt{\mathrm{Var}(q_t)}\sqrt{\mathrm{Var}(P_t^i)}.$$

Suppose that the investor consider a 5-time-period investment with initial wealth $x_0 = 3$ and initial liability $l_0 = 1$.

Example 3.1. An Correlation Example

Assume that the returns of the assets and liability are correlated with $\rho = (\rho_1, \rho_2, \rho_3) = (-0.25, 0.5, 0.25)$. Hence,

$$\begin{aligned} \mathrm{Cov}\left(\begin{pmatrix} \mathbf{P}_t \\ q_t \end{pmatrix}\right) &= \begin{pmatrix} \mathrm{Cov}(\mathbf{P}_t) & \mathrm{Cov}(q_t, \mathbf{P}_t) \\ \mathrm{Cov}(q_t, \mathbf{P}_t') & \mathrm{Var}(q_t) \end{pmatrix} \\ &= \begin{pmatrix} 0.0342 & 0.0355 & 0.0351 & -0.0092 \\ 0.0355 & 0.0900 & 0.0540 & 0.0300 \\ 0.0351 & 0.0540 & 0.0576 & 0.0120 \\ -0.0092 & 0.0300 & 0.0120 & 0.0400 \end{pmatrix} \\ &\succ 0. \end{aligned}$$

Using the above formula of $\mathbb{E}[q_t P_t^i]$, we have $\mathbb{E}[q_t\mathbf{P}_t] = (0.0898, 0.1510, 0.1440)'$. Moreover,

$$\mathbf{K}_1 = \mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t] = \begin{bmatrix} 1.0580 \\ -0.1207 \\ 1.1052 \end{bmatrix}, \quad \mathbf{K}_2 = \mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[q_t\mathbf{P}_t] = \begin{bmatrix} -0.2398 \\ 0.4374 \\ 1.7446 \end{bmatrix}.$$

We seek for the expected terminal target with $d = 3.5$. According to Theorem 3.1, we can derive $\gamma^* = 4.0470$ and the optimal strategy of problem (3.1) is specified as follows,

$$\pi_0^* = -1.05(x_0 - 3.1710)\mathbf{K}_1 + 1.2053\mathbf{K}_2 l_0,$$

$$\pi_1^* = -1.05(x_1 - 3.3295)\mathbf{K}_1 + 1.1503\mathbf{K}_2 l_1,$$

$$\pi_2^* = -1.05(x_2 - 3.4960)\mathbf{K}_1 + 1.0979\mathbf{K}_2 l_2,$$

$$\pi_3^* = -1.05(x_3 - 3.6708)\mathbf{K}_1 + 1.0478\mathbf{K}_2 l_3,$$

$$\pi_4^* = -1.05(x_4 - 3.8543)\mathbf{K}_1 + 1.0000\mathbf{K}_2 l_4.$$

The variance of the final optimal surplus is $\mathrm{Var}(x_5 - l_5) = 0.7289$.

Example 3.2. An Uncorrelation Example

Assume that the returns of the assets and liability are uncorrelated. Hence,

$$\begin{aligned}
\mathrm{Cov}\left(\begin{pmatrix}\mathbf{P}_t \\ q_t\end{pmatrix}\right) &= \begin{pmatrix}\mathrm{Cov}(\mathbf{P}_t) & \mathrm{Cov}(q_t, \mathbf{P}_t) \\ \mathrm{Cov}(q_t, \mathbf{P}_t') & \mathrm{Var}(q_t)\end{pmatrix} \\
&= \begin{pmatrix} 0.0342 & 0.0355 & 0.0351 & 0 \\ 0.0355 & 0.0900 & 0.0540 & 0 \\ 0.0351 & 0.0540 & 0.0576 & 0 \\ 0 & 0 & 0 & 0.04 \end{pmatrix} \\
&\succ 0.
\end{aligned}$$

We still seek for the same expected terminal target with $d = 3.5$. According to Theorem 3.3, we can derive $\gamma^* = 4.0464$ and the optimal strategy of problem (3.1) is specified as follows,

$$\pi_0^* = -1.05(x_0 - 3.1705 + 1.1472 l_0)\mathbf{K}_1,$$

$$\pi_1^* = -1.05(x_1 - 3.3290 + 1.0950 l_1)\mathbf{K}_1,$$

$$\pi_2^* = -1.05(x_2 - 3.4955 + 1.0452 l_2)\mathbf{K}_1,$$

$$\pi_3^* = -1.05(x_3 - 3.6702 + 0.9977 l_3)\mathbf{K}_1,$$

$$\pi_4^* = -1.05(x_4 - 3.8538 + 0.9524 l_4)\mathbf{K}_1,$$

where $\mathbf{K}_1$ is the same as Example 3.1. And the variance of the final optimal surplus is $\mathrm{Var}(x_5 - l_5) = 1.0043$.

3.4 Conclusion

Using the parameterized method, the state variable transformation technique and the dynamic programming approach, we obtain in this chapter the closed-form expressions for the optimal investment strategy and the efficient frontier of our multi-period mean-variance asset-liability management problem. Compared with previous literatures, our method is simpler yet more efficient, and the result is more concise and powerful.

Chapter 4

A Mean-Field Formulation for Multi-Period Mean-Variance Asset-Liability Portfolio Selection with an Uncertain Exit time

Most investors realize that they never know exactly the time exiting the market. That is due to many factors can affect the exit time, for example, the price movement of risky assets, securities markets behavior, exogenous huge consumption such as purchasing a house or accident. Therefore, it seems more realistic to relax the restrictive assumption that the investment horizon is pre-determined with certainty. Many papers (see Yi et al. (2008); Li and Xie (2010); Wu and Li (2011); Zhang and Li (2012)) concerned with multi-period mean- variance model with uncertain exit time and derived analytical optimal strategies for their problems. The main difficulty of the model is the non-separability induced by the variance term. There are several methods to conquer it, such as the embedding technique proposed by Li and Ng (2000), the parameterized method developed by Li et al. (2002), just like Chapter 3, the mean-field formulation presented by Cui et al. (2014) and etc. In fact, when the investor exits the capital market with an uncertain time, the first two methods do not work smoothly and efficiently. This chapter we focus on the mean-field formulation to tackle the multi-period mean-variance portfolio of asset and liability with uncertain exit time. We drive the analytical optimal strategies and efficient frontiers. Numerical examples are presented to show efficiency and accuracy of the mean-field formulation to solve the non-separability of multi-period mean-variance portfolio s-election problems. Compared to the embedding technique, the mean-field approach makes the whole process to derive the optimal strategy simpler and more direct. We first introduce the mean-field formulation to solve an uncertain exit model without liability, then we extend it to the case when liability is concerned. The results can reduce to those derived in Chapter 3 if we fix the expected return and the exit time to the terminal, which suggests further our methods make sense.

4.1 Multi-Period Mean-Variance Portfolio Selection without Liability

In order to see the mean-field formulation tackle multi-period mean-variance model clearly, we consider in this section a problem without liability.

4.1.1 The Model

Assume that an investor joins the market at the beginning of period 0 with an initial wealth x_0. He may be forced to leave the financial market at time τ before T by some uncontrollable reasons. The uncertain exit time τ is supposed to be an exogenous random variable with probability mass function $\tilde{p}_t = \Pr\{\tau = t\}, t = 1, 2, \cdots$. Therefore, the actual exit time of the investor is $T \wedge \tau = \min\{T, \tau\}$, and its probability mass function is

$$\alpha_t \triangleq \Pr\{T \wedge \tau = t\} = \begin{cases} \tilde{p}_t, & t = 1, 2, \cdots, T-1, \\ 1 - \sum_{j=1}^{T-1} \tilde{p}_j, & t = T. \end{cases}$$

The multi-period mean-variance investor with an uncertain exit time is to seek the best strategy, $\pi_t^* = [(\pi_t^1)^*, (\pi_t^2)^*, \cdots, (\pi_t^n)^*]'$, $t = 0, 1, \cdots, T-1$, which is the optimizer of the following stochastic optimal control problem,

$$\begin{cases} \min & \mathrm{Var}^{(\tau)}(x_{T\wedge\tau}) - w\mathbb{E}^{(\tau)}[x_{T\wedge\tau}], \\ \text{s.t.} & x_{t+1} = \sum_{i=1}^{n} e_t^i \pi_t^i + \left(x_t - \sum_{i=1}^{n} \pi_t^i\right) s_t \\ & \quad\;\; = s_t x_t + \mathbf{P}_t' \pi_t, \quad t = 0, 1, \cdots, T-1, \end{cases} \tag{4.1}$$

where $w > 0$ is the trade-off parameter between the mean and the variance, and $\mathbb{E}^{(\tau)}[x_{T\wedge\tau}]$ and $\mathrm{Var}^{(\tau)}(x_{T\wedge\tau})$ are defined as follows,

$$\mathbb{E}^{(\tau)}[x_{T\wedge\tau}] \triangleq \sum_{t=1}^{T} \mathbb{E}\big[x_{T\wedge\tau} \big| T\wedge\tau = t\big] \Pr\{T\wedge\tau = t\} = \sum_{t=1}^{T} \mathbb{E}[x_t]\alpha_t,$$

$$\mathrm{Var}^{(\tau)}(x_{T\wedge\tau}) \triangleq \sum_{t=1}^{T} \mathrm{Var}\big(x_{T\wedge\tau} \big| T\wedge\tau = t\big) \Pr\{T\wedge\tau = t\} = \sum_{t=1}^{T} \mathrm{Var}(x_t)\alpha_t,$$

respectively. Then the multi-period mean-variance model with an uncertain exit time (4.1) can be equivalently re-written into the following problem,

$$\begin{cases} \min & \sum_{t=1}^{T} \alpha_t \Big\{ \mathrm{Var}(x_t) - w\mathbb{E}[x_t] \Big\}, \\ \text{s.t.} & x_{t+1} = s_t x_t + \mathbf{P}_t' \pi_t, \quad t = 0, 1, \cdots, T-1. \end{cases} \tag{4.2}$$

4.1.2 The Mean-Field Formulation

Similar to other dynamic mean-variance problems, model (4.2) cannot be solved by dynamic programming directly, as the variance term does not satisfy the smooth property as mentioned in Chapter 1. In this section, we use the mean-field formulation approach proposed in Cui et al. (2014) to tackle this difficulty. The mean-field type stochastic control problem refers to the problem where either the objective functional or the dynamic system involves state processes and their expectations.

Now, let us build the mean-field formulation for problem (4.2). First, for $t = 0, 1, \cdots, T-1$, the evolution of the expectation of the wealth dynamics specified in (4.2) can be presented as

$$\left\{\begin{array}{rcl} \mathbb{E}[x_{t+1}] &=& s_t\mathbb{E}[x_t] + \mathbb{E}[\mathbf{P}_t']\mathbb{E}[\pi_t], \\ \mathbb{E}[x_0] &=& x_0, \end{array}\right. \tag{4.3}$$

due to the independence between $\mathbf{P}_t$ and π_t. Combining the wealth dynamics specified in (4.2) and dynamic equation (4.3) yields the following for $t = 0, 1, \cdots, T-1$,

$$\left\{\begin{array}{rcl} x_{t+1} - \mathbb{E}[x_{t+1}] &=& s_t\big(x_t - \mathbb{E}[x_t]\big) + \mathbf{P}_t'\pi_t - \mathbb{E}[\mathbf{P}_t']\mathbb{E}[\pi_t] \\ &=& s_t\big(x_t - \mathbb{E}[x_t]\big) + \mathbf{P}_t'\big(\pi_t - \mathbb{E}[\pi_t]\big) + \big(\mathbf{P}_t' - \mathbb{E}[\mathbf{P}_t']\big)\mathbb{E}[\pi_t], \\ x_0 - \mathbb{E}[x_0] &=& 0. \end{array}\right. \tag{4.4}$$

In the language of optimal control, by doing so, we have enlarged the state space (x_t) into $(\mathbb{E}[x_t], x_t - \mathbb{E}[x_t])$ and the control space (π_t) into $(\mathbb{E}[\pi_t], \pi_t - \mathbb{E}[\pi_t])$.

Second, although the new control vectors $\mathbb{E}[\pi_t]$ and $\pi_t - \mathbb{E}[\pi_t]$ can be determined independently at time t, they should be chosen such that

$$\mathbb{E}(\pi_t - \mathbb{E}[\pi_t]) = \mathbf{0}, \quad t = 0, 1, \cdots, T-1.$$

Furthermore, we confine $\mathbb{E}[\pi_t]$ to be an $\mathcal{F}_0$-measurable control and $\pi_t - \mathbb{E}[\pi_t]$ to be an $\mathcal{F}_t$-measurable Markov control. Then, $\mathbb{E}[x_t]$ is $\mathcal{F}_0$-measurable, $x_t - \mathbb{E}[x_t]$ is $\mathcal{F}_t$-measurable, $\{(\mathbb{E}[x_t], x_t - \mathbb{E}[x_t])\}$ is again an adapted Markovian process and $\mathcal{F}_t = \sigma(\mathbb{E}[x_t], x_t - \mathbb{E}[x_t])$.

We need to point out that state $\mathbb{E}[x_{t+1}]$ is not observable in the market. Actually, $\mathbb{E}[x_{t+1}]$ is computed through dynamic equation (4.3) after choosing $\mathcal{F}_0$-measurable control $\mathbb{E}[\pi_t]$ and knowing $\mathbb{E}[x_t]$ at time t. Then, state $x_{t+1} - \mathbb{E}[x_{t+1}]$ is obtained after observing x_{t+1} in the market at time $t+1$. The constraint $\mathbb{E}(\pi_t - \mathbb{E}[\pi_t]) = \mathbf{0}$ makes sure that $\mathcal{F}_0$-measurable control $\mathbb{E}[\pi_t]$ and $\mathcal{F}_t$-measurable Markov control $\pi_t - \mathbb{E}[\pi_t]$ are consistent.

Thus, problem (4.2) can be equivalently reformulated as a mean-field type of linear quadratic optimal stochastic control problem,

$$\left\{\begin{array}{rl} \min & \sum_{t=1}^{T}\alpha_t\Big\{\mathbb{E}\big[(x_t-\mathbb{E}[x_t])^2\big]-w\mathbb{E}[x_t]\Big\}, \\ \text{s.t.} & \mathbb{E}[x_t] \text{ satisfies dynamic equation (4.3)}, \\ & x_t-\mathbb{E}[x_t] \text{ satisfies dynamic equation (4.4)}, \\ & \mathbb{E}(\pi_t-\mathbb{E}[\pi_t])=\mathbf{0}, \quad t=0,1,\cdots,T-1. \end{array}\right. \tag{4.5}$$

In this mean-filed formulation of the multi-period mean-variance model with an uncertain exit time, the objective function becomes separable in the expanded state space $(\mathbb{E}[x_t], x_t-\mathbb{E}[x_t])$, which enables us to solve the problem by dynamic programming. However, an additional linear constraint on the second control vector $\pi_t-\mathbb{E}[\pi_t]$ is imposed, which requires caution during the solution process.

4.1.3 The Optimal Strategy and the Efficient Frontier

In this section, we will derive the optimal strategy of problem (4.5) and its corresponding efficient frontier by dynamic programming. Two useful lemmas are introduced before our main results.

Lemma 4.1. *Suppose that $\mathbb{E}[\mathbf{P}_t\mathbf{P}_t']-\mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t']$ is invertible. Then*

$$\Big(\mathbb{E}[\mathbf{P}_t\mathbf{P}_t']-\mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t']\Big)^{-1}\mathbb{E}[\mathbf{P}_t]=\frac{1}{1-B_t}\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t].$$

Proof. Applying Sherman-Morrison formula (lemma 2.1) directly gives rise to the result. □

Define a cost-to-go function at time $t+1$ as

$$J_{t+1}(\mathbb{E}[x_{t+1}], x_{t+1}-\mathbb{E}[x_{t+1}]) \triangleq \min\sum_{j=t+1}^{T}\alpha_j\Big\{\mathbb{E}\big[(x_j-\mathbb{E}[x_j])^2\big]-w\mathbb{E}[x_j]\Big\}.$$

Lemma 4.2. *If the cost-to-go function $J_{t+1}(\mathbb{E}[x_{t+1}], x_{t+1} - \mathbb{E}[x_{t+1}])$ at time $t+1$ satisfies the following decomposition,*

$$\begin{aligned}
&\mathbb{E}[J_{t+1}(\mathbb{E}[x_{t+1}], x_{t+1} - \mathbb{E}[x_{t+1}])|\mathcal{F}_t] \\
&= G_t^1\big(\mathbb{E}[x_t], x_t - \mathbb{E}[x_t]; \mathbb{E}[\pi_t], \pi_t - \mathbb{E}[\pi_t]\big) + G_t^2\big(\mathbb{E}[x_t], x_t - \mathbb{E}[x_t]; \mathbb{E}[\pi_t], \pi_t - \mathbb{E}[\pi_t]\big)
\end{aligned}$$

where $\mathbb{E}[G_t^2(\mathbb{E}[x_t], x_t - \mathbb{E}[x_t]; \mathbb{E}[\pi_t], \pi_t - \mathbb{E}[\pi_t])|\mathcal{F}_0] = 0$ holds for all admissible $\{\mathbb{E}[\pi_i], \pi_i - \mathbb{E}[\pi_i]\}_{i=0}^t$, then we can choose

$$J_t(\mathbb{E}[x_t], x_t - \mathbb{E}[x_t]) = \min_{(\mathbb{E}[\pi_t], \pi_t - \mathbb{E}[\pi_t])} G_t^1\big(\mathbb{E}[x_t], x_t - \mathbb{E}[x_t]; \mathbb{E}[\pi_t], \pi_t - \mathbb{E}[\pi_t]\big)$$

as the cost-to-go function at time t.

Proof. The principle of optimality implies that

$$\begin{aligned}
&\{\mathbb{E}[\pi_i^*], \pi_i^* - \mathbb{E}[\pi_i^*]\}_{i=0}^t \\
&= \arg\min \mathbb{E}[J_{t+1}(\mathbb{E}[x_{t+1}], x_{t+1} - \mathbb{E}[x_{t+1}])|\mathcal{F}_0] \\
&= \arg\min \mathbb{E}[\mathbb{E}[J_{t+1}(\mathbb{E}[x_{t+1}], x_{t+1} - \mathbb{E}[x_{t+1}])|\mathcal{F}_t]|\mathcal{F}_0] \\
&= \arg\min \mathbb{E}[G_t^1\big(\mathbb{E}[x_t], x_t - \mathbb{E}[x_t]; \mathbb{E}[\pi_t], \pi_t - \mathbb{E}[\pi_t]\big)|\mathcal{F}_0] \\
&= \underset{\{\mathbb{E}[\pi_i], \pi_i - \mathbb{E}[\pi_i]\}_{i=0}^{t-1}}{\operatorname{argmin}} \mathbb{E}[\min_{(\mathbb{E}[\pi_t], \pi_t - \mathbb{E}[\pi_t])} G_t^1\big(\mathbb{E}[x_t], x_t - \mathbb{E}[x_t]; \mathbb{E}[\pi_t], \pi_t - \mathbb{E}[\pi_t]\big)|\mathcal{F}_0].
\end{aligned}$$

Obviously, we can set $\min_{(\mathbb{E}[\pi_t], \pi_t - \mathbb{E}[\pi_t])} G_t^1\big(\mathbb{E}[x_t], x_t - \mathbb{E}[x_t]; \mathbb{E}[\pi_t], \pi_t - \mathbb{E}[\pi_t]\big)$ as the benefit-to-go function at time t. □

Remark 4.1. *Lemma 4.1 is the same as the one in Cui et al. (2014). Lemma 4.2 is a simple version of Lemma 3 in Cui et al. (2014). Presenting them here again is to keep the solution procedure intact.*

Lemma 4.2 suggests that we can simplify the cost-to-go function at time t by removing the terms with a zero unconditional expected value. Before presenting our main theorem, we define the following backward recursions for $\{\xi_t\}$ and $\{\zeta_t\}$,

$$\xi_t = \alpha_t + s_t^2(1 - B_t)\xi_{t+1},$$

$$\zeta_t = \alpha_t + s_t\zeta_{t+1},$$

for $t = T-1, T-2, \cdots, 1$, with $\xi_T = \alpha_T, \zeta_T = \alpha_T$. Also, we set $\prod_{\varnothing}(\cdot) = 1$ and $\sum_{\varnothing}(\cdot) = 0$ for the convenience in this thesis.

Theorem 4.1. *The optimal strategies of problem (4.5) are represented by*

$$\pi_t^* - \mathbb{E}[\pi_t^*] = -s_t\big(x_t - \mathbb{E}[x_t]\big)\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t], \tag{4.6}$$

$$\mathbb{E}[\pi_t^*] = \frac{w\zeta_{t+1}}{2\xi_{t+1}} \cdot \frac{1}{1-B_t}\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t], \tag{4.7}$$

for $t = 0, 1, \cdots, T-1$, where the optimal expected wealth level is

$$\mathbb{E}[x_t] = x_0\prod_{k=0}^{t-1} s_k + \frac{w}{2}\sum_{j=0}^{t-1}\frac{\zeta_{j+1}}{\xi_{j+1}} \cdot \frac{B_j}{1-B_j}\prod_{\ell=j+1}^{t-1} s_\ell, \quad \text{for } t = 1, 2, \cdots, T. \tag{4.8}$$

Proof. We prove by backward induction that, for a given information set $\mathcal{F}_t = \sigma\big(\mathbb{E}[x_t], x_t - \mathbb{E}[x_t]\big)$, we have the following expression,

$$J_t\big(\mathbb{E}[x_t], x_t - \mathbb{E}[x_t]\big) = \xi_t\big(x_t - \mathbb{E}[x_t]\big)^2 - w\zeta_t\mathbb{E}[x_t] - \frac{w^2}{4}\sum_{j=t}^{T-1}\frac{\zeta_{j+1}^2}{\xi_{j+1}}\frac{B_j}{1-B_j}, \tag{4.9}$$

as the cost-to-go function at time t.

When $t = T$,

$$\begin{aligned} J_T\big(\mathbb{E}[x_T], x_T - \mathbb{E}[x_T]\big) &= \alpha_T\big(x_T - \mathbb{E}[x_T]\big)^2 - w\alpha_T\mathbb{E}[x_T] \\ &= \xi_T\big(x_T - \mathbb{E}[x_T]\big)^2 - w\zeta_T\mathbb{E}[x_T]. \end{aligned}$$

Assume that we have expression (4.9) for the cost-to-go function at time $t+1$. We prove that we still have expression (4.9) for the cost-to-go function at time t. For the given information set $\mathcal{F}_t$, i.e., $\big(\mathbb{E}[x_t], x_t - \mathbb{E}[x_t]\big)$, the recursive equation reads as

$$\begin{aligned} &J_t\big(\mathbb{E}[x_t], x_t - \mathbb{E}[x_t]\big) \\ =&\alpha_t\big(x_t - \mathbb{E}[x_t]\big)^2 - w\alpha_t\mathbb{E}[x_t] + \min_{(\mathbb{E}[\pi_t], \pi_t - \mathbb{E}[\pi_t])}\mathbb{E}\big[J_{t+1}\big(\mathbb{E}[x_{t+1}], x_{t+1} - \mathbb{E}[x_{t+1}]\big)\big|\mathcal{F}_t\big]. \end{aligned}$$

Based on dynamics (4.3) and (4.4), we deduce

$$
\begin{aligned}
&\mathbb{E}\Big[J_{t+1}\big(\mathbb{E}[x_{t+1}], x_{t+1}-\mathbb{E}[x_{t+1}]\big)\big|\mathcal{F}_t\Big] \\
=& \ \mathbb{E}\Big[\xi_{t+1}\big(x_{t+1}-\mathbb{E}[x_{t+1}]\big)^2 - w\zeta_{t+1}\mathbb{E}[x_{t+1}]\big|\mathcal{F}_t\Big] - \frac{w^2}{4}\sum_{j=t+1}^{T-1}\frac{\zeta_{j+1}^2}{\xi_{j+1}}\frac{B_j}{1-B_j} \\
=& \xi_{t+1} w \mathbb{E}\Big[s_t^2\big(x_t-\mathbb{E}[x_t]\big)^2 + \Big(\mathbf{P}_t'\big(\pi_t-\mathbb{E}[\pi_t]\big)\Big)^2 + \Big(\big(\mathbf{P}_t'-\mathbb{E}[\mathbf{P}_t']\big)\mathbb{E}[\pi_t]\Big)^2 \\
&+ 2s_t\big(x_t-\mathbb{E}[x_t]\big)\mathbf{P}_t'\big(\pi_t-\mathbb{E}[\pi_t]\big) + 2s_t\big(x_t-\mathbb{E}[x_t]\big)\big(\mathbf{P}_t'-\mathbb{E}[\mathbf{P}_t']\big)\mathbb{E}[\pi_t] \\
&+ 2\big(\pi_t-\mathbb{E}[\pi_t]\big)'\mathbf{P}_t\big(\mathbf{P}_t'-\mathbb{E}[\mathbf{P}_t']\big)\mathbb{E}[\pi_t]\Big|\mathcal{F}_t\Big] - w\zeta_{t+1}\big(s_t\mathbb{E}[x_t]+\mathbb{E}[\mathbf{P}_t']\mathbb{E}[\pi_t]\big) \\
&- \frac{w^2}{4}\sum_{j=t+1}^{T-1}\frac{\zeta_{j+1}^2}{\xi_{j+1}}\frac{B_j}{1-B_j}.
\end{aligned}
$$

Since both $\pi_t - \mathbb{E}[\pi_t]$ and $\mathbb{E}[\pi_t]$ are $\mathcal{F}_t$-measurable and $\mathbf{P}_t$ is independent of $\mathcal{F}_t$, we have

$$
\mathbb{E}\Big[\Big(\mathbf{P}_t'\big[\pi_t-\mathbb{E}[\pi_t]\big]\Big)^2\Big|\mathcal{F}_t\Big] = \big(\pi_t-\mathbb{E}[\pi_t]\big)'\mathbb{E}[\mathbf{P}_t\mathbf{P}_t']\big(\pi_t-\mathbb{E}[\pi_t]\big),
$$

$$
\mathbb{E}\Big[\Big(\big(\mathbf{P}_t'-\mathbb{E}[\mathbf{P}_t']\big)\mathbb{E}[\pi_t]\Big)^2\Big|\mathcal{F}_t\Big] = \mathbb{E}[\pi_t']\big(\mathbb{E}[\mathbf{P}_t\mathbf{P}_t'] - \mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t']\big)\mathbb{E}[\pi_t],
$$

$$
\mathbb{E}\Big[2s_t\big(x_t-\mathbb{E}[x_t]\big)\mathbf{P}_t'\big(\pi_t-\mathbb{E}[\pi_t]\big)\Big|\mathcal{F}_t\Big] = 2s_t\big(x_t-\mathbb{E}[x_t]\big)\mathbb{E}[\mathbf{P}_t']\big(\pi_t-\mathbb{E}[\pi_t]\big),
$$

$$
\mathbb{E}\Big[2s_t\big(x_t-\mathbb{E}[x_t]\big)\big(\mathbf{P}_t'-\mathbb{E}[\mathbf{P}_t']\big)\mathbb{E}[\pi_t]\Big|\mathcal{F}_t\Big] = 0,
$$

$$
\mathbb{E}\Big[2\big(\pi_t-\mathbb{E}[\pi_t]\big)'\mathbf{P}_t\big(\mathbf{P}_t'-\mathbb{E}[\mathbf{P}_t']\big)\mathbb{E}[\pi_t]\Big|\mathcal{F}_t\Big] = 2\big(\pi_t-\mathbb{E}[\pi_t]\big)'\big(\mathbb{E}[\mathbf{P}_t\mathbf{P}_t'] - \mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t']\big)\mathbb{E}[\pi_t],
$$

which further implies,

$$
\begin{aligned}
&\mathbb{E}\big[J_{t+1}\big(\mathbb{E}[x_{t+1}], x_{t+1}-\mathbb{E}[x_{t+1}]\big)\big|\mathcal{F}_t\big] \\
=&\xi_{t+1}\Big[s_t^2\big(x_t-\mathbb{E}[x_t]\big)^2+\big(\pi_t-\mathbb{E}[\pi_t]\big)'\mathbb{E}[\mathbf{P}_t\mathbf{P}_t']\big(\pi_t-\mathbb{E}[\pi_t]\big) \\
&+2s_t\big(x_t-\mathbb{E}[x_t]\big)\mathbb{E}[\mathbf{P}_t']\big(\pi_t-\mathbb{E}[\pi_t]\big)\Big]+\xi_{t+1}\mathbb{E}[\pi_t']\big(\mathbb{E}[\mathbf{P}_t\mathbf{P}_t']-\mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t']\big)\mathbb{E}[\pi_t] \\
&-w\zeta_{t+1}\mathbb{E}[\mathbf{P}_t']\mathbb{E}[\pi_t]-ws_t\zeta_{t+1}\mathbb{E}[x_t]-\frac{w^2}{4}\sum_{j=t+1}^{T-1}\frac{\zeta_{j+1}^2}{\xi_{j+1}}\frac{B_j}{1-B_j} \\
&-2\xi_{t+1}\big(\pi_t-\mathbb{E}[\pi_t]\big)'\big(\mathbb{E}[\mathbf{P}_t\mathbf{P}_t']-\mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t']\big)\mathbb{E}[\pi_t] \\
=&\;G_t^1\big(\mathbb{E}[x_t], x_t-\mathbb{E}[x_t];\mathbb{E}[\pi_t],\pi_t-\mathbb{E}[\pi_t]\big)+G_t^2\big(\mathbb{E}[x_t], x_t-\mathbb{E}[x_t];\mathbb{E}[\pi_t],\pi_t-\mathbb{E}[\pi_t]\big),
\end{aligned}
$$

where

$$
\begin{aligned}
&G_t^1\big(\mathbb{E}[x_t], x_t-\mathbb{E}[x_t];\mathbb{E}[\pi_t],\pi_t-\mathbb{E}[\pi_t]\big) \\
=&\xi_{t+1}\Big[s_t^2\big(x_t-\mathbb{E}[x_t]\big)^2+\big(\pi_t-\mathbb{E}[\pi_t]\big)'\mathbb{E}[\mathbf{P}_t\mathbf{P}_t']\big(\pi_t-\mathbb{E}[\pi_t]\big) \\
&+2s_t\big(x_t-\mathbb{E}[x_t]\big)\mathbb{E}[\mathbf{P}_t']\big(\pi_t-\mathbb{E}[\pi_t]\big)\Big]+\xi_{t+1}\mathbb{E}[\pi_t']\big(\mathbb{E}[\mathbf{P}_t\mathbf{P}_t']-\mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t']\big)\mathbb{E}[\pi_t] \\
&-w\zeta_{t+1}\mathbb{E}[\mathbf{P}_t']\mathbb{E}[\pi_t]-ws_t\zeta_{t+1}\mathbb{E}[x_t]-\frac{w^2}{4}\sum_{j=t+1}^{T-1}\frac{\zeta_{j+1}^2}{\xi_{j+1}}\frac{B_j}{1-B_j},
\end{aligned}
$$

$$
G_t^2\big(\mathbb{E}[x_t], x_t-\mathbb{E}[x_t];\mathbb{E}[\pi_t],\pi_t-\mathbb{E}[\pi_t]\big)=2\xi_{t+1}\big(\pi_t-\mathbb{E}[\pi_t]\big)'\big(\mathbb{E}[\mathbf{P}_t\mathbf{P}_t']-\mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t']\big)\mathbb{E}[\pi_t].
$$

Note that any admissible $\big(\mathbb{E}[\pi_t],\pi_t-\mathbb{E}[\pi_t]\big)$ satisfies $\mathbb{E}\big(\pi_t-\mathbb{E}[\pi_t]\big)=\mathbf{0}$, which implies

$$
\mathbb{E}\big[G_t^2\big(\mathbb{E}[x_t], x_t-\mathbb{E}[x_t];\mathbb{E}[\pi_t],\pi_t-\mathbb{E}[\pi_t]\big)\big|\mathcal{F}_0\big]=0.
$$

Since

$$
\mathbb{E}[\mathbf{P}_t\mathbf{P}_t']-\mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t']=\mathrm{Cov}(\mathbf{P}_t)=\mathrm{Cov}(\mathbf{e}_t-s_t\mathbf{1})=\mathrm{Cov}(\mathbf{e}_t)\succ 0,
$$

we complete the square for $G^1(\cdot)$ directly to derive the optimal strategy at time t,

$$
\begin{aligned}
&G_t^1\big(\mathbb{E}[x_t], x_t-\mathbb{E}[x_t]; \mathbb{E}[\pi_t], \pi_t-\mathbb{E}[\pi_t]\big)\\
=&\xi_{t+1}\Big\{s_t^2(1-B_t)\big(x_t-\mathbb{E}[x_t]\big)^2+\Big[\big(\pi_t-\mathbb{E}[\pi_t]\big)+s_t\big(x_t-\mathbb{E}[x_t]\big)\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t']\Big]'\\
&\cdot\mathbb{E}[\mathbf{P}_t\mathbf{P}_t']\Big[\big(\pi_t-\mathbb{E}[\pi_t]\big)+s_t\big(x_t-\mathbb{E}[x_t]\big)\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t']\Big]\Big\}\\
&+\xi_{t+1}\left[\mathbb{E}[\pi_t]-\frac{w\zeta_{t+1}}{2\xi_{t+1}}\big(\mathbb{E}[\mathbf{P}_t\mathbf{P}_t']-\mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t']\big)^{-1}\mathbb{E}[\mathbf{P}_t]\right]'\big(\mathbb{E}[\mathbf{P}_t\mathbf{P}_t']-\mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t']\big)\\
&\cdot\left[\mathbb{E}[\pi_t]-\frac{w\zeta_{t+1}}{2\xi_{t+1}}\big(\mathbb{E}[\mathbf{P}_t\mathbf{P}_t']-\mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t']\big)^{-1}\mathbb{E}[\mathbf{P}_t]\right]-\frac{w^2}{4}\frac{\zeta_{t+1}^2}{\xi_{t+1}}\frac{B_t}{1-B_t}-ws_t\zeta_{t+1}\mathbb{E}[x_t]\\
&-\frac{w^2}{4}\sum_{j=t+1}^{T-1}\frac{\zeta_{j+1}^2}{\xi_{j+1}}\frac{B_j}{1-B_j}.
\end{aligned}
$$

By means of Lemma 4.1, we further have

$$
\begin{aligned}
&G_t^1\big[\mathbb{E}[x_t], x_t-\mathbb{E}[x_t]; \mathbb{E}[\pi_t], \pi_t-\mathbb{E}[\pi_t]\big]\\
=&\xi_{t+1}\Big\{s_t^2(1-B_t)\big(x_t-\mathbb{E}[x_t]\big)^2+\Big[\big(\pi_t-\mathbb{E}[\pi_t]\big)+s_t\big(x_t-\mathbb{E}[x_t]\big)\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t']\Big]'\\
&\cdot\mathbb{E}[\mathbf{P}_t\mathbf{P}_t']\Big[\big(\pi_t-\mathbb{E}[\pi_t]\big)+s_t\big(x_t-\mathbb{E}[x_t]\big)\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t']\Big]\Big\}\\
&\xi_{t+1}\left[\mathbb{E}[\pi_t]-\frac{w\zeta_{t+1}}{2\xi_{t+1}}\cdot\frac{1}{1-B_t}\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t]\right]'\big(\mathbb{E}[\mathbf{P}_t\mathbf{P}_t']-\mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t']\big)\\
&\cdot\left[\mathbb{E}[\pi_t]-\frac{w\zeta_{t+1}}{2\xi_{t+1}}\cdot\frac{1}{1-B_t}\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t]\right]-\frac{w^2}{4}\frac{\zeta_{t+1}^2}{\xi_{t+1}}\frac{B_t}{1-B_t}-ws_t\zeta_{t+1}\mathbb{E}[x_t]\\
&-\frac{w^2}{4}\sum_{j=t+1}^{T-1}\frac{\zeta_{j+1}^2}{\xi_{j+1}}\frac{B_j}{1-B_j}.
\end{aligned}
$$

We first derive the optimal $(\mathbb{E}[\pi_t^*], \pi_t^*-\mathbb{E}[\pi_t^*])$ by maximizing G_t^1 without the linear constraint $\mathbb{E}(\pi_t-\mathbb{E}[\pi_t])=\mathbf{0}$, and then show the derived optimal strategy satisfies this constraint. More specifically, maximizing G_t^1 yields

$$\pi_t^* - \mathbb{E}[\pi_t^*] = -s_t\big(x_t - \mathbb{E}[x_t]\big)\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t],$$

$$\mathbb{E}[\pi_t^*] = \frac{w\zeta_{t+1}}{2\xi_{t+1}} \cdot \frac{1}{1-B_t}\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t].$$

Therefore, we get

$$\begin{aligned}
&G_t^1\big(\mathbb{E}[x_t], x_t - \mathbb{E}[x_t]; \mathbb{E}[\pi_t^*], \pi_t^* - \mathbb{E}[\pi_t^*]\big) + \alpha_t\big(x_t - \mathbb{E}[x_t]\big)^2 - w\alpha_t\mathbb{E}[x_t] \\
=&\xi_{t+1}s_t^2(1-B_t)\big(x_t - \mathbb{E}[x_t]\big)^2 - ws_t\zeta_{t+1}\mathbb{E}[x_t] + \alpha_t\big(x_t - \mathbb{E}[x_t]\big)^2 - w\alpha_t\mathbb{E}[x_t] \\
=&\xi_t w\big(x_t - \mathbb{E}[x_t]\big)^2 - w\zeta_t\mathbb{E}[x_t] - \frac{w^2}{4}\sum_{j=t}^{T-1}\frac{\zeta_{j+1}^2}{\xi_{j+1}w}\frac{B_j}{1-B_j}
\end{aligned}$$

as the cost-to-go function at time t.

Substituting the optimal expected strategy (4.7) into dynamic equation (4.3), we further deduce the following recursive relationship of the optimal expected wealth level,

$$\mathbb{E}[x_{t+1}] = s_t\mathbb{E}[x_t] + \frac{w}{2}\frac{\zeta_{t+1}}{\xi_{t+1}} \cdot \frac{B_t}{1-B_t},$$

which implies

$$\mathbb{E}[x_t] = x_0\prod_{k=0}^{t-1}s_k + \frac{w}{2}\sum_{j=0}^{t-1}\frac{\zeta_{j+1}}{\xi_{j+1}} \cdot \frac{B_j}{1-B_j}\prod_{\ell=j+1}^{t-1}s_\ell.$$

Finally, we show that this optimal strategy satisfies the linear constraint. At time 0, $\mathbb{E}(\pi_0^* - \mathbb{E}[\pi_0^*]) = \mathbf{0}$ is obvious. Then, according to the dynamic equation (4.4), we have $\mathbb{E}(x_1 - \mathbb{E}(x_1)) = 0$, which further implies $\mathbb{E}(\pi_1^* - \mathbb{E}[\pi_1^*]) = \mathbf{0}$. Repeating this argument, we have $\mathbb{E}(\pi_t^* - \mathbb{E}[\pi_t^*]) = \mathbf{0}$ holds for all t. $\square$

It follows from Theorem 4.1 that we have

$$
\begin{aligned}
\pi_t^* =& - s_t \mathbb{E}^{-1}[\mathbf{P}_t \mathbf{P}_t'] \mathbb{E}[\mathbf{P}_t] x_t + s_t \mathbb{E}^{-1}[\mathbf{P}_t \mathbf{P}_t'] \mathbb{E}[\mathbf{P}_t] \mathbb{E}[x_t] + \mathbb{E}[\pi_t^*] \\
=& - s_t \mathbb{E}^{-1}[\mathbf{P}_t \mathbf{P}_t'] \mathbb{E}[\mathbf{P}_t] x_t + \mathbb{E}^{-1}[\mathbf{P}_t \mathbf{P}_t'] \mathbb{E}[\mathbf{P}_t] \Bigg[x_0 \prod_{k=0}^{t} s_k \\
&+ \sum_{j=0}^{t-1} \frac{w\zeta_{j+1}}{2\xi_{j+1}} \cdot \frac{B_j}{1-B_j} \prod_{\ell=j+1}^{t} s_\ell + \frac{w\zeta_{t+1}}{2\xi_{t+1}} \cdot \frac{1}{1-B_t} \Bigg],
\end{aligned} \tag{4.10}
$$

which is the optimal portfolio strategy obtained in Zhang and Li (2012).

Substituting the optimal strategies (4.6) and (4.7) into dynamic equation (4.4), we further deduce the following recursive relationship,

$$
\begin{aligned}
x_{t+1} - \mathbb{E}[x_{t+1}] =& \; s_t \big(1 - \mathbf{P}_t' \mathbb{E}^{-1}[\mathbf{P}_t \mathbf{P}_t'] \mathbb{E}[\mathbf{P}_t]\big) \big(x_t - \mathbb{E}[x_t]\big) \\
&+ \frac{w\zeta_{t+1}}{2\xi_{t+1}} \cdot \frac{1}{1-B_t} \big(\mathbf{P}_t' - \mathbb{E}[\mathbf{P}_t']\big) \mathbb{E}^{-1}[\mathbf{P}_t \mathbf{P}_t'] \mathbb{E}[\mathbf{P}_t].
\end{aligned}
$$

Completing the square for the above equation yields

$$
\begin{aligned}
\big(x_{t+1} - \mathbb{E}[x_{t+1}]\big)^2 =& \; s_t^2 \big(1 - \mathbf{P}_t' \mathbb{E}^{-1}[\mathbf{P}_t \mathbf{P}_t'] \mathbb{E}[\mathbf{P}_t]\big)^2 \big(x_t - \mathbb{E}[x_t]\big)^2 \\
&+ \frac{w^2}{4} \frac{\zeta_{t+1}^2}{\xi_{t+1}^2} \cdot \frac{1}{(1-B_t)^2} \Big[\big(\mathbf{P}_t' - \mathbb{E}[\mathbf{P}_t']\big) \mathbb{E}^{-1}[\mathbf{P}_t \mathbf{P}_t'] \mathbb{E}[\mathbf{P}_t] \Big]^2 \\
&+ s_t \big(1 - \mathbf{P}_t' \mathbb{E}^{-1}[\mathbf{P}_t \mathbf{P}_t'] \mathbb{E}[\mathbf{P}_t]\big) \big(x_t - \mathbb{E}[x_t]\big) \\
&\cdot \frac{w\zeta_{t+1}}{\xi_{t+1}} \cdot \frac{1}{1-B_t} \big(\mathbf{P}_t' - \mathbb{E}[\mathbf{P}_t']\big) \mathbb{E}^{-1}[\mathbf{P}_t \mathbf{P}_t'] \mathbb{E}[\mathbf{P}_t],
\end{aligned}
$$

which implies

$$
\begin{aligned}
\mathrm{Var}\big(x_{t+1}\big) =& \; \mathbb{E}\Big[\big(x_{t+1} - \mathbb{E}[x_{t+1}]\big)^2\Big] \\
=& \; s_t^2 (1-B_t) \mathbb{E}\Big[\big(x_t - \mathbb{E}[x_t]\big)^2\Big] + \frac{w^2}{4} \frac{\zeta_{t+1}^2}{\xi_{t+1}^2} \cdot \frac{B_t}{1-B_t} \\
=& \; s_t^2 (1-B_t) \mathrm{Var}\big(x_t\big) + \frac{w^2}{4} \frac{\zeta_{t+1}^2}{\xi_{t+1}^2} \cdot \frac{B_t}{1-B_t}.
\end{aligned}
$$

This leads to

$$\operatorname{Var}\big(x_t\big) = \frac{w^2}{4}\sum_{j=0}^{t-1}\frac{\zeta_{j+1}^2}{\xi_{j+1}^2}\cdot\frac{B_j}{1-B_j}\prod_{\ell=j+1}^{t-1}s_\ell^2(1-B_\ell), \quad \text{for } t=1,2,\cdots,T. \tag{4.11}$$

It follows from (4.8) and (4.11) that we have

$$\begin{aligned}
\mathbb{E}^{(\tau)}[x_{T\wedge\tau}] &= \sum_{t=1}^{T}\mathbb{E}[x_t]\alpha_t \\
&= \sum_{t=1}^{T}\left\{x_0\prod_{k=0}^{t-1}s_k + \frac{w}{2}\sum_{j=0}^{t-1}\frac{\zeta_{j+1}}{\xi_{j+1}}\cdot\frac{B_j}{1-B_j}\prod_{\ell=j+1}^{t-1}s_\ell\right\}\alpha_t \\
&= x_0\sum_{t=1}^{T}\alpha_t\prod_{k=0}^{t-1}s_k + \frac{w}{2}\sum_{t=1}^{T}\left\{\alpha_t\sum_{j=0}^{t-1}\frac{\zeta_{j+1}}{\xi_{j+1}}\cdot\frac{B_j}{1-B_j}\prod_{\ell=j+1}^{t-1}s_\ell\right\}
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{Var}^{(\tau)}(x_{T\wedge\tau}) &= \sum_{t=1}^{T}\operatorname{Var}(x_t)\alpha_t \\
&= \sum_{t=1}^{T}\left\{\frac{w^2}{4}\sum_{j=0}^{t-1}\frac{\zeta_{j+1}^2}{\xi_{j+1}^2}\cdot\frac{B_j}{1-B_j}\prod_{\ell=j+1}^{t-1}s_\ell^2(1-B_\ell)\right\}\alpha_t \\
&= \frac{w^2}{4}\sum_{t=1}^{T}\left\{\alpha_t\sum_{j=0}^{t-1}\frac{\zeta_{j+1}^2}{\xi_{j+1}^2}\cdot\frac{B_j}{1-B_j}\prod_{\ell=j+1}^{t-1}s_\ell^2(1-B_\ell)\right\}.
\end{aligned}$$

Thus, we have the following theorem.

Theorem 4.2. *The optimal strategy of problem (4.1) is represented by (4.10) and the efficient frontier is given by*

$$\operatorname{Var}^{(\tau)}(x_{T\wedge\tau}) = \frac{\sum_{t=1}^{T}\left\{\alpha_t\sum_{j=0}^{t-1}\frac{\zeta_{j+1}^2}{\xi_{j+1}^2}\cdot\frac{B_j}{1-B_j}\prod_{\ell=j+1}^{t-1}s_\ell^2(1-B_\ell)\right\}}{\left(\sum_{t=1}^{T}\left\{\alpha_t\sum_{j=0}^{t-1}\frac{\zeta_{j+1}}{\xi_{j+1}}\cdot\frac{B_j}{1-B_j}\prod_{\ell=j+1}^{t-1}s_\ell\right\}\right)^2}\left(\mathbb{E}^{(\tau)}[x_{T\wedge\tau}]-x_0\sum_{t=1}^{T}\alpha_t\prod_{k=0}^{t-1}s_k\right)^2$$

$$\textit{for } \mathbb{E}^{(\tau)}[x_{T\wedge\tau}] \geqslant x_0\sum_{t=1}^{T}\alpha_t\prod_{k=0}^{t-1}s_k.$$

The optimal strategy obtained in Theorem 4.2 is the same as the result established in Zhang and Li (2012) when the return rates in their work are not serially correlated.

4.1.4 Numerical Example

Example 4.1. Consider the example as Section 3.3. Here we ignore the information of liability i.e., ignore the last line and last column of Table 3.1 and do not fix the terminal expectation but balance the variance and expectation by the trade-off parameter.

Assume that an investor plans a five-period investment with an initial wealth $x_0 = 1$ and that the trade-off parameter $w = 1$. But he may exit the market at any time t ($t = 1, 2, 3, 4, 5$). To investigate the impact of uncertain exit time on the optimal policy and efficient frontier clearly, we choose different probability mass function $\alpha^{(i)} = (\alpha_1^{(i)}, \alpha_2^{(i)}, \alpha_3^{(i)}, \alpha_4^{(i)}, \alpha_5^{(i)}), (i = 1, 2, 3, 4)$ of the exit time τ:

$$\alpha^{(1)} = (0.1, 0.15, 0.2, 0.25, 0.3),$$

$$\alpha^{(2)} = (0, 0.1, 0.1, 0.3, 0.5),$$

$$\alpha^{(3)} = (0, 0, 0.1, 0.2, 0.7),$$

$$\alpha^{(4)} = (0, 0, 0, 0, 1),$$

where $\alpha^{(4)}$ means the investor exit the market at the terminal time.

Then the optimal expected wealth level

$$\mathbb{E}[\mathbf{x}]^{(i)} = (\mathbb{E}[x_1]^{(i)}, \mathbb{E}[x_2]^{(i)}, \mathbb{E}[x_3]^{(i)}, \mathbb{E}[x_4]^{(i)}, \mathbb{E}[x_5]^{(i)}), i = 1, 2, 3, 4$$

are given by

$$\mathbb{E}[\mathbf{x}]^{(1)} = (1.2675, 1.5210, 1.7659, 2.0055, 2.2423),$$

$$\mathbb{E}[\mathbf{x}]^{(2)} = (1.3006, 1.5723, 1.8304, 2.0756, 2.3159),$$

$$\mathbb{E}[\mathbf{x}]^{(3)} = (1.3220, 1.6125, 1.8781, 2.1304, 2.3735),$$

$$\mathbb{E}[\mathbf{x}]^{(4)} = (1.3451, 1.6557, 1.9392, 2.2017, 2.4483).$$

Therefore, according to Theorem 4.2, the optimal strategy of problem 4.1 is specified as follows,

$$\pi_0^{(1)*} = (-1.05x_0 + 2.0635)\mathbf{K}_1, \qquad \pi_0^{(2)*} = (-1.05x_0 + 2.2182)\mathbf{K}_1,$$

$$\pi_1^{(1)*} = (-1.05x_1 + 2.2175)\mathbf{K}_1, \qquad \pi_1^{(2)*} = (-1.05x_1 + 2.3291)\mathbf{K}_1,$$

$$\pi_2^{(1)*} = (-1.05x_2 + 2.3841)\mathbf{K}_1, \qquad \pi_2^{(2)*} = (-1.05x_1 + 2.4879)\mathbf{K}_1,$$

$$\pi_3^{(1)*} = (-1.05x_3 + 2.5596)\mathbf{K}_1, \qquad \pi_3^{(2)*} = (-1.05x_1 + 2.6384)\mathbf{K}_1,$$

$$\pi_4^{(1)*} = (-1.05x_4 + 2.7423)\mathbf{K}_1, \qquad \pi_4^{(2)*} = (-1.05x_1 + 2.8159)\mathbf{K}_1,$$

$$\pi_0^{(3)*} = (-1.05x_0 + 2.3182)\mathbf{K}_1, \qquad \pi_0^{(4)*} = (-1.05x_0 + 2.4256)\mathbf{K}_1,$$

$$\pi_1^{(3)*} = (-1.05x_1 + 2.4341)\mathbf{K}_1, \qquad \pi_1^{(4)*} = (-1.05x_1 + 2.5468)\mathbf{K}_1,$$

$$\pi_2^{(3)*} = (-1.05x_2 + 2.5558)\mathbf{K}_1, \qquad \pi_2^{(4)*} = (-1.05x_1 + 2.6742)\mathbf{K}_1,$$

$$\pi_3^{(3)*} = (-1.05x_3 + 2.7103)\mathbf{K}_1, \qquad \pi_3^{(4)*} = (-1.05x_1 + 2.8079)\mathbf{K}_1,$$

$$\pi_4^{(3)*} = (-1.05x_4 + 2.8735)\mathbf{K}_1, \qquad \pi_4^{(4)*} = (-1.05x_1 + 2.9483)\mathbf{K}_1,$$

where $\mathbf{K}_1$ is the same as Example 3.1. The variances of the optimal wealth levels

$$\mathrm{Var}(\mathbf{x})^{(i)} = (\mathrm{Var}[x_1]^{(i)}, \mathrm{Var}[x_2]^{(i)}, \mathrm{Var}[x_3]^{(i)}, \mathrm{Var}[x_4]^{(i)}, \mathrm{Var}[x_5]^{(i)}), i = 1, 2, 3, 4$$

are given as

$$\mathrm{Var}(\mathbf{x})^{(1)} = (0.1731, 0.2824, 0.3489, 0.3860, 0.4026),$$

$$\mathrm{Var}(\mathbf{x})^{(2)} = (0.2299, 0.3555, 0.4260, 0.4554, 0.4626),$$

$$\mathrm{Var}(\mathbf{x})^{(3)} = (0.2710, 0.4190, 0.4882, 0.5146, 0.5140),$$

$$\mathrm{Var}(\mathbf{x})^{(4)} = (0.3188, 0.4930, 0.5744, 0.5978, 0.5860).$$

Finally, we have

$$\mathbb{E}^{(\tau)}[x_{5\wedge\tau}]^{(1)} = 1.8821, \quad \mathrm{Var}^{(\tau)}(x_{5\wedge\tau})^{(1)} = 0.3467,$$

$$\mathbb{E}^{(\tau)}[x_{5\wedge\tau}]^{(2)} = 2.1209, \quad \mathrm{Var}^{(\tau)}(x_{5\wedge\tau})^{(2)} = 0.4461,$$

$$\mathbb{E}^{(\tau)}[x_{5\wedge\tau}]^{(3)} = 2.2753, \quad \mathrm{Var}^{(\tau)}(x_{5\wedge\tau})^{(3)} = 0.5115,$$

$$\mathbb{E}^{(\tau)}[x_{5\wedge\tau}]^{(4)} = 2.4483, \quad \mathrm{Var}^{(\tau)}(x_{5\wedge\tau})^{(4)} = 0.5860,$$

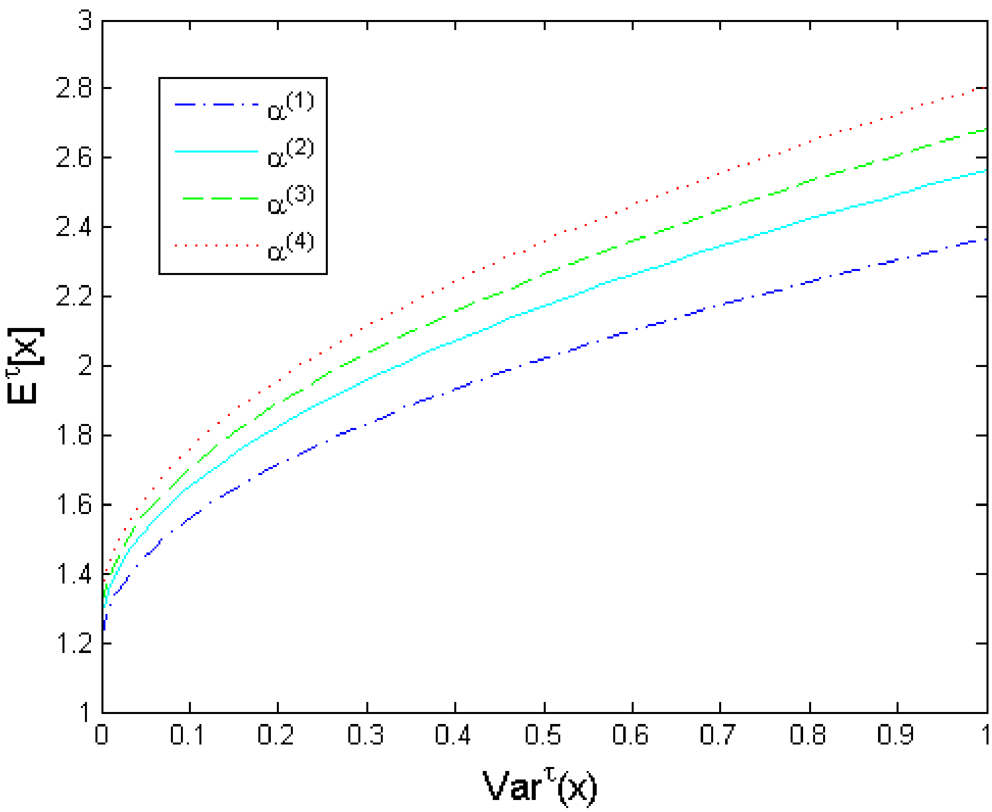

Figure 4.1: Efficient frontiers with different probability mass function of exit time

Figure 4.1 is the efficient frontier with different probability mass function of the exit time. We can see that the one exits at the terminal time gets most expected wealth return at the same risk level compared with others. It also indicates that when the later the investor exits the financial market, the more expected wealth returns he/she will obtain at the same level of the risk, which is consistent with the real life.

4.2 Multi-Period Mean-Variance Asset-Liability Portfolio Selection

This section is concerned with optimal multi-period asset-liability mean-variance portfolio selection with an uncertain exit time in a mean-field formulation. Compared with Section 4.1, we take liability into account.

4.2.1 Formulation

The multi-period asset-liability mean-variance portfolio selection problem with uncertain exit time is to seek the best strategy, $\pi_t^* = [(\pi_t^1)^*, (\pi_t^2)^*, \cdots, (\pi_t^n)^*]'$, $t = 0, 1, \cdots, T-1$, which is the optimizer of the following stochastic optimal control problem with uncertain exit time,

$$\begin{cases} \min & \mathrm{Var}^{(\tau)}(x_{T\wedge\tau} - l_{T\wedge\tau}) - w\mathbb{E}^{(\tau)}[x_{T\wedge\tau} - l_{T\wedge\tau}], \\ s.t. & x_{t+1} = \displaystyle\sum_{i=1}^{n} e_t^i \pi_t^i + \left(x_t - \sum_{i=1}^{n} \pi_t^i\right) s_t \\ & \quad\;\; = s_t x_t + \mathbf{P}_t' \pi_t, \\ & l_{t+1} = q_t l_t, \qquad\qquad t = 0, 1, \cdots, T-1, \end{cases} \tag{4.12}$$

where $\mathbb{E}^{(\tau)}[x_{T\wedge\tau} - l_{T\wedge\tau}]$, $\mathrm{Var}^{(\tau)}(x_{T\wedge\tau} - l_{T\wedge\tau})$ are defined as

$$\begin{aligned} \mathbb{E}^{(\tau)}[x_{T\wedge\tau} - l_{T\wedge\tau}] &\triangleq \sum_{t=0}^{T} \mathbb{E}\big[x_{T\wedge\tau} - l_{T\wedge\tau} \big| T \wedge \tau = t\big] \Pr\{T \wedge \tau = t\} \\ &= \sum_{t=0}^{T} \mathbb{E}[x_t - l_t]\alpha_t, \end{aligned}$$

$$\begin{aligned}\mathrm{Var}^{(\tau)}(x_{T\wedge\tau}-l_{T\wedge\tau}) &\triangleq \sum_{t=0}^{T}\mathrm{Var}\big(x_{T\wedge\tau}-l_{T\wedge\tau}\big|T\wedge\tau=t\big)\Pr\{T\wedge\tau=t\}\\ &=\sum_{t=0}^{T}\mathrm{Var}(x_t-l_t)\alpha_t,\end{aligned}$$

respectively. Similar to Section 4.1, we tackle it by mean-field formulation. For $t=0,1,\cdots,T-1$, taking the expectation operator of the dynamic system specified in (4.12) and noticing that $\mathbf{P}_t$ and π_t, q_t and l_t are independent, we can drive

$$\begin{cases}\mathbb{E}[x_{t+1}]=s_t\mathbb{E}[x_t]+\mathbb{E}[\mathbf{P}_t']\mathbb{E}[\pi_t],\\ \mathbb{E}[l_{t+1}]=\mathbb{E}[q_t]\mathbb{E}[l_t],\\ \mathbb{E}[x_0]=x_0,\\ \mathbb{E}[l_0]=l_0.\end{cases}\tag{4.13}$$

Combining the dynamic systems of (4.12) and (4.13) yields the following, for $t=0,1,\cdots,T-1$,

$$\begin{cases}x_{t+1}-\mathbb{E}[x_{t+1}]=s_t\big(x_t-\mathbb{E}[x_t]\big)+\mathbf{P}_t'\pi_t-\mathbb{E}[\mathbf{P}_t']\mathbb{E}[\pi_t]\\ \qquad\qquad =s_t\big(x_t-\mathbb{E}[x_t]\big)+\mathbf{P}_t'\big(\pi_t-\mathbb{E}[\pi_t]\big)+\big(\mathbf{P}_t'-\mathbb{E}[\mathbf{P}_t']\big)\mathbb{E}[\pi_t],\\ l_{t+1}-\mathbb{E}[l_{t+1}]=q_tl_t-\mathbb{E}[q_t]\mathbb{E}[l_t]\\ \qquad\qquad =q_t\big(l_t-\mathbb{E}[l_t]\big)+\big(q_t-\mathbb{E}[q_t]\big)\mathbb{E}[l_t],\\ x_0-\mathbb{E}[x_0]=0,\\ l_0-\mathbb{E}[l_0]=0.\end{cases}\tag{4.14}$$

Then the state space (x_t,l_t) and the control space (π_t) are enlarged into $(\mathbb{E}[x_t],x_t-\mathbb{E}[x_t],\mathbb{E}[l_t],l_t-\mathbb{E}[l_t])$ and $(\mathbb{E}[\pi_t],\pi_t-\mathbb{E}[\pi_t])$, respectively. Although we can select the control vector $\mathbb{E}[\pi_t]$ and $\pi_t-\mathbb{E}[\pi_t]$ independently at time t, they should be chosen such that

$$\mathbb{E}[\pi_t-\mathbb{E}[\pi_t]]=\mathbf{0},\quad t=0,1,\cdots,T-1,$$

and thus

$$\mathbb{E}[x_t-\mathbb{E}[x_t]]=\mathbf{0},\quad t=0,1,\cdots,T-1,$$

is satisfied. We also confine admissible investment strategies $(\mathbb{E}[\pi_t], \pi_t - \mathbb{E}[\pi_t])$ to be $\mathcal{F}_t$-measurable Markov controls.

The problem (4.12) can be now reformulated as the following mean-filed type of linear quadratic optimal stochastic control problem

$$\begin{cases} \min & \sum_{t=1}^{T} \alpha_t \Big\{ \mathbb{E}\big[\big(x_t - l_t - \mathbb{E}[x_t - l_t]\big)^2\big] - w\mathbb{E}\big[x_t - l_t\big] \Big\}, \\ \text{s.t.} & \{\mathbb{E}[x_t], \mathbb{E}[l_t], \mathbb{E}[\pi_t]\} \text{ satisfy dynamic equation (4.13)}, \\ & \{x_t - \mathbb{E}[x_t], l_t - \mathbb{E}[l_t], \pi_t - \mathbb{E}[\pi_t]\} \text{ satisfy dynamic equation (4.14)}, \\ & \mathbb{E}[\pi_t - \mathbb{E}[\pi_t]] = \mathbf{0}, \quad t = 0, 1, \cdots, T-1. \end{cases} \tag{4.15}$$

Now, it is indeed a separable linear quadratic optimal stochastic control problem which can be solved by classic dynamic programming approach.

4.2.2 The Optimal Strategies with Correlation Between Assets and Liability

In this subsection, assume that the returns of assets and liability are correlated at every period. For simplicity, we define the following backward recursions for seven deterministic sequences of parameters, $\{\xi_t\}$, $\{\eta_t\}$, $\{\epsilon_t\}$, $\{\zeta_t\}$, $\{\theta_t\}$, $\{\delta_t\}$ and $\{\psi_t\}$, as

$$\xi_t = \xi_{t+1}(1 - B_t)s_t^2 + \alpha_t,$$

$$\eta_t = \eta_{t+1}\big(\mathbb{E}[q_t] - \widehat{B}_t\big)s_t + \alpha_t,$$

$$\epsilon_t = \epsilon_{t+1}\mathbb{E}[q_t^2] - \eta_{t+1}^2\xi_{t+1}^{-1}\widetilde{B}_t + \alpha_t,$$

$$\zeta_t = \zeta_{t+1}s_t + \alpha_t,$$

$$\theta_t = \theta_{t+1}\mathbb{E}[q_t] - \frac{\zeta_{t+1}\eta_{t+1}}{\xi_{t+1}}\frac{\widehat{B}_t - \mathbb{E}[q_t]B_t}{1 - B_t} + \alpha_t,$$

$$\delta_t = \delta_{t+1}(\mathbb{E}[q_t])^2 + \epsilon_{t+1}(\mathbb{E}[q_t^2] - (\mathbb{E}[q_t])^2) - \frac{\eta_{t+1}^2}{\xi_{t+1}}\left(\widetilde{B}_t - (\mathbb{E}[q_t])^2 + \frac{(\widehat{B}_t - \mathbb{E}[q_t])^2}{1 - B_t}\right),$$

$$\psi_t = \psi_{t+1} - \frac{\zeta_{t+1}^2}{4\xi_{t+1}}\frac{B_t}{1 - B_t},$$

for $t = T-1, T-2, \cdots, 0$, with terminal conditions

$$\xi_T = \alpha_T,\ \eta_T = \alpha_T,\ \epsilon_T = \alpha_T,\ \zeta_T = \alpha_T,\ \theta_T = \alpha_T,\ \delta_T = 0,\ \psi_T = 0.$$

These parameters can also be expressed as follows,

$$\xi_t = \sum_{k=t}^{T} \alpha_k \prod_{j=t}^{k-1} (1 - B_j) s_j^2,$$

$$\eta_t = \sum_{k=t}^{T} \alpha_k \prod_{j=t}^{k-1} \big(\mathbb{E}[q_j] - \widehat{B}_j\big) s_j,$$

$$\epsilon_t = \sum_{k=t}^{T-1} (\alpha_k - \eta_{k+1}^2 \xi_{k+1}^{-1} \widetilde{B}_k) \prod_{j=t}^{k-1} \mathbb{E}[q_j^2] + \alpha_T \prod_{j=t}^{T-1} \mathbb{E}[q_j^2],$$

$$\zeta_t = \sum_{k=t}^{T} \alpha_k \prod_{j=t}^{k-1} s_j,$$

$$\theta_t = \sum_{k=t}^{T-1} \left(\alpha_k - \frac{\zeta_{k+1}\eta_{k+1}}{\xi_{k+1}} \frac{\widehat{B}_k - \mathbb{E}[q_k] B_k}{1 - B_k}\right) \prod_{j=t}^{k-1} \mathbb{E}[q_j] + \alpha_T \prod_{j=t}^{T-1} \mathbb{E}[q_j],$$

$$\begin{aligned}
\delta_t &= \sum_{k=t}^{T-1} \left[\epsilon_{k+1} (\mathbb{E}[q_k^2] - (\mathbb{E}[q_k])^2) - \frac{\eta_{k+1}^2}{\xi_{k+1}} \left(\widetilde{B}_k - (\mathbb{E}[q_k])^2 + \frac{(\widehat{B}_k - \mathbb{E}[q_k])^2}{1 - B_k}\right)\right] \prod_{j=t}^{k-1} (\mathbb{E}[q_j])^2 \\
&= -\sum_{k=t}^{T-1} \frac{\eta_{k+1}^2}{\xi_{k+1}} \left(\widetilde{B}_k - (\mathbb{E}[q_k])^2 + \frac{(\widehat{B}_k - \mathbb{E}[q_k])^2}{1 - B_k}\right) \prod_{j=t}^{k-1} (\mathbb{E}[q_j])^2 \\
&\quad + \sum_{k=t}^{T-1} (\alpha_k - \eta_{k+1}^2 \xi_{k+1}^{-1} \widetilde{B}_k) \left(\prod_{j=t}^{k-1} \mathbb{E}[q_j^2] - \prod_{j=t}^{k-1} (\mathbb{E}[q_j])^2\right) + \alpha_T \left(\prod_{j=t}^{T-1} \mathbb{E}[q_j^2] - \prod_{j=t}^{T-1} (\mathbb{E}[q_j])^2\right),
\end{aligned}$$

$$\psi_t = -\sum_{k=t}^{T-1} \frac{\zeta_{k+1}^2}{4\xi_{k+1}} \frac{B_k}{1 - B_k}.$$

Theorem 4.3. *Assume that the returns of assets and liability are correlated at every* *period. Then, the optimal strategy of problem (4.15) is given by*

$$\begin{aligned}\pi_t^* = & -\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t]s_t\left[x_t - \mathbb{E}[x_t] - \frac{w\zeta_{t+1} + 2\eta_{t+1}\big(\widehat{B}_t - \mathbb{E}[q_t]\big)\mathbb{E}[l_t]}{2s_t\xi_{t+1}(1-B_t)}\right] \\ & +\frac{\eta_{t+1}}{\xi_{t+1}}\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t q_t]l_t,\end{aligned} \tag{4.16}$$

where

$$\mathbb{E}[x_t] = x_0\prod_{j=0}^{t-1}s_j + \sum_{k=0}^{t-1}\Bigg(\prod_{j=k+1}^{t-1}s_j\Bigg)\Bigg(\frac{w\zeta_{k+1}B_k}{2\xi_{t+1}(1-B_k)} + \frac{\eta_{k+1}}{\xi_{k+1}}\frac{\widehat{B}_k - \mathbb{E}[q_k]B_k}{1-B_k}\prod_{j=0}^{k-1}\mathbb{E}[q_j]l_0\Bigg). \tag{4.17}$$

Proof. We prove the main results by dynamic programming approach. For the information set $\mathcal{F}_t$, the cost-to-go functional at period t is computed by

$$\begin{aligned}& J_t(\mathbb{E}[x_t], x_t - \mathbb{E}[x_t], \mathbb{E}[l_t], l_t - \mathbb{E}[l_t]) \\ = & \min_{\{\pi_t - \mathbb{E}[\pi_t], \mathbb{E}[\pi_t]\}} \mathbb{E}\big[J_{t+1}\big(\mathbb{E}[x_{t+1}], x_{t+1} - \mathbb{E}[x_{t+1}], \mathbb{E}[l_{t+1}], l_{t+1} - \mathbb{E}[l_{t+1}]\big)\big|\mathcal{F}_t\big] \\ & + \alpha_t\big(x_t - l_t - \mathbb{E}[x_t - l_t]\big)^2 - w\alpha_t\mathbb{E}[x_t - l_t].\end{aligned}$$

The cost-to-go functional at terminal time T is

$$\begin{aligned}& J_T\big(\mathbb{E}[x_T], x_T - \mathbb{E}[x_T], \mathbb{E}[l_T], l_T - \mathbb{E}[l_T]\big) \\ = & \alpha_T\big(x_T - l_T - \mathbb{E}[x_T - l_T]\big)^2 - w\alpha_T\mathbb{E}\big[x_T - l_T\big] \\ = & \xi_T\big(x_T - \mathbb{E}[x_T]\big)^2 - 2\eta_T\big(l_T - \mathbb{E}[l_T]\big)\big(x_T - \mathbb{E}[x_T]\big) + \epsilon_T\big(l_T - \mathbb{E}[l_T]\big)^2 \\ & - w\zeta_T\mathbb{E}[x_T] + w\theta_T\mathbb{E}[l_T] + \delta_T(\mathbb{E}[l_T])^2 + w^2\psi_T.\end{aligned}$$

Assume that the cost-to-go functional at time $t+1$ is the following expression

$$\begin{aligned}& J_{t+1}\big(\mathbb{E}[x_{t+1}], x_{t+1} - \mathbb{E}[x_{t+1}], \mathbb{E}[l_{t+1}], l_{t+1} - \mathbb{E}[l_{t+1}]\big) \\ = & \ \xi_{t+1}\big(x_{t+1} - \mathbb{E}[x_{t+1}]\big)^2 - 2\eta_{t+1}\big(l_{t+1} - \mathbb{E}[l_{t+1}]\big)\big(x_{t+1} - \mathbb{E}[x_{t+1}]\big) \\ & +\epsilon_{t+1}\big(l_{t+1} - \mathbb{E}[l_{t+1}]\big)^2 - w\zeta_{t+1}\mathbb{E}[x_{t+1}] + \omega\theta_{t+1}\mathbb{E}[l_{t+1}] + \delta_{t+1}(\mathbb{E}[l_{t+1}])^2 + w^2\psi_{t+1}.\end{aligned}$$

We prove that the above statement still holds at time t. For given information set $\mathcal{F}_t$, i.e., knowing $x_t - \mathbb{E}[x_t]$, $\mathbb{E}[x_t]$, $l_t - \mathbb{E}[l_t]$ and $\mathbb{E}[l_t]$, we have

$$
\begin{aligned}
&\mathbb{E}\Big[J_{t+1}\big(\mathbb{E}[x_{t+1}], x_{t+1} - \mathbb{E}[x_{t+1}], \mathbb{E}[l_{t+1}], l_{t+1} - \mathbb{E}[l_{t+1}]\big)\Big|\mathcal{F}_t\Big] \\
=&\mathbb{E}\Bigg[\xi_{t+1}\Big[s_t\big(x_t - \mathbb{E}[x_t]\big) + \mathbf{P}_t'\big(\pi_t - \mathbb{E}[\pi_t]\big) + \big(\mathbf{P}_t' - \mathbb{E}[\mathbf{P}_t']\big)\mathbb{E}[\pi_t]\Big]^2 \\
&- 2\eta_{t+1}\Big[q_t\big(l_t - \mathbb{E}[l_t]\big) + \big(q_t - \mathbb{E}[q_t]\big)\mathbb{E}[l_t]\Big]\Big[s_t\big(x_t - \mathbb{E}[x_t]\big) + \mathbf{P}_t'\big(\pi_t - \mathbb{E}[\pi_t]\big) \\
&+ \big(\mathbf{P}_t' - \mathbb{E}[\mathbf{P}_t']\big)\mathbb{E}[\pi_t]\Big] + \epsilon_{t+1}\big[q_t(l_t - \mathbb{E}[l_t]) + (q_t - \mathbb{E}[q_t])\mathbb{E}[l_t]\big]^2 \\
&- w\zeta_{t+1}\big(s_t\mathbb{E}[x_t] + \mathbb{E}[\mathbf{P}_t']\mathbb{E}[\pi_t]\big) + w\theta_{t+1}\mathbb{E}[q_t]\mathbb{E}[l_t] + \delta_{t+1}(\mathbb{E}[q_t]\mathbb{E}[l_t])^2 + w^2\psi_{t+1}\Bigg|\mathcal{F}_t\Bigg] \\
=&\xi_{t+1}\Big[s_t^2\big(x_t-\mathbb{E}[x_t]\big)^2+\big(\pi_t-\mathbb{E}[\pi_t]\big)'\mathbb{E}[\mathbf{P}_t\mathbf{P}_t']\big(\pi_t-\mathbb{E}[\pi_t]\big)+2s_t\big(x_t-\mathbb{E}[x_t]\big)\mathbb{E}[\mathbf{P}_t']\big(\pi_t-\mathbb{E}[\pi_t]\big) \\
&+ \mathbb{E}[\pi_t']\big(\mathbb{E}[\mathbf{P}_t\mathbf{P}_t'] - \mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t']\big)\mathbb{E}[\pi_t] + 2\big(\pi_t - \mathbb{E}[\pi_t]\big)'\big(\mathbb{E}[\mathbf{P}_t\mathbf{P}_t'] - \mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t']\big)\mathbb{E}[\pi_t]\Big] \\
&- 2\eta_{t+1}\Big[s_t\mathbb{E}[q_t]\big(l_t - \mathbb{E}[l_t]\big)\big(x_t - \mathbb{E}[x_t]\big) + \mathbb{E}[q_t\mathbf{P}_t']\big(l_t - \mathbb{E}[l_t]\big)\big(\pi_t - \mathbb{E}[\pi_t]\big) \\
&+ \big(\mathbb{E}[q_t\mathbf{P}_t'] - \mathbb{E}[q_t]\mathbb{E}[\mathbf{P}_t']\big)\Big(\mathbb{E}[l_t]\big(\pi_t - \mathbb{E}[\pi_t]\big) + \big(l_t - \mathbb{E}[l_t]\big)\mathbb{E}[\pi_t] + \mathbb{E}[l_t]\mathbb{E}[\pi_t]\Big)\Big] \\
&+ \epsilon_{t+1}\Big[\mathbb{E}[q_t^2](l_t-\mathbb{E}[l_t])^2+2(\mathbb{E}[q_t^2]-(\mathbb{E}[q_t])^2)(l_t-\mathbb{E}[l_t])\mathbb{E}[l_t]+(\mathbb{E}[q_t^2]-(\mathbb{E}[q_t])^2)(\mathbb{E}[l_t])^2\Big] \\
&- w\zeta_{t+1}\big(s_t\mathbb{E}[x_t] + \mathbb{E}[\mathbf{P}_t']\mathbb{E}[\pi_t]\big) + w\theta_{t+1}\mathbb{E}[q_t]\mathbb{E}[l_t] + \delta_{t+1}(\mathbb{E}[q_t]\mathbb{E}[l_t])^2 + w^2\psi_{t+1}.
\end{aligned}
$$

Since any admissible strategy of $(\mathbb{E}[\pi_t], \pi_t - \mathbb{E}[\pi_t])$ satisfies $\mathbb{E}[\pi_t - \mathbb{E}[\pi_t]] = \mathbf{0}$ and $\mathbb{E}[l_t - \mathbb{E}[l_t]] = 0$ holds, we have

$$
\begin{aligned}
&\mathbb{E}\Big[\big(\pi_t - \mathbb{E}[\pi_t]\big)'\big(\mathbb{E}[\mathbf{P}_t\mathbf{P}_t'] - \mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t']\big)\mathbb{E}[\pi_t]\Big|\mathcal{F}_0\Big] = 0, \\
&\mathbb{E}\Big[\big(\mathbb{E}[q_t\mathbf{P}_t'] - \mathbb{E}[q_t]\mathbb{E}[\mathbf{P}_t']\big)\mathbb{E}[l_t]\big(\pi_t - \mathbb{E}[\pi_t]\big)\Big|\mathcal{F}_0\Big] = 0, \\
&\mathbb{E}\Big[\big(\mathbb{E}[q_t\mathbf{P}_t'] - \mathbb{E}[q_t]\mathbb{E}[\mathbf{P}_t']\big)\big(l_t - \mathbb{E}[l_t]\big)\mathbb{E}[\pi_t]\Big|\mathcal{F}_0\Big] = 0, \\
&\mathbb{E}\Big[(\mathbb{E}[q_t^2] - (\mathbb{E}[q_t])^2)(l_t - \mathbb{E}[l_t])\mathbb{E}[l_t]\Big|\mathcal{F}_0\Big] = 0.
\end{aligned}
$$

We first identify optimal $(\mathbb{E}[\pi_t^*], \pi_t^* - \mathbb{E}[\pi_t^*])$ by minimizing the following equivalent cost functional,

$$
\begin{aligned}
&\mathbb{E}\Big[J_{t+1}\big(\mathbb{E}[x_{t+1}], x_{t+1} - \mathbb{E}[x_{t+1}], \mathbb{E}[l_{t+1}], l_{t+1} - \mathbb{E}[l_{t+1}]\big)\Big|\mathcal{F}_t\Big] \\
=&\xi_{t+1}\Big[s_t^2\big(x_t - \mathbb{E}[x_t]\big)^2 + \big(\pi_t - \mathbb{E}[\pi_t]\big)'\mathbb{E}[\mathbf{P}_t\mathbf{P}_t']\big(\pi_t - \mathbb{E}[\pi_t]\big) \\
&+ 2s_t\big(x_t - \mathbb{E}[x_t]\big)\mathbb{E}[\mathbf{P}_t']\big(\pi_t - \mathbb{E}[\pi_t]\big) + \mathbb{E}[\pi_t']\big(\mathbb{E}[\mathbf{P}_t\mathbf{P}_t'] - \mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t']\big)\mathbb{E}[\pi_t]\Big] \\
&- 2\eta_{t+1}\Big[s_t\mathbb{E}[q_t]\big(l_t - \mathbb{E}[l_t]\big)\big(x_t - \mathbb{E}[x_t]\big) + \mathbb{E}[q_t\mathbf{P}_t']\big(l_t - \mathbb{E}[l_t]\big)\big(\pi_t - \mathbb{E}[\pi_t]\big) \\
&+ \big(\mathbb{E}[q_t\mathbf{P}_t'] - \mathbb{E}[q_t]\mathbb{E}[\mathbf{P}_t']\big)\mathbb{E}[l_t]\mathbb{E}[\pi_t]\Big] + \epsilon_{t+1}\Big[\mathbb{E}[q_t^2](l_t - \mathbb{E}[l_t])^2 \\
&+ (\mathbb{E}[q_t^2] - (\mathbb{E}[q_t])^2)(\mathbb{E}[l_t])^2\Big] - w\zeta_{t+1}\big(s_t\mathbb{E}[x_t] + \mathbb{E}[\mathbf{P}_t']\mathbb{E}[\pi_t]\big) \\
&+ w\theta_{t+1}\mathbb{E}[q_t]\mathbb{E}[l_t] + \delta_{t+1}(\mathbb{E}[q_t]\mathbb{E}[l_t])^2 + w^2\psi_{t+1},
\end{aligned}
$$

without considering the linear constraint $\mathbb{E}[\pi_t - \mathbb{E}[\pi_t]] = \mathbf{0}$, and verify then the derived optimal strategy satisfies this constraint automatically.

It is easy to see that $\pi_t^* - \mathbb{E}[\pi_t^*]$ can be expressed by the linear form of states and their expected states, and $\mathbb{E}[\pi_t^*]$ can be constructed by the linear form of the expected states, i.e.,

$$
\pi_t^* - \mathbb{E}[\pi_t^*] = -\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t]s_t\big(x_t - \mathbb{E}[x_t]\big) + \eta_{t+1}\xi_{t+1}^{-1}\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_tq_t]\big(l_t - \mathbb{E}[l_t]\big), \tag{4.18}
$$

$$
\begin{aligned}
\mathbb{E}[\pi_t^*] =& \big(\mathbb{E}[\mathbf{P}_t\mathbf{P}_t'] - \mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t']\big)^{-1}\Big[\frac{w\zeta_{t+1}}{2\xi_{t+1}}\mathbb{E}[\mathbf{P}_t] + \frac{\eta_{t+1}}{\xi_{t+1}}\big(\mathbb{E}[\mathbf{P}_tq_t] - \mathbb{E}[q_t]\mathbb{E}[\mathbf{P}_t]\big)\mathbb{E}[l_t]\Big] \\
=& \frac{w\zeta_{t+1} + 2\eta_{t+1}\big(\widehat{B}_t - \mathbb{E}[q_t]\big)\mathbb{E}[l_t]}{2\xi_{t+1}(1 - B_t)}\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t] + \frac{\eta_{t+1}}{\xi_{t+1}}\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_tq_t]\mathbb{E}[l_t].
\end{aligned} \tag{4.19}
$$

In order to get the explicit expression of the cost-to-go functional at time t, we substitute $\pi_t^* - \mathbb{E}[\pi_t^*]$ and $\mathbb{E}[\pi_t^*]$ back and derive

$$
\begin{aligned}
& J_t\big(\mathbb{E}[x_t], x_t - \mathbb{E}[x_t], \mathbb{E}[l_t], l_t - \mathbb{E}[l_t]\big) \\
= & \min_{\{\pi_t - \mathbb{E}[\pi_t], \mathbb{E}[\pi_t]\}} \mathbb{E}\Big[J_{t+1}\big(\mathbb{E}[x_{t+1}], x_{t+1} - \mathbb{E}[x_{t+1}], \mathbb{E}[l_{t+1}], l_{t+1} - \mathbb{E}[l_{t+1}]\big)\Big|\mathcal{F}_t\Big] \\
& + \alpha_t\big(x_t - l_t - \mathbb{E}[x_t - l_t]\big)^2 - w\alpha_t\mathbb{E}[x_t - l_t] \\
= & \xi_{t+1}s_t^2\big(x_t - \mathbb{E}[x_t]\big)^2 - 2\eta_{t+1}s_t\mathbb{E}[q_t]\big(l_t - \mathbb{E}[l_t]\big)\big(x_t - \mathbb{E}[x_t]\big) - w\zeta_{t+1}s_t\mathbb{E}[x_t] + w\theta_{t+1}\mathbb{E}[q_t]\mathbb{E}[l_t] \\
& + \epsilon_{t+1}\Big[\mathbb{E}[q_t^2](l_t - \mathbb{E}[l_t])^2 + (\mathbb{E}[q_t^2] - (\mathbb{E}[q_t])^2)(\mathbb{E}[l_t])^2\Big] + \delta_{t+1}(\mathbb{E}[q_t]\mathbb{E}[l_t])^2 + w^2\psi_{t+1} \\
& - \xi_{t+1}\Big[-\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t']s_t\big(x_t - \mathbb{E}[x_t]\big) + \eta_{t+1}\xi_{t+1}^{-1}\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[q_t\mathbf{P}_t']\big(l_t - \mathbb{E}[l_t]\big)\Big] \\
& \cdot \mathbb{E}[\mathbf{P}_t\mathbf{P}_t']\Big[-\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t]s_t\big(x_t - \mathbb{E}[x_t]\big) + \eta_{t+1}\xi_{t+1}^{-1}\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_tq_t]\big(l_t - \mathbb{E}[l_t]\big)\Big] \\
& - \xi_{t+1}\left[\frac{w\zeta_{t+1}}{2\xi_{t+1}}\mathbb{E}[\mathbf{P}_t'] + \frac{\eta_{t+1}}{\xi_{t+1}}\big(\mathbb{E}[q_t\mathbf{P}_t'] - \mathbb{E}[q_t]\mathbb{E}[\mathbf{P}_t']\big)\mathbb{E}[l_t]\right] \\
& \cdot \big(\mathbb{E}[\mathbf{P}_t\mathbf{P}_t'] - \mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t']\big)^{-1}\left[\frac{w\zeta_{t+1}}{2\xi_{t+1}}\mathbb{E}[\mathbf{P}_t] + \frac{\eta_{t+1}}{\xi_{t+1}}\big(\mathbb{E}[\mathbf{P}_tq_t] - \mathbb{E}[q_t]\mathbb{E}[\mathbf{P}_t]\big)\mathbb{E}[l_t]\right] \\
& + \alpha_t\big[(x_t - \mathbb{E}[x_t]) - (l_t - \mathbb{E}[l_t])\big]^2 - w\alpha_t\mathbb{E}[x_t - l_t] \\
= & \xi_{t+1}s_t^2(1 - B_t)\big(x_t - \mathbb{E}[x_t]\big)^2 - 2\eta_{t+1}s_t\big(\mathbb{E}[q_t] - \widehat{B}_t\big)\big(l_t - \mathbb{E}[l_t]\big)\big(x_t - \mathbb{E}[x_t]\big) \\
& + \big(\epsilon_{t+1}\mathbb{E}[q_t^2] - \eta_{t+1}^2\xi_{t+1}^{-1}\widetilde{B}_t\big)\big(l_t - \mathbb{E}[l_t]\big)^2 - w\zeta_{t+1}s_t\mathbb{E}[x_t] \\
& + w\left[\theta_{t+1}\mathbb{E}[q_t] - \frac{\zeta_{t+1}\eta_{t+1}}{\xi_{t+1}}\frac{\widehat{B}_t - \mathbb{E}[q_t]B_t}{1 - B_t}\right]\mathbb{E}[l_t] + w^2\left[\psi_{t+1} - \frac{\zeta_{t+1}^2}{4\xi_{t+1}}\frac{B_t}{1 - B_t}\right] \\
& + \left[\delta_{t+1}(\mathbb{E}[q_t])^2 + \epsilon_{t+1}(\mathbb{E}[q_t^2 - (\mathbb{E}[q_t])^2) - \frac{\eta_{t+1}^2}{\xi_{t+1}}\left(\widetilde{B}_t - (\mathbb{E}[q_t])^2 + \frac{(\widehat{B}_t - \mathbb{E}[q_t])^2}{1 - B_t}\right)\right]\big(\mathbb{E}[l_t]\big)^2 \\
& + \alpha_t(x_t - \mathbb{E}[x_t])^2 - 2\alpha_t(l_t - \mathbb{E}[l_t])(x_t - \mathbb{E}[x_t]) + \alpha_t(l_t - \mathbb{E}[l_t])^2 - w\alpha_t\mathbb{E}[x_t] + w\alpha_t\mathbb{E}[l_t] \\
= & \xi_t\big(x_t - \mathbb{E}[x_t]\big)^2 - 2\eta_t\big(l_t - \mathbb{E}[l_t]\big)\big(x_t - \mathbb{E}[x_t]\big) + \epsilon_t\big(l_t - \mathbb{E}[l_t]\big)^2 \\
& - w\zeta_t\mathbb{E}[x_t] + w\theta_t\mathbb{E}[l_t] + \delta_t(\mathbb{E}[l_t])^2 + w^2\psi_t.
\end{aligned}
$$

Substituting $\mathbb{E}[\pi_t^*]$ to dynamics of $\mathbb{E}[x_t]$ in (4.13) yields

$$\mathbb{E}[x_{t+1}] = s_t\mathbb{E}[x_t] + \frac{w}{2}\frac{\zeta_{t+1}}{\xi_{t+1}}\frac{B_t}{1-B_t} + \frac{\eta_{t+1}}{\xi_{t+1}}\frac{\widehat{B}_t - \mathbb{E}[q_t]B_t}{1-B_t}\mathbb{E}[l_t],$$

which implies (4.17). Hence, following from (4.18), (4.19) and (4.17), we derive the desired result, i.e., $\pi_t^* = \big(\pi_t^* - \mathbb{E}[\pi_t^*]\big) + \mathbb{E}[\pi_t^*]$ in (4.16).

Finally, we show that this optimal strategy satisfies the linear constraints. At time 0, $\mathbb{E}[\pi_0^* - \mathbb{E}[\pi_0^*]] = \mathbf{0}$ is obvious due to $x_0 = \mathbb{E}[x_0]$ and $l_0 = \mathbb{E}[l_0]$. Then, according to the dynamic system of (4.14), we have $\mathbb{E}[x_1 - \mathbb{E}[x_1]] = 0$ and $\mathbb{E}[l_1 - \mathbb{E}[l_1]] = 0$, which further implies $\mathbb{E}[\pi_1^* - \mathbb{E}[\pi_1^*]] = \mathbf{0}$. Repeating this argument, we have $\mathbb{E}[\pi_t^* - \mathbb{E}[\pi_t^*]] = \mathbf{0}$ holds for all t. $\square$

Based on the proof of Theorem 4.3, the optimal objective of problem (4.15) is as follows,

$$J_0\big(\mathbb{E}[x_0], 0, \mathbb{E}[l_0], 0\big) = -w\zeta_0\mathbb{E}[x_0] + w\theta_0\mathbb{E}[l_0] + \delta_0(\mathbb{E}[l_0])^2 + w^2\psi_0. \qquad (4.20)$$

In addition, from (4.17), we have

$$\begin{aligned}
& \mathbb{E}^{(\tau)}[x_{T\wedge\tau} - l_{T\wedge\tau}] \\
= & \sum_{t=0}^{T}\mathbb{E}[x_t]\alpha_t - \sum_{t=1}^{T}\mathbb{E}[l_t]\alpha_t \\
= & \sum_{t=0}^{T}\Bigg(x_0\prod_{j=0}^{t-1}s_j + \sum_{k=0}^{t-1}\Bigg(\prod_{j=k+1}^{t-1}s_j\Bigg)\Bigg(\frac{w\zeta_{k+1}B_k}{2\xi_{k+1}(1-B_k)} + \frac{\eta_{k+1}}{\xi_{k+1}}\frac{\widehat{B}_k - \mathbb{E}[q_k]B_k}{1-B_k}\prod_{j=0}^{k-1}\mathbb{E}[q_j]l_0\Bigg) \\
& \quad -\mathbb{E}[l_t]\Bigg)\alpha_t \\
= & \zeta_0x_0 - 2w\psi_0 - \theta_0l_0,
\end{aligned}$$

i.e.,

$$w = -(2\psi_0)^{-1}\big(\mathbb{E}^{(\tau)}[x_{T\wedge\tau} - l_{T\wedge\tau}] - \zeta_0x_0 + \theta_0l_0\big).$$

Hence, according to (4.20), we can derive the variance term as

$$\begin{aligned}
&\operatorname{Var}^{(\tau)}(x_{T\wedge\tau} - l_{T\wedge\tau}) \\
=&w\mathbb{E}^{(\tau)}[x_{T\wedge\tau} - l_{T\wedge\tau}] + J_0\big(x_0, 0, l_0, 0\big) \\
=&-w^2\psi_0 + \delta_0 l_0^2 \\
=&-(4\psi_0)^{-1}\big(\mathbb{E}^{(\tau)}[x_{T\wedge\tau} - l_{T\wedge\tau}] - \zeta_0 x_0 + \theta_0 l_0\big)^2 + \delta_0 l_0^2.
\end{aligned}$$

Theorem 4.4. *Assume that the returns of assets and liability are correlated at every period. Then, the efficient frontier of problem (4.12) is given by*

$$\begin{aligned}
\operatorname{Var}^{(\tau)}(x_{T\wedge\tau} - l_{T\wedge\tau}) = -(4\psi_0)^{-1}\big(\mathbb{E}^{(\tau)}[x_{T\wedge\tau} - l_{T\wedge\tau}] - \zeta_0 x_0 + \theta_0 l_0\big)^2 + \delta_0 l_0^2, \\
\text{for } \mathbb{E}^{(\tau)}[x_{T\wedge\tau} - l_{T\wedge\tau}] \geqslant \zeta_0 x_0 - \theta_0 l_0.
\end{aligned} \tag{4.21}$$

Now we consider the case with terminal exit. We assume that the investment will be stopped at the terminal time T. This means that $\alpha_T = 1$ and $\alpha_t = 0$ for $t = 0, 1, \cdots, T-1$. Then, the seven deterministic parameters are reduced into the following expressions,

$$\begin{aligned}
\xi_t &= \prod_{j=t}^{T-1}(1 - B_j)s_j^2, \\
\eta_t &= \prod_{j=t}^{T-1}\big(\mathbb{E}[q_j] - \widehat{B}_j\big)s_j, \\
\epsilon_t &= -\sum_{k=t}^{T-1}\widetilde{B}_k\prod_{j=k+1}^{T-1}\frac{(\mathbb{E}[q_j] - \widehat{B}_j)^2}{1 - B_j}\prod_{j=t}^{k-1}\mathbb{E}[q_j^2] + \prod_{j=t}^{T-1}\mathbb{E}[q_j^2], \\
\zeta_t &= \prod_{j=t}^{T-1}s_j,
\end{aligned}$$

$$\theta_t = \prod_{j=t}^{T-1} \frac{\mathbb{E}[q_j] - \widehat{B}_j}{1 - B_j},$$

$$\delta_t = -\prod_{j=t}^{T-1} \frac{(\mathbb{E}[q_j] - \widehat{B}_j)^2}{1 - B_j} - \sum_{k=t}^{T-1} \widetilde{B}_k \left(\prod_{j=k+1}^{T-1} \frac{\left(\mathbb{E}[q_j] - \widehat{B}_j\right)^2}{1 - B_j} \right) \left(\prod_{j=t}^{k-1} \mathbb{E}[q_j^2] \right) + \prod_{j=t}^{T-1} \mathbb{E}[q_j^2],$$

$$\psi_t = -\frac{1 - \prod_{j=t}^{T-1}(1 - B_j)}{4 \prod_{j=t}^{T-1}(1 - B_j)}.$$

Furthermore, with the help of Theorem 4.3, we get

$$\begin{aligned}
\mathbb{E}[x_t] =& x_0 \prod_{j=0}^{t-1} s_j + \sum_{k=0}^{t-1} \left(\prod_{j=k+1}^{t-1} s_j \right) \left(\frac{w\zeta_{k+1} B_k}{2\xi_{k+1}(1-B_k)} + \frac{\eta_{k+1}}{\xi_{k+1}} \frac{\widehat{B}_k - \mathbb{E}[q_k] B_k}{1 - B_k} \prod_{j=0}^{k-1} \mathbb{E}[q_j] l_0 \right) \\
=& x_0 \prod_{j=0}^{t-1} s_j + \frac{w}{2} \left(\prod_{j=t}^{T-1} \frac{1}{(1 - B_j)s_j} \right) \sum_{k=0}^{t-1} \frac{B_k}{1 - B_k} \left(\prod_{j=k+1}^{t-1} \frac{1}{1 - B_j} \right) \\
&+ \left(\prod_{j=t}^{T-1} \frac{\mathbb{E}[q_j] - \widehat{B}_j}{(1 - B_j)s_j} \right) \sum_{k=0}^{t-1} \frac{\widehat{B}_k - \mathbb{E}[q_k] B_k}{1 - B_k} \left(\prod_{j=k+1}^{t-1} \frac{\mathbb{E}[q_j] - \widehat{B}_j}{1 - B_j} \right) \prod_{j=0}^{k-1} \mathbb{E}[q_j] l_0 \\
=& x_0 \prod_{j=0}^{t-1} s_j + \frac{w}{2} \left(\prod_{j=t}^{T-1} \frac{1}{(1 - B_j)s_j} \right) \frac{1 - \prod_{j=0}^{t-1}(1 - B_j)}{\prod_{j=0}^{t-1}(1 - B_j)} \\
&+ \left(\prod_{j=t}^{T-1} \frac{\mathbb{E}[q_j] - \widehat{B}_j}{(1 - B_j)s_j} \right) \left(\prod_{j=0}^{t-1} \mathbb{E}[q_j] - \prod_{j=0}^{t-1} \frac{\mathbb{E}[q_j] - \widehat{B}_j}{1 - B_j} \right) l_0.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\mathbb{E}[x_t] + \frac{1}{(1 - B_t)s_t} \left(\frac{w\zeta_{t+1}}{2\xi_{t+1}} - \frac{\eta_{t+1}}{\xi_{t+1}} (\mathbb{E}[q_t] - \widehat{B}_t) \mathbb{E}[l_t] \right) \\
=& \mathbb{E}[x_t] + \frac{w}{2} \prod_{j=t}^{T-1} \frac{1}{(1 - B_j)s_j} - \prod_{j=t}^{T-1} \frac{\mathbb{E}[q_j] - \widehat{B}_j}{(1 - B_j)s_j} \prod_{j=0}^{t-1} \mathbb{E}[q_j] l_0 \\
=& \left(x_0 \prod_{j=0}^{T-1} s_j + \frac{w}{2} \prod_{j=0}^{T-1} \frac{1}{1 - B_j} - l_0 \prod_{j=0}^{T-1} \frac{\mathbb{E}[q_j] - \widehat{B}_j}{1 - B_j} \right) \prod_{j=t}^{T-1} s_j^{-1}.
\end{aligned}$$

Theorem 4.5. *Assume that the returns of assets and liability are correlated at every period. If the exit time is the terminal time, then the optimal strategy of problem*

(4.15) is given by

$$\pi_t^* = -\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t]s_t\Big(x_t - \gamma\prod_{j=t}^{T-1} s_j^{-1}\Big) + \Big(\prod_{j=t+1}^{T-1}\frac{\mathbb{E}[q_j]-\widehat{B}_j}{s_j(1-B_j)}\Big)\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t q_t]l_t,$$

where

$$\gamma = x_0\prod_{j=0}^{T-1} s_j + \frac{w}{2}\prod_{j=0}^{T-1}\frac{1}{1-B_j} - l_0\prod_{j=0}^{T-1}\frac{\mathbb{E}[q_j]-\widehat{B}_j}{1-B_j}.$$

And the efficient frontier of problem (4.15) is given by

$$\mathrm{Var}(x_T - l_T) = \frac{\prod_{j=t}^{T-1}(1-B_j)}{1-\prod_{j=t}^{T-1}(1-B_j)}\Big(\mathbb{E}[x_T-l_T]-\prod_{j=0}^{T-1}s_jx_0+\prod_{j=0}^{T-1}\frac{\mathbb{E}[q_j]-\widehat{B}_j}{1-B_j}l_0\Big)^2+\delta_0 l_0^2, \tag{4.22}$$

$$\text{for } \mathbb{E}[x_T - l_T] \geqslant \prod_{j=0}^{T-1}s_jx_0 - \prod_{j=0}^{T-1}\frac{\mathbb{E}[q_j]-\widehat{B}_j}{1-B_j}l_0,$$

where

$$\delta_0 = -\prod_{j=0}^{T-1}\frac{(\mathbb{E}[q_j]-\widehat{B}_j)^2}{1-B_j} - \sum_{k=0}^{T-1}\widetilde{B}_k\Big(\prod_{j=k+1}^{T-1}\frac{\big(\mathbb{E}[q_j]-\widehat{B}_j\big)^2}{1-B_j}\Big)\Big(\prod_{j=0}^{k-1}\mathbb{E}[q_j^2]\Big) + \prod_{j=0}^{T-1}\mathbb{E}[q_j^2].$$

4.2.3 The Optimal Strategies with Uncorrelation between Assets and Liability

In this subsection, we assume that the returns of assets and liability are uncorrelated at every period.

The seven deterministic parameters can be reduced into the following expressions,

$$\xi_t = \sum_{k=t}^{T}\alpha_k\prod_{j=t}^{k-1}(1-B_j)s_j^2,$$

$$\eta_t = \sum_{k=t}^{T}\alpha_k\prod_{j=t}^{k-1}(1-B_j)\mathbb{E}[q_j]s_j,$$

$$
\begin{aligned}
\epsilon_t &= \sum_{k=t}^{T-1}(\alpha_k - \eta_{k+1}^2\xi_{k+1}^{-1}B_k(\mathbb{E}[q_k])^2)\prod_{j=t}^{k-1}\mathbb{E}[q_j^2] + \alpha_T\prod_{j=t}^{T-1}\mathbb{E}[q_j^2],\\
\zeta_t &= \sum_{k=t}^{T}\alpha_k\prod_{j=t}^{k-1}s_j,\\
\theta_t &= \sum_{k=t}^{T}\alpha_k\prod_{j=t}^{k-1}\mathbb{E}[q_j],\\
\delta_t &= \sum_{k=t}^{T-1}(\alpha_k - \eta_{k+1}^2\xi_{k+1}^{-1}B_k(\mathbb{E}[q_k])^2)\Bigg(\prod_{j=t}^{k-1}\mathbb{E}[q_j^2] - \prod_{j=t}^{k-1}(\mathbb{E}[q_j])^2\Bigg)\\
&\quad + \alpha_T\Bigg(\prod_{j=t}^{T-1}\mathbb{E}[q_j^2] - \prod_{j=t}^{T-1}(\mathbb{E}[q_j])^2\Bigg),\\
\psi_t &= -\sum_{k=t}^{T-1}\frac{\zeta_{k+1}^2}{4\xi_{k+1}}\frac{B_k}{1-B_k}.
\end{aligned}
$$

It follows from Theorem 4.3 and the above notations that we have the following theorem.

Theorem 4.6. *Assume that the returns of assets and liability are uncorrelated at every period. Then, the optimal strategy of problem (4.15) is given by*

$$
\pi_t^* = -\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t]s_t\Bigg[x_t - \mathbb{E}[x_t] - \frac{w\zeta_{t+1}}{2s_t\xi_{t+1}(1-B_t)} - \frac{\eta_{t+1}\mathbb{E}[q_t](l_t - \mathbb{E}[l_t])}{s_t\xi_{t+1}}\Bigg],
$$

where

$$
\mathbb{E}[x_t] = x_0\prod_{j=0}^{t-1}s_j + \frac{w}{2}\sum_{k=0}^{t-1}\Bigg(\prod_{j=k+1}^{t-1}s_j\Bigg)\frac{\zeta_{k+1}}{\xi_{k+1}}\frac{B_k}{1-B_k}.
$$

And the efficient frontier of problem (4.15) is given by

$$\begin{aligned}&\mathrm{Var}(x_T - l_T)\\&= \left(\sum_{k=0}^{T-1}\frac{\zeta_{k+1}^2}{\xi_{k+1}}\frac{B_k}{1-B_k}\right)^{-1}\left(\mathbb{E}[x_T - l_T] - \sum_{k=0}^{T}\alpha_k\prod_{j=0}^{k-1}s_j x_0 + \sum_{k=0}^{T}\alpha_k\prod_{j=0}^{k-1}\mathbb{E}[q_j]l_0\right)^2 + \delta_0 l_0^2,\end{aligned} \tag{4.23}$$

$$\textit{for } \mathbb{E}[x_T - l_T] \geqslant \sum_{k=0}^{T}\alpha_k\prod_{j=0}^{k-1}s_j x_0 - \sum_{k=0}^{T}\alpha_k\prod_{j=0}^{k-1}\mathbb{E}[q_j]l_0,$$

where

$$\delta_0 = \sum_{k=0}^{T-1}(\alpha_k - \eta_{k+1}^2\xi_{k+1}^{-1}B_k(\mathbb{E}[q_k])^2)\left(\prod_{j=0}^{k-1}\mathbb{E}[q_j^2] - \prod_{j=0}^{k-1}(\mathbb{E}[q_j])^2\right) + \alpha_T\left(\prod_{j=0}^{T-1}\mathbb{E}[q_j^2] - \prod_{j=0}^{T-1}(\mathbb{E}[q_j])^2\right).$$

Now we consider the case with terminal exit. Assume that the investment will be stopped at the terminal time T. This means that $\alpha_T = 1$ and $\alpha_t = 0$ for $t = 0, 1, 2, \cdots, T-1$. The seven deterministic parameters are reduced to

$$\xi_t = \prod_{j=t}^{T-1}(1-B_j)s_j^2,$$

$$\eta_t = \prod_{j=t}^{T-1}(1-B_j)\mathbb{E}[q_j]s_j,$$

$$\epsilon_t = -\sum_{k=t}^{T-1}(\mathbb{E}[q_k])^2 B_k\prod_{j=k+1}^{T-1}(\mathbb{E}[q_j])^2(1-B_j)\prod_{j=t}^{k-1}\mathbb{E}[q_j^2] + \prod_{j=t}^{T-1}\mathbb{E}[q_j^2],$$

$$\zeta_t = \prod_{j=t}^{T-1}s_j,$$

$$\theta_t = \prod_{j=t}^{T-1}\mathbb{E}[q_j],$$

$$\delta_t = -\prod_{j=t}^{T-1}(\mathbb{E}[q_j])^2(1-B_j) - \sum_{k=t}^{T-1}(\mathbb{E}[q_k])^2 B_k\left(\prod_{j=k+1}^{T-1}(\mathbb{E}[q_j])^2(1-B_j)\right)$$

$$\cdot\left(\prod_{j=t}^{k-1}\mathbb{E}[q_j^2]\right) + \prod_{j=t}^{T-1}\mathbb{E}[q_j^2],$$

$$\psi_t = -\frac{1-\prod_{j=t}^{T-1}(1-B_j)}{4\prod_{j=t}^{T-1}(1-B_j)}.$$

It follows from Theorem 4.5 that we have the following theorem.

Theorem 4.7. *Assume that the returns of assets and liability are uncorrelated at every period. If the exit time is terminal time, then the optimal strategy of problem (4.15) is given by*

$$\pi_t^* = -\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t]s_t\left(x_t - \gamma\prod_{j=t}^{T-1}s_j^{-1} - \prod_{j=t}^{T-1}\frac{\mathbb{E}[q_j]}{s_j}l_t\right),$$

where

$$\gamma = x_0\prod_{j=0}^{T-1}s_j + \frac{w}{2}\prod_{j=0}^{T-1}\frac{1}{1-B_j} - l_0\prod_{j=0}^{T-1}\mathbb{E}[q_j].$$

And the efficient frontier of problem (4.15) is given by

$$\mathrm{Var}(x_T - l_T) = \frac{\prod_{j=t}^{T-1}(1-B_j)}{1-\prod_{j=0}^{T-1}(1-B_j)}\left(\mathbb{E}[x_T - l_T] - \prod_{j=0}^{T-1}s_j x_0 + \prod_{j=0}^{T-1}\mathbb{E}[q_j]l_0\right)^2 + \delta_0 l_0^2, \tag{4.24}$$

$$\textit{for } \mathbb{E}[x_T - l_T] \geqslant \prod_{j=0}^{T-1}s_j x_0 - \prod_{j=0}^{T-1}\mathbb{E}[q_j]l_0,$$

where

$$\delta_0 = -\prod_{j=0}^{T-1}(\mathbb{E}[q_j])^2(1-B_j) - \sum_{k=0}^{T-1}(\mathbb{E}[q_k])^2 B_k\left(\prod_{j=k+1}^{T-1}(\mathbb{E}[q_j])^2(1-B_j)\right)\left(\prod_{j=0}^{k-1}\mathbb{E}[q_j^2]\right) + \prod_{j=0}^{T-1}\mathbb{E}[q_j^2].$$

4.2.4 Numerical Examples

We consider the example as Section 3.3. The only difference here is that the terminal expectation is not deterministic and we choose the trade-off parameter w=1.

Case 1 Correlation Examples

Assume the same ρ as Example 3.1.

Example 4.2. An example with an Uncertain Exit Time

The probability mass function of an exit time τ is

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (0.10, 0.15, 0.2, 0.25, 0.3),$$

respectively, for $t = 1, 2, 3, 4, 5$. According to Theorem 4.3, we can derive the optimal strategy of problem (4.12) as follows,

$$\begin{aligned}
\pi_0^* &= -1.05(x_0 - 2.7999)\mathbf{K}_1 + 1.1124\mathbf{K}_2 l_0, \\
\pi_1^* &= -1.05(x_1 - 2.9681)\mathbf{K}_1 + 1.0793\mathbf{K}_2 l_1, \\
\pi_2^* &= -1.05(x_2 - 3.1437)\mathbf{K}_1 + 1.0505\mathbf{K}_2 l_2, \\
\pi_3^* &= -1.05(x_3 - 3.3241)\mathbf{K}_1 + 1.0244\mathbf{K}_2 l_3, \\
\pi_4^* &= -1.05(x_4 - 3.5081)\mathbf{K}_1 + 1.0000\mathbf{K}_2 l_4.
\end{aligned}$$

The mean and variance of the final optimal surplus are $\mathbb{E}^{(\tau)}(x_{5\wedge\tau} - l_{5\wedge\tau}) = 2.8521$ and $\mathrm{Var}^{(\tau)}(x_{5\wedge\tau} - l_{5\wedge\tau}) = 0.3644$, respectively.

Example 4.3. An example with the Terminal Exit

The probability mass function of an exit time τ is

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (0, 0, 0, 0, 1),$$

respectively, for $t = 1, 2, 3, 4, 5$. According to Theorem 4.5, we can derive the optimal strategy of problem (4.12) as follows,

$$\pi_0^* = -1.05(x_0 - 3.0477)\mathbf{K}_1 + 1.2053\mathbf{K}_2 l_0,$$

$$\pi_1^* = -1.05(x_1 - 3.2001)\mathbf{K}_1 + 1.1503\mathbf{K}_2 l_1,$$

$$\pi_2^* = -1.05(x_2 - 3.3601)\mathbf{K}_1 + 1.0979\mathbf{K}_2 l_2,$$

$$\pi_3^* = -1.05(x_3 - 3.5281)\mathbf{K}_1 + 1.0478\mathbf{K}_2 l_3,$$

$$\pi_4^* = -1.05(x_4 - 3.7045)\mathbf{K}_1 + 1.0000\mathbf{K}_2 l_4.$$

The mean and variance of the final optimal surplus are $\mathbb{E}(x_5 - l_5) = 3.3897$ and $\mathrm{Var}(x_5 - l_5) = 0.6135$, respectively.

Case 2 Uncorrelation Examples

Example 4.4. An example with an Uncertain Exit Time

The probability mass function of an exit time τ is

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (0.10, 0.15, 0.2, 0.25, 0.3),$$

respectively, for $t = 1, 2, 3, 4, 5$. According to Theorem 4.6, the optimal strategy of problem (4.12) is specified as follows,

$$\pi_0^* = -1.05(x_0 - 2.8005 + 1.0590 l_0)\mathbf{K}_1,$$

$$\pi_1^* = -1.05(x_1 - 2.9689 + 1.0276 l_1)\mathbf{K}_1,$$

$$\pi_2^* = -1.05(x_2 - 3.1446 + 1.0003 l_2)\mathbf{K}_1,$$

$$\pi_3^* = -1.05(x_3 - 3.3252 + 0.9755 l_3)\mathbf{K}_1,$$

$$\pi_4^* = -1.05(x_4 - 3.5094 + 0.9524 l_4)\mathbf{K}_1.$$

The mean and variance of the final optimal surplus are $\mathbb{E}^{(\tau)}(x_{5\wedge\tau} - l_{5\wedge\tau}) = 2.8529$ and $\mathrm{Var}^{(\tau)}(x_{5\wedge\tau} - l_{5\wedge\tau}) = 0.5388$, respectively.

Example 4.5. An example with the Terminal Exit

The probability mass function of an exit time τ is

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (0, 0, 0, 0, 1),$$

respectively, for $t = 1, 2, 3, 4, 5$. According to Theorem 4.7, the optimal strategy of problem (4.12) is specified as follows,

$$\pi_0^* = -1.05(x_0 - 3.0487 + 1.1472 l_0)\mathbf{K}_1,$$
$$\pi_1^* = -1.05(x_1 - 3.2012 + 1.0950 l_1)\mathbf{K}_1,$$
$$\pi_2^* = -1.05(x_2 - 3.3612 + 1.0452 l_2)\mathbf{K}_1,$$
$$\pi_3^* = -1.05(x_3 - 3.5293 + 0.9977 l_3)\mathbf{K}_1,$$
$$\pi_4^* = -1.05(x_4 - 3.7058 + 0.9524 l_4)\mathbf{K}_1.$$

The mean and variance of the final optimal surplus are $\mathbb{E}(x_5 - l_5) = 3.3911$ and $\mathrm{Var}(x_5 - l_5) = 0.8903$, respectively.

Chapter 5

Multi-Period Mean-Variance Asset-Liability Portfolio Selection with Bankruptcy Control

This chapter considers the multi-period asset-liability mean-variance portfolio selection with control over bankruptcy. It is impossible to eliminate the possibility of bankruptcy in multi-period investment setting since the financial market is volatile. We assume in this paper bankruptcy occurs when the surplus (total wealth minus liability) falls below a preset level. Once an investor goes bankruptcy, he/she will suffer a great loss such as retrieve part of his/her wealth (even take nothing back), high liability and low credit. It is crucial for a successful investment take bankruptcy into account. Analytical optimal policy and efficient frontier are obtained by using the mean-field formulation. Numerical examples are presented to show the necessity of considering bankruptcy when an investor builds his/her investment.

5.1 Formulation

We add the constraint on bankruptcy control in this chapter. An investor goes bankruptcy when his/her surplus falls below zero, i.e., his wealth is not more than liability, at any intermediate or the final period. We denote the event of a bankruptcy at period t as A_t. The probability of A_t is

$$\Pr(\mathrm{A}_t) = \Pr(x_t < l_t, x_i \geqslant l_i, i = 1, 2, \cdots, t-1).$$

Since the probabilistic constraint is not easy to conquer in dynamic portfolio selection, we turn it to its upper bound:

$$\begin{aligned}
\Pr(\mathrm{A}_t) &= \Pr(x_t < l_t, x_i \geqslant l_i, i = 1, 2, \cdots, t-1) \\
&\leqslant \Pr(x_t < l_t) \\
&\leqslant \frac{\mathrm{Var}(x_t - l_t)}{(\mathbb{E}[x_t - l_t])^2},
\end{aligned}$$

where the second inequality is due to the Tchebycheff inequality. The mean-variance model for multi-period asset-liability portfolio selection with probability constraints is to seek the best strategy, $\pi_t^* = [(\pi_t^1)^*, (\pi_t^2)^*, \cdots, (\pi_t^n)^*]'$, $t = 0, 1, \cdots, T-1$, which is the optimizer of the following stochastic optimal control problem,

$$\begin{cases}
\min & \mathrm{Var}(x_T - l_T) - w\mathbb{E}[x_T - l_T], \\
\text{s.t.} & x_{t+1} = s_t x_t + \mathbf{P}_t' \pi_t, \\
& l_{t+1} = q_t l_t, \\
& \mathrm{Var}(x_t - l_t) \leqslant a_t (\mathbb{E}[x_t - l_t])^2, \quad t = 1, \cdots, T-1,
\end{cases} \tag{5.1}$$

where $\mathbf{a} = (a_1, ..., a_{T-1})$ is the vector whose components are the levels of a risk control over bankruptcy for the intermediate periods in a dynamic investment. To solve problem (5.1), we consider the following Lagrangian minimum problem,

$$\begin{cases}
\min & \mathrm{Var}(x_T - l_T) - w\mathbb{E}[x_T - l_T] + \sum_{t=1}^{T-1} \lambda_t \Big(\mathrm{Var}(x_t - l_t) - a_t (\mathbb{E}[x_t - l_t])^2 \Big), \\
\text{s.t.} & x_{t+1} = s_t x_t + \mathbf{P}_t' \pi_t, \\
& l_{t+1} = q_t l_t, \quad t = 1, \cdots, T-1,
\end{cases} \tag{5.2}$$

where $\lambda = (\lambda_1, \lambda_2, ..., \lambda_{T-1}) \in \mathbb{R}_+^{T-1}$ is the vector of Lagrangian multipliers.

As in Chapter 4, the problem (5.2) can be now reformulated as the following mean-filed type of linear quadratic optimal stochastic control problem

$$
\left\{
\begin{array}{ll}
\min & \mathbb{E}\Big[\big(x_T - l_T - \mathbb{E}[x_T - l_T]\big)^2\Big] - w\mathbb{E}[x_T - l_T] \\
& \quad + \sum_{t=1}^{T-1} \Big\{\lambda_t \mathbb{E}\Big[\big(x_t - l_t - \mathbb{E}[x_t - l_t]\big)^2\Big] - \lambda_t a_t (\mathbb{E}[x_t - l_t])^2\Big\}, \\
\text{s.t.} & \{\mathbb{E}[x_t], \mathbb{E}[l_t], \mathbb{E}[\pi_t]\} \text{ satisfy dynamic equation (4.13)}, \\
& \{x_t - \mathbb{E}[x_t], l_t - \mathbb{E}[l_t], \pi_t - \mathbb{E}[\pi_t]\} \text{ satisfy dynamic equation (4.14)}, \\
& \mathbb{E}(\pi_t - \mathbb{E}[\pi_t]) = \mathbf{0}, \quad t = 0, 1, \cdots, T-1.
\end{array}
\right.
\tag{5.3}
$$

It is indeed a separable linear quadratic optimal stochastic control problem which can be solved by classic dynamic programming approach.

5.2 The Optimal Strategy

Before deriving the main results, we present two useful lemmas.

Lemma 5.1. *Suppose that $\xi_{t+1}(1 - B_t) + \beta_{t+1}B_t \neq 0$ holds. Then*

$$
\begin{aligned}
&\Big(\xi_{t+1}\mathbb{E}[\mathbf{P}_t\mathbf{P}_t'] - \big(\xi_{t+1} - \beta_{t+1}\big)\mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t']\Big)^{-1}\mathbb{E}[\mathbf{P}_t] \\
&= \frac{1}{\xi_{t+1}(1 - B_t) + \beta_{t+1}B_t}\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t].
\end{aligned}
\tag{5.4}
$$

Proof. Applying Sherman-Morrison formula yields

$$
\begin{aligned}
&\Big(\xi_{t+1}\mathbb{E}[\mathbf{P}_t\mathbf{P}_t'] - \big(\xi_{t+1} - \beta_{t+1}\big)\mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t']\Big)^{-1}\mathbb{E}[\mathbf{P}_t] \\
&= \left(\xi_{t+1}^{-1}\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t'] + \frac{\xi_{t+1}^{-1}\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t'](\xi_{t+1} - \beta_{t+1})\mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t']\xi_{t+1}^{-1}\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']}{1 - \xi_{t+1}^{-1}(\xi_{t+1} - \beta_{t+1})\mathbb{E}[\mathbf{P}_t']\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t]}\right)\mathbb{E}[\mathbf{P}_t] \\
&= \frac{1}{\xi_{t+1}(1 - B_t) + \beta_{t+1}B_t}\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t].
\end{aligned}
$$

□

Lemma 5.2. *Suppose that* $\xi_{t+1}(1-B_t)+\beta_{t+1}B_t \neq 0$ *holds. Then*

$$\begin{aligned}&\Big(\xi_{t+1}\mathbb{E}[\mathbf{P}_t\mathbf{P}_t']-\big(\xi_{t+1}-\beta_{t+1}\big)\mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t']\Big)^{-1}\mathbb{E}[\mathbf{P}_tq_t]\\&=\frac{\big(1-\frac{\beta_{t+1}}{\xi_{t+1}}\big)\widehat{B}_t}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t}\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t]+\frac{1}{\xi_{t+1}}\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_tq_t].\end{aligned}\tag{5.5}$$

Proof. Applying Sherman-Morrison formula yields

$$\Big(\xi_{t+1}\mathbb{E}[\mathbf{P}_t\mathbf{P}_t']-\big(\xi_{t+1}-\beta_{t+1}\big)\mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t']\Big)^{-1}\mathbb{E}[\mathbf{P}_tq_t]$$

$$=\left(\xi_{t+1}^{-1}\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']+\frac{\xi_{t+1}^{-1}\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t'](\xi_{t+1}-\beta_{t+1})\mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t']\xi_{t+1}^{-1}\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']}{1-\xi_{t+1}^{-1}(\xi_{t+1}-\beta_{t+1})\mathbb{E}[\mathbf{P}_t']\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t]}\right)\mathbb{E}[\mathbf{P}_tq_t]$$

$$=\frac{\big(1-\frac{\beta_{t+1}}{\xi_{t+1}}\big)\widehat{B}_t}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t}\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t]+\frac{1}{\xi_{t+1}}\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_tq_t].$$

□

Assume that the returns of assets and liability are correlated at every period. For simplicity, we define the following backward recursions for eight deterministic sequences of parameters, $\{\xi_t\}$, $\{\eta_t\}$, $\{\epsilon_t\}$, $\{\beta_t\}$, $\{\zeta_t\}$, $\{\theta_t\}$, $\{\delta_t\}$ and $\{\psi_t\}$, as

$$\begin{aligned}\xi_t&=\xi_{t+1}s_t^2(1-B_t)+\lambda_t,\\ \eta_t&=\eta_{t+1}s_t(\mathbb{E}[q_t]-\widehat{B}_t)+\lambda_t,\\ \epsilon_t&=\epsilon_{t+1}\mathbb{E}[q_t^2]-\frac{\eta_{t+1}^2}{\xi_{t+1}}\widetilde{B}_t+\lambda_t,\\ \beta_t&=\beta_{t+1}s_t^2-\frac{\beta_{t+1}^2s_t^2B_t}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t}-\lambda_ta_t,\\ \zeta_t&=\zeta_{t+1}s_t-\beta_{t+1}s_t\frac{\zeta_{t+1}B_t+2\eta_{t+1}\big(\widehat{B}_t-\mathbb{E}[q_t]B_t\big)\mathbb{E}[l_t]}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t}+2\lambda_ta_t\mathbb{E}[l_t],\\ \theta_t&=\theta_{t+1}\mathbb{E}[q_t]-\zeta_{t+1}\frac{\eta_{t+1}\big(\widehat{B}_t-\mathbb{E}[q_t]B_t\big)}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t},\end{aligned}$$

$$\delta_t = \delta_{t+1}\big(\mathbb{E}[q_t]\big)^2 + \epsilon_{t+1}\big(\mathbb{E}[q_t^2] - (\mathbb{E}[q_t])^2\big)$$

$$- \eta_{t+1}^2\Big(\frac{(1-\frac{\beta_{t+1}}{\xi_{t+1}})\widehat{B}_t^2 - 2\mathbb{E}[q_t]\widehat{B}_t + \big(\mathbb{E}[q_t]\big)^2 B_t}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t} + \frac{1}{\xi_{t+1}}\widetilde{B}_t\Big) - \lambda_t a_t,$$

$$\psi_t = \psi_{t+1} - \frac{1}{4}\frac{\zeta_{t+1}^2 B_t}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t},$$

for $t = T-1, T-2, \cdots, 0$, with terminal conditions

$$\lambda_T = 0,\ \xi_T = 1,\ \eta_T = -1,\ \epsilon_T = 1,\ \beta_T = 0,\ \zeta_T = -w,\ \theta_T = w,\ \delta_T = 0,\ \psi_T = 0,$$

where $\lambda_0 = 0$.

Remark 5.1. *When the returns of assets and liability are uncorrelated, which is to say, $\widehat{B}_t = \mathbb{E}[q_t]B_t$, $\widetilde{B}_t = (\mathbb{E}[q_t])^2 B_t$, parameters $\{\eta_t\}$, $\{\epsilon_t\}$, $\{\zeta_t\}$, $\{\theta_t\}$ and $\{\delta_t\}$ reduce to*

$$\eta_t = \eta_{t+1}s_t\mathbb{E}[q_t](1-B_t) + \lambda_t,$$

$$\epsilon_t = \epsilon_{t+1}\mathbb{E}[q_t^2] - \frac{\eta_{t+1}^2}{\xi_{t+1}}(\mathbb{E}[q_t])^2 B_t + \lambda_t,$$

$$\zeta_t = \zeta_{t+1}s_t - \frac{\zeta_{t+1}\beta_{t+1}B_t s_t}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t} + 2\lambda_t a_t\mathbb{E}[l_t],$$

$$\theta_t = \theta_{t+1}\mathbb{E}[q_t],$$

$$\delta_t = \delta_{t+1}\big(\mathbb{E}[q_t]\big)^2 + \epsilon_{t+1}\big(\mathbb{E}[q_t^2] - (\mathbb{E}[q_t])^2\big) - \lambda_t a_t.$$

And others are the same as the correlated case.

Theorem 5.1. *Assume that the returns of assets and liability are correlated at every* *period. Then, the optimal strategy of problem (5.2) is given by*

$$\begin{aligned}\pi_t^* = \ & -\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t]\Bigg(s_t x_t - \frac{(\xi_{t+1}-\beta_{t+1})(1-B_t)}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t}s_t\mathbb{E}[x_t] \\ & + \frac{\frac{1}{2}\zeta_{t+1} + \eta_{t+1}\big((1-\frac{\beta_{t+1}}{\xi_{t+1}})\widehat{B}_t - \mathbb{E}[q_t]\big)\mathbb{E}[l_t]}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t}\Bigg) - \frac{\eta_{t+1}}{\xi_{t+1}}\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t q_t]l_t,\end{aligned} \tag{5.6}$$

where

$$\begin{aligned}\mathbb{E}[x_t] = \ & x_0\prod_{j=0}^{t-1}\frac{\xi_{j+1}(1-B_j)s_j}{\xi_{j+1}(1-B_j)+\beta_{j+1}B_j} - \sum_{k=0}^{t-1}\Bigg(\prod_{j=k+1}^{t-1}\frac{\xi_{j+1}(1-B_j)s_j}{\xi_{j+1}(1-B_j)+\beta_{j+1}B_j}\Bigg) \\ & \cdot\frac{\frac{1}{2}\zeta_{k+1}B_k + \eta_{k+1}\big(\widehat{B}_k - \mathbb{E}[q_k]B_k\big)\big(\prod_{j=0}^{k-1}\mathbb{E}[q_j]\big)l_0}{\xi_{k+1}(1-B_k)+\beta_{k+1}B_k}.\end{aligned} \tag{5.7}$$

for $t = 0, 1, \cdots, T-1$.

Proof. We prove the main results by dynamic programming approach. For the information set $\mathcal{F}_t$, the cost-to-go functional at period t is computed by

$$\begin{aligned}& J_t(\mathbb{E}[x_t], x_t - \mathbb{E}[x_t], \mathbb{E}[l_t], l_t - \mathbb{E}[l_t]) \\ = & \min_{\{\pi_t - \mathbb{E}[\pi_t], \mathbb{E}[\pi_t]\}}\mathbb{E}\big[J_{t+1}\big(\mathbb{E}[x_{t+1}], x_{t+1} - \mathbb{E}[x_{t+1}], \mathbb{E}[l_{t+1}], l_{t+1} - \mathbb{E}[l_{t+1}]\big)\big|\mathcal{F}_t\big] \\ & + \lambda_t\big(x_t - l_t - \mathbb{E}[x_t - l_t]\big)^2 - \lambda_t a_t\big(\mathbb{E}[x_t - l_t]\big)^2.\end{aligned}$$

The cost-to-go functional at terminal time T is

$$\begin{aligned}& J_T\big(\mathbb{E}[x_T], x_T - \mathbb{E}[x_T], \mathbb{E}[l_T], l_T - \mathbb{E}[l_T]\big) \\ = & \big(x_T - l_T - \mathbb{E}[x_T - l_T]\big)^2 - w\mathbb{E}\big[x_T - l_T\big] \\ = & \xi_T\big(x_T - \mathbb{E}[x_T]\big)^2 + 2\eta_T\big(l_T - \mathbb{E}[l_T]\big)\big(x_T - \mathbb{E}[x_T]\big) + \epsilon_T\big(l_T - \mathbb{E}[l_T]\big)^2 \\ & + \beta_T(\mathbb{E}[x_T])^2 + \zeta_T\mathbb{E}[x_T] + \theta_T\mathbb{E}[l_T] + \delta_T(\mathbb{E}[l_T])^2 + \psi_T.\end{aligned}$$

Assume that the cost-to-go functional at time $t+1$ is the following expression

$$\begin{aligned}& J_{t+1}\big(\mathbb{E}[x_{t+1}], x_{t+1} - \mathbb{E}[x_{t+1}], \mathbb{E}[l_{t+1}], l_{t+1} - \mathbb{E}[l_{t+1}]\big) \\ = & \xi_{t+1}\big(x_{t+1} - \mathbb{E}[x_{t+1}]\big)^2 + 2\eta_{t+1}\big(l_{t+1} - \mathbb{E}[l_{t+1}]\big)\big(x_{t+1} - \mathbb{E}[x_{t+1}]\big) + \epsilon_{t+1}\big(l_{t+1} - \mathbb{E}[l_{t+1}]\big)^2 \\ & + \beta_{t+1}(\mathbb{E}[x_{t+1}])^2 + \zeta_{t+1}\mathbb{E}[x_{t+1}] + \theta_{t+1}\mathbb{E}[l_{t+1}] + \delta_{t+1}(\mathbb{E}[l_{t+1}])^2 + \psi_{t+1}.\end{aligned}$$

We prove that the above statement still holds at time t. For given information set $\mathcal{F}_t$, i.e., knowing $x_t - \mathbb{E}[x_t]$, $\mathbb{E}[x_t]$, $l_t - \mathbb{E}[l_t]$ and $\mathbb{E}[l_t]$, we have

$$
\begin{aligned}
&\mathbb{E}\Big[J_{t+1}\big(\mathbb{E}[x_{t+1}], x_{t+1}-\mathbb{E}[x_{t+1}], \mathbb{E}[l_{t+1}], l_{t+1}-\mathbb{E}[l_{t+1}]\big)\Big|\mathcal{F}_t\Big] \\
=&\,\mathbb{E}\Bigg[\xi_{t+1}\Big[s_t\big(x_t-\mathbb{E}[x_t]\big)+\mathbf{P}'_t\big(\pi_t-\mathbb{E}[\pi_t]\big)+\big(\mathbf{P}'_t-\mathbb{E}[\mathbf{P}'_t]\big)\mathbb{E}[\pi_t]\Big]^2 \\
&+2\eta_{t+1}\Big[q_t\big(l_t-\mathbb{E}[l_t]\big)+\big(q_t-\mathbb{E}[q_t]\big)\mathbb{E}[l_t]\Big] \\
&\cdot\Big[s_t\big(x_t-\mathbb{E}[x_t]\big)+\mathbf{P}'_t\big(\pi_t-\mathbb{E}[\pi_t]\big)+\big(\mathbf{P}'_t-\mathbb{E}[\mathbf{P}'_t]\big)\mathbb{E}[\pi_t]\Big] \\
&+\epsilon_{t+1}\big[q_t(l_t-\mathbb{E}[l_t])+(q_t-\mathbb{E}[q_t])\mathbb{E}[l_t]\big]^2+\beta_{t+1}\big(s_t\mathbb{E}[x_t]+\mathbb{E}[\mathbf{P}'_t]\mathbb{E}[\pi_t]\big)^2 \\
&+\zeta_{t+1}\big(s_t\mathbb{E}[x_t]+\mathbb{E}[\mathbf{P}'_t]\mathbb{E}[\pi_t]\big)+\theta_{t+1}\mathbb{E}[q_t]\mathbb{E}[l_t]+\delta_{t+1}(\mathbb{E}[q_t]\mathbb{E}[l_t])^2+\psi_{t+1}\Bigg|\mathcal{F}_t\Bigg] \\
=&\,\xi_{t+1}\Big[s_t^2\big(x_t-\mathbb{E}[x_t]\big)^2+\big(\pi_t-\mathbb{E}[\pi_t]\big)'\mathbb{E}[\mathbf{P}_t\mathbf{P}'_t]\big(\pi_t-\mathbb{E}[\pi_t]\big) \\
&+2s_t\big(x_t-\mathbb{E}[x_t]\big)\mathbb{E}[\mathbf{P}'_t]\big(\pi_t-\mathbb{E}[\pi_t]\big)+\mathbb{E}[\pi'_t]\big(\mathbb{E}[\mathbf{P}_t\mathbf{P}'_t]-\mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}'_t]\big)\mathbb{E}[\pi_t] \\
&+2\big(\pi_t-\mathbb{E}[\pi_t]\big)'\big(\mathbb{E}[\mathbf{P}_t\mathbf{P}'_t]-\mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}'_t]\big)\mathbb{E}[\pi_t]\Big]+2\eta_{t+1}\Big[s_t\mathbb{E}[q_t]\big(l_t-\mathbb{E}[l_t]\big) \\
&\big(x_t-\mathbb{E}[x_t]\big)+\mathbb{E}[q_t\mathbf{P}'_t]\big(l_t-\mathbb{E}[l_t]\big)\big(\pi_t-\mathbb{E}[\pi_t]\big)+\big(\mathbb{E}[q_t\mathbf{P}'_t]-\mathbb{E}[q_t]\mathbb{E}[\mathbf{P}'_t]\big) \\
&\cdot\Big(\mathbb{E}[l_t]\big(\pi_t-\mathbb{E}[\pi_t]\big)+\big(l_t-\mathbb{E}[l_t]\big)\mathbb{E}[\pi_t]+\mathbb{E}[l_t]\mathbb{E}[\pi_t]\Big)\Big]+\epsilon_{t+1}\Big[\mathbb{E}[q_t^2](l_t-\mathbb{E}[l_t])^2 \\
&+2(\mathbb{E}[q_t^2]-(\mathbb{E}[q_t])^2)(l_t-\mathbb{E}[l_t])\mathbb{E}[l_t]+(\mathbb{E}[q_t^2]-(\mathbb{E}[q_t])^2)(\mathbb{E}[l_t])^2\Big] \\
&+\beta_{t+1}\Big[s_t^2\big(\mathbb{E}[x_t]\big)^2+2s_t\mathbb{E}[x_t]\mathbb{E}[\mathbf{P}'_t]\mathbb{E}[\pi_t]+\mathbb{E}[\pi'_t]\mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}'_t]\mathbb{E}[\pi_t]\Big] \\
&+\zeta_{t+1}\big(s_t\mathbb{E}[x_t]+\mathbb{E}[\mathbf{P}'_t]\mathbb{E}[\pi_t]\big)+\theta_{t+1}\mathbb{E}[q_t]\mathbb{E}[l_t]+\delta_{t+1}(\mathbb{E}[q_t]\mathbb{E}[l_t])^2+\psi_{t+1}.
\end{aligned}
$$

Using the same technique in Chapter 4, we can obtain optimal $(\mathbb{E}[\pi_t^*], \pi_t^* - \mathbb{E}[\pi_t^*])$

by minimizing the following equivalent cost functional,

$$
\begin{aligned}
&\mathbb{E}\big[J_{t+1}\big(\mathbb{E}[x_{t+1}], x_{t+1}-\mathbb{E}[x_{t+1}], \mathbb{E}[l_{t+1}], l_{t+1}-\mathbb{E}[l_{t+1}]\big)\big|\mathcal{F}_t\big] \\
&= \xi_{t+1}\Big[s_t^2\big(x_t-\mathbb{E}[x_t]\big)^2 + \big(\pi_t-\mathbb{E}[\pi_t]\big)'\mathbb{E}[\mathbf{P}_t\mathbf{P}_t']\big(\pi_t-\mathbb{E}[\pi_t]\big) \\
&\quad + 2s_t\big(x_t-\mathbb{E}[x_t]\big)\mathbb{E}[\mathbf{P}_t']\big(\pi_t-\mathbb{E}[\pi_t]\big) + \mathbb{E}[\pi_t']\big(\mathbb{E}[\mathbf{P}_t\mathbf{P}_t'] - \mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t']\big)\mathbb{E}[\pi_t]\Big] \\
&\quad + 2\eta_{t+1}\Big[s_t\mathbb{E}[q_t]\big(l_t-\mathbb{E}[l_t]\big)\big(x_t-\mathbb{E}[x_t]\big) + \mathbb{E}[q_t\mathbf{P}_t']\big(l_t-\mathbb{E}[l_t]\big)\big(\pi_t-\mathbb{E}[\pi_t]\big) \\
&\quad + \big(\mathbb{E}[q_t\mathbf{P}_t'] - \mathbb{E}[q_t]\mathbb{E}[\mathbf{P}_t']\big)\mathbb{E}[l_t]\mathbb{E}[\pi_t]\Big] \\
&\quad + \epsilon_{t+1}\Big[\mathbb{E}[q_t^2](l_t-\mathbb{E}[l_t])^2 + (\mathbb{E}[q_t^2]-(\mathbb{E}[q_t])^2)(\mathbb{E}[l_t])^2\Big] \\
&\quad + \beta_{t+1}\Big[s_t^2\big(\mathbb{E}[x_t]\big)^2 + 2s_t\mathbb{E}[x_t]\mathbb{E}[\mathbf{P}_t']\mathbb{E}[\pi_t] + \mathbb{E}[\pi_t']\mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t']\mathbb{E}[\pi_t]\Big] \\
&\quad + \zeta_{t+1}\big(s_t\mathbb{E}[x_t] + \mathbb{E}[\mathbf{P}_t']\mathbb{E}[\pi_t]\big) + \theta_{t+1}\mathbb{E}[q_t]\mathbb{E}[l_t] + \delta_{t+1}(\mathbb{E}[q_t]\mathbb{E}[l_t])^2 + \psi_{t+1} \\
&= \xi_{t+1}\Big[s_t^2\big(x_t-\mathbb{E}[x_t]\big)^2 + \big(\pi_t-\mathbb{E}[\pi_t]\big)'\mathbb{E}[\mathbf{P}_t\mathbf{P}_t']\big(\pi_t-\mathbb{E}[\pi_t]\big) \\
&\quad + 2s_t\big(x_t-\mathbb{E}[x_t]\big)\mathbb{E}[\mathbf{P}_t']\big(\pi_t-\mathbb{E}[\pi_t]\big)\Big] \\
&\quad + \mathbb{E}[\pi_t']\big(\xi_{t+1}\mathbb{E}[\mathbf{P}_t\mathbf{P}_t'] - (\xi_{t+1}-\beta_{t+1})\mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t']\big)\mathbb{E}[\pi_t] \\
&\quad + 2\eta_{t+1}\Big[s_t\mathbb{E}[q_t]\big(l_t-\mathbb{E}[l_t]\big)\big(x_t-\mathbb{E}[x_t]\big) + \mathbb{E}[q_t\mathbf{P}_t']\big(l_t-\mathbb{E}[l_t]\big)\big(\pi_t-\mathbb{E}[\pi_t]\big) \\
&\quad + \big(\mathbb{E}[q_t\mathbf{P}_t'] - \mathbb{E}[q_t]\mathbb{E}[\mathbf{P}_t']\big)\mathbb{E}[l_t]\mathbb{E}[\pi_t]\Big] \\
&\quad + \epsilon_{t+1}\Big[\mathbb{E}[q_t^2](l_t-\mathbb{E}[l_t])^2 + (\mathbb{E}[q_t^2]-(\mathbb{E}[q_t])^2)(\mathbb{E}[l_t])^2\Big] \\
&\quad + \beta_{t+1}\Big[s_t^2\big(\mathbb{E}[x_t]\big)^2 + 2s_t\mathbb{E}[x_t]\mathbb{E}[\mathbf{P}_t']\mathbb{E}[\pi_t]\Big] + \zeta_{t+1}\big(s_t\mathbb{E}[x_t] + \mathbb{E}[\mathbf{P}_t']\mathbb{E}[\pi_t]\big) \\
&\quad + \theta_{t+1}\mathbb{E}[q_t]\mathbb{E}[l_t] + \delta_{t+1}(\mathbb{E}[q_t]\mathbb{E}[l_t])^2 + \psi_{t+1}.
\end{aligned}
$$

It is easy to see that $\pi_t^* - \mathbb{E}[\pi_t^*]$ can be expressed by the linear form of states and their expected states, and $\mathbb{E}[\pi_t^*]$ can be constructed by the linear form of the expected states, i.e.,

$$\pi_t^* - \mathbb{E}[\pi_t^*] = -\,\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t]s_t\big(x_t - \mathbb{E}[x_t]\big) - \frac{\eta_{t+1}}{\xi_{t+1}}\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t q_t]\big(l_t - \mathbb{E}[l_t]\big), \tag{5.8}$$

$$\begin{aligned}
\mathbb{E}[\pi_t^*] =& -\Big(\xi_{t+1}\mathbb{E}[\mathbf{P}_t\mathbf{P}_t'] - (\xi_{t+1}-\beta_{t+1})\mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t']\Big)^{-1}\bigg(\beta_{t+1}s_t\mathbb{E}[x_t]\mathbb{E}[\mathbf{P}_t] \\
&+ \frac{1}{2}\zeta_{t+1}\mathbb{E}[\mathbf{P}_t] + \eta_{t+1}\big(\mathbb{E}[\mathbf{P}_t q_t] - \mathbb{E}[q_t]\mathbb{E}[\mathbf{P}_t]\big)\mathbb{E}[l_t]\bigg) \\
=& -\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t]\frac{\beta_{t+1}s_t\mathbb{E}[x_t] + \frac{1}{2}\zeta_{t+1} + \eta_{t+1}\big((1-\frac{\beta_{t+1}}{\xi_{t+1}})\widehat{B}_t - \mathbb{E}[q_t]\big)\mathbb{E}[l_t]}{\xi_{t+1}(1-B_t) + \beta_{t+1}B_t} \\
&- \frac{\eta_{t+1}}{\xi_{t+1}}\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t q_t]\mathbb{E}[l_t].
\end{aligned} \tag{5.9}$$

In order to get the explicit expression of the cost-to-go functional at time t, we substitute $\pi_t^* - \mathbb{E}[\pi_t^*]$ and $\mathbb{E}[\pi_t^*]$ back and derive

$$\begin{aligned}
& J_t\big(\mathbb{E}[x_t], x_t - \mathbb{E}[x_t], \mathbb{E}[l_t], l_t - \mathbb{E}[l_t]\big) \\
=& \min_{\{\pi_t - \mathbb{E}[\pi_t], \mathbb{E}[\pi_t]\}} \mathbb{E}\Big[J_{t+1}\big(\mathbb{E}[x_{t+1}], x_{t+1} - \mathbb{E}[x_{t+1}], \mathbb{E}[l_{t+1}], l_{t+1} - \mathbb{E}[l_{t+1}]\big)\Big|\mathcal{F}_t\Big] \\
&+ \lambda_t\big(x_t - l_t - \mathbb{E}[x_t - l_t]\big)^2 - \lambda_t a_t\big(\mathbb{E}[x_t - l_t]\big)^2 \\
=& \,\xi_{t+1}s_t^2\big(x_t - \mathbb{E}[x_t]\big)^2 + 2\eta_{t+1}s_t\mathbb{E}[q_t]\big(l_t - \mathbb{E}[l_t]\big)\big(x_t - \mathbb{E}[x_t]\big) \\
&+ \beta_{t+1}s_t^2\big(\mathbb{E}[x_t]\big)^2 + \zeta_{t+1}s_t\mathbb{E}[x_t] + \theta_{t+1}\mathbb{E}[q_t]\mathbb{E}[l_t] \\
&+ \epsilon_{t+1}\Big[\mathbb{E}[q_t^2](l_t - \mathbb{E}[l_t])^2 + (\mathbb{E}[q_t^2] - (\mathbb{E}[q_t])^2)(\mathbb{E}[l_t])^2\Big] + \delta_{t+1}(\mathbb{E}[q_t]\mathbb{E}[l_t])^2 + \psi_{t+1} \\
&- \xi_{t+1}\Big[-\mathbb{E}[\mathbf{P}_t]s_t\big(x_t - \mathbb{E}[x_t]\big) - \frac{\eta_{t+1}}{\xi_{t+1}}\mathbb{E}[\mathbf{P}_t q_t]\big(l_t - \mathbb{E}[l_t]\big)\Big]' \\
&\cdot \mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\Big[-\mathbb{E}[\mathbf{P}_t]s_t\big(x_t - \mathbb{E}[x_t]\big) - \frac{\eta_{t+1}}{\xi_{t+1}}\mathbb{E}[\mathbf{P}_t q_t]\big(l_t - \mathbb{E}[l_t]\big)\Big]
\end{aligned}$$

$$
\begin{aligned}
&- \Big[\beta_{t+1} s_t \mathbb{E}[x_t]\mathbb{E}[\mathbf{P}_t] + \frac{1}{2}\zeta_{t+1}\mathbb{E}[\mathbf{P}_t] + \eta_{t+1}\big(\mathbb{E}[\mathbf{P}_t q_t] - \mathbb{E}[q_t]\mathbb{E}[\mathbf{P}_t]\big)\mathbb{E}[l_t]\Big]' \\
&\cdot \big(\xi_{t+1}\mathbb{E}[\mathbf{P}_t\mathbf{P}_t'] - (\xi_{t+1} - \beta_{t+1})\mathbb{E}[\mathbf{P}_t]\mathbb{E}[\mathbf{P}_t']\big)^{-1} \\
&\cdot \Big[\beta_{t+1} s_t \mathbb{E}[x_t]\mathbb{E}[\mathbf{P}_t] + \frac{1}{2}\zeta_{t+1}\mathbb{E}[\mathbf{P}_t] + \eta_{t+1}\big(\mathbb{E}[\mathbf{P}_t q_t] - \mathbb{E}[q_t]\mathbb{E}[\mathbf{P}_t]\big)\mathbb{E}[l_t]\Big] \\
&+ \lambda_t\big(x_t - l_t - \mathbb{E}[x_t - l_t]\big)^2 - \lambda_t a_t\big(\mathbb{E}[x_t - l_t]\big)^2 \\
=& \, \xi_{t+1} s_t^2 (1 - B_t)\big(x_t - \mathbb{E}[x_t]\big)^2 + 2\eta_{t+1} s_t(\mathbb{E}[q_t] - \widehat{B}_t)\big(l_t - \mathbb{E}[l_t]\big)\big(x_t - \mathbb{E}[x_t]\big) \\
&+ \Big(\epsilon_{t+1}\mathbb{E}[q_t^2] - \frac{\eta_{t+1}^2}{\xi_{t+1}}\widetilde{B}_t\Big)(l_t - \mathbb{E}[l_t])^2 + \Big(\beta_{t+1} - \frac{\beta_{t+1}^2 B_t}{\xi_{t+1}(1 - B_t) + \beta_{t+1}B_t}\Big) s_t^2 \big(\mathbb{E}[x_t]\big)^2 \\
&+ \Big(\zeta_{t+1} - \beta_{t+1}\frac{\zeta_{t+1}B_t + 2\eta_{t+1}\big(\widehat{B}_t - \mathbb{E}[q_t]B_t\big)\mathbb{E}[l_t]}{\xi_{t+1}(1 - B_t) + \beta_{t+1}B_t}\Big) s_t \mathbb{E}[x_t] \\
&+ \Big(\theta_{t+1}\mathbb{E}[q_t] - \zeta_{t+1}\frac{\eta_{t+1}\big(\widehat{B}_t - \mathbb{E}[q_t]B_t\big)}{\xi_{t+1}(1 - B_t) + \beta_{t+1}B_t}\Big)\mathbb{E}[l_t] \\
&+ \Big[\epsilon_{t+1}(\mathbb{E}[q_t^2] - (\mathbb{E}[q_t])^2) + \delta_{t+1}(\mathbb{E}[q_t])^2 \\
&- \eta_{t+1}^2\Big(\frac{(1 - \frac{\beta_{t+1}}{\xi_{t+1}})\widehat{B}_t^2 - 2\mathbb{E}[q_t]\widehat{B}_t + \big(\mathbb{E}[q_t]\big)^2 B_t}{\xi_{t+1}(1 - B_t) + \beta_{t+1}B_t} + \frac{1}{\xi_{t+1}}\widetilde{B}_t\Big)\Big]\big(\mathbb{E}[l_t]\big)^2 \\
&+ \psi_{t+1} - \frac{1}{4}\frac{\zeta_{t+1}^2 B_t}{\xi_{t+1}(1 - B_t) + \beta_{t+1}B_t} + \lambda_t\big(x_t - l_t - \mathbb{E}[x_t - l_t]\big)^2 - \lambda_t a_t\big(\mathbb{E}[x_t - l_t]\big)^2 \\
=& \Big(\xi_{t+1} s_t^2(1 - B_t) + \lambda_t\Big)\big(x_t - \mathbb{E}[x_t]\big)^2 \\
&+ 2\Big(\eta_{t+1} s_t(\mathbb{E}[q_t] - \widehat{B}_t) + \lambda_t\Big)\big(l_t - \mathbb{E}[l_t]\big)\big(x_t - \mathbb{E}[x_t]\big) \\
&+ \Big(\epsilon_{t+1}\mathbb{E}[q_t^2] - \frac{\eta_{t+1}^2}{\xi_{t+1}}\widetilde{B}_t + \lambda_t\Big)(l_t - \mathbb{E}[l_t])^2 \\
&+ \Big(\beta_{t+1}s_t^2 - \frac{\beta_{t+1}^2 s_t^2 B_t}{\xi_{t+1}(1 - B_t) + \beta_{t+1}B_t} - \lambda_t a_t\Big)\big(\mathbb{E}[x_t]\big)^2
\end{aligned}
$$

$$
\begin{aligned}
&+\Bigg[\Big(\zeta_{t+1}s_t-\beta_{t+1}s_t\frac{\zeta_{t+1}B_t+2\eta_{t+1}\big(\widehat{B}_t-\mathbb{E}[q_t]B_t\big)\mathbb{E}[l_t]}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t}\Big)+2\lambda_t a_t\mathbb{E}[l_t]\Bigg]\mathbb{E}[x_t]\\
&+\Big(\theta_{t+1}\mathbb{E}[q_t]-\zeta_{t+1}\frac{\eta_{t+1}\big(\widehat{B}_t-\mathbb{E}[q_t]B_t\big)}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t}\Big)\mathbb{E}[l_t]\\
&+\Bigg[\delta_{t+1}\big(\mathbb{E}[q_t]\big)^2+\epsilon_{t+1}\big(\mathbb{E}[q_t^2]-(\mathbb{E}[q_t])^2\big)\\
&-\eta_{t+1}^2\Big(\frac{\big(1-\frac{\beta_{t+1}}{\xi_{t+1}}\big)\widehat{B}_t^2-2\mathbb{E}[q_t]\widehat{B}_t+\big(\mathbb{E}[q_t]\big)^2B_t}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t}+\frac{1}{\xi_{t+1}}\widetilde{B}_t\Big)-\lambda_t a_t\Bigg]\big(\mathbb{E}[l_t]\big)^2\\
&+\psi_{t+1}-\frac{1}{4}\frac{\zeta_{t+1}^2B_t}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t}\\
=&\,\xi_t\big(x_t-\mathbb{E}[x_t]\big)^2+2\eta_t\big(l_t-\mathbb{E}[l_t]\big)\big(x_t-\mathbb{E}[x_t]\big)+\epsilon_t\big(l_t-\mathbb{E}[l_t]\big)^2\\
&+\beta_t(\mathbb{E}[x_t])^2+\zeta_t\mathbb{E}[x_t]+\theta_t\mathbb{E}[l_t]+\delta_t(\mathbb{E}[l_t])^2+\psi_t.
\end{aligned}
$$

Substituting $\mathbb{E}[\pi_t^*]$ to dynamics of $\mathbb{E}[x_t]$ yields

$$
\mathbb{E}[x_{t+1}]=\frac{\xi_{t+1}(1-B_t)s_t}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t}\mathbb{E}[x_t]-\frac{\frac{1}{2}\zeta_{t+1}B_t+\eta_{t+1}\big(\widehat{B}_t-\mathbb{E}[q_t]B_t\big)\mathbb{E}[l_t]}{\xi_{t+1}(1-B_t)+\beta_{t+1}B_t}
$$

which implies

$$
\begin{aligned}
\mathbb{E}[x_t]=x_0\prod_{j=0}^{t-1}\frac{\xi_{j+1}(1-B_j)s_j}{\xi_{j+1}(1-B_j)+\beta_{j+1}B_j}-\sum_{k=0}^{t-1}\Big(\prod_{j=k+1}^{t-1}\frac{\xi_{j+1}(1-B_j)s_j}{\xi_{j+1}(1-B_j)+\beta_{j+1}B_j}\Big)\\
\cdot\frac{\frac{1}{2}\zeta_{k+1}B_k+\eta_{k+1}\big(\widehat{B}_k-\mathbb{E}[q_k]B_k\big)\big(\prod_{j=0}^{k-1}\mathbb{E}[q_j]\big)l_0}{\xi_{k+1}(1-B_k)+\beta_{k+1}B_k}.
\end{aligned}
$$

Hence, combining with (5.8) and (5.9), we derive the desired result (5.6).

Remark 5.2. *When the returns of assets and liability are not correlated,*

$$
\begin{aligned}
\mathbb{E}[x_t]=&\,x_0\prod_{j=0}^{t-1}\frac{\xi_{j+1}(1-B_j)s_j}{\xi_{j+1}(1-B_j)+\beta_{j+1}B_j}\\
&-\frac{1}{2}\sum_{k=0}^{t-1}\Big(\prod_{j=k+1}^{t-1}\frac{\xi_{j+1}(1-B_j)s_j}{\xi_{j+1}(1-B_j)+\beta_{j+1}B_j}\Big)\frac{\zeta_{k+1}B_k}{\xi_{k+1}(1-B_k)+\beta_{k+1}B_k}.
\end{aligned}
$$

$\square$

Based on the proof of Theorem 5.1, the optimal objective of problem (5.2) is as follows,

$$J_0\big(\mathbb{E}[x_0], 0, \mathbb{E}[l_0], 0\big) = \beta_0 x_0^2 + \zeta_0 x_0 + \theta_0 l_0 + \delta_0 l_0^2 + \psi_0. \qquad (5.10)$$

In fact, $J_0(\cdot)$ is convex in λ. Hence, according to (5.10), we can derive the variance term as

$$\mathrm{Var}(x_T - l_T) = \max_{\lambda \in \mathbb{R}_+^{T-1}} J_0\big(x_0, 0, l_0, 0\big) + w\mathbb{E}[x_T - l_T].$$

Theorem 5.2. *Assume that the returns of assets and liability are correlated at every period. Then, the efficient frontier of problem (5.2) is given by*

$$\mathrm{Var}(x_T - l_T) = \max_{\lambda \in \mathbb{R}_+^{T-1}} J_0\big(x_0, 0, l_0, 0\big) + w\mathbb{E}[x_T - l_T]. \qquad (5.11)$$

Remark 5.3. *When we delete the constraint on bankruptcy, all the results reduce to those in Section 4.2 when the exit time is terminal. We can also consider the case when the exit time is random, but we omit it due to the same approach and space limit.*

5.3 Numerical Examples

We consider the example as Section 3.3 but we do not fix the terminal expectation. We further assume that the trade-off parameter $w = 1$ and the probability of bankruptcy $a_t = 0.1$, for $t = 1, 2, 3, 4$. We adopt the Matlab optimization function "fmincon" to identify the optimal multiplier λ^*.

Example 5.1. Assume that the returns of the assets and liability are correlated with the same ρ in Example 3.1. By interior point algorithm of "fmincon" with the initial point $\lambda = (0, 0, 0, 0)$, we can obtain $\lambda^* = (0, 0, 0, 0.4902)$. Then according to

Theorem 5.1, we can derive the optimal strategy of problem (5.2) as follows,

$$\pi_0^* = -1.05(x_0 - 3.6997)\mathbf{K}_1 + 0.3520\mathbf{K}_2 l_0,$$

$$\pi_1^* = -1.05(x_1 - 3.8847)\mathbf{K}_1 + 0.3360\mathbf{K}_2 l_1,$$

$$\pi_2^* = -1.05(x_2 - 4.0789)\mathbf{K}_1 + 0.3207\mathbf{K}_2 l_2,$$

$$\pi_3^* = -1.05(x_3 - 4.2829)\mathbf{K}_1 + 0.3060\mathbf{K}_2 l_3,$$

$$\pi_4^* = -1.05(x_4 - 3.5243)\mathbf{K}_1 + 1.0000\mathbf{K}_2 l_4.$$

The optimal expected surplus levels are $\mathbb{E}(x_5 - l_5) = 3.2005$ and $\mathrm{Var}(x_5 - l_5) = 0.5740$, respectively.

Example 5.2. Assume that the returns of the assets and liability are uncorrelated. Then parameters $\{\xi_t\}$, $\{\eta_t\}$, $\{\epsilon_t\}$, $\{\beta_t\}$, $\{\zeta_t\}$, $\{\theta_t\}$, $\{\delta_t\}$ and $\{\psi_t\}$ are defined in Remark 5.1 By interior point algorithm of "fmincon" with the initial point $\lambda = (0, 0, 0, 0)$, we can obtain $\lambda^* = (0, 0, 0, 0.1775)$. According to Theorem 5.1, the optimal strategy of problem (5.2) is specified as follows,

$$\pi_0^* = -1.05(x_0 - 3.3587 + 0.7658 l_0)\mathbf{K}_1,$$

$$\pi_1^* = -1.05(x_1 - 3.5267 + 0.7310 l_1)\mathbf{K}_1,$$

$$\pi_2^* = -1.05(x_2 - 3.7030 + 0.6977 l_2)\mathbf{K}_1,$$

$$\pi_3^* = -1.05(x_3 - 3.8882 + 0.6660 l_3)\mathbf{K}_1,$$

$$\pi_4^* = -1.05(x_4 - 3.6231 + 0.9524 l_4)\mathbf{K}_1.$$

The mean and variance of the final optimal surplus are $\mathbb{E}(x_5 - l_5) = 3.3042$ and $\mathrm{Var}(x_5 - l_5) = 0.8157$, respectively.

Remark 5.4. *When we do not take bankruptcy into account, which is to say,* $\lambda = (0, 0, 0, 0)$ *in formulation (5.3), Example 5.1 and Example 5.2 reduce to Example 4.3 and Example 4.5 in Section 4.2, respectively. And the results are same.*

Chapter 6

Multi-Period Mean-Variance Asset-Liability Portfolio Selection with Correlated Returns

This chapter reconsiders the problem in Chapter 3 while we allow the return vectors in different time periods, $\{e_t\}_{t=0}^{T-1}$, to be statistically correlated, which is always the case in real financial market. The returns of assets and liability are also correlated at every period, i.e., $\mathbf{P}_t$ and q_t are dependent each other at period $t = 0, 1, \cdots, T-1$. The formulation of the multi-period mean-variance asset-liability portfolio selection is the same as Chapter 3. Since the only difference is the returns in different time periods are statistically correlated, we do not repeat the formulation.

6.1 Preliminary

We use the notation $\mathbb{E}_t[\cdot], \mathrm{Cov}_t[\cdot]$ and $\mathrm{Var}_t[\cdot]$ to denote the conditional expectation $\mathbb{E}[\cdot|\mathcal{F}_t]$, the conditional covariance matrix $\mathrm{Cov}[\cdot|\mathcal{F}_t]$ and the conditional variance $\mathrm{Var}[\cdot|\mathcal{F}_t]$, respectively. It is reasonable to assume that the conditional covariance matrices,

$$\mathrm{Cov}_t\left(\begin{pmatrix}\mathbf{e}_t\\ q_t\end{pmatrix}\right) = \mathbb{E}_t\left[\begin{pmatrix}\mathbf{e}_t\\ q_t\end{pmatrix}\begin{pmatrix}\mathbf{e}_t' & q_t\end{pmatrix}\right] - \mathbb{E}_t\left[\begin{pmatrix}\mathbf{e}_t\\ q_t\end{pmatrix}\right]\mathbb{E}_t\left[\begin{pmatrix}\mathbf{e}_t' & q_t\end{pmatrix}\right],$$

are positive definite for all $t = 0, 1, \cdots, T-1$.

Before presenting the main result, we define seven sequences as follows:

$$\begin{aligned}
\xi_t &= \mathbb{E}_t[\xi_{t+1}] - B_t,\\
\eta_t &= \mathbb{E}_t[q_t\eta_{t+1}] - \widehat{B}_t,\\
\zeta_t &= \mathbb{E}_t[q_t^2\zeta_{t+1}] + \widetilde{B}_t,\\
B_t &= \mathbb{E}_t[\xi_{t+1}\mathbf{P}_t']\mathbb{E}_t^{-1}[\xi_{t+1}\mathbf{P}_t\mathbf{P}_t']\mathbb{E}_t[\xi_{t+1}\mathbf{P}_t],\\
\widehat{B}_t &= \mathbb{E}_t[\xi_{t+1}\mathbf{P}_t']\mathbb{E}_t^{-1}[\xi_{t+1}\mathbf{P}_t\mathbf{P}_t']\mathbb{E}_t[\eta_{t+1}q_t\mathbf{P}_t],\\
\widetilde{B}_t &= \mathbb{E}_t[\eta_{t+1}q_t\mathbf{P}_t']\mathbb{E}_t^{-1}[\xi_{t+1}\mathbf{P}_t\mathbf{P}_t']\mathbb{E}_t[\eta_{t+1}q_t\mathbf{P}_t],\\
\epsilon_t &= \mathbb{E}_t[\epsilon_{t+1}q_t^2],
\end{aligned}$$

with $\xi_T = \eta_T = \epsilon_T = 1, \zeta_T = 0$.

Lemma 6.1. *For any $t = 0, 1, \cdots, T-1$, $0 < \xi_t < 1$.*

Proof. We prove it by mathematical induction. For any stage k, $\mathrm{Cov}_k[\mathbf{e}_k] \succ 0$, implies $\mathrm{Cov}_k[\mathbf{P}_k] \succ 0$. At stage $T-1$, Applying Schur complement theorem to $\mathrm{Cov}_{T-1}[\mathbf{P}_{T-1}] \succ 0$ derives

$$0 < 1 - \mathbb{E}_{T-1}[\mathbf{P}_{T-1}']\mathbb{E}_{T-1}[\mathbf{P}_{T-1}\mathbf{P}_{T-1}']\mathbb{E}_{T-1}[\mathbf{P}_{T-1}] < 1,$$

that is

$$0 < \xi_{T-1} = 1 - B_{T-1} < 1.$$

Assume that $0 < \xi_{k+1} < 1$, we will show that $0 < \xi_k < 1$. Since $\xi_{k+1} > 0$, we have $\mathbb{E}_k[\xi_{k+1}] > 0$. Define a positive random variable $Z \triangleq \frac{\xi_{k+1}}{\mathbb{E}_k[\xi_{k+1}]}$. Obviously, $\mathbb{E}_k[Z] = 1$. Thus we can construct a new probability measure $\widehat{\mathbb{P}}$ as $\widehat{\mathbb{P}} \triangleq \int_A Z(w)d\widehat{\mathbb{P}}(w)$, for any $A \in \mathcal{F}_k$. Under the new probability measure $\widehat{\mathbb{P}}$, we use the notation $\widehat{\mathbb{E}}_k[\cdot], \widehat{\mathrm{Cov}}_k[\cdot]$ and $\widehat{\mathrm{Var}}_k[\cdot]$ to denote the conditional expectation $\widehat{\mathbb{E}}[\cdot|\mathcal{F}_k]$, the conditional covariance matrix $\widehat{\mathrm{Cov}}[\cdot|\mathcal{F}_k]$ and the conditional variance $\widehat{\mathrm{Var}}[\cdot|\mathcal{F}_k]$, respectively. First we prove that $\widehat{\mathrm{Cov}}_k[\mathbf{P}_k] = \widehat{\mathbb{E}}_k[\mathbf{P}_k\mathbf{P}_k'] - \widehat{\mathbb{E}}_k[\mathbf{P}_k]\mathbb{E}[\mathbf{P}_k'] \succ 0$. In fact, suppose that there exists an $x \in \mathbb{R}^n$ with $x \neq 0$ such that $x'\widehat{\mathrm{Cov}}_k[\mathbf{P}_k]x = 0$, which in turn implies that $\widehat{\mathbb{E}}_k\big[(x'\mathbf{P}_k)^2 - \widehat{\mathbb{E}}_k[x'\mathbf{P}_k]^2\big] = 0$. Since the probability measure $\mathbb{P}$ and $\widehat{\mathbb{P}}$ are equivalent in the sense that they are absolute continuous with each other, we have $\mathbb{E}_k\big[(x'\mathbf{P}_k)^2 - \mathbb{E}_k[x'\mathbf{P}_k]^2\big] = 0$, which is a contradiction to $\mathrm{Cov}_k[\mathbf{P}_k] \succ 0$. Applying the Schur's complement theorem to it yields

$$1 - \widehat{\mathbb{E}}_k[\mathbf{P}'_k]\widehat{\mathbb{E}}_k^{-1}[\mathbf{P}_k\mathbf{P}'_k]\widehat{\mathbb{E}}_k[\mathbf{P}_k] > 0.$$

Since $\widehat{\mathbb{E}}_k[\mathbf{P}_k] = \mathbb{E}_k[Z\mathbf{P}_k]$, the above equality gives rise to

$$1 - \frac{1}{\mathbb{E}_k[\xi_{k+1}]}\mathbb{E}_k[\xi_{k+1}\mathbf{P}'_k]\mathbb{E}_k^{-1}[\xi_{k+1}\mathbf{P}_k\mathbf{P}'_k]\mathbb{E}_k[\xi_{k+1}\mathbf{P}_k] > 0.$$

Multiplying $\mathbb{E}_k[\xi_{k+1}] > 0$ on both sides of the above inequality yields

$$\xi_k = \mathbb{E}_k[\xi_{k+1}] - \mathbb{E}_k[\xi_{k+1}\mathbf{P}'_k]\mathbb{E}_k^{-1}[\xi_{k+1}\mathbf{P}_k\mathbf{P}'_k]\mathbb{E}_k[\xi_{k+1}\mathbf{P}_k] > 0. \tag{6.1}$$

Note that $\widehat{\mathrm{Cov}}_k[\mathbf{P}_k] \succ 0$ also implies $\mathbb{E}_k^{-1}[\xi_{k+1}\mathbf{P}_k\mathbf{P}'_k] \succ 0$. On the other hand, the induction assumption $\xi_{k+1} < 1$ implies that $\mathbb{E}_k[\xi_{k+1}] < 1$. Thus

$$\xi_k = \mathbb{E}_k[\xi_{k+1}] - \mathbb{E}_k[\xi_{k+1}\mathbf{P}'_k]\mathbb{E}_k^{-1}[\xi_{k+1}\mathbf{P}_k\mathbf{P}'_k]\mathbb{E}_k[\xi_{k+1}\mathbf{P}_k] < 1,$$

which together with inequality (6.1) derives $0 < \xi_k < 1$. This completes the proof.

Lemma 6.2. *If* $\mathbb{E}_k\left[\begin{pmatrix}\xi_{k+1}\mathbf{P}_k \\ \eta_{k+1}q_k\end{pmatrix}\begin{pmatrix}\xi_{k+1}\mathbf{P}'_k & \eta_{k+1}q_k\end{pmatrix}\right]$ *is positive definite for* $k = 0, 1, \cdots, T-1$, *then*

$$-\frac{\eta_k^2}{\xi_k} - \zeta_k + \epsilon_k \geqslant 0. \tag{6.2}$$

Proof. We prove it by mathematical induction. When $t = T-1$, it is obvious. Assume that the inequality holds at the stage $t = k+1$, i.e.

$$\eta_{k+1}^2 \leqslant (\epsilon_{k+1} - \zeta_{k+1})\xi_{k+1}. \tag{6.3}$$

We will show it is still true at stage k, i.e.

$$\eta_k^2 \leqslant (\epsilon_k - \zeta_k)\xi_k. \tag{6.4}$$

Let $L_k = \begin{pmatrix} \mathbf{P}_k \\ 1 \end{pmatrix}$ and $Q_k = \begin{pmatrix} \xi_{k+1}\mathbf{P}_k \\ \eta_{k+1}q_k \end{pmatrix}$, then

$$\begin{pmatrix} \mathbb{E}_k[\xi_{k+1}\mathbf{P}_k\mathbf{P}_k'] & \mathbb{E}_k[\xi_{k+1}\mathbf{P}_k] \\ \mathbb{E}_k[\xi_{k+1}\mathbf{P}_k'] & \mathbb{E}_k[\xi_{k+1}] \end{pmatrix} = \mathbb{E}_k\left[\begin{pmatrix} \xi_{k+1}\mathbf{P}_k \\ \xi_{k+1} \end{pmatrix}\begin{pmatrix} \mathbf{P}_k' & 1 \end{pmatrix}\right] = \mathbb{E}_k[\xi_{k+1}L_kL_k'], \tag{6.5}$$

$$\begin{pmatrix} \mathbb{E}_k[\xi_{k+1}\mathbf{P}_k\mathbf{P}_k'] & \mathbb{E}_k[\eta_{k+1}q_k\mathbf{P}_k] \\ \mathbb{E}_k[\eta_{k+1}q_k\mathbf{P}_k'] & \mathbb{E}_k\left[\frac{\eta_{k+1}^2}{\xi_{k+1}}q_k^2\right] \end{pmatrix} = \mathbb{E}\left[\begin{pmatrix} \xi_{k+1}\mathbf{P}_k \\ \eta_{k+1}q_k \end{pmatrix}\begin{pmatrix} \mathbf{P}_k' & \frac{\eta_{k+1}}{\xi_{k+1}}q_k \end{pmatrix}\right] = \mathbb{E}_k\left[Q_k\frac{Q_k'}{\xi_{k+1}}\right], \tag{6.6}$$

$$\begin{pmatrix} \mathbb{E}_k[\xi_{k+1}\mathbf{P}_k\mathbf{P}_k'] & \mathbb{E}_k[\xi_{k+1}\mathbf{P}_k] \\ \mathbb{E}_k[\eta_{k+1}q_k\mathbf{P}_k'] & \mathbb{E}_k[\eta_{k+1}q_k] \end{pmatrix} = \mathbb{E}_k\left[\begin{pmatrix} \xi_{k+1}\mathbf{P}_k \\ \eta_{k+1}q_k \end{pmatrix}\begin{pmatrix} \mathbf{P}_k' & 1 \end{pmatrix}\right] = \mathbb{E}_k[Q_kL_k']. \tag{6.7}$$

By lemma 6.1 we have $\mathbb{E}_k[\xi_{k+1}\mathbf{P}_k\mathbf{P}_k'] \succ 0$. Taking determinant on both sides for (6.5)-(6.7) we get

$$|\mathbb{E}_k[\xi_{k+1}L_kL_k']| = (\mathbb{E}_k[\xi_{k+1}] - B_k)\,|\mathbb{E}_k[\xi_{k+1}\mathbf{P}_k\mathbf{P}_k']|\,, \tag{6.8}$$

$$\left|\mathbb{E}_k\left[Q_k\frac{Q_k'}{\xi_{k+1}}\right]\right| = \left(\mathbb{E}_k\left[\frac{\eta_{k+1}^2}{\xi_{k+1}}q_k^2\right] - \widetilde{B}_k\right)|\mathbb{E}_k[\xi_{k+1}\mathbf{P}_k\mathbf{P}_k']|\,, \tag{6.9}$$

$$|\mathbb{E}_k[Q_kL_k']| = \left(\mathbb{E}_k[\eta_{k+1}q_k] - \widehat{B}_k\right)|\mathbb{E}_k[\xi_{k+1}\mathbf{P}_k\mathbf{P}_k']|\,. \tag{6.10}$$

From lemma 6.2, we have $0 < \xi_{k+1} < 1$. By the assumption of $\mathbb{E}_k[Q_kQ_k'] \succ 0$, the inverse $\mathbb{E}_k^{-1}\left[Q_k\frac{Q_k'}{\xi_{k+1}}\right]$ of $\mathbb{E}_k\left[Q_k\frac{Q_k'}{\xi_{k+1}}\right]$ exists. Since

$$\mathbb{E}_k\left[\begin{pmatrix} \xi_{k+1}L_k \\ Q_k \end{pmatrix}\begin{pmatrix} L_k' & \frac{Q_k'}{\xi_{k+1}} \end{pmatrix}\right] = \begin{pmatrix} \mathbb{E}_k[\xi_{k+1}L_kL_k'] & \mathbb{E}_k[L_kQ_k'] \\ \mathbb{E}_k[Q_kL_k'] & \mathbb{E}_k\left[Q_k\frac{Q_k'}{\xi_{k+1}}\right] \end{pmatrix} \succcurlyeq 0, \tag{6.11}$$

it follows from Lemma 2.4 that

$$\mathbb{E}_k[\xi_{k+1}L_kL_k'] - \mathbb{E}_k[L_kQ_k']\mathbb{E}_k^{-1}\left[Q_k\frac{Q_k'}{\xi_{k+1}}\right]\mathbb{E}[Q_kL_k'] \succcurlyeq 0.$$

Consequently,

$$\mathbb{E}_k[\xi_{k+1} L_k L_k'] \geqslant \mathbb{E}_k[L_k Q_k'] \mathbb{E}_k^{-1}\left[Q_k \frac{Q_k'}{\xi_{k+1}}\right] \mathbb{E}_k[Q_k L_k']. \tag{6.12}$$

Then according to (6.12) and Lemma 2.3, it follows that

$$\begin{aligned} |\mathbb{E}_k[\xi_{k+1} L_k L_k']| &\geqslant \left|\mathbb{E}_k[L_k Q_k'] \mathbb{E}_k^{-1}\left[Q_k \frac{Q_k'}{\xi_{k+1}}\right] \mathbb{E}_k[Q_k L_k']\right| \\ &= |\mathbb{E}_k[L_k Q_k']| \left|\mathbb{E}_k^{-1}\left[Q_k \frac{Q_k'}{\xi_{k+1}}\right]\right| |\mathbb{E}_k[Q_k L_k']| \\ &= |\mathbb{E}_k[Q_k L_k']|^2 \left|\mathbb{E}_k\left[Q_k \frac{Q_k'}{\xi_{k+1}}\right]\right|^{-1}, \end{aligned} \tag{6.13}$$

i.e.

$$|\mathbb{E}_k[Q_k L_k']|^2 \leqslant \left|\mathbb{E}_k\left[Q_k \frac{Q_k'}{\xi_{k+1}}\right]\right| |\mathbb{E}_k[\xi_{k+1} L_k L_k']|. \tag{6.14}$$

By (6.8)-(6.10) and (6.14), we obtain

$$\left(\mathbb{E}_k[\eta_{k+1} q_k] - \widehat{B}_k\right)^2 \leqslant \left(\mathbb{E}_k\left[\frac{\eta_{k+1}^2}{\xi_{k+1}} q_k^2\right] - \widetilde{B}_k\right) (\mathbb{E}_k[\xi_{k+1}] - B_k).$$

Namely,

$$\eta_k^2 \leqslant \left(\mathbb{E}_k\left[\frac{\eta_{k+1}^2}{\xi_{k+1}} q_k^2\right] - \widetilde{B}_k\right) \xi_k.$$

In order to prove inequality (6.4), we just need to show

$$\mathbb{E}_k\left[\frac{\eta_{k+1}^2}{\xi_{k+1}} q_k^2\right] - \widetilde{B}_k \leqslant \epsilon_k - \zeta_k.$$

In fact, the assumption (6.3) is the same as

$$\frac{\eta_{k+1}^2}{\xi_{k+1}} \leqslant \epsilon_{k+1} - \zeta_{k+1},$$

then

$$\mathbb{E}_k\left[\frac{\eta_{k+1}^2}{\xi_{k+1}}q_k^2\right] \leqslant \mathbb{E}_k\big[(\epsilon_{k+1}-\zeta_{k+1})q_k^2\big],$$

thus

$$\mathbb{E}_k\left[\frac{\eta_{k+1}^2}{\xi_{k+1}}q_k^2\right] - \widetilde{B}_k \leqslant \mathbb{E}_k[\epsilon_{k+1}q_k^2] - \mathbb{E}_k[\zeta_{k+1}q_k^2] - \widetilde{B}_k,$$

which is

$$\eta_k^2 \leqslant (\epsilon_k - \zeta_k)\xi_k.$$

This completes the proof.

6.2 Optimal Strategy

Theorem 6.1. *Assume that the returns of assets and liability are correlated at every period and the returns in different time periods are correlated too. Then the optimal strategy of problem (3.1) is given by*

$$\pi_t^* = -\mathbb{E}_t^{-1}[\xi_{t+1}\mathbf{P}_t\mathbf{P}_t']\left(\mathbb{E}_t[\xi_{t+1}\mathbf{P}_t]s_t\left(x_t - \gamma^*\prod_{k=t}^{T-1}s_k^{-1}\right) - \left(\prod_{k=t+1}^{T-1}s_k^{-1}\right)\mathbb{E}_t[\eta_{t+1}q_t\mathbf{P}_t]l_t\right), \tag{6.15}$$

where

$$\gamma^* = \frac{\xi_0 x_0 \prod_{k=0}^{T-1} s_k - d - \eta_0 l_0}{\xi_0 - 1}. \tag{6.16}$$

And the efficient frontier is

$$\mathrm{Var}(x_T - l_T) = \frac{\xi_0}{1-\xi_0}\left(x_0\prod_{k=0}^{T-1}s_k - d - \frac{\eta_0 l_0}{\xi_0}\right)^2 + \left(\frac{-\eta_0^2}{\xi_0} - \zeta_0 + \epsilon_0\right)l_0^2. \tag{6.17}$$

Proof. We prove it by making use of the dynamic programming approach. For the information set $\mathcal{F}_t$, the cost-to-go functional of problem (3.8) at period t is

$$J_t(y_t, l_t) = \min_{\pi_t}\mathbb{E}\big[J_{t+1}(y_{t+1}, l_{t+1})\big|\mathcal{F}_t\big],$$

where the terminal condition $J_T(y_T, l_T) = y_T^2 - 2l_T y_T$.

Assume that the cost-to-go functional at period $t+1$ is the following expression

$$J_{t+1}(y_{t+1}, l_{t+1}) = \Bigg(\prod_{k=t+1}^{T-1} s_k^2\Bigg)\xi_{t+1}y_{t+1}^2 - 2\Bigg(\prod_{k=t+1}^{T-1} s_k\Bigg)\eta_{t+1}l_{t+1}y_{t+1} - \zeta_{t+1}l_{t+1}^2,$$

Then we shall prove that it still holds at time t. For the given information set $\mathcal{F}_t$, we have

$$\begin{aligned}
&\mathbb{E}\big[J_{t+1}(y_{t+1}, l_{t+1})\big|\mathcal{F}_t\big] \\
&= \mathbb{E}\Bigg[\Bigg(\prod_{k=t+1}^{T-1} s_k^2\Bigg)\xi_{t+1}y_{t+1}^2 - 2\Bigg(\prod_{k=t+1}^{T-1} s_k\Bigg)\eta_{t+1}l_{t+1}y_{t+1} - \zeta_{t+1}l_{t+1}^2\Bigg|\mathcal{F}_t\Bigg] \\
&= \mathbb{E}\Bigg[\Bigg(\prod_{k=t+1}^{T-1} s_k^2\Bigg)\xi_{t+1}\big(s_t^2y_t^2 + 2s_ty_t\mathbf{P}'_t\pi_t + \pi'_t\mathbf{P}_t\mathbf{P}'_t\pi_t\big) \\
&\quad - 2\Bigg(\prod_{k=t+1}^{T-1} s_k\Bigg)\eta_{t+1}\big(q_ts_tl_ty_t + q_t\mathbf{P}'_tl_t\pi_t\big) - \zeta_{t+1}q_t^2l_t^2\Bigg|\mathcal{F}_t\Bigg] \\
&= \Bigg(\prod_{k=t+1}^{T-1} s_k^2\Bigg)\mathbb{E}_t[\xi_{t+1}]s_t^2y_t^2 + 2\Bigg(\prod_{k=t+1}^{T-1} s_k^2\Bigg)\mathbb{E}_t[\xi_{t+1}\mathbf{P}'_t]s_ty_t\pi_t \\
&\quad + \pi'_t\Bigg(\prod_{k=t+1}^{T-1} s_k^2\Bigg)\mathbb{E}_t[\xi_{t+1}\mathbf{P}_t\mathbf{P}'_t]\pi_t - 2\Bigg(\prod_{k=t+1}^{T-1} s_k\Bigg)\mathbb{E}_t[\eta_{t+1}q_t]s_tl_ty_t \\
&\quad - 2\Bigg(\prod_{k=t+1}^{T-1} s_k\Bigg)\mathbb{E}_t[\eta_{t+1}q_t\mathbf{P}'_t]l_t\pi_t - \mathbb{E}_t[\zeta_{t+1}q_t^2]l_t^2.
\end{aligned}$$

Minimizing the above functional with respect to π_t, we get the optimal strategy decision at time t as follows

$$\pi_t^* = -\mathbb{E}_t^{-1}[\xi_{t+1}\mathbf{P}_t\mathbf{P}'_t]\Bigg(\mathbb{E}_t[\xi_{t+1}\mathbf{P}_t]s_ty_t - \Bigg(\prod_{k=t+1}^{T-1} s_k^{-1}\Bigg)\mathbb{E}_t[\eta_{t+1}q_t\mathbf{P}_t]l_t\Bigg).$$

Substituting it to $\mathbb{E}\big[J_{t+1}(y_{t+1}, l_{t+1})\big|\mathcal{F}_t\big]$ yields

$$J_t(y_t, l_t) = \min_{\pi_t} \mathbb{E}\big[J_{t+1}(y_{t+1}, l_{t+1})\big|\mathcal{F}_t\big]$$

$$= \bigg(\prod_{k=t+1}^{T-1} s_k^2\bigg)\mathbb{E}_t[\xi_{t+1}]s_t^2 y_t^2 - 2\bigg(\prod_{k=t+1}^{T-1} s_k\bigg)\mathbb{E}_t[\eta_{t+1}q_t]s_t l_t y_t - \mathbb{E}_t[\zeta_{t+1}q_t^2]l_t^2$$

$$- \mathbb{E}_t[\xi_{t+1}\mathbf{P}_t']\mathbb{E}_t^{-1}[\xi_{t+1}\mathbf{P}_t\mathbf{P}_t']\mathbb{E}_t[\xi_{t+1}\mathbf{P}_t]s_t^2 y_t^2$$

$$+ 2\mathbb{E}_t[\xi_{t+1}\mathbf{P}_t']\mathbb{E}_t^{-1}[\xi_{t+1}\mathbf{P}_t\mathbf{P}_t']\mathbb{E}_t[\eta_{t+1}q_t\mathbf{P}_t]\bigg(\prod_{k=t+1}^{T-1} s_k^2\bigg)\bigg(\prod_{k=t+1}^{T-1} s_k^{-1}\bigg)s_t l_t y_t$$

$$- \mathbb{E}_t[\eta_{t+1}q_t\mathbf{P}_t']\mathbb{E}_t^{-1}[\xi_{t+1}\mathbf{P}_t\mathbf{P}_t']\mathbb{E}_t[\eta_{t+1}q_t\mathbf{P}_t]l_t^2$$

$$= \bigg(\prod_{k=t}^{T-1} s_k^2\bigg)\big(\mathbb{E}_t[\xi_{t+1}] - B_t\big)y_t^2 - 2\bigg(\prod_{k=t}^{T-1} s_k\bigg)\big(\mathbb{E}_t[q_t\eta_{t+1}] - \widehat{B}_t\big)l_t y_t - \big(\mathbb{E}_t[q_t^2\zeta_{t+1}] + \widetilde{B}_t\big)l_t^2$$

$$= \bigg(\prod_{k=t}^{T-1} s_k^2\bigg)\xi_t y_t^2 - 2\bigg(\prod_{k=t}^{T-1} s_k\bigg)\eta_t l_t y_t - \zeta_t l_t^2.$$

To derive the expression (6.16) of γ, we first consider the value of the optimal objective function in (3.8). In fact,

$$\mathbb{E}\big[y_T^2 - 2l_T y_T\big] = \mathbb{E}\big[y_T^2 - 2l_T y_T\big|\mathcal{F}_0\big] = J_0(y_0, l_0)$$

$$= \bigg(\prod_{k=0}^{T-1} s_k^2\bigg)\xi_0 y_0^2 - 2\bigg(\prod_{k=0}^{T-1} s_k\bigg)\eta_0 l_0 y_0 - \zeta_0 l_0^2.$$

Then

$$\mathrm{Var}(x_T - l_T) = \mathbb{E}[y_T^2 - 2l_T y_T] + \mathbb{E}[l_T^2] - \omega^2$$

$$= \bigg(\prod_{k=0}^{T-1} s_k^2\bigg)\xi_0 y_0^2 - 2\bigg(\prod_{k=0}^{T-1} s_k\bigg)\eta_0 l_0 y_0 - \zeta_0 l_0^2 + \epsilon_0 l_0^2 - \omega^2$$

$$= \bigg(\prod_{k=0}^{T-1} s_k^2\bigg)\xi_0\bigg(x_0 - (d+\omega)\prod_{k=0}^{T-1} s_k^{-1}\bigg)^2 - 2\bigg(\prod_{k=0}^{T-1} s_k\bigg)\eta_0 l_0\bigg(x_0 - (d+\omega)\prod_{k=0}^{T-1} s_k^{-1}\bigg)$$

$$- \zeta_0 l_0^2 + \epsilon_0 l_0^2 - \omega^2$$

$$
\begin{aligned}
&= (\xi_0 - 1)\omega^2 - 2\xi_0\left(x_0\prod_{k=0}^{T-1} s_k - d\right)\omega + \xi_0\left(x_0\prod_{k=0}^{T-1} s_k - d\right)^2 \\
&\quad + 2\eta_0 l_0\omega - 2\eta_0 l_0\left(x_0\prod_{k=0}^{T-1} s_k - d\right) - \zeta_0 l_0^2 + \epsilon_0 l_0^2 \\
&= (\xi_0 - 1)\left(\omega - \frac{\xi_0\big(x_0\prod_{k=0}^{T-1} s_k - d\big) - \eta_0 l_0\big)}{\xi_0 - 1}\right)^2 \\
&\quad - \frac{\xi_0^2\big(x_0\prod_{k=0}^{T-1} s_k - d\big)^2 - 2\xi_0\eta_0 l_0\big(x_0\prod_{k=0}^{T-1} s_k - d\big) + \eta_0^2 l_0^2}{\xi_0 - 1} \\
&\quad + \xi_0\left(x_0\prod_{k=0}^{T-1} s_k - d\right)^2 - 2\eta_0 l_0\left(x_0\prod_{k=0}^{T-1} s_k - d\right) - \zeta_0 l_0^2 + \epsilon_0 l_0^2 \\
&= (\xi_0 - 1)\left(\omega - \frac{\xi_0\big(x_0\prod_{k=0}^{T-1} s_k - d\big) - \eta_0 l_0\big)}{\xi_0 - 1}\right)^2 \\
&\quad + \frac{\xi_0}{1-\xi_0}\left(x_0\prod_{k=0}^{T-1} s_k - d\right)^2 - \frac{2\eta_0 l_0}{1-\xi_0}\left(x_0\prod_{k=0}^{T-1} s_k - d\right) - \zeta_0 l_0^2 + \epsilon_0 l_0^2 + \frac{\eta_0^2}{1-\xi_0} l_0^2 \\
&= (\xi_0 - 1)\left(\omega - \frac{\xi_0\big(x_0\prod_{k=0}^{T-1} s_k - d\big) + \eta_0 l_0}{\xi_0 - 1}\right)^2 \\
&\quad + \frac{\xi_0}{1-\xi_0}\left(x_0\prod_{k=0}^{T-1} s_k - d - \frac{\eta_0 l_0}{\xi_0}\right)^2 + \big(\frac{-\eta_0^2}{\xi_0} - \zeta_0 + \epsilon_0\big) l_0^2.
\end{aligned}
$$

By lemma 6.1, we have

$$0 < \xi_0 < 1,$$

the variance $\mathrm{Var}(x_T - l_T)$ is concave in ω. Similar to Chapter 3, we can drive

$$\omega^* = \frac{\xi_0\big(x_0\prod_{k=0}^{T-1} s_k - d\big) - \eta_0 l_0}{\xi_0 - 1}.$$

and the expression of $\mathrm{Var}(x_T - l_T)$ (6.17).

Remark 6.1. *If the returns e_t and q_t are independent among different periods, the conditional expectations degenerates to the unconditional expectation. In particular, we have* $\mathbb{E}_t[\xi_{t+1}\mathbf{P}_t] = \xi_{t+1}\mathbb{E}[\mathbf{P}_t]$, $\mathbb{E}_t[\xi_{t+1}\mathbf{P}_t\mathbf{P}_t'] = \xi_{t+1}\mathbb{E}[\mathbf{P}_t\mathbf{P}_t']$, $\mathbb{E}_t[\eta_{t+1}q_t] = \eta_{t+1}\mathbb{E}[q_t]$, $\mathbb{E}_t[\eta_{t+1}q_t\mathbf{P}_t] = \eta_{t+1}\mathbb{E}[q_t\mathbf{P}_t]$, $\mathbb{E}_t[\zeta_{t+1}q_t^2] = \zeta_{t+1}\mathbb{E}[q_t^2]$, $\mathbb{E}_t[\epsilon_{t+1}q_t^2] = \epsilon_{t+1}\mathbb{E}[q_t^2]$. *Then Theorem 6.1 reduces to Theorem 3.4.*

6.3 An Example

In this section, we use a simple example to illustrate the computational procedure. We consider a 2 periods investment case with one riskless asset, one risky asset and one liability.

Let

$$\begin{cases} \mathbf{P}_0 = \alpha_0, & \mathbf{P}_1 = \beta\mathbf{P}_0 + \alpha_1, \\ q_0 = \bar{\alpha}_0, & q_1 = \bar{\beta}q_0 + \bar{\alpha}_1. \end{cases}$$

To be simple, we assume further α_t and $\bar{\alpha}_t$ are independent, for $t = 0, 1$. Obviously, the excess return of the asset $\mathbf{P}_1$ and the return of the liability l_1 follow AR(1) models at period $t = 1$. Since this is a two-period model, we have

$$\xi_2 = 1, \quad \eta_2 = 1, \quad \zeta_2 = 0, \quad \epsilon_2 = 1.$$

Before deriving the strategy at period time $t = 1$, we estimate the following parameters:

$$\begin{aligned}
\mathbb{E}_1[\xi_2\mathbf{P}_1] &= \mathbb{E}_1[\beta\mathbf{P}_0 + \alpha_1] = \beta\mathbf{P}_0 + \mathbb{E}[\alpha_1], \\
\mathbb{E}_1[\xi_2\mathbf{P}_1^2] &= \mathbb{E}_1[(\beta\mathbf{P}_0 + \alpha_1)^2] = \beta^2\mathbf{P}_0^2 + 2\beta\mathbf{P}_0\mathbb{E}[\alpha_1] + \mathbb{E}[\alpha_1^2], \\
\mathbb{E}_1[\eta_2 q_1\mathbf{P}_1] &= \mathbb{E}_1[(\bar{\beta}q_0 + \bar{\alpha}_1)(\beta\mathbf{P}_0 + \alpha_1)] = (\bar{\beta}q_0 + \mathbb{E}[\bar{\alpha}_1])(\beta\mathbf{P}_0 + \mathbb{E}[\alpha_1]), \\
\mathbb{E}_1[\eta_2 q_1] &= \mathbb{E}_1[\bar{\beta}q_0 + \bar{\alpha}_1] = \bar{\beta}q_0 + \mathbb{E}[\bar{\alpha}_1],
\end{aligned}$$

$$B_1 = \frac{(\mathbb{E}_1[\xi_2 \mathbf{P}_1])^2}{\mathbb{E}_1[\xi_2 \mathbf{P}_1^2]} = \frac{(\beta \mathbf{P}_0 + \mathbb{E}[\alpha_1])^2}{\beta^2 \mathbf{P}_0^2 + 2\beta \mathbf{P}_0 \mathbb{E}[\alpha_1] + \mathbb{E}[\alpha_1^2]},$$

$$\begin{aligned}
\widehat{B}_1 &= \frac{\mathbb{E}_1[\xi_2 \mathbf{P}_1]\mathbb{E}_1[\eta_2 q_1 \mathbf{P}_1]}{\mathbb{E}_1[\xi_2 \mathbf{P}_1^2]} \\
&= \frac{(\beta \mathbf{P}_0 + \mathbb{E}[\alpha_1])^2(\bar{\beta} q_0 + \mathbb{E}[\bar{\alpha}_1])}{\beta^2 \mathbf{P}_0^2 + 2\beta \mathbf{P}_0 \mathbb{E}[\alpha_1] + \mathbb{E}[\alpha_1^2]},
\end{aligned}$$

$$\widetilde{B}_1 = \frac{(\mathbb{E}_1[\eta_2 q_1 \mathbf{P}_1])^2}{\mathbb{E}_1[\xi_2 \mathbf{P}_1^2]} = \frac{(\bar{\beta} q_0 + \mathbb{E}[\bar{\alpha}_1])^2(\beta \mathbf{P}_0 + \mathbb{E}[\alpha_1])^2}{\beta^2 \mathbf{P}_0^2 + 2\beta \mathbf{P}_0 \mathbb{E}[\alpha_1] + \mathbb{E}[\alpha_1^2]},$$

$$\xi_1 = \mathbb{E}_1[\xi_2] - B_1 = 1 - B_1 = \frac{\mathbb{E}[\alpha_1^2] - (\mathbb{E}[\alpha_1])^2}{\beta^2 \mathbf{P}_0^2 + 2\beta \mathbf{P}_0 \mathbb{E}[\alpha_1] + \mathbb{E}[\alpha_1^2]},$$

$$\begin{aligned}
\eta_1 &= \mathbb{E}_1[\eta_2 q_1] - \widehat{B}_1 \\
&= \bar{\beta} q_0 + \mathbb{E}[\bar{\alpha}_1] - \frac{(\beta \mathbf{P}_0 + \mathbb{E}[\alpha_1])(\bar{\beta} q_0 + \mathbb{E}[\bar{\alpha}_1])}{\beta^2 \mathbf{P}_0^2 + 2\beta \mathbf{P}_0 \mathbb{E}[\alpha_1] + \mathbb{E}[\alpha_1^2]} \\
&= \frac{(\bar{\beta} q_0 + \mathbb{E}[\bar{\alpha}_1])\big(\mathbb{E}[\alpha_1^2] - (\mathbb{E}[\alpha_1])^2\big)}{\beta^2 \mathbf{P}_0^2 + 2\beta \mathbf{P}_0 \mathbb{E}[\alpha_1] + \mathbb{E}[\alpha_1^2]},
\end{aligned}$$

$$\zeta_1 = \mathbb{E}_1[\zeta_2 q_1^2] + \widetilde{B}_1 = \frac{(\bar{\beta} q_0 + \mathbb{E}[\bar{\alpha}_1])^2(\beta \mathbf{P}_0 + \mathbb{E}[\alpha_1])^2}{\beta^2 \mathbf{P}_0^2 + 2\beta \mathbf{P}_0 \mathbb{E}[\alpha_1] + \mathbb{E}[\alpha_1^2]},$$

$$\epsilon_1 = \mathbb{E}_1[\epsilon_2 q_1^2] = \mathbb{E}_1[(\bar{\beta} q_0 + \bar{\alpha}_1)^2] = \bar{\beta}^2 q_0^2 + 2\bar{\beta} q_0 \mathbb{E}[\bar{\alpha}_1] + \mathbb{E}[\bar{\alpha}_1^2].$$

Hence, using Theorem 6.1 yields

$$\begin{aligned}
\pi_1^* &= -\mathbb{E}_1^{-1}[\xi_2 \mathbf{P}_1^2]\left[\mathbb{E}_1[\xi_2 \mathbf{P}_1] s_1 \left(x_1 - \frac{\gamma^*}{s_1}\right) - \mathbb{E}_1[\eta_2 q_1 \mathbf{P}_1] l_1\right] \\
&= -\frac{1}{\beta^2 \mathbf{P}_0^2 + 2\beta \mathbf{P}_0 \mathbb{E}[\alpha_1] + \mathbb{E}[\alpha_1^2]}\Bigg[s_1\Big(\beta \mathbf{P}_0 + \mathbb{E}[\alpha_1]\Big)\left(x_1 - \frac{\gamma^*}{s_1}\right) \\
&\quad - \Big(\beta\bar{\beta} \mathbf{P}_0 q_0 + \bar{\beta} q_0 \mathbb{E}[\alpha_1] + \beta \mathbb{E}[\bar{\alpha}_1] \mathbf{P}_0 + \mathbb{E}[\alpha_1]\mathbb{E}[\bar{\alpha}_1]\Big) l_1\Bigg].
\end{aligned}$$

Simulating the deriving procedure of period time $t = 1$, we estimate the following parameters while time $t = 0$,

$$
\begin{aligned}
\mathbb{E}_0[\xi_1 \mathbf{P}_0] &= \mathbb{E}_0\bigg[\frac{\mathbb{E}[\alpha_1^2] - (\mathbb{E}[\alpha_1])^2}{\beta^2 \mathbf{P}_0^2 + 2\beta \mathbf{P}_0 \mathbb{E}[\alpha_1] + \mathbb{E}[\alpha_1^2]} \mathbf{P}_0\bigg] \\
&= \big(\mathbb{E}[\alpha_1^2] - (\mathbb{E}[\alpha_1])^2\big) \mathbb{E}\bigg[\frac{\alpha_0}{\beta^2 \alpha_0^2 + 2\beta \alpha_0 \mathbb{E}[\alpha_1] + \mathbb{E}[\alpha_1^2]}\bigg], \\
\mathbb{E}_0[\xi_1 \mathbf{P}_0^2] &= \mathbb{E}_0\bigg[\frac{\mathbb{E}[\alpha_1^2] - (\mathbb{E}[\alpha_1])^2}{\beta^2 \mathbf{P}_0^2 + 2\beta \mathbf{P}_0 \mathbb{E}[\alpha_1] + \mathbb{E}[\alpha_1^2]} \mathbf{P}_0^2\bigg] \\
&= \big(\mathbb{E}[\alpha_1^2] - (\mathbb{E}[\alpha_1])^2\big) \mathbb{E}\bigg[\frac{\alpha_0^2}{\beta^2 \alpha_0^2 + 2\beta \alpha_0 \mathbb{E}[\alpha_1] + \mathbb{E}[\alpha_1^2]}\bigg], \\
\mathbb{E}_0[\eta_1 q_0 \mathbf{P}_0] &= \mathbb{E}_0\bigg[\frac{(\bar{\beta} q_0 + \mathbb{E}[\bar{\alpha}_1])\big(\mathbb{E}[\alpha_1^2] - (\mathbb{E}[\alpha_1])^2\big)}{\beta^2 \mathbf{P}_0^2 + 2\beta \mathbf{P}_0 \mathbb{E}[\alpha_1] + \mathbb{E}[\alpha_1^2]} q_0 \mathbf{P}_0\bigg] \\
&= \big(\mathbb{E}[\alpha_1^2] - (\mathbb{E}[\alpha_1])^2\big) \mathbb{E}\bigg[\frac{(\bar{\beta} \alpha_0 \bar{\alpha}_0^2 + \alpha_0 \bar{\alpha}_0 \mathbb{E}[\bar{\alpha}_1])}{\beta^2 \alpha_0^2 + 2\beta \alpha_0 \mathbb{E}[\alpha_1] + \mathbb{E}[\alpha_1^2]}\bigg], \\
\mathbb{E}_0[\eta_1 q_0] &= \mathbb{E}_0\bigg[\frac{(\bar{\beta} q_0 + \mathbb{E}[\bar{\alpha}_1])\big(\mathbb{E}[\alpha_1^2] - (\mathbb{E}[\alpha_1])^2\big)}{\beta^2 \mathbf{P}_0^2 + 2\beta \mathbf{P}_0 \mathbb{E}[\alpha_1] + \mathbb{E}[\alpha_1^2]} q_0\bigg] \\
&= \big(\mathbb{E}[\alpha_1^2] - (\mathbb{E}[\alpha_1])^2\big) \mathbb{E}\bigg[\frac{\bar{\beta} \bar{\alpha}_0^2 + \mathbb{E}[\bar{\alpha}_1] \bar{\alpha}_0}{\beta^2 \alpha_0^2 + 2\beta \alpha_0 \mathbb{E}[\alpha_1] + \mathbb{E}[\alpha_1^2]}\bigg], \\
B_0 &= \frac{(\mathbb{E}_0[\xi_1 \mathbf{P}_0])^2}{\mathbb{E}_0[\xi_1 \mathbf{P}_0^2]}, \\
\widehat{B}_0 &= \frac{\mathbb{E}_0[\xi_1 \mathbf{P}_0] \mathbb{E}_0[\eta_1 q_0 \mathbf{P}_0]}{\mathbb{E}_0[\xi_1 \mathbf{P}_0^2]}, \\
\widetilde{B}_0 &= \frac{(\mathbb{E}_0[\eta_1 q_0 \mathbf{P}_0])^2}{\mathbb{E}_0[\xi_1 \mathbf{P}_0^2]}, \\
\xi_0 &= \mathbb{E}_0[\xi_1] - B_0 \\
&= \mathbb{E}_0\bigg[\frac{\mathbb{E}[\alpha_1^2] - (\mathbb{E}[\alpha_1])^2}{\beta^2 \mathbf{P}_0^2 + 2\beta \mathbf{P}_0 \mathbb{E}[\alpha_1] + \mathbb{E}[\alpha_1^2]}\bigg] - B_0 \\
&= \big(\mathbb{E}[\alpha_1^2] - (\mathbb{E}[\alpha_1])^2\big) \mathbb{E}\bigg[\frac{1}{\beta^2 \alpha_0^2 + 2\beta \alpha_0 \mathbb{E}[\alpha_1] + \mathbb{E}[\alpha_1^2]}\bigg] - B_0,
\end{aligned}
$$

$$
\begin{aligned}
\eta_0 &= \mathbb{E}_0[\eta_1 q_0] - \widehat{B}_0 \\
&= \big(\mathbb{E}[\alpha_1^2] - (\mathbb{E}[\alpha_1])^2\big)\mathbb{E}\left[\frac{\bar{\beta}\bar{\alpha}_0^2 + \mathbb{E}[\bar{\alpha}_1]\bar{\alpha}_0}{\beta^2\alpha_0^2 + 2\beta\alpha_0\mathbb{E}[\alpha_1] + \mathbb{E}[\alpha_1^2]}\right] - \widehat{B}_0, \\
\zeta_0 &= \mathbb{E}_0[\zeta_1 q_0^2] + \widetilde{B}_0 = \mathbb{E}_0\left[\frac{(\bar{\beta}q_0 + \mathbb{E}[\bar{\alpha}_1])^2(\beta\mathbf{P}_0 + \mathbb{E}[\alpha_1])^2}{\beta^2\mathbf{P}_0^2 + 2\beta\mathbf{P}_0\mathbb{E}[\alpha_1] + \mathbb{E}[\alpha_1^2]} q_0^2\right] + \widetilde{B}_0 \\
&= \mathbb{E}\left[\frac{(\bar{\beta}q_0 + \mathbb{E}[\bar{\alpha}_1])^2(\beta\mathbf{P}_0 + \mathbb{E}[\alpha_1])^2}{\beta^2\alpha_0^2 + 2\beta\alpha_0\mathbb{E}[\alpha_1] + \mathbb{E}[\alpha_1^2]}\bar{\alpha}_0^2\right] + \widetilde{B}_0, \\
\epsilon_0 &= \mathbb{E}_0[\epsilon_1 q_0^2] = \mathbb{E}_0[(\bar{\beta}^2 q_0^2 + 2\bar{\beta}q_0\mathbb{E}[\bar{\alpha}_1] + \mathbb{E}[\bar{\alpha}_1^2])q_0^2] \\
&= \mathbb{E}[(\bar{\beta}^2\bar{\alpha}_0^2 + 2\bar{\beta}\bar{\alpha}_0\mathbb{E}[\bar{\alpha}_1] + \mathbb{E}[\bar{\alpha}_1^2])\bar{\alpha}_0^2].
\end{aligned}
$$

Hence, using Theorem 6.1 yields

$$
\pi_0^* = -\mathbb{E}_0^{-1}[\xi_1\mathbf{P}_0^2]\left[\mathbb{E}_0[\xi_1\mathbf{P}_0]s_0\left(x_0 - \frac{\gamma^*}{s_0 s_1}\right) - s_1^{-1}\mathbb{E}_0[\eta_1 q_0\mathbf{P}_0]l_0\right].
$$

In order to calculate the above parameters, we introduce the following lemma.

Lemma 6.3. *Let X and Y be two random variables for which the mean of functions of X and Y exists. Then*

$$
\begin{aligned}
\mathbb{E}[g(X,Y)] \approx g(\mathbb{E}[X],\mathbb{E}[Y]) &+ \frac{1}{2}\mathrm{Var}(X)\frac{\partial^2}{\partial x^2}g(x,y)\bigg|_{\mathbb{E}[X],\mathbb{E}[Y]} \\
&+ \frac{1}{2}\mathrm{Var}(Y)\frac{\partial^2}{\partial y^2}g(x,y)\bigg|_{\mathbb{E}[X],\mathbb{E}[Y]} + \mathrm{Cov}[X,Y]\frac{\partial^2}{\partial y\partial x}g(x,y)\bigg|_{\mathbb{E}[X],\mathbb{E}[Y]}.
\end{aligned}
$$

Let

$$
\begin{aligned}
g_1(X) &= \frac{X}{aX^2 + bX + c}, \\
g_2(X) &= \frac{X^2}{aX^2 + bX + c}, \\
g_3(X) &= \frac{1}{aX^2 + bX + c},
\end{aligned}
$$

$$g_4(Y) = (\bar{a}Y^2 + \bar{b}Y + \bar{c})Y^2,$$

$$g_5(X,Y) = \frac{mXY^2 + nXY}{aX^2 + bX + c},$$

$$g_6(X,Y) = \frac{mY^2 + nY}{aX^2 + bX + c},$$

$$g_7(X,Y) = \frac{(sX+t)^2(mY+n)^2Y^2}{aX^2 + bX + c},$$

and let $a = \beta^2, b = 2\beta\mathbb{E}[\alpha_1], c = \mathbb{E}[\alpha_1^2], \bar{a} = \bar{\beta}^2, \bar{b} = 2\bar{\beta}\mathbb{E}[\bar{\alpha}_1], \bar{c} = \mathbb{E}[\bar{\alpha}_1^2]$, then according to Lemma 6.3, we have

$$\begin{aligned}
\mathbb{E}[g_1(\alpha_0)] &= \mathbb{E}\left[\frac{\alpha_0}{\beta^2\alpha_0^2 + 2\beta\mathbb{E}[\alpha_1]\alpha_0 + \mathbb{E}[\alpha_1^2]}\right] \\
&\approx \frac{\mathbb{E}[\alpha_0]}{\beta^2\mathbb{E}[\alpha_0]^2 + 2\beta\mathbb{E}[\alpha_1]\mathbb{E}[\alpha_0] + \mathbb{E}[\alpha_1^2]} \\
&\quad + \frac{\mathrm{Var}(\alpha_0)}{2}\frac{2\beta^4\mathbb{E}[\alpha_0]^3 - 6\beta^2\mathbb{E}[\alpha_1^2]\mathbb{E}[\alpha_0] - 4\beta\mathbb{E}[\alpha_1]\mathbb{E}[\alpha_1^2]}{(\beta^2\mathbb{E}[\alpha_0]^2 + 2\beta\mathbb{E}[\alpha_1]\mathbb{E}[\alpha_0] + \mathbb{E}[\alpha_1^2])^3},
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}[g_2(\alpha_0)] &= \mathbb{E}\left[\frac{\alpha_0^2}{\beta^2\alpha_0^2 + 2\beta\mathbb{E}[\alpha_1]\alpha_0 + \mathbb{E}[\alpha_1^2]}\right] \\
&\approx \frac{\mathbb{E}[\alpha_0]^2}{\beta^2\mathbb{E}[\alpha_0]^2 + 2\beta\mathbb{E}[\alpha_1]\mathbb{E}[\alpha_0] + \mathbb{E}[\alpha_1^2]} \\
&\quad + \frac{\mathrm{Var}(\alpha_0)}{2}\frac{-4\beta^3\mathbb{E}[\alpha_1]\mathbb{E}[\alpha_0]^3 - 6\beta^2\mathbb{E}[\alpha_1^2]\mathbb{E}[\alpha_0]^2 + 2\mathbb{E}[\alpha_1^2]^2}{(\beta^2\mathbb{E}[\alpha_0]^2 + 2\beta\mathbb{E}[\alpha_1]\mathbb{E}[\alpha_0] + \mathbb{E}[\alpha_1^2])^3},
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}[g_3(\alpha_0)] &= \mathbb{E}\left[\frac{1}{\beta^2\alpha_0^2 + 2\beta\mathbb{E}[\alpha_1]\alpha_0 + \mathbb{E}[\alpha_1^2]}\right] \\
&\approx \frac{1}{\beta^2\mathbb{E}[\alpha_0]^2 + 2\beta\mathbb{E}[\alpha_1]\mathbb{E}[\alpha_0] + \mathbb{E}[\alpha_1^2]} \\
&\quad + \frac{\mathrm{Var}(\alpha_0)}{2}\frac{6\beta^4\mathbb{E}[\alpha_0]^2 + 12\beta^3\mathbb{E}[\alpha_1]\mathbb{E}[\alpha_0] - 2\beta^2\mathbb{E}[\alpha_1^2] + 4\beta^2\mathbb{E}[\alpha_1]^2}{(\beta^2\mathbb{E}[\alpha_0]^2 + 2\beta\mathbb{E}[\alpha_1]\mathbb{E}[\alpha_0] + \mathbb{E}[\alpha_1^2])^3},
\end{aligned}$$

$$
\begin{aligned}
\mathbb{E}[g_4(\bar{\alpha}_0)] &= \mathbb{E}[(\bar{\beta}^2\bar{\alpha}_0^2 + 2\bar{\beta}\mathbb{E}[\bar{\alpha}_1]\bar{\alpha}_0 + \mathbb{E}[\bar{\alpha}_1^2])\bar{\alpha}_0^2] \\
&\approx (\bar{\beta}^2\mathbb{E}[\bar{\alpha}_0]^2 + 2\bar{\beta}\mathbb{E}[\bar{\alpha}_1]\mathbb{E}[\bar{\alpha}_0] + \mathbb{E}[\bar{\alpha}_1^2])\mathbb{E}[\bar{\alpha}_0]^2 \\
&\quad + \frac{\mathrm{Var}(\bar{\alpha}_0)}{2} \cdot \big(12\bar{\beta}^2\mathbb{E}[\bar{\alpha}_0]^2 + 12\bar{\beta}\mathbb{E}[\bar{\alpha}_1]\mathbb{E}[\bar{\alpha}_0] + 2\mathbb{E}[\bar{\alpha}_1^2]\big),
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}[g_5(\alpha_0, \bar{\alpha}_0)] &= \mathbb{E}\left[\frac{(\bar{\beta}\alpha_0\bar{\alpha}_0^2 + \alpha_0\bar{\alpha}_0\mathbb{E}[\bar{\alpha}_1])}{\beta^2\alpha_0^2 + 2\beta\alpha_0\mathbb{E}[\alpha_1] + \mathbb{E}[\alpha_1^2]}\right] \\
&\approx \frac{(\bar{\beta}\mathbb{E}[\alpha_0]\mathbb{E}[\bar{\alpha}_0]^2 + \mathbb{E}[\alpha_0]\mathbb{E}[\bar{\alpha}_0]\mathbb{E}[\bar{\alpha}_1])}{\beta^2\mathbb{E}[\alpha_0]^2 + 2\beta\mathbb{E}[\alpha_0]\mathbb{E}[\alpha_1] + \mathbb{E}[\alpha_1^2]} \\
&\quad + \frac{\mathrm{Var}(\alpha_0)}{2}\left(- \frac{(\bar{\beta}\mathbb{E}[\bar{\alpha}_0]^2 + \mathbb{E}[\bar{\alpha}_1]\mathbb{E}[\bar{\alpha}_0])}{(\beta^2\mathbb{E}[\alpha_0]^2 + 2\beta\mathbb{E}[\alpha_1]\mathbb{E}[\alpha_0] + \mathbb{E}[\alpha_1^2])^3}\right) \\
&\quad \cdot \big(- 2\beta^4\mathbb{E}[\alpha_0]^3 + 6\beta^2\mathbb{E}[\alpha_1^2]\mathbb{E}[\alpha_0] + 4\beta\mathbb{E}[\alpha_1]\mathbb{E}[\alpha_1^2]\big) \\
&\quad + \frac{\mathrm{Var}(\bar{\alpha}_0)}{2}\frac{2\bar{\beta}\mathbb{E}[\alpha_0]}{\beta^2\mathbb{E}[\alpha_0]^2 + 2\beta\mathbb{E}[\alpha_1]\mathbb{E}[\alpha_0] + \mathbb{E}[\alpha_1^2]}
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}[g_6(\alpha_0, \bar{\alpha}_0)] &= \mathbb{E}\left[\frac{\bar{\beta}\bar{\alpha}_0^2 + \mathbb{E}[\bar{\alpha}_1]\bar{\alpha}_0}{,}\beta^2\alpha_0^2 + 2\beta\alpha_0\mathbb{E}[\alpha_1] + \mathbb{E}[\alpha_1^2]\right] \\
&\approx \frac{\bar{\beta}\mathbb{E}[\bar{\alpha}_0]^2 + \mathbb{E}[\bar{\alpha}_1]\mathbb{E}[\bar{\alpha}_0]}{\beta^2\mathbb{E}[\alpha_0]^2 + 2\beta\mathbb{E}[\alpha_0]\mathbb{E}[\alpha_1] + \mathbb{E}[\alpha_1^2]} \\
&\quad + \frac{\mathrm{Var}(\alpha_0)}{2}\left(\frac{(\bar{\beta}\mathbb{E}[\bar{\alpha}_0]^2 + \mathbb{E}[\bar{\alpha}_1]\mathbb{E}[\bar{\alpha}_0])}{(\beta^2\mathbb{E}[\alpha_0]^2 + 2\beta\mathbb{E}[\alpha_1]\mathbb{E}[\alpha_0] + \mathbb{E}[\alpha_1^2])^3}\right) \\
&\quad \cdot (6\beta^4\mathbb{E}[\alpha_0]^2 + 12\beta^3\mathbb{E}[\alpha_1]\mathbb{E}[\alpha_0] + 4\beta^2\mathbb{E}[\alpha_1]^2 - 2\beta^2\mathbb{E}[\alpha_1^2]) \\
&\quad + \frac{\mathrm{Var}(\bar{\alpha}_0)}{2}\left(\frac{2\bar{\beta}}{\beta^2\mathbb{E}[\alpha_0]^2 + 2\beta\mathbb{E}[\alpha_1]\mathbb{E}[\alpha_0] + \mathbb{E}[\alpha_1^2]}\right),
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}[g_7(\alpha_0, \bar{\alpha}_0)] &= \mathbb{E}\left[\frac{(\beta\alpha_0 + \mathbb{E}[\alpha_1])^2(\bar{\beta}\bar{\alpha}_0 + \mathbb{E}[\bar{\alpha}_1])^2}{\beta^2\alpha_0^2 + 2\beta\alpha_0\mathbb{E}[\alpha_1] + \mathbb{E}[\alpha_1^2]}\bar{\alpha}_0^2\right] \\
&\approx \frac{(\beta\mathbb{E}[\alpha_0] + \mathbb{E}[\alpha_1])^2(\bar{\beta}\mathbb{E}[\bar{\alpha}_0] + \mathbb{E}[\bar{\alpha}_1])^2\mathbb{E}[\bar{\alpha}_0]^2}{\beta^2\mathbb{E}[\alpha_0]^2 + 2\beta\mathbb{E}[\alpha_0]\mathbb{E}[\alpha_1] + \mathbb{E}[\alpha_1^2]} \\
&\quad + \frac{\mathrm{Var}(\alpha_0)}{2}\frac{2(\bar{\beta}\mathbb{E}[\bar{\alpha}_0] + \mathbb{E}[\bar{\alpha}_1])^2\mathbb{E}[\bar{\alpha}_0]^2}{(\beta^2\mathbb{E}[\alpha_0]^2 + 2\beta\mathbb{E}[\alpha_1]\mathbb{E}[\alpha_0] + \mathbb{E}[\alpha_1^2])^3} \\
&\quad \cdot \Big(\mathbb{E}[\alpha_1^2]^2\beta^2 + 4\beta^2\mathbb{E}[\alpha_1]^2\mathbb{E}[\alpha_1]^2 + \beta^4\mathbb{E}[\alpha_1]\mathbb{E}[\alpha_0]^2(3\mathbb{E}[\alpha_1] + 2\beta\mathbb{E}[\alpha_0]) \\
&\quad + \beta^3\mathbb{E}[\alpha_1]\mathbb{E}[\alpha_0](3\mathbb{E}[\alpha_1]^2 - \beta^2\mathbb{E}[\alpha_0]^2)\mathbb{E}[\alpha_1^2]\big(2\beta^2\mathbb{E}[\alpha_1]\mathbb{E}[\alpha_1] \\
&\quad + \beta^2(\mathbb{E}[\alpha_1]^2 + 6\beta\mathbb{E}[\alpha_1]\mathbb{E}[\alpha_0] + 3\beta^2\mathbb{E}[\alpha_0]^2)\big)\Big) \\
&\quad + \frac{\mathrm{Var}(\bar{\alpha}_0)}{2}\frac{(\beta\mathbb{E}[\alpha_0] + \mathbb{E}[\alpha_1])^2}{\beta^2\mathbb{E}[\alpha_0]^2 + 2\beta\mathbb{E}[\alpha_0]\mathbb{E}[\alpha_1] + \mathbb{E}[\alpha_1^2]} \\
&\quad \big(12\bar{\beta}^2\mathbb{E}[\bar{\alpha}_0]^2 + 12\bar{\beta}\mathbb{E}[\bar{\alpha}_1]\mathbb{E}[\bar{\alpha}_0] + 2\mathbb{E}[\bar{\alpha}_1]^2\big).
\end{aligned}
$$

Suppose that the correlation parameters are $\beta = 0.2, \bar{\beta} = 0.9$. We still use the same data as Example 3.1. This time we just consider one asset, the S&P 500. That is to say, the return of riskless asset is $s_t = 1.05$, the first and second moment of the disturbance variables are $\mathbb{E}[\alpha_t] = 0.09, \mathbb{E}[\bar{\alpha}_t] = 1.1, \mathbb{E}[\alpha_t^2] = 0.0423$, and $\mathbb{E}[\bar{\alpha}_t^2] = 1.25$, for $t = 0, 1, 2$. The initial wealth and liability are $x_0 = 3$ and $l_0 = 1$ respectively, and $d = 3.5$. Then we have

$$
\begin{aligned}
\mathbb{E}_0[\xi_1\mathbf{P}_0] &= \big(\mathbb{E}[\alpha_1^2] - (\mathbb{E}[\alpha_1])^2\big)\mathbb{E}[g_1(\alpha_0)] = 0.0431, \\
\mathbb{E}_0[\xi_1\mathbf{P}_0^2] &= \big(\mathbb{E}[\alpha_1^2] - (\mathbb{E}[\alpha_1])^2\big)\mathbb{E}[g_2(\alpha_0)] = 0.0272, \\
\mathbb{E}_0[\eta_1 q_0\mathbf{P}_0] &= \big(\mathbb{E}[\alpha_1^2] - (\mathbb{E}[\alpha_1])^2\big)\mathbb{E}[g_5(\alpha_0, \bar{\alpha}_0)] = 0.0576,
\end{aligned}
$$

$$\xi_0 = \big(\mathbb{E}[\alpha_1^2] - (\mathbb{E}[\alpha_1])^2\big)\left(\mathbb{E}[g_3(\alpha_0)] - \frac{\mathbb{E}[g_1]^2}{\mathbb{E}[g_2(\alpha_0)]}\right) = 0.6777,$$

$$\eta_0 = \big(\mathbb{E}[\alpha_1^2] - (\mathbb{E}[\alpha_1])^2\big)\left(\mathbb{E}[g_6(\alpha_0,\bar{\alpha}_0)] - \frac{\mathbb{E}[g_1(\alpha_0)]\mathbb{E}[g_5(\alpha_0,\bar{\alpha}_0)]}{\mathbb{E}[g_2(\alpha_0)]}\right) = 0.9043,$$

$$\zeta_0 = \mathbb{E}[g_7(\alpha_0,\bar{\alpha}_0)] - \big(\mathbb{E}[\alpha_1^2] - (\mathbb{E}[\alpha_1])^2\big)\frac{(\mathbb{E}[g_5(\alpha_0,\bar{\alpha}_0)])^2}{\mathbb{E}[g_2(\alpha_0)]} = 0.5893,$$

$$\epsilon_0 = \mathbb{E}[g_4(\bar{\alpha}_0)] = 1.8900.$$

By Theorem 6.1, the optimal strategies are given by

$$\pi_1^* = -\frac{\big(0.21\alpha_0 + 0.0945\big)\big(x_1 - 6.3913\big) - \big(0.02\alpha_0\bar{\alpha}_0 + 0.22\bar{\alpha}_0 + 0.009\bar{\alpha}_0 + 0.099\big)l_0}{0.04\alpha_0^2 + 0.036\alpha_0 + 0.0423},$$

$$\pi_0^* = -1.6634(x_0 - 6.08698) + 2.0162 l_0,$$

and $\mathrm{Var}(x_2 - l_2) = 4.9964$.

Remark 6.2. *If we consider a 2 periods investment in Example 3.1 and just consider the asset S&P, then the results are the same with the example here when we take* $\beta = \bar{\beta} = 0$. *This further prove that.*

Chapter 7

Conclusions and Future work

This chapter draws conclusions on the thesis, and points out some possible research directions related to the work done in this thesis.

7.1 Conclusions

The focus of the thesis has been placed on multi-period asset-liability mean-variance portfolio selection. It is a nonseparable dynamic programming problem since it cannot be decomposed by a stage-wise backward recursion. In this thesis, we first formulate the problem in deterministic terminal expectation and solved it by parameterized method. By introducing a Lagrangian multiplier and applying Lagrangian relaxation and state variable substitution, we turn it to a solvable stochastic control problem. Second, we put weights on the variance and the expectation to transform the bi-objective optimization problem to single-objective problem and tackle it using mean-field formulation. By these two methods, we derive the analytical optimal strategies and efficient frontiers of multi-period asset-liability mean-variance portfolio selection problems with various kinds of constraints, such as uncertain exit time, bankruptcy control, correlated returns. The relation of them are given and the effects of different constraints are illustrated by numerical examples. Our methods are showed to be much more efficient and accuracy compared with other methods in the literature.

7.2 Future Work

Related topics for the future research work are listed below.

1. This thesis suppose that there is only one deterministic market state. However, the underlying market environment is random and there are various market states in the real world. In recent years, regime-switching models have become popular for reflecting the various states of the financial market. In the future, using mean-field formulation to tackle mean-variance model with regime-switching is worthwhile and challenging.

2. Although asset-liability mean-variance portfolio selection is an important issue in modern finance theory, the time-consistent problem has not attract much attention. In the future work, seeking for time-consistent optimal strategy and efficient frontier for asset-liability management is indeed meaningful.

Bibliography

Albert, A. (1969), "Conditions for positive and nonnegative definiteness in terms of pseudoinverses," *SIAM Journal on Applied Mathematics*, 17, 434–440.

Balvers, R. J. and Mitchell, D. W. (1997), "Autocorrelated returns and optimal intertemporal portfolio choice," *Management Science*, 43, 1537–1551.

Bellman, R. (2010), *Dynamic programming*, Princeton University Press.

Bielecki, T. R., Jin, H., Pliska, S. R., and Zhou, X. Y. (2005), "Continuous-time mean-variance portfolio selection with bankruptcy prohibition," *Mathematical Finance*, 15, 213–244.

Černỳ, A. and Kallsen, J. (2009), "Hedging by sequential regressions revisited," *Mathematical Finance*, 19, 591–617.

Chen, P. and Yang, H. (2011), "Markowitz's mean-variance asset–liability management with regime switching: A multi-period model," *Applied Mathematical Finance*, 18, 29–50.

Chiu, M. C. and Li, D. (2006), "Asset and liability management under a continuous-time mean–variance optimization framework," *Insurance: Mathematics and Economics*, 39, 330–355.

Costa, O. and Nabholz, R. (2007), "Multiperiod mean-variance optimization with intertemporal restrictions," *Journal of Optimization Theory and Applications*, 134, 257–274.

Cui, X., Li, X., and Li, D. (2014), "Unified Framework of Mean-Field Formulations for Optimal Multi-period Mean-Variance Portfolio Selection," *Automatic Control, IEEE Transactions on*, 59, 1833–1844.

Czichowsky, C. (2013), "Time-consistent mean-variance portfolio selection in discrete and continuous time," *Finance and Stochastics*, 17, 227–271.

Elton, E. J., Gruber, M. J., Brown, S. J., and Goetzmann, W. N. (2009), *Modern portfolio theory and investment analysis*, John Wiley & Sons.

Fu, C., Lari-Lavassani, A., and Li, X. (2010), "Dynamic mean–variance portfolio selection with borrowing constraint," *European Journal of Operational Research*, 200, 312–319.

Gao, J. and Li, D. (2014), "Multiperiod Mean-Variance Portfolio Optimization with General Correlated Returns?" in *World Congress*, vol. 19, pp. 9007–9012.

Hakansson, N. H. (1969), "Optimal investment and consumption strategies under risk, an uncertain lifetime, and insurance," *International Economic Review*, 10, 443–466.

Krouse, C. G. (1970), "Portfolio balancing corporate assets and liabilities with special application to insurance management," *Journal of Financial and Quantitative Analysis*, 5, 77–104.

Leippold, M., Trojani, F., and Vanini, P. (2004), "A geometric approach to multiperiod mean variance optimization of assets and liabilities," *Journal of Economic Dynamics and Control*, 28, 1079–1113.

Li, D. and Ng, W.-L. (2000), "Optimal dynamic portfolio selection: Multiperiod mean-variance formulation," *Mathematical Finance*, 10, 387–406.

Li, X. and Zhou, X. Y. (2006), "Continuous-time mean-variance efficiency: the 80% rule," *The Annals of Applied Probability*, 16, 1751–1763.

Li, X., Zhou, X. Y., and Lim, A. E. (2002), "Dynamic mean-variance portfolio selection with no-shorting constraints," *SIAM Journal on Control and Optimization*, 40, 1540–1555.

Li, Z. and Xie, S. (2010), "Mean-variance portfolio optimization under Stochastic income and uncertain exit time," *Dynamics of Continuous, Discrete and Impulsive Systems B: Applications and Algorithms*, 17, 131–147.

Markowitz, H. (1952), "Portfolio selection," *The journal of finance*, 7, 77–91.

Merton, R. C. (1971), "Optimum consumption and portfolio rules in a continuous-time model," *Journal of economic theory*, 3, 373–413.

Merton, R. C. (1972), "An analytic derivation of the efficient portfolio frontier," *Journal of financial and quantitative analysis*, 7, 1851–1872.

Nourian, M., Caines, P. E., Malhame, R. P., and Huang, M. (2013), "Nash, social and centralized solutions to consensus problems via mean field control theory," *Automatic Control, IEEE Transactions on*, 58, 639–653.

Pang, W.-K., Ni, Y.-H., Li, X., and Yiu, K.-F. C. (2014), "Continuous-Time Mean-Variance Portfolio Selection with Partial Information," *Journal of Mathematical Finance*, 4, 353.

Schweizer, M. et al. (1996), "Approximation pricing and the variance-optimal martingale measure," *The Annals of Probability*, 24, 206–236.

Sharpe, W. F. and Tint, L. G. (1990), "Liabilities-a new approach," *The journal of portfolio management*, 16, 5–10.

Wei, S.-z. and Ye, Z.-x. (2007), "Multi-period optimization portfolio with bankruptcy control in stochastic market," *Applied Mathematics and Computation*, 186, 414–425.

Wu, H. and Li, Z. (2011), "Multi-period mean-variance portfolio selection with Markov regime switching and uncertain time-horizon," *Journal of Systems Science and Complexity*, 24, 140–155.

Wu, H. and Li, Z. (2012), "Multi-period mean–variance portfolio selection with regime switching and a stochastic cash flow," *Insurance: Mathematics and Economics*, 50, 371–384.

Wu, H. and Zeng, Y. (2013), "Multi-period mean-variance portfolio selection in a regime-switching market with a bankruptcy state," *Optimal Control Applications and Methods*, 34, 415–432.

Xia, J. and Yan, J.-A. (2006), "Markowitz's portfolio optimization in an incomplete market," *Mathematical Finance*, 16, 203–216.

Xie, S., Li, Z., and Wang, S. (2008), "Continuous-time portfolio selection with liability: Mean–variance model and stochastic LQ approach," *Insurance: Mathematics and Economics*, 42, 943–953.

Xu, Y. and Li, Z. (2008), "Dynamic portfolio selection based on serially correlated return–Dynamic mean-variance formulation," *System Engineering Theory and Practice*, 28, 123–131.

Yaari, M. E. (1965), "Uncertain lifetime, life insurance, and the theory of the consumer," *The Review of Economic Studies*, 32, 137–150.

Yao, H., Zeng, Y., and Chen, S. (2013), "Multi-period mean–variance asset–liability management with uncontrolled cash flow and uncertain time-horizon," *Economic Modelling*, 30, 492–500.

Yi, L., Li, Z., and Li, D. (2008), "Multi-period portfolio selection for asset-liability management with uncertain investment horizon," *Journal of industrial and management optimization*, 4, 535–552.

Yin, G. and Zhou, X. Y. (2004), "Markowitz's mean-variance portfolio selection with regime switching: from discrete-time models to their continuous-time limits," *Automatic Control, IEEE Transactions on*, 49, 349–360.

Yong, J. (2013), "Linear-quadratic optimal control problems for mean-field stochastic differential equations," *SIAM Journal on Control and Optimization*, 51, 2809–2838.

Zeng, Y. and Li, Z. (2011), "Asset-liability management under benchmark and mean-variance criteria in a jump diffusion market," *Journal of Systems Science and Complexity*, 24, 317–327.

Zhang, F. (2011), *Matrix theory: basic results and techniques*, Springer.

Zhang, L. and Li, Z. (2012), "Multi-Period Mean-Variance Portfolio Selection with Uncertain Time Horizon When Returns Are Serially Correlated," *Mathematical Problems in Engineering*, 2012.

Zhou, X. Y. and Li, D. (2000), "Continuous-time mean-variance portfolio selection: A stochastic LQ framework," *Applied Mathematics & Optimization*, 42, 19–33.

Zhu, S.-S., Li, D., and Wang, S.-Y. (2004), "Risk control over bankruptcy in dynamic portfolio selection: A generalized mean-variance formulation," *Automatic Control, IEEE Transactions on*, 49, 447–457.