LINEAR QUADRATIC MEAN FIELD GAMES OF FORWARD-BACKWARD STOCHASTIC SYSTEMS

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________________________(Signed)

WANG Shujun (Name of student)

Dedicate to my family.

Abstract

The thesis is concerned with the linear quadratic (LQ) mean field games (MFGs) involving forward-backward stochastic differential equations (FBSDEs). Five topics are under consideration:

1. The large-population dynamic optimization in *forward-backward* setting.
2. The *backward* LQ games of stochastic large-population systems.
3. The large-population systems in *major-minor* framework.
4. The combination problems of *leader-follower* and *major-minor* large-population systems.
5. The dynamic optimization of large-population systems with *partial information.*

For the first topic, a class of dynamic optimization problems of large-population are formulated. The most significant feature in this setup is the dynamics of individual agents follow the FBSDEs in which the forward and backward states are coupled at the terminal time. The related LQMFG, in its forward-backward sense, is also formulated to seek the decentralized strategies. Unlike the forward case, the consistency conditions of the forward-backward MFGs involve six Riccati and force rate equations. Moreover, their initial and terminal conditions are mixed which requires some special decoupling technique. The ϵ-Nash equilibrium property of the derived

decentralized strategies is also verified. To this end, some estimates to backward stochastic system are employed. In addition, due to the adaptiveness requirement to forward-backward system, all arguments here are not parallel to those in its forward case.

For the second topic, the backward LQMFGs of weakly coupled stochastic large-population system are studied. In contrast to the well-studied forward LQMFGs, the individual state in this large-population system follows the backward stochastic differential equation (BSDE) whose *terminal* instead of *initial* condition should be prescribed. The individual agents of large-population system are weakly coupled in their state dynamics and the full information is accessible to all agents. The explicit form of the limiting process and ϵ-Nash equilibrium of the decentralized control strategy are investigated. To this end, some estimates to BSDE, are presented in the large-population setting.

For the third topic, the backward-forward LQ games with major and minor players are investigated. In this topic, the dynamics of major player is given by a BSDE; while dynamics of minor players are described by (forward) SDEs. A backward-forward stochastic differential equation (BFSDE) system is established in which a large number of negligible agents are coupled in their dynamics via state average. The problem when major player takes into account the relative performance by comparison to minor players is under consideration. Some auxiliary mean field (MF) SDEs and a 3×2 mixed FBSDE system are considered and analyzed instead of involving the fixed-point analysis. The decentralized strategies are derived, which are also shown to satisfy the ϵ-Nash equilibrium property.

For the fourth topic, the combination problems of leader-follower and major-minor large-population systems are proposed. In the entire system, the major and minor agents are together regarded as the leaders, which are called major-leader and minor-leaders, respectively. The major-leader tracks a convex combination of the centroid of the minor-leaders and the followers; the minor-leaders track a convex combination of their own centroid and the major-leader's dynamics; and the followers track a convex combination of their own centroid and the centroid of the minor-leaders or a convex combination of the centroid of the minor-leaders and the major-leader's dynamics. As the applications of leader-follower and major-minor theory, the analysis of this problem is only presented as a framework and three consistency condition systems are obtained.

For the fifth topic, the dynamic optimization of large-population systems with partial information is considered. In this topic, the individual agents can only access the filtration generated by one observable component of the underlying Brownian motion. The state-average limit in this setup turns out to be some stochastic process driven by the common Brownian motion. Two classes of MFGs are proposed in this framework: one is governed by forward dynamics, and the other involves the backward one. In the forward case, the associated MFG is formulated and its consistency condition is equivalent to the wellposedness of some Riccati equation system. In the backward case, the explicit forms of the decentralized strategies and some BSDE (satisfied by the limiting process) are obtained. In both cases, the ϵ-Nash equilibrium properties are presented.

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Contents

List of Notations

$\mathbb{R}$	real number space
$\mathbb{R}^m$	m-dimensional Euclidean space
$\mathbb{R}^{m\times n}$	$m \times n$ real matrix space
$\lvert\cdot\rvert$	norm of a given Euclidean space
T	a finite time point with $T > 0$
$\mathbb{E}x(t),\ \mathbb{E}\xi$	expectation of a given stochastic process $x(t)$, expectation of a given random variable ξ
$C(0,T;\mathbb{R}^m)$	the space of all continuous functions defined on a finite time horizon $[0,T]$ with values in $\mathbb{R}^m$
$L^\infty(0,T;\mathbb{R}^m)$	the space of all uniformly bounded functions defined on $[0,T]$ with values in $\mathbb{R}^m$
$L^2(0,T;\mathbb{R}^m)$	the space of all deterministic functions defined on $[0,T]$ with values in $\mathbb{R}^m$ satisfying $\int_0^T \lvert x(t)\rvert^2 dt < +\infty$
$L^2_{\mathcal{F}_t}(0,T;\mathbb{R}^m)$	the space of all $\mathcal{F}_t$-progressively measurable processes defined on $[0,T]$ with values in $\mathbb{R}^m$ satisfying $\mathbb{E}\int_0^T \lvert x(t)\rvert^2 dt < +\infty$ for a given filtration $\{\mathcal{F}_t\}_{0\leqslant t\leqslant T}$
$\lVert f(t)\rVert_\infty$	$:= \sup_{0\leqslant t\leqslant T} \lvert f(t)\rvert$ for $f(t) \in C(0,T;\mathbb{R}^m)$
N	the population size of large-population system
$\mathcal{A}_i,\ \mathcal{B}_j$	the individual player (agent)

Chapter 1

Introduction

1.1 Background

Game theory is the study of strategic decision making. Generally speaking, it is the study of mathematical models of cooperation and conflict among intelligent rational decision-makers. It is mainly used in economics, political science, psychology, logic, computer science, biology, etc. The subject first addresses zero-sum games, in which one person's gains exactly equal net losses of the other participant or participants. Today, however, game theory applies to a wide range of behavioral relations, and has developed into an umbrella term for the logical side of decision science. In many social, economic and engineering models, the individuals or agents involved have conflicting objectives. Therefore it is more appropriate to consider the optimization problem based upon individual payoffs or costs. This gives rise to noncooperative game theoretic approaches partly based upon the vast corpus of relevant work within economics, social sciences, etc.

In the literature, studies of stochastic dynamic games and team problems may be traced to the 1960s (see e.g.,[1, 2, 3, 4]). The optimal control context weakly interconnected systems were studied in [5], and in a two player noncooperative nonlinear dynamic games setting Nash equilibria were analyzed in [6]. In recent years, the controlled stochastic large-population (also called multi-agent) system is evidently of importance due to its wide range of appearance in these areas. Afterwards, the dynamic optimization or control of this kind of system has attracted consistent and intense attentions by research communities. The most special feature of controlled large-population system lies in the existence of considerable insignificant agents who are individually negligible but their collective behaviors will impose some significant impact on all agents. This feature can be captured by the weakly-coupling structure in the individual dynamics and (or) cost functionals via the state-average across the whole population. In this way, the individual behaviors of all agents in micro-scale, can be connected to their mass effects in the macro-scale. This kind of weak-coupling in both dynamics and costs is used to model the mutual impact of agents during competitive decision-making. In particular, the dynamic coupling specifies the impact of the environment on an individual's decision-making, and the underlying model takes the form of weakly coupled diffusion subject to individual controls.

It is remarkable that the classical strategies by consolidating all agent's exact states, turn out to be infeasible and ineffective due to the highly complicated coupling structure in large-population system. Alternatively, it is more tractable and effective to study the related strategies by considering its own individual state and some off-line quantities only. For large-population stochastic dynamic games with MF couplings, Nash certainty equivalence theory was originally developed in a series of papers by Huang togeter with Caines and Malhamé. The optimization of large-scale linear control systems wherein many agents are coupled with each other via their individual dynamics and the costs are in an "individual to the mass" form, was presented in [7]. Then a general formulation of nonlinear McKean-Vlasov Markov process models was developed in [8, 9, 10].

This thesis mainly focuses on the study of large-population system in its LQ case where the state equations are linear in the state with nonhomogeneous terms, and the cost functionals are quadratic. Recall the linear system and its related LQ control have already been extensively investigated. Such a control problem is called a linear quadratic optimal control problem. Readers can refer to [11] for some classic results of deterministic LQ problems. For the stochastic case, the problems were addressed in [12, 13]. One systematic introduction of stochastic LQ optimal control problem can be found in the monograph [14] and the references therein. Other related literature includes [15, 16, 17], etc. Due to the nice structure of LQ problem, there is also rich literature on large-population problem modeled by LQ system. LQ games in large-population systems where the agents evolve according to nonuniform dynamics were considered and an ϵ-Nash equilibrium property was proved in [18]. In [19], the author solved an Hamilton-Jacobi-Bellman and Kolmogorov-Fokker-Plank equations and found explicit Nash equilibria in the form of linear feedbacks. [20] aimed to study a class of LQ control problems with N decision makers, where the basic objective is to minimize a social cost as the sum of N individual costs containing MF coupling. Later on, [21] provided a comprehensive study of a general class of MF games in the LQ framework. For more literature about LQ problem with large-population, see [22, 23, 24] etc.

As a new branch of game theory, MFGs arises from variety of areas, such as particle physics, economics, etc. In many situations of particle physics, it is possible to construct an excellent approximation to the situation by introducing one or more "mean fields" that serve as mediators for describing inter-particle interactions. In this kind of model, one describes the contribution of each particle to the creation of a mean field and the effect of the mean field on each particle, by conceiving each particle as infinitesimal, i.e. by carrying out a kind of limit process on the number N of particles ($N \to +\infty$). In game theory, from a mathematical standpoint it involves of studying the limit of a large class of N-player games when N tends to infinity. Usually, differential games with N-players turn out to be untractable. Fortunately things are simplified, as least for a wide range of games that are symmetrical as far as players are concerned, as the number of players increases. Indeed, interindividual complex strategies can no longer be implemented by the players, for each player is progressively lost in the crowd in the eyes of other players when the number of players increases.

During the last few decades, there is a growing literature to the study of MFGs and their applications. For this class of game problems a closely related approach was independently developed in [25, 26, 27]. Based on these, considerable research attention has been drawn along this research line. Some recent literature include [28, 29, 30, 31] for recent progress in MFG theory. Introductions and some models to MFGs were given in [28]. In [29], the authors provided a complete probabilistic analysis of a large class of stochastic differential games with MF interactions. [30] was devoted to discussing and comparing two investigation methods of the asymptotic regime of stochastic differential games with a finite number of players as the number of players tends to the infinity. In addition, in [31], a model of inter-bank borrowing and lending was proposed, and systemic risk was analyzed.

MF type control has also been extensively studied recently. In [32], the authors obtained MF BSDEs associated with a MF SDE as a limit of a high dimensional system of forward and backward SDEs, corresponding to a large number of agents. Later on, [33] deepened the investigation of such MF BSDEs with general coefficients and presented the related partial differential equations. Based upon these, [34] and [35] independently studied the optimal control of a SDE of MF type when the action space is convex, which was a partial result of [36]. Moreover, [37] provided an existence result for the solution of fully coupled FBSDEs of the MF type. For more literature about MF games and controls, see [38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50], etc. It is worth pointing out that there are differences between MFG and MF type control. Generally speaking, as addressed in [29, 30, 31], the MFG and MF type control are essentially different in their methods applied and the equilibriums derived.

To be precise, the method of MFG will be "asynchronous"-styled. It will first fix or *freeze* the state-average $x_t^{(N)}$ (in its linear case) or empirical measure μ_{x_t} (in its nonlinear case) to reduce the initial problem into some standard problem but parameterized by such frozen term. Note that such frozen term is still undetermined in this step. Next, this parameterized standard problem can be solved and the optimal state can be obtained. This can be called the *decentralized control*. With this in hand, the frozen state-average or empirical measure can be further determined through some fixed-point analysis and *consistency condition* concerning the obtained optimality system. In this sense, the state-average (or, empirical measure) and the underlying control in MFG will change "asynchronously".

By contrast, in MF type control problems, the state-average (i.e., the expectation $\mathbb{E}x_t = \lim_{N\to+\infty} x_t^{(N)}$) or empirical measure will not be fixed or frozen beforehand. Actually, they will change depending on the underlying control applied. In this way, the state-average term and state itself will be treated in "synchronous" style. Moreover, the equilibrium derived in MF type control will be the *franchised* equilibrium (see [31]) whereas the equilibrium from MFG will be in the "ϵ-Nash sense" (see [29]). These two equilibriums are both approximating equilibriums to handle the large-population systems but as discussed in [31], they are very different in their dynamic properties.

It is remarkable that all agents in above literature are comparably negligible in that they are not able to affect the whole population in separable manner. By contrast, their impacts are imposed in a unified manner through the population state-average. In this sense, all agents can be viewed as peers. One real example is the market price formation in which there are considerable producers or firms to produce the same type product. Each firm is so small thus its individual production behavior can't affect its peers' states. However, the average production of all firms will determine the market price of this product. All small firms are price-takers thus they are further interacted and coupled via such price formation mechanism. The above discussion assumes all agents are equally participating in the market price formation. However, in reality we point out the status and roles of agents may illustrate significant differences in various realistic situations. For instance, the decision making of small individuals are always influenced by some “leading” agent or “dominated” institutions. In our price formation example, such “leading” agent can be interpreted as some monopoly firm which takes considerable production capacity thus imposes more significant affects to the price formation. As to the “dominated” institution, it can be viewed as the local government as its industrial policy will greatly affect the production behaviors of all firms. Conversely, the small firms will also affect the local government through the market price. One channel is the production tax revenue, an important factor to calibrate the local government’s state, will depend on the formulated market price.

The above discussion suggests the so-called major-minor agent models. To be more precise, let us figure out the following oil production example. In the crude oil exploration, each individual oil production company aims to explore more oil and thus pursue more profits. In this sense, their production plans always intend to take less account of some macro-factors such as the limited oil resources, the possible environmental costs as well as the long-term benefits in their exploration. On the other hand, these factors are mainly the concerns of relevant supervisory department or local government. Unlike the individual oil company, they are more concerned about the factors such as sustainable development, and the overall benefit of oil sector. Thus, they will always execute some macro-control policy by assuming the responsibility of major agent. All small firms (as the minor agents) should obey the policies when the production plan is making. Consequently, the set of all individual small producers consists of our minor-agent part, and it is further coupled with the local government (the major agent) via their state-average. The major-minor large-population system and related MFGs are extensively studied. Looking back to previous work, [51] discussed large-population systems with major and minor players by analyzing the case in an infinite set where the minor players are from a total of K classes. Later on, [52] considered a LQ problem with major and minor players by directly treating the mean field z in the population limit as a random process with random coefficients. Recently, [53] studied large-population dynamic games involving nonlinear stochastic dynamical systems with a major agent and a population of N minor agents and derived the ϵ_N-Nash equilibrium property where $\epsilon_N = O(1/\sqrt{N})$. In addition, [54] derived a game problem in a weak formulation; this means in particular that the game was of the type "feedback control against feedback control". Then payoff/cost functional was defined through a controlled BSDE, for which the driving coefficient was assumed to satisfy strict concavity-convexity with respect to the control parameters.

In addition, [55] investigated a leader-follower hierarchical game. The feature of this kind of game is as follows. For any choice u_2 of the leader, the follower would like to choose a strategy u_1 to minimize his/her cost. Knowing the follower would take such an optimal strategy $\bar{u}_1$ (supposing it exist, which depends on the choice u_2 of the leader, in general), the leader would like to choose some $\bar{u}_2$ to minimize his/her cost, in which the strategy of the follower ($\bar{u}_1$) is already optimal. Based on some discussions of stochastic Riccati equations, the author obtained an open-loop solution of the leader-follower differential game. For the approach of the large-population leader-follower model, [56] was devoted to developing a general model and presenting the main adaptation result of the uniform cost coupling model in the case that the leaders' costs are based on a tradeoff between a certain reference trajectory and the centroid of themselves, while the followers "only" track the centroid of the leaders. In [57], the authors completely analyzed a more general scenario where the followers are tracking a convex combination of their own centroid and the centroid of the leaders. Besides, in [57], the leaders observe no one and the followers have limited observations on the leaders. [58] is concerned with a leader-follower stochastic differential game with asymmetric information. Stochastic maximum principles and verification theorems with partial information are obtained, to represent the Stackelberg equilibrium. It is also realistic to consider the combinations of major-minor and leader-follower manners. Take the above production model for example. As referred, the "leading" agent that can be viewed as the local government or supervisory department, will greatly affect the production behaviors of all firms. Conversely, the small firms will also affect the local government through some factor like the market price. However, in this industry chain, there may also exist the downstream industry which can be viewed as the suppliers of raw material or manufacturers of primary commodity. There is no doubt that the behaviors of suppliers or manufacturers (downstream industry) are affected directly by that of all small firms (upstream industry). And when making the industrial policy, the "leading" agent should also consider the price of raw materials or productions of primary commodities sufficiently. Therefore, the behaviors of suppliers or manufacturers (downstream industry) will affect the policy making by the "leading" agent. In addition, in many practical cases, the government or supervisory will also make some policies about the raw material or primary commodity by considering the resource factor, environmental factor, etc. In this way, the "leading" agent imposes the impacts to the suppliers of raw material and manufacturers of primary commodity directly. Consequently, the leader-follower involving major-minor models are proposed to characterize this kind of problem.

In most control problems, the information is assumed to be completely observed. However, it may be not reasonable in reality. It turns out that various stochastic control problems fit into the partial information framework due to the factors such as finite datum, latent process or noisy observation, etc. An extensive review of stochastic control with partial information was provided in [59]. There is other rich literature on partially observed stochastic control systems (see e.g. [60, 61, 62, 63, 64, 65, 66, 67] for previous work and [68, 69, 70, 71, 72, 73, 74, 75, 76, 77] for recent work). For the literature on partially observed stochastic games, please refer to [78, 79, 80] and references therein. Remark that a class of LQMFGs with noisy observations was also addressed in [81] but defined on an infinite-time horizon so the algebra Riccati equations were involved. Moreover, the limiting state-average in [81] was deterministic as there was no common noise.

It is worth pointing out in all above works involving large-population system, all agents' states are formulated by (forward) SDEs with the initial conditions as a priori. As a sequel, the agents' objectives are minimizations of cost functionals involving their terminal states. As the BSDE are well-defined stochastic systems with broad-range applications, it is very natural to study its dynamic optimization in large-population setup. Indeed, the dynamic optimization of backward large-population system is inspired by a variety of scenarios. For example, the dynamic economic models for which the participants are of some recursive utilities or nonlinear expectations, or some production planning problems with some tracking terminal objectives but affected by the market price via production average. Another example arises from the risk management when considering the relative or comparable criteria based on the average performance of all other peers through the whole sector. This is the case for a given pension fund to evaluate its own performance by setting the average performance (say, average hedging cost or initial deposit, surplus) as its benchmark. In addition, the controlled forward large-population systems, which are subjected to some terminal constraints, can be reformulated by some backward large-population systems, as motivated by [82]. Different to SDE, the terminal instead of initial condition of BSDE should be specified as the priori. As a consequence, the BSDE will admit one adapted solution pair (y_t, z_t) where the second solution component z_t (it is also called the diffusion component) is naturally presented here due to the martingale representation and the adaptiveness requirement. It is remarkable that there exist rich literature concerning the theories and applications of BSDE. The linear BSDEs were first introduced by [83] when studying stochastic optimal control problems. [84] first proved the existence and uniqueness of solution for nonlinear BSDEs, which have been extensively used in stochastic control and mathematical finance. Independently, [85] presented a stochastic differential recursive utility, which is a generalization of a standard additive utility with an instantaneous utility depending not only on an instantaneous consumption rate, but also on a future utility. As found by [86], the utility process can be regarded as a solution of a special BSDE. [86] also gave the formulations of recursive utilities and their properties from the point of view of BSDE. A BSDE coupled with a SDE in their terminal condition formulates the FBSDE. The forward-backward large-population dynamic optimization problems arise naturally in many practical situations. A typical situation is from the large-population system with constrained terminal condition (see e.g., [87]). In this situation, the standard forward stochastic control problem can be well approximated by some forward-backward stochastic control problem.

In the last few decades, FBSDE has been well studied. There are several methods to solve FBSDE. The method of contraction mapping was first used by [88] and later detailed by [89]. It works well when the duration T is relatively small. Another method called the "four step scheme" ([90]) was the first solution method that removed restriction on the time duration for Markovian FBSDEs. The third is the method of continuation. This is a method that can treat non-Markovian FBSDEs with arbitrary duration, initiated by [91] and [92], and later developed by [93] and [94]. Please refer to the book [95] for the detailed accounts for all the three methods. Recently, in [96] the authors find a unified scheme which combines all existing methodology, and overcome some fundamental difficulties that have been longstanding problems for non-Markovian FBSDEs. For more theoretical and practical results on FBSDE, please refer to [97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110] and references therein. Due to the interdependence of the states, FBSDE can be divided into two kinds: the partially-coupled FBSDE and fully-coupled FBSDE. The former means that the backward state component y_t depends explicitly on the forward x_t, but x_t doesn't explicitly depend on the backward (y_t, z_t), which is more accepted to represent the recursive utility or nonlinear expectation (see, e.g. [87, 72, 41]). In fact, the forward state x_t usually represents the dynamics of some underlying asset, the backward state y_t stands for the nonlinear expectation or recursive utility of decision maker. Thus it is reasonable and natural that the recursively utility will depend on the underlying state. But conversely, the forward underlying state will not be affected by the recursive utility adopted. Mathematically, there's great value to formulate and study the fully-coupled FBSDE (namely, the forward state also depends on the backward state).

1.2 Contributions and Organization of the Thesis

As the novelty, this thesis mainly considers the LQMFGs in forward-backward framework. Details can be summarized as follows.

- The large-population dynamic optimization in *forward-backward* setting is formulated and the related LQMFGs for partially-coupled FBSDEs are investigated in Chapter 2. The optimal control of auxiliary track system is studied. The decoupling procedure of the Hamiltonian system (due to its two-dimensional feature) will involve six Riccati equations or ordinary differential equations (ODEs). Moreover, the decentralized control strategies are derived from the consistency condition and approximation scheme. In addition, the ϵ-Nash equilibrium property of our original problem is also verified based on some FBSDE estimates.

- The *backward* LQ games of large-population systems are studied, for which the individual states follow some BSDEs. Chapter 3 focuses on this problem. This feature makes the setting very different to the existing works of LQMFGs wherein the individual states evolve by some SDEs. The individual state dynamics are weakly coupled through the state-average and the full information structure is assumed thus the individual agent can access the central information of all other agents. The explicit form of the limiting process and ϵ-Nash equilibrium of the decentralized control strategy are investigated.

- The large-population system in *major-minor* framework is considered in Chapter 4, in which the major agent's dynamics is characterized by some BSDE with prescribed terminal condition while the minor agents' dynamics are governed by SDEs with prescribed initial condition. In this way, the major agent's objective turns to minimize the cost functional depending on initial state and the minor agents want to minimize the cost functionals depending on terminal states. The problem when major player takes into account the relative performance by comparison to minor players is under consideration. The related LQMFGs are discussed and the decentralized strategies are derived. A stochastic process which relates to the state of major player is introduced here to be the approximation of the state-average process. An auxiliary MF SDE and a 3×2 FBSDE system are considered and analyzed. Here, the 3×2 FBSDE, which is also called a triple FBSDE (TFBSDE), is composed by three forward and three backward equations. With the help of the monotonic method in [92] and [108], the wellposedness of this FBSDE is obtained. Finally, the ϵ-Nash equilibrium property of decentralized control strategy is derived with $\epsilon = O(1/\sqrt{N})$.

- In Chapter 5, the combination problems of *leader-follower* and *major-minor* large-population systems are proposed. In the entire system, the major and minor agents are together regarded as the leaders, which are called major-leader and minor-leaders, respectively. The major-leader tracks a convex combination of the centroid of the minor-leaders and the followers; the minor-leaders track a convex combination of their own centroid and the major-leader's dynamics; and the followers track a convex combination of their own centroid and the centroid of the minor-leaders or a convex combination of the centroid of the minor-leaders and the major-leader's dynamics. Although the analysis of this problem in this chapter is only presented as a framework, it is divided into three topics due to the tracking structure and processing ways. Three consistency condition systems are obtained for all the topics.

- Chapter 6 is devoted to the dynamic optimizations of large-population systems with *partial information* structure. Here, the individual agents can only access the filtration generated by one observable component of underlying Brownian motion. The state-average limit in this setup turns out to be some stochastic process driven by the common Brownian motion. Two classes of MFGs are proposed in this framework: one is governed by forward dynamics, and the other involves the backward one. In the forward case, the associated MFG and some Riccati equation system are formulated. In the backward case, the explicit forms of the decentralized strategies and some BSDE (satisfied by the limiting process) are obtained. In both cases, the ϵ-Nash equilibrium properties are presented.

- Chapter 7 concludes the whole thesis and plans for the future work.

Chapter 2

LQMFGs of FBSDEs

This chapter studies a new class of dynamic optimization problems of large-population system. The most significant feature in this setup is the dynamics of individual agents follow the FBSDEs in which the forward and backward states are coupled at the terminal time. This work is hence different to most existing large-population literature where the individual states are typically modeled by the SDEs only including the forward state ([8, 18, 20], etc.). The associated LQMFG, in its forward-backward sense, is also formulated to seek the decentralized strategies. Unlike the forward case, the consistency conditions of the forward-backward MFGs involve six Riccati and force rate equations. Moreover, their initial and terminal conditions are mixed thus some special decoupling technique is applied here. The fixed-point analysis and the asymptotic near-optimality property (namely, ϵ-Nash equilibrium) of the derived decentralized strategies are also investigated. To this end, some estimates to forward-backward stochastic systems are employed. In addition, due to the adaptiveness requirement to forward-backward system, the arguments here are not parallel to those in its forward case. Anyway, for notational simplicity, in this chapter and the following chapters we focus on the cases where all processes are 1-dimensional. Actually, for higher dimensional we can also derive the corresponding results in the same way.

2.1 Problem Formulation

Suppose $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leqslant t\leqslant T}, P)$ is a complete filtered probability space on which a standard N-dimensional Brownian motion $\{W_i(t),\ 1 \leqslant i \leqslant N\}_{0\leqslant t\leqslant T}$ is defined. Denote by $\{\mathcal{F}_t^{w_i}\}_{0\leqslant t\leqslant T}$ the filtration generated by $\{W_i(s), 0 \leqslant s \leqslant t\}$ but augmented by all P-null sets.

Consider a large-population system with N individual agents, denoted by $\{\mathcal{A}_i\}_{1\leqslant i\leqslant N}$. The dynamics for individual agent involves three components. The forward components $\{x_i\}_{1\leqslant i\leqslant N}$ of $\{\mathcal{A}_i\}_{1\leqslant i\leqslant N}$ satisfy

$$\begin{cases} dx_i(t) =\Big[Ax_i(t) + Bu_i(t) + Fx^{(N)}(t)\Big]dt + \sigma x_i(t)dW_i(t), \\ x_i(0) =x_{i0} \end{cases} \tag{2.1}$$

where $\{x_{i0}\}_{i=1}^N$ are initial conditions of the forward system (2.1), and the backward states are

$$\begin{cases} -dy_i(t) =\Big[Cy_i(t) + Du_i(t) + Hx_i(t) + Lx^{(N)}(t)\Big]dt - \sum\limits_{j=1}^N z_{ij}(t)dW_j(t), \\ y_i(T) =Kx_i(T) \end{cases} \tag{2.2}$$

where $x^{(N)}(t) = \frac{1}{N}\sum\limits_{i=1}^N x_i(t)$ is the (forward) state-average. Here, $A, B, F, C, D, H, L, K, \sigma$ are scalar constants. Equation (2.1) and (2.2) together become a partially-coupled FBSDE. By "partially-coupled", we mean the dynamics of forward state does not depend on the backward components. Introduce $\mathcal{F}_t := \sigma\{W_i(s), x_{i0}; 0 \leqslant s \leqslant t, 1 \leqslant i \leqslant N\}$ as the full information accessible to the large-population system up to time t. Different to forward large-population system, the backward diffusion term $\sum\limits_{j=1}^N z_{ij}(t)dW_j(t)$ driving by all Brownian motions (not W_i only), should be introduced in the dynamics of $\mathcal{A}_i$ by considering $x^{(N)}(t) \in \mathcal{F}_t$ (even through Eq.(2.1), the forward state of $\mathcal{A}_i$ is only driven by W_i only). Let $U_i,\ i=1,2,\ldots,N$ be subsets of $\mathbb{R}$. The admissible control $u_i \in \mathcal{U}_i$ where the admissible control set $\mathcal{U}_i$ is defined as

$$\mathcal{U}_i := \Big\{ u_i \Big| u_i(t) \in U_i, 0 \leqslant t \leqslant T;\ u_i(\cdot) \in L^2_{\mathcal{F}_t}(0,T;\mathbb{R}) \Big\},\ 1 \leqslant i \leqslant N.$$

Let $u = (u_1,\ldots,u_N)$ denote the set of control strategies of all N agents; $u_{-i} = (u_1,\ldots,u_{i-1},\, u_{i+1},\ldots,u_N)$ the control strategies except i^{th} agent $\mathcal{A}_i$. The individual cost functional is given by

$$\mathcal{J}_i(u_i(\cdot),u_{-i}(\cdot)) = \frac{1}{2}\mathbb{E}\left\{ \int_0^T \left[Q\Big(x_i(t) - \big(Sx^{(N)}(t)+\eta\big)\Big)^2 + Ru_i^2(t)\right] dt + N_0 y_i^2(0) \right\} \tag{2.3}$$

where S,η are scalar constants and $Q \geqslant 0, R > 0, N_0 \geqslant 0$.

It is worthy pointing out that the system (2.1)-(2.2) is well motivated by various real examples in decision making or mathematical finance, such as the recursive utility optimization or the principle-agent problem, etc. For illustration, let us consider the following recursive utility optimization in large-population system built on the model of ([111], [72]).

Example 2.1. *(Recursive Utility Optimization) Suppose there is an economy or market which consists of N individual participants. Each participant has its own individual underlying state (asset) with dynamics $x_i(\cdot)$ as follows:*

$$\begin{cases} dx_i(t) = \Big[A_1 x_i(t) + A_2 \pi_i(t) + A_3 x^{(N)}(t)\Big]dt + \delta dW_i(t), \\ x_i(0) = x_{i0} > 0. \end{cases}$$

Here, $x^{(N)}(\cdot) = \frac{1}{N}\sum_{i=1}^{N} x_i(\cdot)$ is the asset-average which represents some common economy primitive (e.g., the price index); A_1, A_2, A_3, δ are constants; $W_i(\cdot),\ i=1,\ldots,N$ are standard Brownian motions; $\pi_i(\cdot) \in \mathbb{R}$ is regarded as some idiosyncratic economic factor such as the individual investment strategies.

Now we consider the given participant may consume continuously from 0 to T. Let $c_i(\cdot),\ i=1,\ldots,N$ be continuous consumption rate processes and suppose that *there exist terminal rewards $Kx_i(T)$ at time T. By [86], the recursive utility of the investor is a solution of a BSDE, which is denoted by $y_i^{c_i,\pi_i}(\cdot)$. We assume it satisfies*

$$\begin{cases} -dy_i(t) = \Big[B_1 y_i(t) + B_2 c_i(t) + B_3 x^{(N)}(t)\Big]dt - \displaystyle\sum_{j=1}^{N} z_{ij}(t)dW_j(t), \\ \\ \quad y_i(T) = Kx_i(T). \end{cases}$$

Define $\mathcal{F}_t := \sigma\{W_i(s); 0 \leqslant s \leqslant t,\ 1 \leqslant i \leqslant N\}$. In this setting, to select a $\mathcal{F}_t$-adapted process $(\bar{c}_i(\cdot), \bar{\pi}_i(\cdot))$ such that $y_i^{\bar{c}_i,\bar{\pi}_i}(0) = \max\limits_{(c_i,\pi_i)} y_i^{c_i,\pi_i}(0)$ is recognized as a recursive optimal control problem. Based on this motivation, we formulate the large-population LQ system (2.1)-(2.3) *in FBSDE setting. For more applications, please refer [112, 110, 41], etc.*

Remark 2.1. *As referred before, unlike the forward large-population literature, the new term of backward state $N_0 y_i^2(0)$ is introduced in* (2.3) *to denote some recursive evaluation or nonlinear expectation. Another practical meaning of it is the initial hedging deposits in the pension fund industry. In addition, one explanation of above forward-backward system* (2.1) *and* (2.2) *is as follows: the forward state x_i in* (2.1) *represents some underlying asset/product dynamics while the state-average $x^{(N)}(t)$ denotes some average market index on it; the control u_i stands for a economic factor (for example, a dividend rate, a consumption rate, a tax rate); and the backward state y_i denotes the dynamics of some derivative asset on x_i (for example, the option on real product such as crude oil). In this case,* (2.3) *implies the minimization of the average deviation from market price, and the initial hedging cost for some future commitment at the same time.*

We introduce the following assumption:

(H2.1) $\{x_{i0}\}_{i=1}^N$ are independent and identically distributed (i.i.d) with $\mathbb{E}|x_{i0}|^2 < +\infty$, and also independent of $\{W_i, 1 \leqslant i \leqslant N\}$.

Now, we formulate the large-population dynamic optimization problem of forward-backward stochastic system.

Problem (FB-MFG). Find a control strategy set $\bar{u} = (\bar{u}_1, \dots, \bar{u}_N)$ which satisfies

$$\mathcal{J}_i(\bar{u}_i(\cdot), \bar{u}_{-i}(\cdot)) = \inf_{u_i \in \mathcal{U}_i} \mathcal{J}_i(u_i(\cdot), \bar{u}_{-i}(\cdot))$$

where $\bar{u}_{-i}$ represents $(\bar{u}_1, \dots, \bar{u}_{i-1}, \bar{u}_{i+1}, \dots, \bar{u}_N)$.

2.2 The Limiting Control Problem

To study Problem **(FB-MFG)**, one efficient approach is to discuss the associated MFGs via limiting problem when the agent number N tends to infinity. To obtain the feedback control and the desired results, we assume $U_i = \mathbb{R}$ for $i = 1, 2, \dots, N$. As $N \to +\infty$, suppose $x^{(N)}$ can be approximated by a deterministic continuous function $\bar{x}$ and introduce the following auxiliary (forward) state dynamics

$$\begin{cases} dx_i(t) = \Big[Ax_i(t) + Bu_i(t) + F\bar{x}(t)\Big]dt + \sigma x_i(t)dW_i(t), \\ x_i(0) = x_{i0} \end{cases} \tag{2.4}$$

and

$$\begin{cases} -dy_i(t) = \Big[Cy_i(t) + Du_i(t) + Hx_i(t) + L\bar{x}(t)\Big]dt - z_i(t)dW_i(t), \\ y_i(T) = Kx_i(T). \end{cases} \tag{2.5}$$

The associated limiting cost functional becomes

$$J_i(u_i(\cdot)) = \frac{1}{2}\mathbb{E}\left\{\int_0^T \left[Q\Big(x_i(t) - \big(S\bar{x}(t) + \eta\big)\Big)^2 + Ru_i^2(t)\right] dt + N_0 y_i^2(0)\right\}. \tag{2.6}$$

Thus, we formulate the limiting LQ game **(L-FB-MFG)** as follows.

Problem (L-FB-MFG). For the i^{th} agent $\mathcal{A}_i$, $i = 1, 2, \dots, N$, find $\bar{u}_i \in \mathcal{U}_i$ satisfying

$$J_i(\bar{u}_i(\cdot)) = \inf_{u_i \in \mathcal{U}_i} J_i(u_i(\cdot)). \tag{2.7}$$

$\bar{u}_i$ satisfying (2.7) is called an optimal control for **(L-FB-MFG)**. Applying the s-tandard variational method, we have:

Lemma 2.1. *Under* (H2.1), *the optimal control for Problem* **(L-FB-MFG)** *is given by*

$$\bar{u}_i(t) = R^{-1}\Big[D\hat{k}_i(t) - B\hat{p}_i(t)\Big] \tag{2.8}$$

where the adjoint process $(\hat{k}_i, \hat{p}_i, \hat{q}_i)$ *and the optimal trajectory* $(\hat{x}_i, \hat{y}_i, \hat{z}_i)$ *satisfy the SDE*

$$\begin{cases} d\hat{x}_i(t) =\Big[A\hat{x}_i(t) + R^{-1}BD\hat{k}_i(t) - R^{-1}B^2\hat{p}_i(t) + F\bar{x}(t)\Big]dt + \sigma\hat{x}_i(t)dW_i(t), \\ d\hat{k}_i(t) =C\hat{k}_i(t)dt, \\ \hat{x}_i(0) =x_{i0}, \quad \hat{k}_i(0) = -N_0\hat{y}_i(0) \end{cases} \tag{2.9}$$

and BSDE

$$\begin{cases} -d\hat{y}_i(t) =\Big[C\hat{y}_i(t) + R^{-1}D^2\hat{k}_i(t) - R^{-1}BD\hat{p}_i(t) + H\hat{x}_i(t) + L\bar{x}(t)\Big]dt \\ \qquad\qquad - \hat{z}_i(t)dW_i(t), \\ -d\hat{p}_i(t) =\Big[A\hat{p}_i(t) - H\hat{k}_i(t) + Q\hat{x}_i(t) - QS\bar{x}(t) - Q\eta + \sigma\hat{q}_i(t)\Big]dt \\ \qquad\qquad - \hat{q}_i(t)dW_i(t), \\ \hat{y}_i(T) =K\hat{x}_i(T), \quad \hat{p}_i(T) = -K\hat{k}_i(T). \end{cases} \tag{2.10}$$

The proof is similar to that of [108]. In the following, we aim to decouple the FBSDE system (2.9)-(2.10). Let $\beta(t)$ be the unique solution of the Riccati equation

$$\begin{cases} \dfrac{d\beta(t)}{dt} + \big(2A + \sigma^2\big)\beta(t) - R^{-1}B^2\beta^2(t) + Q = 0, \\ \beta(T) = 0, \end{cases} \tag{2.11}$$

$\alpha(t)$ the unique solution of the ordinary differential equation (ODE)

$$\begin{cases} \dfrac{d\alpha(t)}{dt} + \big(A + C - R^{-1}B^2\beta(t)\big)\alpha(t) + R^{-1}BD\beta(t) - H = 0, \\ \alpha(T) = -K, \end{cases} \tag{2.12}$$

$\zeta(t)$ the unique solution of the ODE

$$\begin{cases} \dfrac{d\zeta(t)}{dt} + \big(A + C - R^{-1}B^2\beta(t)\big)\zeta(t) - \big(R^{-1}BD\beta(t) - H\big) = 0, \\ \zeta(T) = K, \end{cases} \tag{2.13}$$

and $\xi(t)$ the unique solution of the ODE

$$\begin{cases} \dfrac{d\xi(t)}{dt} + 2C\xi(t) + \big(R^{-1}BD - R^{-1}B^2\alpha(t)\big)\zeta(t) + R^{-1}D^2 - R^{-1}BD\alpha(t) = 0, \\ \xi(T) = 0. \end{cases} \tag{2.14}$$

Introduce

$$\hat{p}_i(t) = \alpha(t)\hat{k}_i(t) + \beta(t)\hat{x}_i(t) + \gamma(t), \tag{2.15}$$

and

$$\hat{y}_i(t) = \xi(t)\hat{k}_i(t) + \zeta(t)\hat{x}_i(t) + \tau(t) \tag{2.16}$$

where $\gamma(t)$ and $\tau(t)$ are to be determined. By Itô's formula, it follows that (2.10) is equivalent to the following BSDEs

$$\begin{cases} -d\gamma(t) = \Big[\big(A - R^{-1}B^2\beta(t)\big)\gamma(t) + \big(F\beta(t) - QS\big)\bar{x}(t) - Q\eta\Big]dt \\ \qquad\qquad - \Big[\hat{q}_i(t) - \sigma\beta(t)\hat{x}_i(t)\Big]dW_i(t), \\ \gamma(T) = 0 \end{cases} \tag{2.17}$$

and

$$\begin{cases} -d\tau(t) = \Big[C\tau(t) - \big(R^{-1}B^2\zeta(t) + R^{-1}BD\big)\gamma(t) + \big(F\zeta(t) + L\big)\bar{x}(t)\Big]dt \\ \qquad\qquad - \Big[\hat{z}_i(t) - \sigma\zeta(t)\hat{x}_i(t)\Big]dW_i(t), \\ \tau(T) = 0. \end{cases} \tag{2.18}$$

In terms of the existence and uniqueness of solutions of BSDEs (see [84]), (2.17)-

(2.18) are equivalent to the following equations

$$\begin{cases} \dfrac{d\gamma(t)}{dt} + \big(A - R^{-1}B^2\beta(t)\big)\gamma(t) + \big(F\beta(t) - QS\big)\bar{x}(t) - Q\eta = 0, \\ \gamma(T) = 0, \end{cases} \tag{2.19}$$

$$\begin{cases} \dfrac{d\tau(t)}{dt} + C\tau(t) - \big(R^{-1}B^2\zeta(t) + R^{-1}BD\big)\gamma(t) + \big(F\zeta(t) + L\big)\bar{x}(t) = 0, \\ \tau(T) = 0, \end{cases} \tag{2.20}$$

$$\hat{q}_i(t) = \sigma\beta(t)\hat{x}_i(t) \tag{2.21}$$

and

$$\hat{z}_i(t) = \sigma\zeta(t)\hat{x}_i(t). \tag{2.22}$$

Note that both (2.19) and (2.20) are the ODEs. Letting $t = 0$ in (2.16), we have

$$\hat{y}_i(0) = \xi(0)\hat{k}_i(0) + \zeta(0)\hat{x}_i(0) + \tau(0). \tag{2.23}$$

From (2.9), we know that

$$\hat{k}_i(0) = -N_0\hat{y}_i(0) \quad \text{and} \quad \hat{x}_i(0) = x_{i0}. \tag{2.24}$$

Supposing $1 + \xi(0)N_0 \neq 0$ and substituting (2.24) into (2.23) yield

$$\hat{y}_i(0) = \frac{\zeta(0)x_{i0} + \tau(0)}{1 + \xi(0)N_0}. \tag{2.25}$$

Then computing $\hat{k}_i(t)$ in (2.9), we obtain the unique solution

$$\hat{k}_i(t) = -\frac{N_0\big(\zeta(0)x_{i0} + \tau(0)\big)e^{Ct}}{1 + \xi(0)N_0}. \tag{2.26}$$

Based on (2.8), (2.15) and (2.26), we can rewrite (2.8) and the first equation in (2.9) as

$$\begin{aligned} \bar{u}_i(t) = &- R^{-1}B\beta(t)\hat{x}_i(t) + \frac{\big(R^{-1}B\alpha(t) - R^{-1}D\big)N_0\big(\zeta(0)x_{i0} + \tau(0)\big)e^{Ct}}{1 + \xi(0)N_0} \\ &- R^{-1}B\gamma(t) \end{aligned} \tag{2.27}$$

and

$$
\begin{cases}
d\hat{x}_i(t) = \Bigg[\big(A - R^{-1}B^2\beta(t)\big)\hat{x}_i(t) + \dfrac{\Big(R^{-1}B^2\alpha(t) - R^{-1}BD\Big)N_0\big(\zeta(0)x_{i0} + \tau(0)\big)e^{Ct}}{1+\xi(0)N_0} \\
\qquad\qquad - R^{-1}B^2\gamma(t) + F\bar{x}(t) \Bigg] dt + \sigma\hat{x}_i(t)dW_i(t), \\
\hat{x}_i(0) = x_{i0}.
\end{cases}
\tag{2.28}
$$

Equation (2.28) admits a unique solution $\hat{x}_i(\cdot)$, which together with (2.26) in turn determines unique solutions $\hat{p}_i(\cdot)$ and $\hat{y}_i(\cdot)$ of equations (2.15) and (2.16), respectively. Meanwhile, $\hat{q}_i(\cdot)$ and $\hat{z}_i(\cdot)$ are uniquely determined by (2.21) and (2.22), respectively.

Remark 2.2. *From* (2.11)-(2.14), (2.19)-(2.20)*, it follows that* $(\beta, \alpha, \zeta, \xi)$ *is independent of the undetermined limiting state-average* $\bar{x}$ *whereas* (γ, τ) *depends on* $\bar{x}$.

Remark 2.3. *It is required that* $1 + \xi(0)N_0 \neq 0$*. One special case is that* $N_0 = 0$*, and in this case, our problem is reduced to the forward large-population problem by considering system* (2.28) *only. On the other hand, a direct calculation implies*

$$
\xi(0) = \int_0^T e^{2Cv}R^{-1}\big(-2BD\alpha(v) + B^2\alpha^2(v) + D^2\big)dv = \int_0^T e^{2Cv}R^{-1}\big(B\alpha(v) - D\big)^2 dv \geqslant 0.
$$

Therefore, $1 + \xi(0)N_0 \neq 0$ *whenever* $N_0 > 0$*. In summary,* $1 + \xi(0)N_0 \neq 0$ *is always true provided* $N_0 \geqslant 0$.

2.3 The Consistency Condition System

For simplicity of presentation, we introduce the following notations

$$
\begin{aligned}
&\mathbb{A}(t) := A - R^{-1}B^2\beta(t), \quad \Gamma_s^t := e^{\int_s^t \mathbb{A}(r)dr}, \ t \geqslant s, \quad \bar{\Gamma} := e^{\int_0^T |\mathbb{A}(r)|dr},\\
&\Theta_1(s) := \frac{\big(R^{-1}B^2\alpha(s) - R^{-1}BD\big)N_0}{1+\xi(0)N_0}, \quad \Theta_2(s) := -\big(R^{-1}B^2\zeta(s) + R^{-1}BD\big),\\
&\Theta_3(s) := F\beta(s) - QS, \qquad\qquad\qquad \Theta_4(s) := F\zeta(s) + L,\\
&\Theta_5(s) := \frac{\big(R^{-1}B\alpha(s) - R^{-1}D\big)N_0}{1+\xi(0)N_0}, \qquad \Theta_6(s) := R^{-1}BD\beta(s) - H,\\
&\bar{\Theta}_i := \int_0^T |\Theta_i(s)|ds, \ i = 1, \dots, 4.
\end{aligned}
\tag{2.29}
$$

Note that the terms defined in (2.29) are not dependent on $\bar{x}(\cdot)$. We present the following result.

Proposition 2.1. *Assume A, B, Q are nonzero, then $\bar{\Theta}_i, i = 1, \dots, 4$ is bounded.*

Proof. Denote by $\mathcal{A} = \begin{pmatrix} A + \frac{\sigma^2}{2} & -\frac{B^2}{R} \\ -Q & -A - \frac{\sigma^2}{2} \end{pmatrix}$, and $\lambda = \sqrt{(A + \frac{\sigma^2}{2})^2 + \frac{B^2Q}{R}}$ as the positive eigenvalue of $\mathcal{A}$. Then we have

$$
\begin{pmatrix} 0 & 1 \end{pmatrix} e^{\mathcal{A}t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2\lambda}\Big[\Big(\lambda - A - \frac{\sigma^2}{2}\Big)e^{\lambda t} + \Big(\lambda + A + \frac{\sigma^2}{2}\Big)e^{-\lambda t}\Big] > 0.
$$

According to [95], we get the explicit expression of $\beta(t)$ as follows

$$
\begin{aligned}
\beta(t) &= -\Big[\begin{pmatrix} 0 & 1 \end{pmatrix} e^{\mathcal{A}(T-t)} \begin{pmatrix} 0 \\ 1 \end{pmatrix}\Big]^{-1} \begin{pmatrix} 0 & 1 \end{pmatrix} e^{\mathcal{A}(T-t)} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\\
&= Q\big(e^{2\lambda(T-t)} - 1\big)\Big[\Big(\lambda - A - \frac{\sigma^2}{2}\Big)e^{2\lambda(T-t)} + \Big(\lambda + A + \frac{\sigma^2}{2}\Big)\Big]^{-1}
\end{aligned}
\tag{2.30}
$$

and we can see $\beta'(t) < 0,\ t \in [0,T]$. Thus, for $\forall\ t \in [0,T]$

$$
\begin{aligned}
0 \leqslant \beta(t) \leqslant \beta(0) &= \frac{Q\big(e^{2\lambda T}-1\big)}{(\lambda - A - \frac{\sigma^2}{2})e^{2\lambda T} + (\lambda + A + \frac{\sigma^2}{2})} \leqslant \frac{Q\big(e^{2\lambda T}-1\big)}{(\lambda - |A+\frac{\sigma^2}{2}|)(e^{2\lambda T}+1)} \\
&< \frac{Q}{\lambda - |A+\frac{\sigma^2}{2}|} = \frac{R}{B^2}\Big(\lambda + |A + \frac{\sigma^2}{2}|\Big) < \frac{R}{B^2}\cdot 2\lambda \leqslant \frac{R}{B^2}(1+\lambda^2) \\
&= Q + \frac{R}{B^2}\Big[1 + \big(A + \frac{1}{2}\sigma^2\big)^2\Big].
\end{aligned}
$$

Then we get

$$
\sup_{0\leqslant t\leqslant T} |\mathbb{A}(t)| = \sup_{0\leqslant t\leqslant T} |A - R^{-1}B^2\beta(t)| < 1 + |A| + (A + \frac{1}{2}\sigma^2)^2 + R^{-1}B^2 Q
$$

and

$$
\bar{\Gamma} = e^{\int_0^T |\mathbb{A}(r)|dr} < e^{\Big[1+|A|+(A+\frac{1}{2}\sigma^2)^2+R^{-1}B^2Q\Big]T}.
$$

Based on (2.30), we can directly solve the ODEs (2.12)-(2.14) as follows

$$
\begin{cases}
\alpha(t) = -Ke^{C(T-t)}\Gamma_t^T + \displaystyle\int_t^T e^{C(v-t)}\Gamma_t^v \Theta_6(v)dv, \\
\zeta(t) = -\alpha(t), \\
\xi(t) = \displaystyle\int_t^T e^{2C(v-t)}\Big[\big(R^{-1}BD - R^{-1}B^2\alpha(v)\big)\zeta(v) + R^{-1}D^2 - R^{-1}BD\alpha(v)\Big]dv.
\end{cases}
\tag{2.31}
$$

Thus, we obtain

$$
\begin{cases}
\sup\limits_{0\leqslant t\leqslant T} |\alpha(t)| = \sup\limits_{0\leqslant t\leqslant T} |\zeta(t)| \\
\qquad \leqslant \Big[|K| + T\Big(\dfrac{|BD|Q}{R} + \dfrac{|D|}{|B|}[1 + (A + \dfrac{1}{2}\sigma^2)^2] + |H|\Big)\Big] \\
\qquad \cdot\, e^{\Big[1+|A|+(A+\frac{1}{2}\sigma^2)^2+|C|+R^{-1}B^2Q\Big]T}, \\
R(1+\xi(0)N_0) = R + N_0\displaystyle\int_0^T e^{2Cv}(B\alpha(v) - D)^2 dv.
\end{cases}
\tag{2.32}
$$

In addition, we get

$$\begin{cases} \bar{\Theta}_1 = \displaystyle\int_0^T \frac{N_0|B||B\alpha(s)-D|}{R+N_0\int_0^T e^{2Cv}(B\alpha(v)-D)^2dv}ds, \\ \bar{\Theta}_2 = \displaystyle\int_0^T \frac{|B||B\alpha(s)-D|}{R}ds, \\ \bar{\Theta}_3 = \displaystyle\int_0^T |F\beta(s)-QS|ds \leqslant T\left(|F|Q+\frac{|F|R}{B^2}[1+(A+\frac{1}{2}\sigma^2)^2]+Q|S|\right), \\ \bar{\Theta}_4 = \displaystyle\int_0^T |F\alpha(s)-L|ds \end{cases} \tag{2.33}$$

which yields the boundness of $\bar{\Theta}_i, i=1,\ldots,4$. The proof is completed. □

For the given deterministic continuous function $\bar{x}$ defined on $[0,T]$, solving the ODEs (2.19) and (2.20),

$$\begin{cases} \gamma(t) = \displaystyle\int_t^T \Gamma_t^v\big(\Theta_3(v)\bar{x}(v)-Q\eta\big)dv, \\ \tau(t) = \displaystyle\int_t^T e^{C(r-t)}\Theta_2(r)\left(\int_r^T \Gamma_r^v\Big(\Theta_3(v)\bar{x}(v)-Q\eta\Big)dv\right)dr \\ \qquad\quad + \displaystyle\int_t^T e^{C(r-t)}\Theta_4(r)\bar{x}(r)dr. \end{cases} \tag{2.34}$$

Now we can introduce the decentralized feedback strategy for $\mathcal{A}_i$ as follows:

$$\bar{u}_i(t) = -R^{-1}B\beta(t)x_i(t) + \big(\zeta(0)x_{i0}+\tau(0)\big)\Theta_5(t)e^{Ct} - R^{-1}B\gamma(t). \tag{2.35}$$

Applying the decentralized control law (2.35) to $\mathcal{A}_i$, its realized closed-loop state becomes

$$\begin{cases} dx_i(t) = \Big[\mathbb{A}(t)x_i(t) + \big(\zeta(0)x_{i0}+\tau(0)\big)\Theta_1(t)e^{Ct} - R^{-1}B^2\gamma(t) + Fx^{(N)}(t)\Big]dt \\ \qquad\qquad + \sigma x_i(t)dW_i(t), \\ x_i(0) = x_{i0} \end{cases} \tag{2.36}$$

and

$$
\begin{cases}
-dy_i(t) =\Big[Cy_i(t) + \big(H - R^{-1}BD\beta(t)\big)x_i(t) + D\big(\zeta(0)x_{i0} + \tau(0)\big)\Theta_5(t)e^{Ct} \\
\qquad\qquad - R^{-1}BD\gamma(t) + Lx^{(N)}(t)\Big]dt - \sum\limits_{j=1}^{N} z_{ij}(t)dW_j(t), \\
y_i(T) = Kx_i(T).
\end{cases}
\tag{2.37}
$$

Taking summation of the above N equations of (2.36) and dividing by N, we get

$$
\begin{cases}
dx^{(N)}(t) =\Big[\mathbb{A}(t)x^{(N)}(t) + \big(\zeta(0)x_0^{(N)} + \tau(0)\big)\Theta_1(t)e^{Ct} - R^{-1}B^2\gamma(t) \\
\qquad\qquad + Fx^{(N)}(t)\Big]dt + \dfrac{1}{N}\sum\limits_{i=1}^{N} \sigma x_i(t)dW_i(t), \\
x^{(N)}(0) =x_0^{(N)}
\end{cases}
\tag{2.38}
$$

where $x^{(N)}(t) = \frac{1}{N}\sum\limits_{i=1}^{N} x_i(t)$, $x_0^{(N)} = \frac{1}{N}\sum\limits_{i=1}^{N} x_{i0}$. On the other hand, by Itô's isometry and basic theory of stochastic process, we have

$$
\mathbb{E}\left(\int_0^t \sigma x_i(s)dW_i(s)\cdot\int_0^t \sigma x_j(s)dW_j(s)\right) = \begin{cases} \mathbb{E}\int_0^t \sigma^2 x_i^2(s)ds, & j = i; \\ 0, & j \neq i. \end{cases}
$$

Thus, it follows that

$$
\mathbb{E}\Big|\frac{1}{N}\sum_{i=1}^{N}\int_0^t \sigma x_i(s)dW_i(s)\Big|^2 = \frac{1}{N^2}\sum_{i=1}^{N}\mathbb{E}\int_0^t \sigma^2 x_i^2(s)ds = O\Big(\frac{1}{N}\Big).
$$

Then we get

$$
\lim_{N\to+\infty}\frac{1}{N}\sum_{i=1}^{N}\int_0^t \sigma x_i(s)dW_i(s) = 0, \quad \text{in } L^2_{\mathcal{F}_t}(0,T;\mathbb{R}).
$$

Letting $N \to +\infty$ and replacing $x^{(N)}$ by $\bar{x}$, we obtain the following limiting system

$$
\begin{cases}
d\bar{x}(t) = \Big[\big(\mathbb{A}(t) + F\big)\bar{x}(t) + \big(\zeta(0)x_0 + \tau(0)\big)\Theta_1(t)e^{Ct} - R^{-1}B^2\gamma(t)\Big]dt, \\
\bar{x}(0) = x_0.
\end{cases}
\tag{2.39}
$$

We call (2.34) and (2.39) the consistency condition system by which the limiting state-average process can be determined through the fixed-point analysis, as discussed below. Solving the ODE (2.39) directly and noting (2.31) and (2.34), we have

$$
\begin{aligned}
\bar{x}(t) =& x_0\Gamma_0^t e^{Ft} + \int_0^t \Gamma_s^t e^{F(t-s)} x_0 \Theta_1(s) e^{Cs} \cdot K\Gamma_0^T e^{CT} ds \\
&- \int_0^t \Gamma_s^t e^{F(t-s)} x_0 \Theta_1(s) e^{Cs} ds \cdot \int_0^T e^{Cr}\Gamma_0^r \Theta_6(r) dr \\
&+ \int_0^t \Gamma_s^t e^{F(t-s)} \Theta_1(s) e^{Cs} ds \cdot \int_0^T e^{Cr}\Theta_2(r) \left(\int_r^T \Gamma_r^v \Big(\Theta_3(v)\bar{x}(v) - Q\eta \Big) dv \right) dr \\
&+ \int_0^t \Gamma_s^t e^{F(t-s)} \Theta_1(s) e^{Cs} ds \cdot \int_0^T e^{Cr}\Theta_4(r)\bar{x}(r) dr \\
&- \int_0^t \Gamma_s^t e^{F(t-s)} R^{-1}B^2 \left(\int_s^T \Gamma_s^v \Big(\Theta_3(v)\bar{x}(v) - Q\eta \Big) dv \right) ds \\
:=& (\mathcal{T}\bar{x})(t).
\end{aligned}
\tag{2.40}
$$

To apply the contraction mapping, hereafter we introduce the following assumption:

(H2.2) $e^{(2|C|+|F|)T}\bar{\Gamma}^2\bar{\Theta}_1\bar{\Theta}_2\bar{\Theta}_3 + e^{(2|C|+|F|)T}\bar{\Gamma}\bar{\Theta}_1\bar{\Theta}_4 + e^{|F|T}R^{-1}B^2T\bar{\Gamma}^2\bar{\Theta}_3 < 1.$

Then the following theorem is obtained.

Theorem 2.1. *Under* (H2.2)*, the map* $\mathcal{T} : C(0,T;\mathbb{R}) \to C(0,T;\mathbb{R})$ *described by* (2.40) *has a unique fixed point. Moreover, the decentralized feedback strategy* $\bar{u}_i$, $1 \leqslant i \leqslant N$ *in* (2.35) *is uniquely determined .*

Proof. For any $x, y \in C(0,T;\mathbb{R})$, we have

$$\begin{aligned}
&\big\|(\mathcal{T}x-\mathcal{T}y)(t)\big\|_\infty\\
=&\bigg\|\int_0^t \Gamma_s^t e^{F(t-s)}\Theta_1(s)e^{Cs}ds\cdot\int_0^T e^{Cr}\Theta_2(r)\bigg[\int_r^T \Gamma_r^v\Theta_3(v)\big(x(v)-y(v)\big)dv\bigg]dr\\
&+\int_0^t \Gamma_s^t e^{F(t-s)}\Theta_1(s)e^{Cs}ds\cdot\int_0^T e^{Cr}\Theta_4(r)\big(x(r)-y(r)\big)dr \qquad (2.41)\\
&-\int_0^t \Gamma_s^t e^{F(t-s)}R^{-1}B^2\left(\int_s^T \Gamma_s^v\Theta_3(v)\big(x(v)-y(v)\big)dv\right)ds\bigg\|_\infty\\
\leqslant&\|x-y\|_\infty\Big(e^{(2|C|+|F|)T}\bar{\Gamma}^2\bar{\Theta}_1\bar{\Theta}_2\bar{\Theta}_3+e^{(2|C|+|F|)T}\bar{\Gamma}\bar{\Theta}_1\bar{\Theta}_4+e^{|F|T}R^{-1}B^2T\bar{\Gamma}^2\bar{\Theta}_3\Big).
\end{aligned}$$

From (H2.2), $\mathcal{T}$ defined by (2.40) is a contraction and has a unique fixed point $\bar{x} \in C(0,T;\mathbb{R})$ which is equivalently given by (2.39) and in turn uniquely determines γ and τ in (2.34). Meanwhile, the solutions γ and τ to (2.19) and (2.20) are equivalently given by (2.34), respectively. Then $\bar{u}_i$ is uniquely determined, which completes the proof. □

Remark 2.4. *(1) From Theorem 2.1, there exists a unique deterministic function $\bar{x}$ in $C(0,T;\mathbb{R})$ to approximate the state-average of forward system. In next section, we specify more details of their difference when applying the system* (2.39)*.*

(2) The limit process $\bar{x}$ in forward equation (2.39) *only involves $\tau(0)$ and $\gamma(t)$. On the other hand,* (2.34) *satisfies the backward system* (2.19) *and* (2.20) *which actually depends on $\bar{x}$. Thus* (2.39) *and* (2.34) *constitute a forward-backward ordinary differential equation (FBODE) system. Here, we focus on the fixed point analysis in Theorem 2.1 which provides one sufficient condition for the well-posedness of FBODE system* (2.39) *and* (2.34)*.*

Remark 2.5. *By Proposition 2.1, if R is large enough and $|F|$ is small enough (it corresponds to the weak-coupling of state-average, see e.g., [8]), we get that $\bar{\Theta}_1\bar{\Theta}_2\bar{\Theta}_3$, $\bar{\Theta}_1\bar{\Theta}_4$ and $R^{-1}\bar{\Theta}_3$ should be small enough hence* (H2.2) *follows.*

Remark 2.6. *(1) One interesting special case is when* $N_0 = 0$ *which corresponds to the forward large-population problem only. In this case, we have* $\bar{\Theta}_1 = 0$, *and* (H2.2) *reads as below:*

(H2.2)' $e^{|F|T} R^{-1} B^2 T \bar{\Gamma}^2 \bar{\Theta}_3 < 1$

which is similar to that of [18] but noting our diffusion term in (2.1) *depends on state itself while in [18] the diffusion term is constant. In addition, different to* (H2.2), (H2.2)' *does not depend on* C. *This is because the dynamic system in this case is irrelevant with the backward one.*

(2) Another interesting special case is when $N_0 > 0$ *but* $Q = 0$. *In this case, the cost functional becomes*

$$\mathcal{J}_i(u_i(\cdot), u_{-i}(\cdot)) = \frac{1}{2}\mathbb{E}\left\{\int_0^T Ru_i^2(t)dt + N_0 y_i^2(0)\right\}$$

which takes into account the initial hedging cost via $N_0 y_i^2(0)$, *and we have* $\beta(t) \equiv 0$ *and thus* $\bar{\Theta}_3 = 0$. *Now* (H2.2) *reads as follows*

(H2.2)" $e^{(2|C|+|F|)T} \bar{\Gamma} \bar{\Theta}_1 \bar{\Theta}_4 < 1.$

To get a more clear result, further assume $H = K = 0, AC \neq 0, A \pm C \neq 0$. *In this case, we have* $\mathbb{A}(t) \equiv A$, $\Gamma_s^t = e^{A(t-s)}$, $\bar{\Gamma} = e^{|A|T}$, $\Theta_6(t) \equiv 0$ *and* $\alpha(t) \equiv 0$. *Then we obtain*

$$\int_0^T e^{2Cv}\big(B\alpha(v) - D\big)^2 dv = \frac{D^2}{2C}(e^{2CT} - 1),$$

$$\bar{\Theta}_1 = \frac{2C|B||D|N_0 T}{2CR + D^2 N_0 (e^{2CT} - 1)},$$

$$\bar{\Theta}_4 = |L|T.$$

Thus, (H2.2)" *implies*

$$\frac{2C|B||D||L|N_0 T^2}{2CR + D^2 N_0(e^{2CT} - 1)} e^{(|A|+2|C|+|F|)T} < 1.$$

If, in system (2.1)-(2.3)*, we fix*

$$\big[A, B, C, D, F, H, L, K, Q, R, N_0\big] = \Big[1, \frac{1}{2}, \frac{1}{2}, 1, 1, 0, \frac{1}{2}, 0, 0, 350, 100\Big]$$

and $T = 1$*,* (H2.2) *implies*

$$\begin{aligned}
&e^{(2|C|+|F|)T}\bar{\Gamma}^2\bar{\Theta}_1\bar{\Theta}_2\bar{\Theta}_3 + e^{(2|C|+|F|)T}\bar{\Gamma}\bar{\Theta}_1\bar{\Theta}_4 + e^{|F|T}R^{-1}B^2T\bar{\Gamma}^2\bar{\Theta}_3 \\
=&\frac{2C|B||D||L|N_0T^2}{2CR + D^2N_0(e^{2CT} - 1)}e^{(|A|+2|C|+|F|)T} \\
=&\frac{e^3}{4e + 10}
\end{aligned}$$

which is obviously less than 1. Thus, the fix-point assumption (H2.2) [(H2.2)”] *holds.*

2.4 ϵ-Nash Equilibrium Analysis for (FB-MFG)

In above sections, we obtained the optimal control $\bar{u}_i(\cdot), 1 \leqslant i \leqslant N$ of Problem **(L-FB-MFG)** through the consistency condition system. Now we turn to verify the ϵ-Nash equilibrium of Problem **(FB-MFG)**. Due to its own forward-backward structure, our analysis here is not simple extension of that in the forward large-population system. More details are as follows. To start, we first present the definition of ϵ-Nash equilibrium.

Definition 2.1. *A set of controls* $u_k(\cdot) \in \mathcal{U}_k,\ 1 \leqslant k \leqslant N$*, for* N *agents is called an* ϵ*-Nash equilibrium with respect to the costs* $\mathcal{J}_k,\ 1 \leqslant k \leqslant N$*, if there exists* $\epsilon \geqslant 0$ *such that for any fixed* $1 \leqslant i \leqslant N$*, we have*

$$\mathcal{J}_i(u_i, u_{-i}) \leqslant \mathcal{J}_i(u_i', u_{-i}) + \epsilon \tag{2.42}$$

when any alternative control $u_i'(\cdot) \in \mathcal{U}_i$ *is applied by* $\mathcal{A}_i$*.*

Now, we state the following result and its proof will be given later.

Theorem 2.2. *Under* (H2.1)-(H2.2), $(\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_N)$ *in Problem* **(FB-MFG)** *satisfies the ϵ-Nash equilibrium where, for $1 \leqslant i \leqslant N$, $\tilde{u}_i$ is given by*

$$\tilde{u}_i(t) = -R^{-1}B\beta(t)\tilde{x}_i(t) + \big(\zeta(0)x_{i0} + \tau(0)\big)\Theta_5(t)e^{Ct} - R^{-1}B\gamma(t) \tag{2.43}$$

for $\tilde{x}_i(\cdot)$ satisfying (2.36), *the decentralized state trajectory for $\mathcal{A}_i$.*

The proof of above theorem needs several lemmas which are presented later. We first introduce the optimal control and state of auxiliary limiting system as

$$\bar{u}_i(t) = -R^{-1}B\beta(t)\hat{x}_i(t) + \big(\zeta(0)x_{i0} + \tau(0)\big)\Theta_5(t)e^{Ct} - R^{-1}B\gamma(t).$$

Note that $\{\tilde{u}_i(\cdot)\}_{i=1}^N$ are different from $\{\bar{u}_i(\cdot)\}_{i=1}^N$, as $\tilde{x}_i(\cdot)$ differs from $\hat{x}_i(\cdot)$ which is the decentralized state of auxiliary system. Applying $\tilde{u}_i(\cdot)$ for $\mathcal{A}_i$, we have the following close-loop system

$$\begin{cases} d\tilde{x}_i(t) =\Big[\mathbb{A}(t)\tilde{x}_i(t) + \big(\zeta(0)x_{i0} + \tau(0)\big)\Theta_1(t)e^{Ct} - R^{-1}B^2\gamma(t) + F\tilde{x}^{(N)}(t)\Big]dt \\ \qquad\qquad + \sigma\tilde{x}_i(t)dW_i(t), \\ \tilde{x}_i(0) =x_{i0} \end{cases} \tag{2.44}$$

and

$$\begin{cases} -d\tilde{y}_i(t) = \Big[C\tilde{y}_i(t) + \big(H - R^{-1}BD\beta(t)\big)\tilde{x}_i(t) + D\big(\zeta(0)x_{i0} + \tau(0)\big)\Theta_5(t)e^{Ct} \\ \qquad\qquad - R^{-1}BD\gamma(t) + L\tilde{x}^{(N)}(t)\Big]dt - \displaystyle\sum_{j=1}^N \tilde{z}_{ij}(t)dW_j(t), \\ \tilde{y}_i(T) = K\tilde{x}_i(T) \end{cases} \tag{2.45}$$

with the cost functional

$$\mathcal{J}_i(\tilde{u}_i(\cdot), \tilde{u}_{-i}(\cdot))=\frac{1}{2}\mathbb{E}\left\{\int_0^T \Big[Q\Big(\tilde{x}_i(t)-\big(S\tilde{x}^{(N)}(t)+\eta\big)\Big)^2+R\tilde{u}_i^2(t)\Big]dt+N_0\tilde{y}_i^2(0)\right\} \tag{2.46}$$

where $\tilde{x}^{(N)}(t) = \frac{1}{N}\sum\limits_{i=1}^{N}\tilde{x}_i(t)$. The auxiliary system (of limiting problem) is given by

$$\begin{cases} d\hat{x}_i(t) = \Big[\mathbb{A}(t)\hat{x}_i(t) + \big(\zeta(0)x_{i0} + \tau(0)\big)\Theta_1(t)e^{Ct} - R^{-1}B^2\gamma(t) + F\bar{x}(t)\Big]dt \\ \qquad\qquad + \sigma\hat{x}_i(t)dW_i(t), \\ \hat{x}_i(0) = x_{i0} \end{cases} \tag{2.47}$$

and

$$\begin{cases} -d\hat{y}_i(t) = \Big[C\hat{y}_i(t) + \big(H - R^{-1}BD\beta(t)\big)\hat{x}_i(t) + D\big(\zeta(0)x_{i0} + \tau(0)\big)\Theta_5(t)e^{Ct} \\ \qquad\qquad - R^{-1}BD\gamma(t) + L\bar{x}(t)\Big]dt - \hat{z}_i(t)dW_i(t), \\ \hat{y}_i(T) = K\hat{x}_i(T) \end{cases} \tag{2.48}$$

with the cost functional

$$J_i(\bar{u}_i(\cdot)) = \frac{1}{2}\mathbb{E}\left\{\int_0^T \Big[Q\Big(\hat{x}_i(t) - \big(S\bar{x}(t) + \eta\big)\Big)^2 + R\bar{u}_i^2(t)\Big]dt + N_0\hat{y}_i^2(0)\right\}. \tag{2.49}$$

We have

Lemma 2.2.

$$\sup_{0\leqslant t\leqslant T}\mathbb{E}\Big|\tilde{x}^{(N)}(t) - \bar{x}(t)\Big|^2 = O\Big(\frac{1}{N}\Big). \tag{2.50}$$

Proof. By (2.44), we have

$$\begin{cases} d\tilde{x}^{(N)}(t) = \Big[\big(\mathbb{A}(t) + F\big)\tilde{x}^{(N)}(t) + \big(\zeta(0)x_0^{(N)} + \tau(0)\big)\Theta_1(t)e^{Ct} - R^{-1}B^2\gamma(t)\Big]dt \\ \qquad\qquad + \dfrac{1}{N}\sum\limits_{i=1}^{N}\sigma\tilde{x}_i(t)dW_i(t), \\ \tilde{x}^{(N)}(0) = x_0^{(N)} \end{cases}$$

where $x_0^{(N)}$ is given in (2.38). Noting (2.39), we get

$$
\begin{cases}
d\Big(\tilde{x}^{(N)}(t)-\bar{x}(t)\Big) = & \Big[\big(\mathbb{A}(t)+F\big)\big(\tilde{x}^{(N)}(t)-\bar{x}(t)\big)+\zeta(0)\Theta_1(t)e^{Ct}\big(x_0^{(N)}-x_0\big)\Big]dt \\
& \quad + \dfrac{1}{N}\displaystyle\sum_{i=1}^{N}\sigma\tilde{x}_i(t)dW_i(t), \\
\tilde{x}^{(N)}(0)-\bar{x}(0) = & x_0^{(N)}-x_0.
\end{cases}
\tag{2.51}
$$

Thus

$$
\begin{aligned}
\Big|\tilde{x}^{(N)}(t)-\bar{x}(t)\Big|^2 \leqslant & 3\Big|x_0^{(N)}-x_0\Big|^2+6t\int_0^t\bigg(\Big|\mathbb{A}(s)+F\Big|^2\Big|\tilde{x}^{(N)}(s)-\bar{x}(s)\Big|^2 \\
& +\Big|\zeta(0)\Theta_1(s)e^{Cs}\Big|^2\Big|x_0^{(N)}-x_0\Big|^2\bigg)ds+3\Big|\int_0^t\frac{1}{N}\sum_{i=1}^{N}\sigma\tilde{x}_i(s)dW_i(s)\Big|^2.
\end{aligned}
$$

By (H2.1), we have

$$
\mathbb{E}\Big|x_0^{(N)}-x_0\Big|^2=\mathbb{E}\Big|\frac{1}{N}\sum_{i=1}^{N}x_{i0}-x_0\Big|^2=O\Big(\frac{1}{N}\Big).
$$

Noting $\sup\limits_{0\leqslant t\leqslant T}\mathbb{E}\tilde{x}_i^2(t)<+\infty$, we have

$$
\mathbb{E}\left|\int_0^T\frac{1}{N}\sum_{i=1}^{N}\sigma\tilde{x}_i(s)dW_i(s)\right|^2=O\Big(\frac{1}{N}\Big).
$$

Thus, (2.50) follows by Gronwall's inequality. □

Considering the difference between the decentralized and centralized states and controls, we have the following estimates:

Lemma 2.3.

$$
\sup_{1\leqslant i\leqslant N}\left[\sup_{0\leqslant t\leqslant T}\mathbb{E}\Big|\tilde{x}_i(t)-\hat{x}_i(t)\Big|^2\right]=O\Big(\frac{1}{N}\Big),
\tag{2.52}
$$

$$
\sup_{1\leqslant i\leqslant N}\left[\sup_{0\leqslant t\leqslant T}\mathbb{E}\Big|\tilde{u}_i(t)-\bar{u}_i(t)\Big|^2\right]=O\Big(\frac{1}{N}\Big),
\tag{2.53}
$$

$$
\sup_{1\leqslant i\leqslant N}\left[\sup_{0\leqslant t\leqslant T}\mathbb{E}\Big|\tilde{y}_i(t)-\hat{y}_i(t)\Big|^2\right]=O\Big(\frac{1}{N}\Big).
\tag{2.54}
$$

Proof. For $\forall\ 1 \leqslant i \leqslant N$, by (2.44) and (2.47), we get

$$\sup_{0\leqslant t\leqslant T}\mathbb{E}\Big|\tilde{x}_i(t)-\hat{x}_i(t)\Big|^2 \leqslant 3\Big[T\|\mathbb{A}(t)\|_\infty^2+\sigma^2\Big]\int_0^T\mathbb{E}\Big|\tilde{x}_i(s)-\hat{x}_i(s)\Big|^2 ds$$

$$+\,3T|F|^2\int_0^T\mathbb{E}\Big|\tilde{x}^{(N)}(s)-\bar{x}(s)\Big|^2 ds.$$

Then (2.52) follows from Lemma 2.2. Noting the difference between $\tilde{u}_i(\cdot)$ and $\bar{u}_i(\cdot)$, (2.53) is obtained by (2.52). From (2.45) and (2.48), we have

$$\begin{cases} -d\Big(\tilde{y}_i(t)-\hat{y}_i(t)\Big)=\Big[C\big(\tilde{y}_i(t)-\hat{y}_i(t)\big)+\big(H-R^{-1}BD\beta(t)\big)\big(\tilde{x}_i(t)-\hat{x}_i(t)\big) \\ \qquad +L\big(\tilde{x}^{(N)}(t)-\bar{x}(t)\big)\Big]dt-\big(\tilde{z}_{ii}(t)-\hat{z}_i(t)\big)dW_i(t)-\displaystyle\sum_{j=1,j\neq i}^{N}\tilde{z}_{ij}(t)dW_j(t), \\ \tilde{y}_i(T)-\hat{y}_i(T)=K\big(\tilde{x}_i(T)-\hat{x}_i(T)\big). \end{cases} \tag{2.55}$$

Applying the basic estimate of BSDE, we get

$$\mathbb{E}\left[\sup_{0\leqslant t\leqslant T}\Big|\tilde{y}_i(t)-\hat{y}_i(t)\Big|^2\right]+\mathbb{E}\int_0^T\Big|\tilde{z}_{ii}(t)-\hat{z}_i(t)\Big|^2dt+\sum_{j=1,j\neq i}^{N}\mathbb{E}\int_0^T\Big|\tilde{z}_{ij}(t)\Big|^2dt$$

$$\leqslant C_1\left\{\mathbb{E}\Big|\tilde{x}_i(T)-\hat{x}_i(T)\Big|^2+\mathbb{E}\int_0^T\Big|H-R^{-1}BD\beta(t)\Big|^2\Big|\tilde{x}_i(t)-\hat{x}_i(t)\Big|^2dt\right.$$

$$\left.+\mathbb{E}\int_0^T\Big|\tilde{x}^{(N)}(t)-\bar{x}(t)\Big|^2dt\right\},$$

where C_1 is a positive constant. Thus, we get (2.54) by Lemma 2.2 and (2.52). □

Lemma 2.4. *For* $\forall\ 1 \leqslant i \leqslant N$,

$$\Big|\mathcal{J}_i(\tilde{u}_i,\tilde{u}_{-i})-J_i(\bar{u}_i)\Big|=O\Big(\frac{1}{\sqrt{N}}\Big). \tag{2.56}$$

Proof. For $\forall\ 1 \leqslant i \leqslant N$, by (2.39) and (2.47), we easily get

$\sup\limits_{0\leqslant t\leqslant T}\mathbb{E}\big|\hat{x}_i(t)-\big(S\bar{x}(t)+\eta\big)\big|^2<+\infty$. Applying Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
&\sup_{0\leqslant t\leqslant T}\mathbb{E}\Big|\big|\tilde{x}_i(t)-\big(S\tilde{x}^{(N)}(t)+\eta\big)\big|^2-\big|\hat{x}_i(t)-\big(S\bar{x}(t)+\eta\big)\big|^2\Big|\\
\leqslant&\sup_{0\leqslant t\leqslant T}\mathbb{E}\big|\tilde{x}_i(t)-\big(S\tilde{x}^{(N)}(t)+\eta\big)-\hat{x}_i(t)+\big(S\bar{x}(t)+\eta\big)\big|^2\\
&+2\sup_{0\leqslant t\leqslant T}\mathbb{E}\Big[\big|\hat{x}_i(t)-\big(S\bar{x}(t)+\eta\big)\big|\big|\tilde{x}_i(t)-\big(S\tilde{x}^{(N)}(t)+\eta\big)-\hat{x}_i(t)+\big(S\bar{x}(t)+\eta\big)\big|\Big]\\
\leqslant&\sup_{0\leqslant t\leqslant T}\mathbb{E}\big|\tilde{x}_i(t)-\hat{x}_i(t)-S\big(\tilde{x}^{(N)}(t)-\bar{x}(t)\big)\big|^2\\
&+2\Big(\sup_{0\leqslant t\leqslant T}\mathbb{E}\big|\hat{x}_i(t)-\big(S\bar{x}(t)+\eta\big)\big|^2\Big)^{\frac{1}{2}}\Big(\sup_{0\leqslant t\leqslant T}\mathbb{E}\big|\tilde{x}_i(t)-\hat{x}_i(t)-S\big(\tilde{x}^{(N)}(t)-\bar{x}(t)\big)\big|^2\Big)^{\frac{1}{2}}\\
=&O\Big(\frac{1}{\sqrt{N}}\Big)
\end{aligned}
$$

where the last equality is obtained by using the fact

$$
\begin{aligned}
&\sup_{0\leqslant t\leqslant T}\mathbb{E}\big|\tilde{x}_i(t)-\hat{x}_i(t)-S\big(\tilde{x}^{(N)}(t)-\bar{x}(t)\big)\big|^2\\
&\qquad\qquad\leqslant 2\sup_{0\leqslant t\leqslant T}\mathbb{E}\big|\tilde{x}_i(t)-\hat{x}_i(t)\big|^2+2S^2\sup_{0\leqslant t\leqslant T}\mathbb{E}\big|\tilde{x}^{(N)}(t)-\bar{x}(t)\big|^2
\end{aligned}
$$

and Lemma 2.2, 2.3. Similarly, by (2.53) and (2.54), we get

$$
\sup_{0\leqslant t\leqslant T}\mathbb{E}\Big|\big|\tilde{u}_i(t)\big|^2-\big|\bar{u}_i(t)\big|^2\Big|=O\Big(\frac{1}{\sqrt{N}}\Big),
$$

and

$$
\sup_{0\leqslant t\leqslant T}\mathbb{E}\Big|\big|\tilde{y}_i(t)\big|^2-\big|\hat{y}_i(t)\big|^2\Big|=O\Big(\frac{1}{\sqrt{N}}\Big).
$$

Further,

$$
\mathbb{E}\Big|\big|\tilde{y}_i(0)\big|^2-\big|\hat{y}_i(0)\big|^2\Big|=O\Big(\frac{1}{\sqrt{N}}\Big).
$$

Then

$$
\begin{aligned}
&\Big|\mathcal{J}_i(\tilde{u}_i, \tilde{u}_{-i}) - J_i(\bar{u}_i)\Big| \\
\leqslant&\frac{1}{2}\mathbb{E}\int_0^T \Bigg[Q\bigg|\Big(\tilde{x}_i(t) - \big(S\tilde{x}^{(N)}(t) + \eta\big)\Big)^2 - \Big(\hat{x}_i(t) - \big(S\bar{x}(t) + \eta\big)\Big)^2\bigg| \\
&\quad + R\Big|\tilde{u}_i^2(t) - \bar{u}_i^2(t)\Big|\Bigg]\, dt + \frac{1}{2}N_0\mathbb{E}\Big|\tilde{y}_i^2(0) - \hat{y}_i^2(0)\Big| \\
=&O\Big(\frac{1}{\sqrt{N}}\Big),
\end{aligned}
$$

which completes the proof. □

Now, we already present some estimates of states and costs corresponding to control $\tilde{u}_i$ and $\bar{u}_i$,$1 \leqslant i \leqslant N$. Our next work is to prove that the control strategies set $(\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_N)$ is an ϵ-Nash equilibrium for **(FB-MFG)**. For any fixed i, $1 \leqslant i \leqslant N$, consider a perturbed control $u_i \in \mathcal{U}_i$ for $\mathcal{A}_i$ and introduce

$$
\begin{cases}
dl_i(t) =\Big[Al_i(t) + Bu_i(t) + Fl^{(N)}(t)\Big]dt + \sigma l_i(t)dW_i(t), \\
l_i(0) =x_{i0}
\end{cases}
\tag{2.57}
$$

whereas other agents keep the control $\tilde{u}_j, 1 \leqslant j \leqslant N, j \neq i$, i.e.,

$$
\begin{cases}
dl_j(t) =\Big[\mathbb{A}(t)l_j(t) + \big(\zeta(0)x_{j0} + \tau(0)\big)\Theta_1(t)e^{Ct} - R^{-1}B^2\gamma(t) + Fl^{(N)}(t)\Big]dt \\
\qquad\qquad + \sigma l_j(t)dW_j(t), \\
l_j(0) =x_{j0}
\end{cases}
\tag{2.58}
$$

where $l^{(N)}(t) = \frac{1}{N}\sum\limits_{k=1}^{N} l_k(t)$. Similar to the forward system, the backward system is introduced as

$$
\begin{cases}
-dm_i(t) =\Big[Cm_i(t) + Du_i(t) + Hl_i(t) + Ll^{(N)}(t)\Big]dt - \sum\limits_{k=1}^{N} n_{ik}(t)dW_k(t), \\
m_i(T) =Kl_i(T)
\end{cases}
\tag{2.59}
$$

while for $j \neq i$,

$$
\begin{cases}
-dm_j(t) = \Big[Cm_j(t) + \big(H - R^{-1}BD\beta(t)\big)l_j(t) + D\big(\zeta(0)x_{j0} + \tau(0)\big)\Theta_5(t)e^{Ct} \\
\qquad\qquad - R^{-1}BD\gamma(t) + Ll^{(N)}(t)\Big]dt - \sum_{k=1}^{N} n_{jk}(t)dW_k(t), \\
m_j(T) = Kl_j(T).
\end{cases}
\tag{2.60}
$$

If $\tilde{u}_i,\ 1 \leqslant i \leqslant N$ is an ϵ-Nash equilibrium with respect to cost $\mathcal{J}_i$, it holds that

$$
\mathcal{J}_i(\tilde{u}_i, \tilde{u}_{-i}) \geqslant \inf_{u_i \in \mathcal{U}_i} \mathcal{J}_i(u_i, \tilde{u}_{-i}) \geqslant \mathcal{J}_i(\tilde{u}_i, \tilde{u}_{-i}) - \epsilon.
$$

Then, when making the perturbation, we just need to consider $u_i \in \mathcal{U}_i$ such that $\mathcal{J}_i(u_i, \tilde{u}_{-i}) \leqslant \mathcal{J}_i(\tilde{u}_i, \tilde{u}_{-i})$, which implies

$$
\frac{1}{2}\mathbb{E}\int_0^T Ru_i^2(t)dt \leqslant \mathcal{J}_i(u_i, \tilde{u}_{-i}) \leqslant \mathcal{J}_i(\tilde{u}_i, \tilde{u}_{-i}) = J_i(\bar{u}_i) + O\Big(\frac{1}{\sqrt{N}}\Big),
$$

i.e.,

$$
\mathbb{E}\int_0^T u_i^2(t)dt \leqslant C_2 \tag{2.61}
$$

where C_2 is a positive constant which is independent of N. Then we have the following proposition.

Proposition 2.2. $\sup_{1 \leqslant j \leqslant N} \Big[\sup_{0 \leqslant t \leqslant T} \mathbb{E}\big|l_j(t)\big|^2\Big]$ *is bounded.*

Proof. By (2.57) and (2.58), it holds that

$$
|l_i(t)|^2 \leqslant C_3 \left\{|x_{i0}|^2 + \int_0^t \Big[|l_i(s)|^2 + |u_i(s)|^2 + \frac{1}{N}\sum_{k=1}^{N}|l_k(s)|^2\Big]ds + \Big|\int_0^t \sigma l_i(s)dW_i(s)\Big|^2\right\}
$$

and for $j \neq i$,

$$
|l_j(t)|^2 \leqslant C_3 \left\{|x_{j0}|^2 + \int_0^t \Big[|l_j(s)|^2 + |\tilde{u}_j(s)|^2 + \frac{1}{N}\sum_{k=1}^{N}|l_k(s)|^2\Big]ds + \Big|\int_0^t \sigma l_j(s)dW_j(s)\Big|^2\right\}
$$

where C_3 is a positive constant. Thus,

$$\begin{aligned}
\mathbb{E}\Big[\sum_{k=1}^{N}|l_k(t)|^2\Big] \leqslant & C_3\left\{\mathbb{E}\Big[\sum_{k=1}^{N}|x_{k0}|^2\Big]+\mathbb{E}\int_0^t\Big[\sum_{k=1}^{N}|l_k(s)|^2+|u_i(s)|^2+\sum_{k=1,k\neq i}^{N}|\tilde{u}_k(s)|^2\right.\\
&\left.+\sum_{k=1}^{N}|l_k(s)|^2\Big]ds+\sum_{k=1}^{N}\mathbb{E}\Big|\int_0^t\sigma l_k(s)dW_k(s)\Big|^2\right\}\\
\leqslant & C_3\left\{\sum_{k=1}^{N}\mathbb{E}|x_{k0}|^2+\int_0^t\Big[2\sum_{k=1}^{N}\mathbb{E}|l_k(s)|^2+\mathbb{E}|u_i(s)|^2\right.\\
&\left.+\sum_{k=1,k\neq i}^{N}\mathbb{E}|\tilde{u}_k(s)|^2\Big]ds+\int_0^t\sum_{k=1}^{N}\mathbb{E}|l_k(s)|^2ds\right\}.
\end{aligned}$$

By (2.61), we can see that $u_i(\cdot)$ is L^2-bounded. Besides, the decentralized optimal controls $\tilde{u}_k(\cdot), k\neq i$ are L^2-bounded. Then by Gronwall's inequality, it follows that

$$\sup_{0\leqslant t\leqslant T}\mathbb{E}\left[\sum_{k=1}^{N}|l_k(t)|^2\right]=O(N),$$

and for any $1\leqslant j\leqslant N$, $\sup\limits_{0\leqslant t\leqslant T}\mathbb{E}|l_j(t)|^2$ is bounded. □

Correspondingly, the system for agent $\mathcal{A}_i$ under control u_i in **(L-FB-MFG)** is as follows

$$\begin{cases} dl_i^0(t)=\Big[Al_i^0(t)+Bu_i(t)+F\bar{x}(t)\Big]dt+\sigma l_i^0(t)dW_i(t),\\ l_i^0(0)=x_{i0}\end{cases}\tag{2.62}$$

and for agent $\mathcal{A}_j,\ j\neq i$,

$$\begin{cases} d\hat{l}_j(t)=\Big[\mathbb{A}(t)\hat{l}_j(t)+\big(\zeta(0)x_{j0}+\tau(0)\big)\Theta_1(t)e^{Ct}-R^{-1}B^2\gamma(t)+F\bar{x}(t)\Big]dt\\ \qquad\qquad+\sigma\hat{l}_j(t)dW_j(t),\\ \hat{l}_j(0)=x_{j0}\end{cases}\tag{2.63}$$

coupled with the backward systems

$$
\begin{cases}
-dm_i^0(t) = \Big[Cm_i^0(t) + Du_i(t) + Hl_i^0(t) + L\bar{x}(t)\Big]dt - n_i^0(t)dW_i(t), \\
\quad m_i^0(T) = Kl_i^0(T)
\end{cases}
\tag{2.64}
$$

for $j \neq i$,

$$
\begin{cases}
-d\hat{m}_j(t) = \Big[C\hat{m}_j(t) + \big(H - R^{-1}BD\beta(t)\big)\hat{l}_j(t) + D\big(\zeta(0)x_{j0} + \tau(0)\big)\Theta_5(t)e^{Ct} \\
\qquad\qquad - R^{-1}BD\gamma(t) + L\bar{x}(t)\Big]dt - \hat{n}_j(t)dW_j(t), \\
\quad \hat{m}_j(T) = K\hat{l}_j(T).
\end{cases}
\tag{2.65}
$$

In order to give necessary estimates in Problem **(FB-MFG)** and **(L-FB-MFG)**, we introduce the intermediate states as

$$
\begin{cases}
d\check{l}_i(t) = \left[A\check{l}_i(t) + Bu_i(t) + \dfrac{N-1}{N}F\check{l}^{(N-1)}(t)\right]dt + \sigma\check{l}_i(t)dW_i(t), \\
\quad \check{l}_i(0) = x_{i0}
\end{cases}
\tag{2.66}
$$

and for $j \neq i$,

$$
\begin{cases}
d\check{l}_j(t) = \Big[\mathbb{A}(t)\check{l}_j(t) + \big(\zeta(0)x_{j0} + \tau(0)\big)\Theta_1(t)e^{Ct} - R^{-1}B^2\gamma(t) \\
\qquad\qquad + \dfrac{N-1}{N}F\check{l}^{(N-1)}(t)\Big]dt + \sigma\check{l}_j(t)dW_j(t), \\
\quad \check{l}_j(0) = x_{j0}
\end{cases}
\tag{2.67}
$$

where $\check{l}^{(N-1)}(t) = \frac{1}{N-1}\sum\limits_{j=1,j\neq i}^{N}\check{l}_j(t)$. Denoting $l^{(N-1)}(t) = \frac{1}{N-1}\sum\limits_{j=1,j\neq i}^{N}l_j(t)$, by (2.58) and (2.67), we get

$$
\begin{cases}
dl^{(N-1)}(t) = \left[\Big(\mathbb{A}(t) + \dfrac{N-1}{N}F\Big)l^{(N-1)}(t) + \big(\zeta(0)x_0^{(N-1)} + \tau(0)\big)\Theta_1(t)e^{Ct} \right. \\
\qquad\qquad \left. -R^{-1}B^2\gamma(t) + \dfrac{F}{N}l_i(t)\right]dt + \dfrac{1}{N-1}\displaystyle\sum_{j=1,j\neq i}^{N}\sigma l_j(t)dW_j(t), \\
\quad l^{(N-1)}(0) = x_0^{(N-1)}
\end{cases}
\tag{2.68}
$$

and

$$\begin{cases} d\check{l}^{(N-1)}(t) =& \Big[\big(\mathbb{A}(t) + \dfrac{N-1}{N}F\big)\check{l}^{(N-1)}(t) + \big(\zeta(0)x_0^{(N-1)} + \tau(0)\big)\Theta_1(t)e^{Ct} \\ & - R^{-1}B^2\gamma(t)\Big]dt + \dfrac{1}{N-1}\displaystyle\sum_{j=1,j\neq i}^{N}\sigma\check{l}_j(t)dW_j(t), \\ \check{l}^{(N-1)}(0) =& x_0^{(N-1)} \end{cases} \tag{2.69}$$

where $x_0^{(N-1)} = \frac{1}{N-1}\sum\limits_{j=1,j\neq i}^{N} x_{j0}$. We have the following estimates on these states.

Proposition 2.3.

$$\sup_{0\leqslant t\leqslant T}\mathbb{E}\Big|l^{(N-1)}(t) - \check{l}^{(N-1)}(t)\Big|^2 = O\Big(\frac{1}{N}\Big), \tag{2.70}$$

$$\sup_{0\leqslant t\leqslant T}\mathbb{E}\Big|l^{(N)}(t) - l^{(N-1)}(t)\Big|^2 = O\Big(\frac{1}{N}\Big), \tag{2.71}$$

$$\sup_{0\leqslant t\leqslant T}\mathbb{E}\Big|\check{l}^{(N-1)}(t) - \bar{x}(t)\Big|^2 = O\Big(\frac{1}{N}\Big). \tag{2.72}$$

Proof. By (2.68)-(2.69), we have

$$\begin{cases} d\Big(l^{(N-1)}(t) - \check{l}^{(N-1)}(t)\Big) =& \Big[\big(\mathbb{A}(t) + \dfrac{N-1}{N}F\big)\big(l^{(N-1)}(t) - \check{l}^{(N-1)}(t)\big) + \dfrac{F}{N}l_i(t)\Big]dt \\ & + \dfrac{1}{N-1}\displaystyle\sum_{j=1,j\neq i}^{N}\sigma\big(l_j(t) - \check{l}_j(t)\big)dW_j(t), \\ l^{(N-1)}(0) - \check{l}^{(N-1)}(0) =& 0. \end{cases}$$

Then by Proposition 2.2 and Gronwall's inequality, the assertion (2.70) holds. (2.71) follows from assumption (H2.2) and the L^2-boundness of controls $u_i(\cdot)$ and $\tilde{u}_j(\cdot), j\neq i$. From (2.39) and (2.69), we get

$$\begin{aligned} d\Big(\check{l}^{(N-1)}(t) - \bar{x}(t)\Big) = & \Big[\big(\mathbb{A}(t) + \frac{N-1}{N}F\big)\big(\check{l}^{(N-1)}(t) - \bar{x}(t)\big) - \frac{F}{N}\bar{x}(t) \\ & + \zeta(0)\big(x_0^{(N-1)} - x_0\big)\Theta_1(t)e^{Ct}\Big]dt + \frac{1}{N-1}\sum_{j=1,j\neq i}^{N}\sigma\check{l}_j(t)dW_j(t) \end{aligned}$$

with $\check{l}^{(N-1)}(0) - \bar{x}(0) = x_0^{(N-1)} - x_0$. Thus, (2.72) is obtained. $\square$

In addition, based on Proposition 2.3, we have

Lemma 2.5.

$$\sup_{0\leqslant t\leqslant T} \mathbb{E}\Big|l^{(N)}(t) - \bar{x}(t)\Big|^2 = O\Big(\frac{1}{N}\Big), \tag{2.73}$$

$$\sup_{0\leqslant t\leqslant T} \mathbb{E}\Big|l_i(t) - l_i^0(t)\Big|^2 = O\Big(\frac{1}{N}\Big), \tag{2.74}$$

$$\sup_{0\leqslant t\leqslant T} \mathbb{E}\Big||m_i(t)|^2 - |m_i^0(t)|^2\Big| = O\Big(\frac{1}{\sqrt{N}}\Big), \tag{2.75}$$

$$\Big|\mathcal{J}_i(u_i, \tilde{u}_{-i}) - J_i(u_i)\Big| = O\Big(\frac{1}{\sqrt{N}}\Big). \tag{2.76}$$

Proof. (2.73) follows from Proposition 2.3 directly. By (2.57), (2.62), and using (2.73), we get (2.74). Noting (2.59) and (2.64), we have

$$\begin{cases} -d\Big(m_i(t) - m_i^0(t)\Big) = & \Big[C\big(m_i(t) - m_i^0(t)\big) + H\big(l_i(t) - l_i^0(t)\big) + L\big(l^{(N)}(t) - \bar{x}(t)\big)\Big]dt \\ & \quad - \big(n_{ii}(t) - n_i^0(t)\big)dW_i(t) - \displaystyle\sum_{k=1,k\neq i}^{N} n_{ik}(t)dW_k(t), \\ m_i(T) - m_i^0(T) = & K\big(l_i(T) - l_i^0(T)\big). \end{cases}$$

Applying the estimate of BSDE, we get

$$\begin{aligned} &\mathbb{E}\left[\sup_{0\leqslant t\leqslant T}\Big|m_i(t) - m_i^0(t)\Big|^2\right] + \mathbb{E}\int_0^T \Big|n_{ii}(t) - n_i^0(t)\Big|^2 dt + \sum_{k=1,k\neq i}^{N} \mathbb{E}\int_0^T \Big|n_{ik}(t)\Big|^2 dt \\ \leqslant & C_4\left\{\mathbb{E}\Big|l_i(T) - l_i^0(T)\Big|^2 + \mathbb{E}\int_0^T \Big|l_i(t) - l_i^0(t)\Big|^2 dt + \mathbb{E}\int_0^T \Big|l^{(N)}(t) - \bar{x}(t)\Big|^2 dt\right\}. \end{aligned}$$

Then by (2.73) and (2.74), we have

$$\mathbb{E}\left[\sup_{0\leqslant t\leqslant T}\Big|m_i(t) - m_i^0(t)\Big|^2\right] = O\Big(\frac{1}{N}\Big).$$

We can see that both $\sup\limits_{0\leqslant t\leqslant T}\mathbb{E}|m_i^0(t)|^2$ and $\sup\limits_{0\leqslant t\leqslant T}\mathbb{E}\Big|l_i^0(t)-\big(S\bar{x}(t)+\eta\big)\Big|^2$ are bounded. Similar to the proof in Lemma 2.4, we get

$$\begin{aligned}
&\sup_{0\leqslant t\leqslant T}\mathbb{E}\Big||m_i(t)|^2-|m_i^0(t)|^2\Big|\\
\leqslant&\sup_{0\leqslant t\leqslant T}\mathbb{E}|m_i(t)-m_i^0(t)|^2+2\Big(\sup_{0\leqslant t\leqslant T}\mathbb{E}|m_i^0(t)|^2\Big)^{\frac{1}{2}}\Big(\sup_{0\leqslant t\leqslant T}\mathbb{E}|m_i(t)-m_i^0(t)|^2\Big)^{\frac{1}{2}}\\
=&O\Big(\frac{1}{\sqrt{N}}\Big),
\end{aligned}$$

which is (2.75). Further, we have

$$\mathbb{E}\Big||m_i(0)|^2-|m_i^0(0)|^2\Big|=O\Big(\frac{1}{\sqrt{N}}\Big).$$

Moreover,

$$\begin{aligned}
&\sup_{0\leqslant t\leqslant T}\mathbb{E}\left|\Big(l_i(t)-\big(Sl^{(N)}(t)+\eta\big)\Big)^2-\Big(l_i^0(t)-\big(S\bar{x}(t)+\eta\big)\Big)^2\right|\\
\leqslant&\sup_{0\leqslant t\leqslant T}\mathbb{E}\Big|l_i(t)-l_i^0(t)-S\big(l^{(N)}(t)-\bar{x}(t)\big)\Big|^2\\
&+2\Big(\sup_{0\leqslant t\leqslant T}\mathbb{E}\big|l_i^0(t)-\big(S\bar{x}(t)+\eta\big)\big|^2\Big)^{\frac{1}{2}}\Big(\sup_{0\leqslant t\leqslant T}\mathbb{E}\big|l_i(t)-l_i^0(t)-S\big(l^{(N)}(t)-\bar{x}(t)\big)\big|^2\Big)^{\frac{1}{2}}\\
=&O\Big(\frac{1}{\sqrt{N}}\Big),
\end{aligned}$$

then

$$\begin{aligned}
&\Big|\mathcal{J}_i(u_i,\tilde{u}_{-i})-J_i(u_i)\Big|\\
\leqslant&\ \frac{1}{2}\mathbb{E}\int_0^T Q\Big|\Big(l_i(t)-\big(Sl^{(N)}(t)+\eta\big)\Big)^2-\Big(l_i^0(t)-\big(S\bar{x}(t)+\eta\big)\Big)^2\Big|dt\\
&+\frac{1}{2}N_0\mathbb{E}\Big|m_i^2(0)-\big(m_i^0(0)\big)^2\Big|\\
=&O\Big(\frac{1}{\sqrt{N}}\Big)
\end{aligned}$$

which implies (2.76). $\square$

Proof of Theorem 2.2: Now, we consider the ϵ-Nash equilibrium for $\mathcal{A}_i$. Combining Lemma 2.4 and 2.5, we have

$$\begin{aligned}\mathcal{J}_i(\tilde{u}_i, \tilde{u}_{-i}) &= J_i(\bar{u}_i) + O\Big(\frac{1}{\sqrt{N}}\Big)\\ &\leqslant J_i(u_i) + O\Big(\frac{1}{\sqrt{N}}\Big)\\ &= \mathcal{J}_i(u_i, \tilde{u}_{-i}) + O\Big(\frac{1}{\sqrt{N}}\Big).\end{aligned}$$

Thus, Theorem 2.2 follows by taking $\epsilon = O\Big(\frac{1}{\sqrt{N}}\Big)$. $\square$

Chapter 3

LQMFGs of BSDEs

This chapter focuses on the backward LQMFGs of weakly coupled stochastic large-population system. In contrast to the well-studied forward LQMFGs, the individual state in this large-population system follows the BSDE whose terminal instead of initial condition should be prescribed. This work also differs from that in Chapter 2, because there are neither forward dynamics nor Riccati equations to be derived. In this chapter, to get the explicit forms, the individual agents of large-population system are assumed to be weakly coupled in their state dynamics. Some estimates to BSDE are presented in the large-population setting. In the end, the ϵ-Nash equilibrium property of decentralized strategies is verified.

3.1 Problem Formulation

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leqslant t\leqslant T}, P)$ be the complete probability space on which a standard N-dimensional Brownian motion $\{W_i(t),\ 1 \leqslant i \leqslant N\}_{0\leqslant t\leqslant T}$ is defined. We denote by $\mathcal{F}_t := \bigcup_{i=1}^{N} \mathcal{F}_t^{w_i}$ the full information of large-population system where $\mathcal{F}_t^{w_i} := \sigma\{W_i(s); 0 \leqslant s \leqslant t\}$ is the natural filtration generated by i^{th} Brownian motion W_i but augmented by all P-null sets. Now we are ready to formulate our backward LQMFGs.

Now, we first introduce the backward LQMFGs in which the large-population system is weakly-coupled in the states of individual agents. For short, the problem is given by

$$
\textbf{(B-MFG)}\begin{cases} \text{state} : \begin{cases} -dy_i(t) = \big[Ay_i(t) + Bu_i(t) + Cy^{(N)}(t)\big]dt - z_i(t)dW_i(t) \\ \qquad\qquad - \sum\limits_{j=1, j\neq i}^{N} z_{ij}(t)dW_j(t), \\ y_i(T) = \xi_i, \end{cases} \\ \text{cost functional} : \mathcal{J}_i\big(u_i(\cdot), u_{-i}(\cdot)\big) = \mathbb{E}\left[\int_0^T Ru_i^2(t)dt + Hy_i^2(0)\right]. \end{cases} \tag{3.1}
$$

Here, we assume the full information (hence (B-MFG) for short) structure. That is, each agent can access the states of all other agents; the dynamics of agent $\mathcal{A}_i$ is denoted by y_i which satisfies the above controlled linear backward stochastic differential equation (LBSDE). It is remarkable that $(z_i, z_{ij}, 1 \leqslant j \leqslant N, j \neq i)$ is also part of our solution of (3.1) which are introduced here to enable y_i to satisfy the adaptation requirement; A, B, C are scalar constants, $R > 0, H \geqslant 0$; $y^{(N)}(t) = \frac{1}{N}\sum\limits_{i=1}^{N} y_i(t)$ is the state average across the whole population. It stands for the global population effects in macro-scale. $\xi_i \in \mathcal{F}_T,\ i = 1, 2, \ldots, N$, are the terminal conditions for individual agents which stand for the future objective or tracking target. Let $U_i,\ i = 1, 2, \ldots, N$ be subsets of $\mathbb{R}$. The admissible control $u_i \in \mathcal{U}_i$ where the admissible control set $\mathcal{U}_i$ is defined as

$$
\mathcal{U}_i := \Big\{u_i \Big| u_i(t) \in U_i, 0 \leqslant t \leqslant T;\ u_i(\cdot) \in L^2_{\mathcal{F}_t}(0, T; \mathbb{R})\Big\},\ 1 \leqslant i \leqslant N.
$$

Let $u = (u_1, \ldots, u_i, \ldots, u_N)$ denote the set of control strategies of all N agents; $u_{-i} = (u_1, \ldots, u_{i-1},\ u_{i+1}, \ldots, u_N)$ the control strategies except the i^{th} agent $\mathcal{A}_i$. Here, we write the cost functional as $\mathcal{J}_i(u_i, u_{-i})$ to emphasize that it depends on both u_i and u_{-i} due to the weakly coupling structure in dynamics.

In full information structure, we make the following assumption:

(H3) The terminal conditions $\{\xi_i\}_{i=1}^{N}$ are independent identically distributed (i.i.d) with $\mathbb{E}|\xi_i|^2 < +\infty$.

It follows that under (H3), the state equation in (3.1) admits a unique solution for all $u_i \in \mathcal{U}_i$. In fact, if we denote by

$$Y = \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix}, U = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix}, Z = \begin{pmatrix} z_1 & z_{12} & \cdots & z_{1,N-1} & z_{1N} \\ z_{21} & z_2 & \cdots & z_{2,N-1} & z_{2N} \\ \vdots & \vdots & & \vdots & \vdots \\ z_{N1} & z_{N2} & \cdots & z_{N,N-1} & z_N \end{pmatrix}, \tilde{W} = \begin{pmatrix} W_1 \\ \vdots \\ W_N \end{pmatrix},$$

$$\Xi = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_N \end{pmatrix}, J_N = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{pmatrix}.$$

Then the state equation in (3.1) can be rewritten as

$$-dY(t) = \Big[AY(t) + BU(t) + \frac{C}{N} J_N Y(t)\Big]dt - Z(t)d\tilde{W}(t), \quad Y(T) = \Xi$$

which is a LBSDE of vector value and admits a unique solution $(Y, Z) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^N) \times L^2_{\mathcal{F}}(0,T;\mathbb{R}^{N\times N})$ for $U \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^N)$, (see [84]). Thus, for any $1 \leqslant i \leqslant N$, the state equation in (3.1) admits a unique solution $\big(y_i, z_i, z_{ij}(j \neq i)\big) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}) \times \cdots \times L^2_{\mathcal{F}}(0,T;\mathbb{R})$.

Remark 3.1. *(1) We now give some remarks to the real meaning of system* (3.1)*. In reality, the LBSDE in* (3.1) *stands for the dynamics of some investment behaviors such as in stocks and bonds in a self-financed market, that is, there is no infusion or withdrawal of funds over* $[0,T]$*. In recursive or hedging problems (finance, optimal control, etc.), the BSDE dynamics have been deeply studied in the existing literature, such as [86], [72] and so on. The cost used to be applied in some terminal hedging problems with possible nonlinear expectation, taking mean variance model as an example. In particular, the initial state* $y_i(0)$ *in our cost can be viewed as the initial hedging cost (or, cash surplus), which aim to reach some future payoff or obligation target* ξ_i *at given time* T*. Besides, the constrained forward LQ control problem with* *state average coupling in state dynamics can also be transferred to the backward LQ control with state given by the linear BSDE, as given in* (3.1)*.*

(2) For simplicity of analysis, the state average in system (3.1) *is coupled in dynamics only. Actually, our analysis can be extended to the problem with coupling in cost functional. Applying similar procedures, we can obtain the corresponding optimal control and fixed point principle, and analyze the properties of ϵ-Nash equilibrium.*

(3) In this system, there are N individual agents coupled together to be investigated for the hedging strategies. Actually, problems to get optimal strategies in forward setup with small players have been well studied by the existing literature, including [113], [18], [20], [8], etc. In this setting, we analyze the limit when the number of players N goes to infinity where the situation considerably simplifies in the spirit of MFGs, see [27].

3.2 The Optimal Control of (L-B-MFG)

Now, we study the problem **(B-MFG)**: the backward LQMFGs with full information (B-MFG). A key component in our analysis is to study the associated LQMFGs via limiting state average, as the number of agents tends to infinity. To obtain the feedback control and the desired results, we suppose $U_i = \mathbb{R}$ for $i = 1, 2, \ldots, N$.

We assume $y^{(N)}$ is approximated by a deterministic continuous function y^0 satisfying

$$\begin{cases} -dy^0(t) = [\widetilde{A}(t)y^0(t) + m(t)]dt, \\ \quad y^0(T) = \xi_0 \end{cases} \tag{3.2}$$

where ξ_0 is some deterministic constant, $\widetilde{A}(t)$ and $m(t)$ are some continuous functions to be determined. Actually, by (H3) and strong law of large numbers (LLN), $\lim\limits_{N\to+\infty} \xi^{(N)}$ exists and ξ_0 is determined by

$$\xi_0 = \lim_{N\to+\infty} \xi^{(N)} = \mathbb{E}\xi_i, \ a.s., \ i = 1, 2, \cdots, N \tag{3.3}$$

where $\xi^{(N)} = \frac{1}{N}\sum_{i=1}^{N}\xi_i$. Now, we introduce the limiting full-information system

$$\begin{cases} -dy_i(t) =\big[Ay_i(t) + Bu_i(t) + Cy^0(t)\big]dt - z_i(t)dW_i(t) - \sum\limits_{j=1,j\neq i}^{N} z_{ij}(t)dW_j(t), \\ \\ \quad y_i(T) =\xi_i \end{cases} \tag{3.4}$$

with the cost functional

$$J_i\big(u_i(\cdot)\big) = \mathbb{E}\left[\int_0^T Ru_i^2(t)dt + Hy_i^2(0)\right] \tag{3.5}$$

where $y^0(\cdot)$ is given by (3.2).

Now we formulate the limiting backward full information (L-B-MFG) problem of our large-population system as follows.

Problem (L-B-MFG). For the i^{th} agent, $i = 1, 2, \dots, N$, find $\bar{u}_i \in \mathcal{U}_i$ satisfying

$$J_i(\bar{u}_i) = \inf_{u_i \in \mathcal{U}_i} J_i(u_i).$$

Then $\bar{u}_i$ is called the optimal control for problem **(L-B-MFG)**.

In the following, we apply the variational method to get the optimal control $\bar{u}_i$. First, introduce the variational equation

$$\begin{cases} -d\zeta_i(t) = \big[A\zeta_i(t) + B\delta u_i(t)\big]dt - \theta_i(t)dW_i(t) - \sum\limits_{j=1,j\neq i}^{N} \theta_{ij}(t)dW_j(t), \\ \\ \quad \zeta_i(T) = 0, \ i = 1, 2, \dots, N \end{cases} \tag{3.6}$$

where $\zeta_i(t) \in L^2_{\mathcal{F}_t}(0, T; \mathbb{R})$, $\delta u_i(\cdot)$ denotes the variation of $\bar{u}_i(\cdot)$. Then the following proposition holds true.

Proposition 3.1. *Let* (H3) *hold. Then the optimal control of* **(L-B-MFG)** *is*

$$\bar{u}_i(t) = -R^{-1}Bp_i(t)$$

where $p_i(t) \in L^2(0,T;\mathbb{R})$ satisfies the following ordinary differential equation (ODE):

$$\begin{cases} dp_i(t) = Ap_i(t)dt, \\ p_i(0) = H\bar{y}_i(0), \ i = 1, 2, \dots, N. \end{cases} \tag{3.7}$$

Proof. Suppose $(\bar{y}_i, \bar{z}_i, \bar{z}_{ij}(j \neq i), \bar{u}_i)$ is an optimal solution. Then for any variation δu_i of $\bar{u}_i$, the associated first order variation of cost functional $J_i(\bar{u}_i)$ satisfies

$$0 = \frac{1}{2}\delta J_i(\bar{u}_i) = \mathbb{E}\left[\int_0^T R\delta u_i(t)\bar{u}_i(t)dt + H\zeta_i(0)\bar{y}_i(0)\right]. \tag{3.8}$$

Applying Itô's formula, we have

$$\begin{aligned} & d\big(\zeta_i(t)p_i(t)\big) \\ =& \left\{ -\big[A\zeta_i(t) + B\delta u_i(t)\big]dt + \theta_i(t)dW_i(t) + \sum_{j=1, j\neq i}^N \theta_{ij}(t)dW_j(t)\right\}p_i(t) + \zeta_i(t)Ap_i(t)dt \\ =& -B\delta u_i(t)p_i(t)dt + p_i(t)\left[\theta_i(t)dW_i(t) + \sum_{j=1, j\neq i}^N \theta_{ij}(t)dW_j(t)\right]. \end{aligned}$$

Combining this identity with $\zeta_i(T) = 0$ and $p_i(0) = H\bar{y}_i(0)$ yields

$$\mathbb{E}\big[\zeta_i(0)H\bar{y}_i(0)\big] = \mathbb{E}\int_0^T B\delta u_i(t)p_i(t)dt. \tag{3.9}$$

It follows from (3.8)-(3.9) that for any $\delta u_i(\cdot) \in L^2_{\mathcal{F}^{w_i}}(0,T;\mathbb{R})$,

$$\mathbb{E}\int_0^T \Big(R\delta u_i(t)\bar{u}_i(t) + B\delta u_i(t)p_i(t)\Big)dt = 0.$$

This implies that $\bar{u}_i(t) = -R^{-1}Bp_i(t)$. On the other hand, the sufficiency of optimal control can also be obtained via the convexity of $J_i(\cdot)$. $\square$

3.3 The Explicit Representation

Now, we aim to study the properties of the given function $y^0(\cdot)$. For $\forall\ 1 \leqslant i \leqslant N$, solving ODE (3.7) directly, we have

$$p_i(t) = H\bar{y}_i(0)e^{At}.$$

Thus, the optimal control $\bar{u}_i(t)$ is given by

$$\bar{u}_i(t) = -R^{-1}BH\bar{y}_i(0)e^{At}. \tag{3.10}$$

Applying the decentralized control law (3.10) for the i^{th} agent $\mathcal{A}_i$, the closed-loop state in system (3.1) becomes

$$\begin{cases} -dy_i(t) = \Big[Ay_i(t) - B^2R^{-1}H\bar{y}_i(0)e^{At} + Cy^{(N)}(t)\Big]dt - z_i(t)dW_i(t) \\ \qquad\qquad - \sum\limits_{j=1, j\neq i}^{N} z_{ij}(t)dW_j(t), \\ y_i(T) = \xi_i \end{cases} \tag{3.11}$$

where $y^{(N)}(t) = \frac{1}{N}\sum\limits_{i=1}^{N} y_i(t)$. Denote by $\bar{y}^{(N)}(0) = \frac{1}{N}\sum\limits_{i=1}^{N} \bar{y}_i(0)$. Summing the above N equations of (3.11) and dividing by N, we get

$$\begin{cases} -dy^{(N)}(t) = \Big[Ay^{(N)}(t) - B^2R^{-1}He^{At}\bar{y}^{(N)}(0) + Cy^{(N)}(t)\Big]dt \\ \qquad\qquad - \frac{1}{N}\sum\limits_{i=1}^{N}\Big[z_i(t)dW_i(t) + \sum\limits_{j=1, j\neq i}^{N} z_{ij}(t)dW_j(t)\Big], \\ y^{(N)}(T) = \xi^{(N)}. \end{cases} \tag{3.12}$$

Letting $N \to +\infty$, replacing $y^{(N)}$ ($\bar{y}^{(N)}$) by y^0 and noting (3.3), we obtain the following limiting system

$$\begin{cases} -dy^0(t) = \Big[\big(A + C\big)y^0(t) - B^2R^{-1}He^{At}y^0(0)\Big]dt, \\ y^0(T) = \xi_0. \end{cases} \tag{3.13}$$

Comparing the coefficients with (3.2), we have

$$\begin{cases} \tilde{A}(t) \equiv A + C, \\ m(t) = -B^2 R^{-1} H e^{At} y^0(0). \end{cases} \tag{3.14}$$

Solving the ODE (3.2), we get

$$y^0(t) = \xi_0 e^{\int_t^T \tilde{A}(s)ds} + \int_t^T m(s) e^{\int_t^s \tilde{A}(u)du} ds.$$

Taking $t = 0$ and noting (3.14), we have

$$y^0(0) = \xi_0 e^{(A+C)T} + \int_0^T m(s) e^{(A+C)s} ds.$$

Thus, $m(t)$ in (3.14) has the following expression:

$$m(t) = -B^2 R^{-1} H e^{At} \xi_0 e^{(A+C)T} - B^2 R^{-1} H e^{At} \int_0^T m(s) e^{(A+C)s} ds. \tag{3.15}$$

We have the following explicit representation of $m(t)$. As a sequel, $y^0(\cdot)$ in (3.2) can be determined.

Proposition 3.2. *$m(\cdot)$ can be explicitly solved as*

$$m(t) = \begin{cases} -\dfrac{B^2 H(2A+C)\xi_0 e^{At+(A+C)T}}{R(2A+C) + B^2 H\big(e^{(2A+C)T} - 1\big)}, & \textit{if } 2A + C \neq 0; \\ -\dfrac{B^2 H[R + B^2 H(T-1)]\xi_0 e^{-A(T-t)}}{R(R + B^2 HT)}, & \textit{if } 2A + C = 0. \end{cases} \tag{3.16}$$

Proof. Denote $K := \int_0^T m(s) e^{(A+C)s} ds$, which is a constant depending on T. Then (3.15) can be rewritten as

$$m(t) = -B^2 R^{-1} H e^{At} \xi_0 e^{(A+C)T} - B^2 R^{-1} H e^{At} K.$$

Multiplying with $e^{(A+C)t}$ on both sides and taking integral from 0 to T w.r.t t, we have

$$
\begin{aligned}
K &= \int_0^T m(t)e^{(A+C)t}dt \\
&= -B^2R^{-1}H\xi_0 e^{(A+C)T}\int_0^T e^{(2A+C)t}dt - B^2R^{-1}HK\int_0^T e^{(2A+C)t}dt.
\end{aligned}
$$

Then we get

$$
K = \begin{cases} -\dfrac{B^2H\xi_0 e^{(A+C)T}\big(e^{(2A+C)T}-1\big)}{R(2A+C)+B^2H\big(e^{(2A+C)T}-1\big)}, & \text{if } 2A+C\neq 0; \\ -\dfrac{B^2H\xi_0 e^{-AT}}{R+B^2HT}, & \text{if } 2A+C=0. \end{cases}
$$

Thus, (3.16) is obtained. Noting (3.14), $y^0(\cdot)$ is also determined. □

Remark 3.2. *(1) By Proposition 3.2, it follows that there exists a unique deterministic function y^0 in $C(0,T;\mathbb{R})$ to approximate the state average $y^{(N)}$. Applying the limiting function y^0, we get the optimal control for* **(L-B-MFG)***, which plays an important role in obtaining the decentralized control and analyzing the properties of ϵ-Nash equilibrium.*

(2) Actually, in (3.16) *if $2A+C>0(<0)$, $e^{(2A+C)T}-1>0(<0)$. Noting $R>0, H\geqslant 0$, we get $R(2A+C)+B^2H\big(e^{(2A+C)T}-1\big)>0(<0)$. Meanwhile, we have $R(R+B^2HT)>0$. Thus, the representation* (3.16) *is meaningful.*

3.4 ϵ-Nash Equilibrium Analysis for (B-MFG)

In previous sections, we obtained the optimal control $\bar{u}_i(\cdot), 1\leqslant i\leqslant N$ of **(L-B-MFG)**. In this section, we analyze the asymptotic property of the decentralized control strategies and verify the ϵ-Nash equilibrium property for **(B-MFG)**.

We state one main result of this paper and its proof will be given later.

Theorem 3.1. *Let* (H3) *hold. Then* $(\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_N)$ *satisfies the* ϵ*-Nash equilibrium of* **(B-MFG)**, *with* ϵ *is of order* $1/\sqrt{N}$. *Here, for* $1 \leqslant i \leqslant N$, $\bar{u}_i$ *is given by*

$$\bar{u}_i(t) = -R^{-1}BHy^0(0)e^{At}. \tag{3.17}$$

Before proving the theorem, some analysis is needed. Applying the optimal control (3.10) to (3.4), we have

$$\begin{cases} -d\bar{y}_i(t) = \Big[A\bar{y}_i(t) - B^2R^{-1}H\bar{y}_i(0)e^{At} + Cy^0(t)\Big]dt - \bar{z}_i(t)dW_i(t) - \sum\limits_{j=1, j\neq i}^{N} \bar{z}_{ij}(t)dW_j(t), \\ \bar{y}_i(T) = \xi_i. \end{cases}$$

Taking expectation and solving the corresponding backward ODE, we get

$$\mathbb{E}\bar{y}_i(t) = \xi_0 e^{A(T-t)} - \int_t^T \Big[BR^{-1}BH\bar{y}_i(0)e^{As} - Cy^0(s)\Big]e^{A(s-t)}ds.$$

Taking $t = 0$ and noting $\bar{y}_i(0) = \mathbb{E}\bar{y}_i(0)$, we obtain

$$\bar{y}_i(0) = \begin{cases} \Big[1 + \frac{B^2HT}{R}\Big]^{-1}\Big[\xi_0 + C\int_0^T y^0(s)ds\Big], & \text{if } A = 0; \\ \Big[1 + \frac{B^2H}{2AR}\big(e^{2AT} - 1\big)\Big]^{-1}\Big[\xi_0 e^{AT} + C\int_0^T y^0(s)e^{As}ds\Big], & \text{if } A \neq 0. \end{cases}$$

Thus, $\bar{y}_i(0)$ is a constant which can be determined by $y^0(\cdot)$ and ξ_0. Further, we have $\bar{y}_i(0) = y^0(0),\ i = 1, 2, \ldots, N$. For simplicity, we use the notation $y^0(0)$ in $\bar{u}_i(\cdot)$ instead of $\bar{y}_i(0)$ hereafter. Now, we formulate the dynamic systems as follows

$$\begin{cases} -dy_i(t) = \Big[Ay_i(t) - B^2R^{-1}Hy^0(0)e^{At} + Cy^{(N)}(t)\Big]dt - z_i(t)dW_i(t) - \sum\limits_{j=1, j\neq i}^{N} z_{ij}(t)dW_j(t), \\ y_i(T) = \xi_i \end{cases} \tag{3.18}$$

and

$$\begin{cases} -d\bar{y}_i(t) = \Big[A\bar{y}_i(t) - B^2R^{-1}Hy^0(0)e^{At} + Cy^0(t)\Big]dt - \bar{z}_i(t)dW_i(t) - \sum\limits_{j=1, j\neq i}^{N} \bar{z}_{ij}(t)dW_j(t), \\ \bar{y}_i(T) = \xi_i. \end{cases} \tag{3.19}$$

Then we have

Lemma 3.1.

$$\sup_{0\leqslant t\leqslant T} \mathbb{E}\Big|y^{(N)}(t) - y^0(t)\Big|^2 = O\Big(\frac{1}{N}\Big), \tag{3.20}$$

$$\sup_{1\leqslant i\leqslant N} \Big[\sup_{0\leqslant t\leqslant T} \mathbb{E}\Big|y_i(t) - \bar{y}_i(t)\Big|^2\Big] = O\Big(\frac{1}{N}\Big). \tag{3.21}$$

Proof. By (3.18) and (3.13), we have

$$\begin{cases} -d\Big(y^{(N)}(t) - y^0(t)\Big) = \Big[(A+C)(y^{(N)}(t) - y^0(t))\Big]dt \\ \qquad\qquad - \frac{1}{N}\sum\limits_{i=1}^{N}\Big[z_i(t)dW_i(t) + \sum\limits_{j=1, j\neq i}^{N} z_{ij}(t)dW_j(t)\Big], \\ y^{(N)}(T) - y^0(T) = \xi^{(N)} - \xi_0. \end{cases} \tag{3.22}$$

Introduce a 1-dimensional dual process $X(s,t)$ for (3.22), which satisfies

$$\begin{cases} dX(s,t) = (A+C)X(s,t)ds, \\ X(t,t) = 1, \ t \leqslant s \leqslant T. \end{cases}$$

$X(s,t)$ is deterministic and belongs to $L^2(0,T;\mathbb{R})$. Applying Itô's formula to $\langle y^{(N)}(s) - y^0(s), X(s,t)\rangle$, we get

$$y^{(N)}(t) - y^0(t) = X(T,t)\mathbb{E}\Big(\xi^{(N)} - \xi_0\Big|\mathcal{F}_t\Big).$$

By (H3), we have

$$\mathbb{E}\Big|\xi^{(N)} - \xi_0\Big|^2 = \mathbb{E}\Big|\frac{1}{N}\sum_{i=1}^{N}\xi_i - \xi_0\Big|^2 = O\Big(\frac{1}{N}\Big).$$

Then (3.20) follows. Noting (3.18) and (3.19), applying the similar method, we can get (3.21). □

Lemma 3.2. *For* $\forall\ 1 \leqslant i \leqslant N$,

$$\Big|\mathcal{J}_i(\bar{u}_i, \bar{u}_{-i}) - J_i(\bar{u}_i)\Big| = O\Big(\frac{1}{\sqrt{N}}\Big).$$

Proof. For $\forall\ 1 \leqslant i \leqslant N$, by (3.19), we get $\sup\limits_{0\leqslant t\leqslant T}\mathbb{E}\big|\bar{y}_i(t)\big|^2 < +\infty$. Applying Cauchy-Schwarz inequality and noting (3.21), we have

$$\sup_{0\leqslant t\leqslant T}\mathbb{E}\Big||y_i(t)|^2 - |\bar{y}_i(t)|^2\Big| = O\Big(\frac{1}{\sqrt{N}}\Big).$$

Further,

$$\mathbb{E}\Big||y_i(0)|^2 - |\bar{y}_i(0)|^2\Big| = O\Big(\frac{1}{\sqrt{N}}\Big).$$

Then

$$\Big|\mathcal{J}_i(\bar{u}_i, \bar{u}_{-i}) - J_i(\bar{u}_i)\Big| \leqslant H\mathbb{E}\Big|y_i^2(0) - \bar{y}_i^2(0)\Big| = O\Big(\frac{1}{\sqrt{N}}\Big),$$

which completes the proof. □

Now, we have addressed some estimates of states and costs corresponding to control $\bar{u}_i$,$1 \leqslant i \leqslant N$. Our remaining analysis is to prove the control strategies set $(\bar{u}_1, \bar{u}_2, \dots, \bar{u}_N)$ is an ϵ-Nash equilibrium for **(B-MFG)**. For any fixed i, $1 \leqslant i \leqslant N$, consider an admissible alternative control $u_i \in \mathcal{U}_i$ for $\mathcal{A}_i$ and introduce the dynamics

$$\begin{cases} -dx_i(t) =\Big[Ax_i(t) + Bu_i(t) + Cx^{(N)}(t)\Big]dt - q_i(t)dW_i(t) - \sum\limits_{k=1,k\neq i}^{N} q_{ik}(t)dW_k(t), \\ \quad x_i(T) =\xi_i \end{cases} \tag{3.23}$$

whereas other agents keep the control $\bar{u}_j, 1 \leqslant j \leqslant N, j \neq i$, i.e.,

$$\begin{cases} -dx_j(t) = \Big[Ax_j(t) - B^2R^{-1}Hy^0(0)e^{At} + Cx^{(N)}(t)\Big]dt - q_j(t)dW_j(t) \\ \qquad\qquad\qquad - \sum\limits_{k=1,k\neq j}^{N} q_{jk}(t)dW_k(t), \\ x_j(T) = \xi_j \end{cases} \tag{3.24}$$

where $x^{(N)}(t) = \frac{1}{N}\sum\limits_{j=1}^{N} x_j(t)$.

If $\bar{u}_i,\ 1 \leqslant i \leqslant N$ is an ϵ-Nash equilibrium with respect to the cost $\mathcal{J}_i$, we have

$$\mathcal{J}_i(\bar{u}_i, \bar{u}_{-i}) \geqslant \inf_{u_i \in \mathcal{U}_i} \mathcal{J}_i(u_i, \bar{u}_{-i}) \geqslant \mathcal{J}_i(\bar{u}_i, \bar{u}_{-i}) - \epsilon.$$

Then, when making the perturbation, we just need to consider $u_i \in \mathcal{U}_i$ such that $\mathcal{J}_i(u_i, \bar{u}_{-i}) \leqslant \mathcal{J}_i(\bar{u}_i, \bar{u}_{-i})$, which implies

$$\mathbb{E}\int_0^T Ru_i^2(t)dt \leqslant \mathcal{J}_i(u_i, \bar{u}_{-i}) \leqslant \mathcal{J}_i(\bar{u}_i, \bar{u}_{-i}) = J_i(\bar{u}_i) + O\Big(\frac{1}{\sqrt{N}}\Big),$$

i.e.,

$$\mathbb{E}\int_0^T u_i^2(t)dt \leqslant C_0, \tag{3.25}$$

where C_0 is a positive constant which is independent of N.

Proposition 3.3. $\sup\limits_{1\leqslant i\leqslant N}\Big[\sup\limits_{0\leqslant t\leqslant T}\mathbb{E}|x_i(t)|^2\Big]$ *is bounded.*

Proof. Similar to the proof of Proposition 2.2, by (3.23) and (3.24), it holds that

$$\mathbb{E}\Bigg[\sum_{k=1}^{N}|x_k(t)|^2\Bigg] \leqslant C_1\Bigg\{\mathbb{E}\Bigg[\sum_{k=1}^{N}|\xi_k|^2\Bigg] + \mathbb{E}\int_t^T\Bigg[2\sum_{k=1}^{N}|x_k(s)|^2 + |u_i(s)|^2 + \sum_{k=1,k\neq i}^{N}|\bar{u}_k(s)|^2\Bigg]ds\Bigg\}$$

where C_1 is a positive constant. By (3.25), we can see $u_i(t)$ is L^2-bounded. Besides, the optimal controls $\bar{u}_k(t), k \neq i$ are L^2-bounded. Then by Gronwall's inequality, we get

$$\sup_{0\leqslant t\leqslant T}\mathbb{E}\left[\sum_{k=1}^{N}|x_k(t)|^2\right]=O(N),$$

and for any $1\leqslant i\leqslant N$, $\sup_{0\leqslant t\leqslant T}\mathbb{E}|x_i(t)|^2$ is bounded. $\square$

For the i^{th} agent $\mathcal{A}_i$, consider the perturbation in **(L-B-MFG)** and introduce a new system

$$\begin{cases} -dx_i^0(t)=&\Big[Ax_i^0(t)+Bu_i(t)+Cy^0(t)\Big]dt-q_i^0(t)dW_i(t)-\displaystyle\sum_{k=1,k\neq i}^{N}q_{ik}^0(t)dW_k(t),\\ x_i^0(T)=&\xi_i \end{cases} \tag{3.26}$$

and for the j^{th} agent $\mathcal{A}_j, j\neq i$,

$$\begin{cases} -d\bar{x}_j(t)=&\Big[A\bar{x}_j(t)-B^2R^{-1}Hy^0(0)e^{At}+Cy^0(t)\Big]dt-\bar{q}_j(t)dW_j(t)\\ &-\displaystyle\sum_{k=1,k\neq j}^{N}\bar{q}_{jk}(t)dW_k(t),\\ \bar{x}_j(T)=&\xi_j. \end{cases} \tag{3.27}$$

In order to obtain necessary estimates for **(B-MFG)** and **(L-B-MFG)**, we need introduce some intermediate states as follows

$$\begin{cases} -d\check{x}_i(t)=&\left[A\check{x}_i(t)+Bu_i(t)+\dfrac{N-1}{N}C\check{x}^{(N-1)}(t)\right]dt-\check{q}_i(t)dW_i(t)\\ &-\displaystyle\sum_{k=1,k\neq i}^{N}\check{q}_{ik}(t)dW_k(t),\\ \check{x}_i(T)=&\xi_i \end{cases} \tag{3.28}$$

and for $j \neq i$,

$$\begin{cases} -d\check{x}_j(t) = \Big[A\check{x}_j(t) - B^2R^{-1}Hy^0(0)e^{At} + \dfrac{N-1}{N}C\check{x}^{(N-1)}(t)\Big]dt \\ \qquad\qquad - \check{q}_j(t)dW_j(t) - \displaystyle\sum_{k=1,k\neq j}^{N} \check{q}_{jk}(t)dW_k(t), \\ \check{x}_j(T) = \xi_j \end{cases} \tag{3.29}$$

where $\check{x}^{(N-1)}(t) = \frac{1}{N-1}\sum_{j=1,j\neq i}^{N} \check{x}_j(t)$. Denote $x^{(N-1)}(t) = \frac{1}{N-1}\sum_{j=1,j\neq i}^{N} x_j(t)$, by (3.24) and (3.29), we get

$$\begin{cases} -dx^{(N-1)}(t) = \Big[\Big(A + \dfrac{N-1}{N}C\Big)x^{(N-1)}(t) - B^2R^{-1}Hy^0(0)e^{At} + \dfrac{C}{N}x_i(t)\Big]dt \\ \qquad\qquad - \dfrac{1}{N-1}\displaystyle\sum_{j=1,j\neq i}^{N}\Big[q_j(t)dW_j(t) + \sum_{k=1,k\neq j}^{N} q_{jk}(t)dW_k(t)\Big], \\ x^{(N-1)}(T) = \xi^{(N-1)} \end{cases} \tag{3.30}$$

and

$$\begin{cases} -d\check{x}^{(N-1)}(t) = \Big[\Big(A + \dfrac{N-1}{N}C\Big)\check{x}^{(N-1)}(t) - B^2R^{-1}Hy^0(0)e^{At}\Big]dt \\ \qquad\qquad - \dfrac{1}{N-1}\displaystyle\sum_{j=1,j\neq i}^{N}\Big[\check{q}_j(t)dW_j(t) + \sum_{k=1,k\neq j}^{N} \check{q}_{jk}(t)dW_k(t)\Big], \\ \check{x}^{(N-1)}(T) = \xi^{(N-1)} \end{cases} \tag{3.31}$$

where $\xi^{(N-1)} = \frac{1}{N-1}\sum_{j=1,j\neq i}^{N} \xi_j$.

We have the following estimates.

Proposition 3.4.

$$\sup_{0\leqslant t\leqslant T}\mathbb{E}\Big|x^{(N-1)}(t)-\check{x}^{(N-1)}(t)\Big|^2=O\Big(\frac{1}{N^2}\Big), \tag{3.32}$$

$$\sup_{0\leqslant t\leqslant T}\mathbb{E}\Big|x^{(N)}(t)-x^{(N-1)}(t)\Big|^2=O\Big(\frac{1}{N}\Big), \tag{3.33}$$

$$\sup_{0\leqslant t\leqslant T}\mathbb{E}\Big|\check{x}^{(N-1)}(t)-y^0(t)\Big|^2=O\Big(\frac{1}{N}\Big). \tag{3.34}$$

Proof. By (3.30) and (3.31), we have

$$\begin{cases} -d\big(x^{(N-1)}(t)-\check{x}^{(N-1)}(t)\big)=\bigg[\Big(A+\dfrac{N-1}{N}C\Big)\big(x^{(N-1)}(t)-\check{x}^{(N-1)}(t)\big)+\dfrac{C}{N}x_i(t)\bigg]dt \\ \qquad\qquad -\dfrac{1}{N-1}\displaystyle\sum_{j=1,j\neq i}^{N}\bigg[\big(q_j(t)-\check{q}_j(t)\big)dW_j(t)+\sum_{k=1,k\neq j}^{N}\big(q_{jk}(t)-\check{q}_{jk}(t)\big)dW_k(t)\bigg], \\ x^{(N-1)}(T)-\check{x}^{(N-1)}(T)=0. \end{cases}$$

By the estimates of BSDE, Proposition 3.3, and Gronwall's inequality, the assertion (3.32) holds. (3.33) follows from assumption (H3) and the L^2-boundness of controls $u_i(\cdot)$ and $\tilde{u}_j(\cdot), j\neq i$. By (3.13) and (3.31), making similar analysis, we get (3.34). □

In addition, based on Proposition 3.4, we obtain more direct estimates to prove Theorem 3.1.

Lemma 3.3.

$$\sup_{0\leqslant t\leqslant T}\mathbb{E}\Big||x_i(t)|^2-|x_i^0(t)|^2\Big|=O\Big(\frac{1}{\sqrt{N}}\Big), \tag{3.35}$$

$$\Big|\mathcal{J}_i(u_i,\bar{u}_{-i})-J_i(u_i)\Big|=O\Big(\frac{1}{\sqrt{N}}\Big). \tag{3.36}$$

Proof. By Proposition 3.4, we get $\sup_{0\leqslant t\leqslant T}\mathbb{E}\Big|x^{(N)}(t)-y^0(t)\Big|^2=O\Big(\frac{1}{N}\Big)$. Besides, by (3.23) and (3.26), we obtain $\sup_{0\leqslant t\leqslant T}\mathbb{E}\Big|x_i(t)-x_i^0(t)\Big|^2=O\Big(\frac{1}{N}\Big)$. Noting $\sup_{0\leqslant t\leqslant T}\mathbb{E}|x_i^0(t)|^2<+\infty$, applying Cauchy-Schwarz inequality, we have (3.35). Further, it follows

$$\mathbb{E}\Big||x_i(0)|^2-|x_i^0(0)|^2\Big|=O\Big(\frac{1}{\sqrt{N}}\Big).$$

Thus, (3.36) is obtained. $\square$

Proof of Theorem 3.1: Now, we consider the ϵ-Nash equilibrium of $\mathcal{A}_i$ for **(B-MFG)**. Combining Lemma 3.2 and 3.3, we have

$$\begin{aligned}\mathcal{J}_i(\bar{u}_i, \bar{u}_{-i}) =& J_i(\bar{u}_i) + O\Big(\frac{1}{\sqrt{N}}\Big) \\ \leqslant& J_i(u_i) + O\Big(\frac{1}{\sqrt{N}}\Big) \\ =& \mathcal{J}_i(u_i, \bar{u}_{-i}) + O\Big(\frac{1}{\sqrt{N}}\Big).\end{aligned}$$

Thus, Theorem 3.1 follows by taking $\epsilon = O\Big(\frac{1}{\sqrt{N}}\Big)$. $\square$

Chapter 4

Backward-Forward LQMFGs with Major and Minor Agents

This chapter aims to investigate the backward-forward LQ games of large-population systems with major and minor agents (players). In the last few years, there is some great work to study the large-population systems with major and minor players, like [51, 52, 53], etc. It is remarkable that in above works, all agents' states are formulated by (forward) SDEs. As the novelty, this chapter turns to consider the major-minor framework in which the major agent's dynamics is characterized by some BSDE with prescribed terminal condition; while dynamics of minor players are described by SDEs. In this way, a BFSDE system is established in which a large number of negligible agents are coupled in their dynamics via state average. The problem when major player takes into account the relative performance by comparison to minor players is also under consideration. Some auxiliary MF SDEs and a 3×2 mixed FBSDE system are considered and analyzed instead of involving the fixed-point analysis as in Chapter 2. The decentralized strategies are derived, which are also shown to satisfy the ϵ-Nash equilibrium property.

4.1 Problem Formulation

Throughout this chapter, suppose $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leqslant t\leqslant T}, P)$ is a complete filtered probability space on which a standard $(1+N)$-dimensional Brownian motion $\{W_0(t), W_i(t),\ 1 \leqslant i \leqslant N\}_{0\leqslant t\leqslant T}$ is defined. $\mathcal{F}_t^{w_0} := \sigma\{W_0(s), 0 \leqslant s \leqslant t\}$, $\mathcal{F}_t^{w_i} := \sigma\{W_i(s), 0 \leqslant s \leqslant t\}$, $\mathcal{F}_t^i := \sigma\{W_0(s), W_i(s); 0 \leqslant s \leqslant t\}$. Here, $\{\mathcal{F}_t^{w_0}\}_{0\leqslant t\leqslant T}$ stands for the information of the major player; while $\{\mathcal{F}_t^{w_i}\}_{0\leqslant t\leqslant T}$ the individual information of i^{th} minor player.

Consider a large-population system with $(1 + N)$ individual agents, denoted by $\mathcal{A}_0$ and $\{\mathcal{A}_i\}_{1\leqslant i\leqslant N}$, where $\mathcal{A}_0$ stands for the major player, while $\mathcal{A}_i$ stands for i^{th} minor player. The dynamics of $\mathcal{A}_0$ is given by a BSDE as follows:

$$\begin{cases} dx_0(t) =\Big[A_0x_0(t) + B_0u_0(t) + C_0z_0(t)\Big]dt + z_0(t)dW_0(t), \\ x_0(T) =\xi \end{cases} \tag{4.1}$$

where $\xi \in \mathcal{F}_T^{w_0}$ satisfies $\mathbb{E}|\xi|^2 < +\infty$. The state of minor player $\mathcal{A}_i$ is a SDE satisfying

$$\begin{cases} dx_i(t) =\Big[Ax_i(t) + Bu_i(t) + Dx^{(N)}(t) + \alpha x_0(t)\Big]dt + \sigma dW_i(t), \\ x_i(0) =x_{i0} \end{cases} \tag{4.2}$$

where $x^{(N)}(t) = \frac{1}{N}\sum_{i=1}^{N} x_i(t)$ is the state-average of minor players; x_{i0} is the initial value of $\mathcal{A}_i$. Here, $A_0, B_0, C_0, A, B, D, \alpha, \sigma$ are scalar constants. Assume that $\mathcal{F}_t$ is the augmentation of $\sigma\{W_0(s), W_i(s), x_{i0}; 0 \leqslant s \leqslant t, 1 \leqslant i \leqslant N\}$ by all the P-null sets of $\mathcal{F}$, which is the full information accessible to the large-population system up to time t. Let $U_i,\ i = 0, 1, 2, \ldots, N$ be subsets of $\mathbb{R}$. The admissible control strategy $u_0 \in \mathcal{U}_0, u_i \in \mathcal{U}_i$ where

$$\mathcal{U}_0 := \Big\{u_0 \Big| u_0(t) \in U_0, 0 \leqslant t \leqslant T;\ u_0(\cdot) \in L^2_{\mathcal{F}_t^{w_0}}(0, T; \mathbb{R})\Big\},$$

and

$$\mathcal{U}_i := \Big\{u_i \Big| u_i(t) \in U_i, 0 \leqslant t \leqslant T;\ u_i(\cdot) \in L^2_{\mathcal{F}_t}(0, T; \mathbb{R})\Big\},\ 1 \leqslant i \leqslant N.$$

Let $u = (u_0, u_1, \dots, u_N)$ denote the set of control strategies of all $(1+N)$ agents; $u_{-0} = (u_1, u_2, \dots, u_N)$ the control strategies except $\mathcal{A}_0$; $u_{-i} = (u_0, u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_N)$ the control strategies except i^{th} agent $\mathcal{A}_i, 1 \leqslant i \leqslant N$. The cost functional for $\mathcal{A}_0$ is given by

$$\mathcal{J}_0(u_0(\cdot), u_{-0}(\cdot)) = \frac{1}{2}\mathbb{E}\left\{\int_0^T \left[Q_0\Big(x_0(t) - x^{(N)}(t)\Big)^2 + \tilde{Q}x_0^2(t) + R_0 u_0^2(t)\right]dt + H_0 x_0^2(0)\right\} \tag{4.3}$$

where $Q_0 \geqslant 0, \tilde{Q} \geqslant 0, R_0 > 0, H_0 \geqslant 0$. The individual cost functional for $\mathcal{A}_i, 1 \leqslant i \leqslant N$, is

$$\mathcal{J}_i(u_i(\cdot), u_{-i}(\cdot)) = \frac{1}{2}\mathbb{E}\left\{\int_0^T \left[Q\Big(x_i(t) - x^{(N)}(t)\Big)^2 + R u_i^2(t)\right]dt + H x_i^2(T)\right\} \tag{4.4}$$

where $Q \geqslant 0, R > 0, H \geqslant 0$.

Remark 4.1. *Unlike [51, 52, 53], the dynamics of major agent in our work is a BSDE with terminal condition as a priori. The term $H_0 x_0^2(0)$ is thus introduced in* (4.3) *to represent some recursive evaluation. One of its practical meanings is the initial hedging deposits in pension fund industry. For sake of simplicity, behaviors of major agent (e.g., the government, as presented in our above example) affect the state of minor agents (which can be understood as considerable individual and negligible firms or producers). Moreover, the major and minor agents are further coupled via the state-average.*

Remark 4.2. *The cost functional* (4.3) *takes some linear combination weighted by Q_0 and $\tilde{Q}$. Regarding this point,* (4.3) *enables us to represent some trade-off between the absolute quadratic cost $x_0^2(t)$ and relative quadratic deviation $\Big(x_0(t) - x^{(N)}(t)\Big)^2$. This functional combination can be interpreted as some balance between the minimization of its own cost and the benchmark index tracking to minor agents' average. Moreover, such tracking can be framed into the relative performance setting. Similar works can be found in [85] where the relative performance is formulated by some convex combination $\lambda\Big(x_i(t) - x^{(N)}(t)\Big)^2 + (1-\lambda)x_0^2(t),\ \lambda \in [0,1]$.*

We introduce the following assumption:

(H4.1) $\{x_{i0}\}_{i=1}^N$ are independent and identically distributed (i.i.d) with $\mathbb{E}x_{i0} = x$, $\mathbb{E}|x_{i0}|^2 < +\infty$, and also independent of $\{W_0, W_i, 1 \leqslant i \leqslant N\}$.

It follows that (4.1) admits a unique solution for all $u_0 \in \mathcal{U}_0$, (see [84]). It is also well known that under (H4.1), (4.2) admits a unique solution for all $u_i \in \mathcal{U}_i, 1 \leqslant i \leqslant N$. Now, we formulate the large-population dynamic optimization problem.

Problem (BF-MM). Find a control strategies set $\bar{u} = (\bar{u}_0, \bar{u}_1, \ldots, \bar{u}_N)$ which satisfies

$$\mathcal{J}_i(\bar{u}_i(\cdot), \bar{u}_{-i}(\cdot)) = \inf_{u_i \in \mathcal{U}_i} \mathcal{J}_i(u_i(\cdot), \bar{u}_{-i}(\cdot)),\ 0 \leqslant i \leqslant N$$

where $\bar{u}_{-0}$ represents $(\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_N)$ and $\bar{u}_{-i}$ represents $(\bar{u}_0, \bar{u}_1, \ldots, \bar{u}_{i-1}, \bar{u}_{i+1}, \ldots, \bar{u}_N)$, for $1 \leqslant i \leqslant N$.

4.2 The Limiting Optimal Control and NCE Equation System

To study Problem **(BF-MM)**, one efficient approach is to discuss the associated MFGs via limiting problem when the agent number N tends to infinity. The key ingredient in this approach is to specify some suitable representation of state-average limit. With such limit representation, we can figure out a family of approximating problems and the decentralized strategies of individual agents can be derived based on them. Now we present some straightforward analysis to determine the limit representation in our current work. To start, we first recall the standard procedures of MFGs. As discussed in [29] and [53], the implementation of MFGs breaks into the following main steps:

- (**i**) Fix or freeze the limit state-average by a given process, say $\bar{x}$ which maybe deterministic or random.

- (**ii**) Solve the standard stochastic control problem by replacing $x^{(N)}$ using $\bar{x}$.

- (**iii**) Determine the frozen term $\bar{x}$ such that the the resulting optimal states will replicate our state limit.

Due to the major-minor agent feature, Step (ii) can be further divided into two sub-steps:

- (**ii-a**) Solve the decentralized standard stochastic control problem of major agent, by replacing $x^{(N)}$ using $\bar{x}$. We can obtain the decentralized optimal control and state of major agent.

- (**ii-b**) Given the decentralized optimal state and control of major agent, solve the stochastic control problem facing by the minor agents. Here, the state is augmented by consisting the individual minor agent's state, the derived optimal state of major agent, as well as the limiting state-average.

We have the following basic observation by noting the above MFG procedures and our backward major's state. First, as addressed in (**i**), the limit state average of minor agents will be frozen and denoted by $\bar{x}$. Then, by (**ii-a**), the optimal state of major agent will be characterized by some BFSDE. This is because the state of major agent is some BSDE, thus its adjoint process will be some forward SDE but these two equations will be further coupled in the initial condition. Therefore, we will get some BFSDE instead the classical FBSDE. Next, by (**ii-b**), the given minor agent will solve some standard stochastic control problem with the augmented state: its own state, the limiting state-average, the optimal state of major agent from (**ii-a**) which is a BFSDE. The minor's optimal control should involve some feedback of this augmented state. In this way, the minor's optimal state will be represented through some coupled system of its own state, the major's agent, the limiting state-average as well as one forcing state equation (which is another BSDE because the limit state-average depends on major's agent thus it should be a random process in general). Last, as specified in (**iii**), we need to make summation of all individual minor agents' states, take average and send it to limit. This will enable us to replicate the limiting state-average frozen in (**i**). In sending limit step, the equations of forward SDE (the minor's state) and the limiting state-average will reduce to the same one. Combining with the major's state and forcing equation (BSDE with null terminal condition), we naturally have the following formulation of limit representation. To obtain the feedback control and the desired results, we assume $U_i = \mathbb{R}$ for $i = 0, 1, 2, \ldots, N$.

Suppose $x^{(N)}(\cdot)$ is approximated by $\bar{x}(\cdot)$ as $N \to +\infty$. Introduce the following auxiliary dynamics of major and minor players, still denoted by $x_0(\cdot), x_i(\cdot)$ respectively:

$$\begin{cases} dx_0(t) = \big[A_0 x_0(t) + B_0 u_0(t) + C_0 z_0(t)\big]dt + z_0(t)dW_0(t), \\ x_0(T) = \xi, \\ d\bar{x}(t) = \big[\bar{A}(t)\bar{x}(t) + \bar{B}(t)x_0(t) + \bar{C}(t)k(t)\big]dt, \\ \bar{x}(0) = x, \\ dk(t) = \big[\tilde{A}(t)k(t) + \tilde{B}(t)\bar{x}(t) + \tilde{C}(t)x_0(t)\big]dt + \theta(t)dW_0(t), \\ k(T) = 0 \end{cases} \tag{4.5}$$

and

$$\begin{cases} dx_i(t) = \big[Ax_i(t) + Bu_i(t) + D\bar{x}(t) + \alpha x_0(t)\big]dt + \sigma dW_i(t), \\ x_i(0) = x_{i0}. \end{cases} \tag{4.6}$$

Note that the coefficients $(\bar{A}(\cdot), \bar{B}(\cdot), \bar{C}(\cdot), \tilde{A}(\cdot), \tilde{B}(\cdot), \tilde{C}(\cdot)) \in L^2(0, T; \mathbb{R}^6)$ are still to be determined. The associated limiting cost functionals become

$$J_0(u_0(\cdot)) = \frac{1}{2}\mathbb{E}\left\{\int_0^T \left[Q_0\Big(x_0(t) - \bar{x}(t)\Big)^2 + \tilde{Q}x_0^2(t) + R_0 u_0^2(t)\right]dt + H_0 x_0^2(0)\right\} \tag{4.7}$$

and

$$J_i(u_i(\cdot)) = \frac{1}{2}\mathbb{E}\left\{\int_0^T \left[Q\Big(x_i(t) - \bar{x}(t)\Big)^2 + Ru_i^2(t)\right]dt + Hx_i^2(T)\right\}. \tag{4.8}$$

Thus, we formulate the limiting LQ game as follows.

Problem (L-BF-MM). For i^{th} agent $\mathcal{A}_i$, $i = 0, 1, 2, \ldots, N$, find $\bar{u}_i \in \mathcal{U}_i$ satisfying

$$J_i(\bar{u}_i(\cdot)) = \inf_{u_i \in \mathcal{U}_i} J_i(u_i(\cdot)). \tag{4.9}$$

$\bar{u}_i$ satisfying (4.9) is called an optimal control for **(L-BF-MM)**.

Remark 4.3. *Since $\bar{x}(t)$ is regarded as the approximated process of state average $x^{(N)}(t)$, we replace $x^{(N)}(t)$ by $\bar{x}(t)$ in Problem* **(L-BF-MM)**. *In what follows,* **(L-BF-MM)** *is called the limiting problem of* **(BF-MM)** *as $N \to +\infty$. As referred at the beginning of this section, we are going to deal with this limiting problem first. Then, we will focus on the $\epsilon-$Nash equilibrium between* **(BF-MM)** *and* **(L-BF-MM)***, which is the biggest difference with the usual Nash equilibrium problem.*

Remark 4.4. *By noting that each minor player's state $x_i(t)$ in* (4.2) *depends on the major player's state $x_0(t)$ explicitly, we claim that the limiting process $\bar{x}(t)$ also depends on $x_0(t)$ explicitly. In fact, the third process $k(t)$ is also meaningful, which is a stochastic process introduced in decoupling the Hamilton system. Hereinafter we will show it.*

Remark 4.5. *Since the state-average of minor players appears only in the cost functional of major player, the first equation in* (4.5) *has the same form with* (4.1) *actually. However, for regularity, we still write it out.*

To get the optimal control of Problem **(L-BF-MM)**, we should obtain the optimal control of $\mathcal{A}_0$ first. We have the following lemma.

Lemma 4.1. *Corresponding to the forward-backward system* (4.5) *and* (4.7)*, the optimal control of $\mathcal{A}_0$ for* **(L-BF-MM)** *is given by*

$$\bar{u}_0(t) = -B_0 R_0^{-1} p_0(t) \tag{4.10}$$

where the adjoint process $p_0(\cdot)$ and the corresponding optimal trajectory $(\hat{x}_0(\cdot), \hat{z}_0(\cdot))$

satisfy the following Hamilton system

$$\begin{cases} d\hat{x}_0(t) = \big[A_0\hat{x}_0(t) - B_0^2R_0^{-1}p_0(t) + C_0\hat{z}_0(t)\big]dt + \hat{z}_0(t)dW_0(t), \\ d\bar{x}(t) = \big[\bar{A}(t)\bar{x}(t) + \bar{B}(t)\hat{x}_0(t) + \bar{C}(t)k(t)\big]dt, \\ dk(t) = \big[\tilde{A}(t)k(t) + \tilde{B}(t)\bar{x}(t) + \tilde{C}(t)\hat{x}_0(t)\big]dt + \theta(t)dW_0(t), \\ dp_0(t) = \big[-A_0p_0(t) - Q_0(\hat{x}_0(t) - \bar{x}(t)) - \tilde{Q}\hat{x}_0(t) - \bar{B}(t)p(t) - \tilde{C}(t)q(t)\big]dt \\ \qquad\qquad - C_0p_0(t)dW_0(t), \\ dp(t) = \big[-\bar{A}(t)p(t) + Q_0(\hat{x}_0(t) - \bar{x}(t)) - \tilde{B}(t)q(t)\big]dt + \bar{\theta}(t)dW_0(t), \\ dq(t) = \big(-\tilde{A}(t)q(t) - \bar{C}(t)p(t)\big)dt, \\ \hat{x}_0(T) = \xi,\ \bar{x}(0) = x,\ k(T) = 0,\ p_0(0) = -H_0\hat{x}_0(0),\ p(T) = 0,\ q(0) = 0 \end{cases} \tag{4.11}$$

where $\theta(\cdot), \bar{\theta}(\cdot) \in L^2_{\mathcal{F}_t^{w_0}}(0,T;\mathbb{R})$.

Proof. For the variation of control $\delta u_0(\cdot) \in L^2_{\mathcal{F}_t^{w_0}}(0,T;\mathbb{R})$, introduce the following variational equations:

$$\begin{cases} d\delta x_0(t) = \big[A_0\delta x_0(t) + B_0\delta u_0(t) + C_0\delta z_0(t)\big]dt + \delta z_0(t)dW_0(t), \\ d\delta\bar{x}(t) = \big[\bar{A}(t)\delta\bar{x}(t) + \bar{B}(t)\delta x_0(t) + \bar{C}(t)\delta k(t)\big]dt, \\ d\delta k(t) = \big[\tilde{A}(t)\delta k(t) + \tilde{B}(t)\delta\bar{x}(t) + \tilde{C}(t)\delta x_0(t)\big]dt + \delta\theta(t)dW_0(t), \\ \delta x_0(T) = 0,\ \delta\bar{x}(0) = 0,\ \delta k(T) = 0. \end{cases} \tag{4.12}$$

Applying Itô's formula to $p_0(t)\delta x_0(t)+p(t)\delta\bar{x}(t)+q(t)\delta k(t)$ and noting the associated first order variation of cost functional :

$$0=\delta J_0(\bar{u}_0)=\mathbb{E}\bigg\{\int_0^T \Big[Q_0\big(\hat{x}_0(t)-\bar{x}(t)\big)\big(\delta x_0(t)-\delta\bar{x}(t)\big) + \tilde{Q}\hat{x}_0(t)\delta x_0(t) + R_0\bar{u}_0(t)\delta u_0(t)\Big]\,dt$$

$$+ H_0\hat{x}_0(0)\delta x_0(0)\bigg\},$$

we obtain the optimal control (4.10). Combining all state equations and adjoint equations, and applying $\bar{u}_0(\cdot)$ to $\mathcal{A}_0$, we get Hamilton system (4.11). □

After obtaining the optimal control of major player $\mathcal{A}_0$, in what follows we aim to get the optimal control $\bar{u}_i$ of minor player $\mathcal{A}_i$, with corresponding optimal trajectory $\hat{x}_i(\cdot)$.

Lemma 4.2. *Under* **(H4.1)***, the optimal control of* $\mathcal{A}_i$ *for* (**L-BF-MM**) *is*

$$\bar{u}_i(t) = -BR^{-1}p_i(t) \tag{4.13}$$

where the adjoint process $p_i(\cdot)$ *and the corresponding optimal trajectory* $\hat{x}_i(\cdot)$ *satisfy BSDE*

$$\begin{cases} dp_i(t) = \big[-Ap_i(t) - Q\big(\hat{x}_i(t) - \bar{x}(t)\big)\big]dt + \theta_0(t)dW_0(t) + \theta_i(t)dW_i(t), \\ p_i(T) = H\hat{x}_i(T) \end{cases} \tag{4.14}$$

and SDE

$$\begin{cases} d\hat{x}_i(t) = \big[A\hat{x}_i(t) - B^2R^{-1}p_i(t) + D\bar{x}(t) + \alpha\hat{x}_0(t)\big]dt + \sigma(t)dW_i(t), \\ \hat{x}_i(0) = x_{i0}. \end{cases} \tag{4.15}$$

Here $\theta_0(\cdot), \theta_i(\cdot) \in L^2_{\mathcal{F}^i_t}(0,T;\mathbb{R})$; $\hat{x}_0(\cdot)$ and $\bar{x}(\cdot)$ are given by (4.11). The proof is similar to that of Lemma 4.1 and omitted. For the coupled BFSDE (4.14) and (4.15), we are going to decouple it and trying to derive the Nash certainty equivalence (NCE) system satisfied by the decentralized control policy. Then we have

Lemma 4.3. *Suppose* $P(\cdot)$ *is the unique solution of the following Riccati equation system*

$$\begin{cases} \dot{P}(t) + 2AP(t) - B^2R^{-1}P^2(t) + Q = 0, \\ P(T) = H, \end{cases} \tag{4.16}$$

then we obtain the following Hamilton system:

$$
\begin{cases}
d\hat{x}_0(t) = \big[A_0\hat{x}_0(t) - B_0^2R_0^{-1}p_0(t) + C_0\hat{z}_0(t)\big]dt + \hat{z}_0(t)dW_0(t), \\
d\bar{x}(t) = \big[\big(A + D - B^2R^{-1}P(t)\big)\bar{x}(t) - B^2R^{-1}k(t) + \alpha\hat{x}_0(t)\big]dt, \\
dk(t) = \big[\big(-A + B^2R^{-1}P(t)\big)k(t) + \big(Q - DP(t)\big)\bar{x}(t) - \alpha P(t)\hat{x}_0(t)\big]dt \\
\qquad\qquad + \theta_0(t)dW_0(t), \\
dp_0(t) = \big[-A_0p_0(t) - Q_0(\hat{x}_0(t) - \bar{x}(t)) - \tilde{Q}\hat{x}_0(t) - \alpha p(t) + \alpha P(t)q(t)\big]dt \\
\qquad\qquad - C_0p_0(t)dW_0(t), \\
dp(t) = \big[-\big(A + D - B^2R^{-1}P(t)\big)p(t) + Q_0(\hat{x}_0(t) - \bar{x}(t)) \\
\qquad\qquad - \big(Q - DP(t)\big)q(t)\big]dt + \bar{\theta}(t)dW_0(t), \\
dq(t) = \big[\big(A - B^2R^{-1}P(t)\big)q(t) + B^2R^{-1}p(t)\big]dt, \\
\hat{x}_0(T) = \xi,\ \bar{x}(0) = x,\ k(T) = 0,\ p_0(0) = -H_0\hat{x}_0(0),\ p(T) = 0,\ q(0) = 0
\end{cases}
\tag{4.17}
$$

which is a 3×2 *FBSDE (TFBSDE).*

Proof. Suppose

$$p_i(t) = P_i(t)\hat{x}_i(t) + f_i(t), \quad 1 \leqslant i \leqslant N$$

where $P_i(\cdot), f_i(\cdot)$ are to be determined. The terminal condition $p_i(T) = H\hat{x}_i(T)$ implies that

$$P_i(T) = H, \quad f_i(T) = 0.$$

Applying Itô's formula to $P_i(t)\hat{x}_i(t) + f_i(t)$, we have

$$
\begin{aligned}
dp_i(t) =& \big[\dot{P}_i(t) + AP_i(t) - B^2R^{-1}P_i^2(t)\big]\hat{x}_i(t)dt \\
&+ \big[DP_i(t)\bar{x}(t) - B^2R^{-1}P_i(t)f_i(t) + \alpha P_i(t)\hat{x}_0(t)\big]dt + df_i(t) + \sigma P_i(t)dW_i(t).
\end{aligned}
$$

Comparing the coefficients with (4.14), we get $\theta_i(t) = \sigma P_i(t)$,

$$
\begin{cases}
\dot{P}_i(t) + 2AP_i(t) - B^2R^{-1}P_i^2(t) + Q = 0, \\
P_i(T) = H
\end{cases}
\tag{4.18}
$$

and

$$\begin{cases} df_i(t) = \big[\big(-A + B^2R^{-1}P_i(t)\big)f_i(t) + \big(Q - DP_i(t)\big)\bar{x}(t) - \alpha P_i(t)\hat{x}_0(t)\big]dt \\ \qquad\qquad + \theta_0(t)dW_0(t), \\ f_i(T) = 0. \end{cases} \tag{4.19}$$

Noting that Riccati equation (4.18) is symmetric, it is well known that (4.18) admits a unique nonnegative bounded solution $P_i(\cdot)$ (see [95]). Further we get that $P_1(\cdot) = P_2(\cdot) = \cdots = P_N(\cdot) := P(\cdot)$. Thus, (4.18) coincides with (4.16). Besides, for given $\bar{x}(\cdot), \hat{x}_0(\cdot) \in L^2_{\mathcal{F}_t^{w_0}}(0,T;\mathbb{R})$, LBSDE (4.19) admits a unique solution $f_i(\cdot) \in L^2_{\mathcal{F}_t^{w_0}}(0,T;\mathbb{R})$. We denote $f_i(\cdot) := f(\cdot), i = 1, 2, \dots, N$.

Therefore, the decentralized feedback strategy for $\mathcal{A}_i, 1 \leqslant i \leqslant N$ is written as

$$u_i(t) = -BR^{-1}\big(P(t)x_i(t) + f(t)\big) \tag{4.20}$$

where $x_i(\cdot)$ is the state of minor player $\mathcal{A}_i$. Plugging (4.20) into (4.2) implies the centralized closed-loop state:

$$\begin{cases} dx_i(t) = \Big[\big(A - B^2R^{-1}P(t)\big)x_i(t) - B^2R^{-1}f(t) + Dx^{(N)}(t) + \alpha x_0(t)\Big]dt + \sigma dW_i(t), \\ x_i(0) = x_{i0}. \end{cases} \tag{4.21}$$

Taking summation, dividing by N and letting $N \to +\infty$, we get

$$\begin{cases} d\bar{x}(t) = \big[\big(A + D - B^2R^{-1}P(t)\big)\bar{x}(t) - B^2R^{-1}f(t) + \alpha x_0(t)\big]dt, \\ \bar{x}(0) = x. \end{cases} \tag{4.22}$$

Comparing the coefficients with the second equation of (4.5), we have

$$\bar{A}(\cdot) = A + D - B^2R^{-1}P(\cdot), \quad \bar{B}(\cdot) = \alpha, \quad \bar{C}(\cdot) = -B^2R^{-1}, \quad k(\cdot) = f(\cdot).$$

Then we obtain

$$\begin{cases} dk(t) = \big[\big(-A + B^2R^{-1}P(t)\big)k(t) + \big(Q - DP(t)\big)\bar{x}(t) - \alpha P(t)x_0(t)\big]dt + \theta_0(t)dW_0(t), \\ k(T) = 0. \end{cases}$$

Noting the third equation of (4.5), it follows that

$$\tilde{A}(\cdot) = -A + B^2R^{-1}P(\cdot), \quad \tilde{B}(\cdot) = Q - DP(\cdot), \quad \tilde{C}(\cdot) = -\alpha P(\cdot), \quad \theta(\cdot) = \theta_0(\cdot).$$

Then (4.17) is obtained, which completes the proof. $\square$

Remark 4.6. *The proof of Lemma 4.3 implies that* $k(\cdot) = f(\cdot)$. *Thus,* $k(\cdot)$, *which is first introduced in* (4.5) *has some specific meanings that it is indeed a force function when decoupling* (4.14) *and* (4.15).

To get the wellposedness of (4.17), we give the following assumption.

(H4.2) $B_0 \neq 0,\ H_0 > 0,\ \tilde{Q} > 0.$

Theorem 4.1. *Under* (H4.2), *TFBSDE* (4.17) *is uniquely solvable.*

Proof. Uniqueness.

It is easy checked that (4.16) admits a unique nonnegative bounded solution (see [95]). For sake of notation convenience, in (4.17) we denote by $b(\phi), \sigma(\phi)$ the coefficients of drift and diffusion terms respectively, for $\phi = p_0, \bar{x}, q$; denote by $f(\psi)$ the generator for $\psi = \hat{x}_0, p, k$.

Define $\Delta := (p_0, \bar{x}, q, \hat{x}_0, p, k, \hat{z}_0, \bar{\theta}, \theta_0)$. Similar to the notations in [92], we denote by

$$\mathbb{A}(t, \Delta) := \Big(- f(\hat{x}_0), -f(p), -f(k), b(p_0), b(\bar{x}), b(q), \sigma(p_0), \sigma(\bar{x}), \sigma(q)\Big),$$

which implies $\mathbb{A}(t, \Delta) = \Big(A_0\hat{x}_0 - B_0^2R_0^{-1}p_0 + C_0\hat{z}_0, -\big(A + D - B^2R^{-1}P(t)\big)p + Q_0(\hat{x}_0 - \bar{x}) - \big(Q - DP(t)\big)q, \big(- A + B^2R^{-1}P(t)\big)k + \big(Q - DP(t)\big)\bar{x} - \alpha P(t)\hat{x}_0, -A_0p_0 - Q_0(\hat{x}_0 - \bar{x}) - \tilde{Q}\hat{x}_0 - \alpha p + \alpha P(t)q, \big(A + D - B^2R^{-1}P(t)\big)\bar{x} - B^2R^{-1}k + \alpha\hat{x}_0, \big(A - B^2R^{-1}P(t)\big)q + B^2R^{-1}p, -C_0p_0, 0, 0\Big)$.

Then for any $\Delta^i = (p_0^i, \bar{x}^i, q^i, \hat{x}_0^i, p^i, k^i, \hat{z}_0^i, \bar{\theta}^i, \theta_0^i), i = 1, 2$, we have

$$\begin{aligned}
&\langle \mathbb{A}(t, \Delta^1) - \mathbb{A}(t, \Delta^2), \Delta^1 - \Delta^2\rangle \\
=& - B_0^2R_0^{-1}(p_0^1 - p_0^2)^2 - Q_0\big[(\bar{x}^1 - \bar{x}^2) - (\hat{x}_0^1 - \hat{x}_0^2)\big]^2 - \tilde{Q}(\hat{x}_0^1 - \hat{x}_0^2)^2 \\
\leqslant& - B_0^2R_0^{-1}(p_0^1 - p_0^2)^2 - \tilde{Q}(\hat{x}_0^1 - \hat{x}_0^2)^2 := -\beta_1(p_0^1 - p_0^2)^2 - \beta_2(\hat{x}_0^1 - \hat{x}_0^2)^2.
\end{aligned}$$

In the following, firstly we are going to show that (4.17) admits at most one adapted solution. Suppose Δ and $\Delta' = (p_0', \bar{x}', q', \hat{x}_0', p', k', \hat{z}_0', \bar{\theta}', \theta_0')$ are two solutions of (4.17). Setting $\hat{\Delta} = (\hat{p}_0, \hat{\bar{x}}, \hat{q}, \hat{\hat{x}}_0, \hat{p}, \hat{k}, \hat{\hat{z}}_0, \hat{\bar{\theta}}, \hat{\theta}_0) = (p_0 - p_0', \bar{x} - \bar{x}', q - q', \hat{x}_0 - \hat{x}_0', p - p', k - k', \hat{z}_0 - \hat{z}_0', \bar{\theta} - \bar{\theta}', \theta_0 - \theta_0')$ and applying Itô's formula to $\langle \hat{p}_0, \hat{\hat{x}}_0 \rangle + \langle \hat{\bar{x}}, \hat{p} \rangle + \langle \hat{q}, \hat{k} \rangle$, we have

$$
\begin{aligned}
-\mathbb{E}\langle \hat{p}_0(0), \hat{\hat{x}}_0(0) \rangle &= \mathbb{E}\int_0^T \langle \mathbb{A}(s, \Delta) - \mathbb{A}(s, \Delta'), \hat{\Delta} \rangle ds \\
&\leqslant -\beta_1 \mathbb{E}\int_0^T (p_0(s) - p_0'(s))^2 ds - \beta_2 \mathbb{E}\int_0^T (\hat{x}_0(s) - \hat{x}_0'(s))^2 ds.
\end{aligned}
$$

It follows that

$$
\beta_1 \mathbb{E}\int_0^T |\hat{p}_0(s)|^2 ds + \beta_2 \mathbb{E}\int_0^T \left|\hat{\hat{x}}_0(s)\right|^2 ds + H_0 \mathbb{E}\left|\hat{\hat{x}}_0(0)\right|^2 \leqslant 0.
$$

By (H4.2) we get $\beta_1 > 0$ and $\beta_2 > 0$. Then $\hat{p}_0(s) \equiv 0$, $\hat{\hat{x}}_0(s) \equiv 0$. Further $\hat{\hat{z}}_0(s) \equiv 0$. Applying the basic technique to $\hat{\bar{x}}(s)$ and $\hat{k}(s)$, and using Gronwall's inequality, we obtain $\hat{\bar{x}}(s) \equiv 0$, $\hat{k}(s) \equiv 0$ and $\hat{\theta}_0(s) \equiv 0$. Similarly, we have $\hat{q}(s) \equiv 0$, $\hat{p}(s) \equiv 0$ and $\hat{\bar{\theta}}(s) \equiv 0$. Therefore, (4.17) admits at most one adapted solution.

Existence. In order to prove the existence of the solution we first consider the following family of FBSDEs parameterized by $\gamma \in [0, 1]$:

$$
\begin{cases}
dp_0^\gamma(t) = \big[-(1-\gamma)\hat{x}_0^\gamma(t)\beta_2 + \gamma b(p_0^\gamma) + \varphi_t^1\big]dt + \big[\gamma\sigma(p_0^\gamma) + \lambda_t\big]dW_0(t), \\
d\hat{x}_0^\gamma(t) = \big[-(1-\gamma)p_0^\gamma(t)\beta_1 - \gamma f(\hat{x}_0^\gamma) + \kappa_t^1\big]dt + \hat{z}_0^\gamma(t)dW_0(t), \\
d\bar{x}^\gamma(t) = \big[\gamma b(\bar{x}^\gamma) + \varphi_t^2\big]dt, \\
dp^\gamma(t) = \big[-\gamma f(p^\gamma) + \kappa_t^2\big]dt + \bar{\theta}^\gamma(t)dW_0(t), \\
dq^\gamma(t) = \big[\gamma b(q^\gamma) + \varphi_t^3\big]dt, \\
dk^\gamma(t) = \big[-\gamma f(k^\gamma) + \kappa_t^3\big]dt + \theta_0^\gamma(t)dW_0(t), \\
p_0^\gamma(0) = -(1-\gamma)\hat{x}_0^\gamma(0) - \gamma H_0\hat{x}_0^\gamma(0) + a,\ \hat{x}_0^\gamma(T) = \gamma\xi,\ \bar{x}^\gamma(0) = \gamma x, \\
p^\gamma(T) = 0,\ q^\gamma(0) = 0,\ k^\gamma(T) = 0
\end{cases}
\tag{4.23}
$$

where $(\varphi^1,\varphi^2,\varphi^3,\lambda,\kappa^1,\kappa^2,\kappa^3)\in L^2_{\mathcal{F}}(0,T;\mathbb{R}^7)$, $a\in L^2(\Omega,\mathcal{F}_0,P;\mathbb{R})$. Clearly, when $\gamma=1$, the existence of (4.23) implies that of (4.17). When $\gamma=0$, it is easy to get that (4.23) admits a unique solution (see [92] and [105] for details).

If, a priori, for each $(\varphi^1,\varphi^2,\varphi^3,\lambda,\kappa^1,\kappa^2,\kappa^3)\in L^2_{\mathcal{F}}(0,T;\mathbb{R}^7)$, a $\gamma_0\in[0,1)$ there exists a unique tuple $(p_0^{\gamma_0},\bar{x}^{\gamma_0},q^{\gamma_0},\hat{x}_0^{\gamma_0},p^{\gamma_0},k^{\gamma_0},\hat{z}_0^{\gamma_0},\bar{\theta}^{\gamma_0},\theta_0^{\gamma_0})$ of (4.23), then for each

$$u_s=\big(p_0(s),\bar{x}(s),q(s),\hat{x}_0(s),p(s),k(s),\hat{z}_0(s),\bar{\theta}(s),\theta_0(s)\big)\in L^2_{\mathcal{F}_s}(0,T;\mathbb{R}^9),$$

there exists a unique tuple $U_s=\big(P_0(s),\bar{X}(s),Q(s),\hat{X}_0(s),P(s),K(s),\hat{Z}_0(s),\bar{\Theta}(s),\Theta_0(s)\big)$ $\in L^2_{\mathcal{F}_s}(0,T;\mathbb{R}^9)$ satisfying the following FBSDEs

$$\begin{cases}
dP_0(t)=\big[-(1-\gamma_0)\hat{X}_0(t)\beta_2+\gamma_0 b(P_0)+\delta(\hat{x}_0(t)\beta_2+b(p_0))+\varphi^1_t\big]dt\\
\qquad\qquad+\big[\gamma_0\sigma(P_0)+\lambda_t\big]dW_0(t),\\
d\hat{X}_0(t)=\big[-(1-\gamma_0)P_0(t)\beta_1-\gamma_0 f(\hat{X}_0)+\delta(p_0(t)\beta_1-f(\hat{x}_0))+\kappa^1_t\big]dt\\
\qquad\qquad+\hat{Z}_0(t)dW_0(t),\\
d\bar{X}(t)=\big[\gamma_0 b(\bar{X})+\delta b(\bar{x})+\varphi^2_t\big]dt,\\
dP(t)=\big[-\gamma_0 f(P)-\delta f(p)+\kappa^2_t\big]dt+\bar{\Theta}(t)dW_0(t),\\
dQ(t)=\big[\gamma_0 b(Q)+\delta b(q)+\varphi^3_t\big]dt,\\
dK(t)=\big[-\gamma_0 f(K)-\delta f(k)+\kappa^3_t\big]dt+\Theta_0(t)dW_0(t),\\
P_0(0)=-(1-\gamma_0)\hat{X}_0(0)-\gamma_0 H_0\hat{X}_0(0)+\delta(1-H_0)\hat{x}_0(0)+a,\\
\hat{X}_0(T)=\gamma_0\xi+\delta\xi,\ \bar{X}(0)=\gamma_0 x+\delta x,\ P(T)=0,\ Q(0)=0,\ K(T)=0.
\end{cases}\tag{4.24}$$

In the following we aim to prove that the mapping defined by

$$I_{\gamma_0+\delta}(u\times\hat{x}_0(0))=U\times\hat{X}_0(0):L^2_{\mathcal{F}}(0,T;\mathbb{R}^9)\times L^2(\Omega,\mathcal{F}_0,P)\to L^2_{\mathcal{F}}(0,T;\mathbb{R}^9)\times L^2(\Omega,\mathcal{F}_0,P)$$

is a contraction.

Introduce $u' = (p_0', \bar{x}', q', \hat{x}_0', p', k', \hat{z}_0', \bar{\theta}', \theta_0') \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^9)$, $U' \times \hat{X}_0'(0) = I_{\gamma_0+\delta}(u' \times \hat{x}_0'(0))$ and set

$$
\begin{aligned}
\hat{u} &= (\hat{p}_0, \hat{\bar{x}}, \hat{q}, \hat{\hat{x}}_0, \hat{p}, \hat{k}, \hat{\hat{z}}_0, \hat{\bar{\theta}}, \hat{\theta}_0) \\
&= (p_0 - p_0', \bar{x} - \bar{x}', q - q', \hat{x}_0 - \hat{x}_0', p - p', k - k', \hat{z}_0 - \hat{z}_0', \bar{\theta} - \bar{\theta}', \theta_0 - \theta_0') \\
\hat{U} &= (\hat{P}_0, \hat{\bar{X}}, \hat{Q}, \hat{\hat{X}}_0, \hat{P}, \hat{K}, \hat{\hat{Z}}_0, \hat{\bar{\Theta}}, \hat{\Theta}_0) \\
&= (P_0 - P_0', \bar{X} - \bar{X}', Q - Q', \hat{X}_0 - \hat{X}_0', P - P', K - K', \hat{Z}_0 - \hat{Z}_0', \bar{\Theta} - \bar{\Theta}', \Theta_0 - \Theta_0').
\end{aligned}
$$

Applying Itô's formula to $\langle \hat{P}_0, \hat{\hat{X}}_0 \rangle + \langle \hat{\bar{X}}, \hat{P} \rangle + \langle \hat{Q}, \hat{K} \rangle$, we have

$$
\begin{aligned}
&\big(\gamma_0 H_0 + (1-\gamma_0)\big)\mathbb{E}\Big|\hat{\hat{X}}_0(0)\Big|^2 + \mathbb{E}\int_0^T \Big(\beta_1\big|\hat{P}_0(s)\big|^2 + \beta_2\big|\hat{\hat{X}}_0(s)\big|^2\Big)ds \\
\leqslant &\delta C_1 \mathbb{E}\int_0^T \Big(|\hat{u}_s|^2 + |\hat{U}_s|^2\Big)ds + \delta C_1 \mathbb{E}\Big|\hat{\hat{x}}_0(0)\Big|^2.
\end{aligned}
\tag{4.25}
$$

On the other hand, since P_0 and P_0' are solutions of SDEs with Itô's type, applying the usual technique, the estimate for the difference $\hat{P}_0 = P_0 - P_0'$ is obtained by

$$
\begin{aligned}
\mathbb{E}\int_0^T |\hat{P}_0(s)|^2 ds \leqslant\ & C_1 T\delta \mathbb{E}\int_0^T |\hat{u}_s|^2 ds + C_1 T \mathbb{E}\Big|\hat{\hat{X}}_0(0)\Big|^2 + C_1 T\delta \mathbb{E}\Big|\hat{\hat{x}}_0(0)\Big|^2 \\
&+ C_1 T \mathbb{E}\int_0^T \Big(|\hat{\hat{X}}_0(s)|^2 + |\hat{\bar{X}}(s)|^2 + |\hat{P}(s)|^2 + |\hat{Q}(s)|^2\Big)ds.
\end{aligned}
\tag{4.26}
$$

Similarly, estimates for the difference $\hat{\bar{X}} = \bar{X} - \bar{X}'$ and $\hat{Q} = Q - Q'$ are given by

$$
\sup_{0\leqslant s\leqslant r} \mathbb{E}\big|\hat{\bar{X}}(s)\big|^2 \leqslant C_1\delta \mathbb{E}\int_0^r |\hat{u}_s|^2 ds + C_1 \mathbb{E}\int_0^r \Big(|\hat{K}(s)|^2 + |\hat{\hat{X}}_0(s)|^2\Big)ds \tag{4.27}
$$

and

$$
\sup_{0\leqslant s\leqslant r} \mathbb{E}\big|\hat{Q}(s)\big|^2 \leqslant C_1\delta \mathbb{E}\int_0^r |\hat{u}_s|^2 ds + C_1 \mathbb{E}\int_0^r \Big(|\hat{K}(s)|^2 + |\hat{P}(s)|^2\Big)ds \tag{4.28}
$$

respectively, for $\forall\ 0 \leqslant r \leqslant T$. In the same way, for the difference of the solutions $(\hat{\hat{X}}_0, \hat{\hat{Z}}_0) = (\hat{X}_0 - \hat{X}_0', \hat{Z}_0 - \hat{Z}_0'), (\hat{P}, \hat{\bar{\Theta}}) = (P - P', \bar{\Theta} - \bar{\Theta}')$ and $(\hat{K}, \hat{\Theta}_0) = (K - K', \Theta_0 - \Theta_0')$, applying the usual technique to the BSDEs, we have

$$\mathbb{E}\int_0^T \Big(|\hat{\hat{X}}_0(s)|^2 + |\hat{\hat{Z}}_0(s)|^2\Big) ds \leqslant C_1 \delta \mathbb{E}\int_0^T |\hat{u}_s|^2 ds + C_1 \mathbb{E}\int_0^T |\hat{P}_0(s)|^2 ds, \tag{4.29}$$

$$\begin{aligned} &\mathbb{E}\int_0^r \Big(|\hat{P}(s)|^2 + |\hat{\bar{\Theta}}(s)|^2\Big) ds \\ &\qquad \leqslant C_1 \delta \mathbb{E}\int_0^r |\hat{u}_s|^2 ds + C_1 \mathbb{E}\int_0^r \Big(|\hat{\hat{X}}_0(s)|^2 + |\hat{\bar{X}}(s)|^2 + |\hat{Q}(s)|^2\Big) ds \end{aligned} \tag{4.30}$$

and

$$\begin{aligned} &\mathbb{E}\int_0^r \Big(|\hat{K}(s)|^2 + |\hat{\Theta}_0(s)|^2\Big) ds \\ &\qquad \leqslant C_1 \delta \mathbb{E}\int_0^r |\hat{u}_s|^2 ds + C_1 \mathbb{E}\int_0^r \Big(|\hat{\hat{X}}_0(s)|^2 + |\hat{\bar{X}}(s)|^2\Big) ds \end{aligned} \tag{4.31}$$

for $\forall\ 0 \leqslant r \leqslant T$. Here the constant C_1 depends on the coefficients of (4.1)-(4.2), $P(\cdot),\ \beta_1,\ \beta_2$ and T. $\gamma_0 H_0 + (1-\gamma_0) \geqslant \mu,\ \mu = \min(1, H_0) > 0$.

Under (H4.2), combining (4.25), (4.27)-(4.28), (4.30)-(4.31) and applying Gronwall's inequality, we obtain

$$\mathbb{E}\int_0^T |\hat{U}_s|^2 ds + \mathbb{E}\Big|\hat{\hat{X}}_0(0)\Big|^2 \leqslant C_2 \delta \Big(\mathbb{E}\int_0^T |\hat{u}_s|^2 ds + \mathbb{E}\big|\hat{\hat{x}}_0(0)\big|^2\Big)$$

where C_2 depends on $C_1,\ \mu$ and T. Choosing $\delta_0 = \frac{1}{2C_2}$, we get that for each fixed $\delta \in [0, \delta_0]$, the mapping $I_{\gamma_0+\delta}$ is a contraction in sense that

$$\mathbb{E}\int_0^T |\hat{U}_s|^2 ds + \mathbb{E}\Big|\hat{\hat{X}}_0(0)\Big|^2 \leqslant \frac{1}{2}\Big(\mathbb{E}\int_0^T |\hat{u}_s|^2 ds + \mathbb{E}\big|\hat{\hat{x}}_0(0)\big|^2\Big).$$

Then it follows that there exists a unique fixed point

$$U^{\gamma_0+\delta} = \big(P_0^{\gamma_0+\delta}, \bar{X}^{\gamma_0+\delta}, Q^{\gamma_0+\delta}, \hat{X}_0^{\gamma_0+\delta}, P^{\gamma_0+\delta}, K^{\gamma_0+\delta}, \hat{Z}_0^{\gamma_0+\delta}, \bar{\Theta}^{\gamma_0+\delta}, \Theta_0^{\gamma_0+\delta}\big)$$

which is the solution of (4.23) for $\gamma = \gamma_0 + \delta$. Since δ_0 depends only on (C_1, μ, T), we can repeat this process for N times with $1 \leqslant N\delta_0 < 1 + \delta_0$.

Then it follows that, in particular, as $\gamma = 1$ corresponding to $\varphi_t^i \equiv 0, \lambda_t \equiv 0, \kappa_t^i \equiv 0, a = 0$ $(i = 1, 2, 3)$, (4.23) admits a unique solution, which implies the wellposedness of (4.17) (also (4.11)). The proof is complete. □

Remark 4.7. *In what follows* (4.17) *is called the Nash certainty equivalence (NCE) equation system (see [51, 18, 20, 8]). By Theorem 4.1 we know that there exists a unique 9-tuple solution* $(p_0, \bar{x}, q, \hat{x}_0, p, k, \hat{z}_0, \bar{\theta}, \theta_0)$ *which can be obtained off-line. Thus it is equivalent with the fixed point principle. To our best knowledge, it is the first time to focus on the wellposedness of TFBSDE in large-population problems. It is of great feature and meaningful.*

4.3 ϵ-Nash Equilibrium Analysis for (BF-MM)

In above sections, we obtained the optimal control $\bar{u}_i(\cdot), 0 \leqslant i \leqslant N$ of Problem **(L-BF-MM)** through the consistency condition system. Now we turn to verify the ϵ-Nash equilibrium of Problem **(BF-MM)**. To start, we first present the definition of ϵ-Nash equilibrium for $(N+1)$ agents.

Definition 4.1. *A set of controls* $u_k \in \mathcal{U}_k,\ 0 \leqslant k \leqslant N$, *for* $(N+1)$ *agents is called an* ϵ*-Nash equilibrium with respect to the costs* $\mathcal{J}_k,\ 0 \leqslant k \leqslant N$, *if there exists* $\epsilon \geqslant 0$ *such that for any fixed* $0 \leqslant i \leqslant N$, *we have*

$$\mathcal{J}_i(u_i, u_{-i}) \leqslant \mathcal{J}_i(u_i', u_{-i}) + \epsilon \tag{4.32}$$

when any alternative control $u_i' \in \mathcal{U}_i$ *is applied by* $\mathcal{A}_i$.

Now, we state the main result of this paper and its proof will be given later.

Theorem 4.2. *Under* (H4.1)-(H4.2), $(\tilde{u}_0, \tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_N)$ *satisfies the* ϵ*-Nash equilibrium of* **(BF-MM)**. *Here,* $\tilde{u}_0$ *is given by*

$$\tilde{u}_0(t) = -B_0 R_0^{-1} p_0(t) \tag{4.33}$$

where $p_0(\cdot)$ *is obtained outline by* (4.17)*; while for* $1 \leqslant i \leqslant N$, $\tilde{u}_i$ *is*

$$\tilde{u}_i(t) = -BR^{-1}P(t)\tilde{x}_i(t) - BR^{-1}k(t) \tag{4.34}$$

where $\tilde{x}_i(\cdot)$*, the state trajectory for* $\mathcal{A}_i$*, satisfies* (4.21).

The proof of above theorem needs several lemmas which are presented later. Denote by $(\tilde{x}_0(\cdot), \tilde{z}_0(\cdot))$ the centralized state trajectory; $(\hat{x}_0(\cdot), \hat{z}_0(\cdot))$ the decentralized one. Applying $\tilde{u}_0(\cdot)$ to $\mathcal{A}_0$ and using the notations above, it is easy to know that $(\tilde{x}_0(\cdot), \tilde{z}_0(\cdot)) \equiv (\hat{x}_0(\cdot), \hat{z}_0(\cdot))$. Further, $(\bar{x}(\cdot), k(\cdot))_{\tilde{x}_0} = (\bar{x}(\cdot), k(\cdot))_{\hat{x}_0}$. Hereafter, for any $h_j(\cdot) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}), j = 1,2,3$, denote by $(h_1(\cdot), h_2(\cdot))_{h_3}$ the stochastic process pair $(h_1(\cdot), h_2(\cdot))$ which is determined by $h_3(\cdot)$. The cost functionals for **(BF-MM)** and **(L-BF-MM)** are given by

$$\mathcal{J}_0(\tilde{u}_0(\cdot), \tilde{u}_{-0}(\cdot)) = \frac{1}{2}\mathbb{E}\left\{\int_0^T \left[Q_0\Big(\tilde{x}_0(t) - \tilde{x}^{(N)}(t)\Big)^2 + \tilde{Q}\tilde{x}_0^2(t) + R_0\tilde{u}_0^2(t)\right]dt + H_0\tilde{x}_0^2(0)\right\} \tag{4.35}$$

and

$$J_0(\bar{u}_0(\cdot)) = \frac{1}{2}\mathbb{E}\left\{\int_0^T \left[Q_0\Big(\hat{x}_0(t) - \bar{x}(t)_{\hat{x}_0}\Big)^2 + \tilde{Q}\hat{x}_0^2(t) + R_0\bar{u}_0^2(t)\right]dt + H_0\hat{x}_0^2(0)\right\} \tag{4.36}$$

respectively. For $\mathcal{A}_i, 1 \leqslant i \leqslant N$, we have the following close-loop system

$$\begin{cases} d\tilde{x}_i(t) = \Big[(A - B^2R^{-1}P(t))\tilde{x}_i(t) - B^2R^{-1}k(t)_{\tilde{x}_0} + D\tilde{x}^{(N)}(t) + \alpha\tilde{x}_0(t)\Big]dt + \sigma dW_i(t), \\ \tilde{x}_i(0) = x_{i0} \end{cases} \tag{4.37}$$

with the cost functional

$$\mathcal{J}_i(\tilde{u}_i(\cdot), \tilde{u}_{-i}(\cdot)) = \frac{1}{2}\mathbb{E}\left\{\int_0^T \left[Q\Big(\tilde{x}_i(t) - \tilde{x}^{(N)}(t)\Big)^2 + R\tilde{u}_i^2(t)\right]dt + H\tilde{x}_i^2(T)\right\} \tag{4.38}$$

where $\tilde{x}^{(N)}(t) = \frac{1}{N}\sum\limits_{i=1}^{N}\tilde{x}_i(t)$. The auxiliary system (limiting problem) is given by

$$\begin{cases} d\hat{x}_i(t) = \Big[(A - B^2R^{-1}P(t))\hat{x}_i(t) - B^2R^{-1}k(t)_{\hat{x}_0} + D\bar{x}(t)_{\hat{x}_0} + \alpha\hat{x}_0(t)\Big]dt + \sigma dW_i(t), \\ \hat{x}_i(0) = x_{i0} \end{cases} \tag{4.39}$$

with the cost functional

$$J_i(\bar{u}_i(\cdot)) = \frac{1}{2}\mathbb{E}\left\{\int_0^T \left[Q\Big(\hat{x}_i(t) - \bar{x}(t)_{\hat{x}_0}\Big)^2 + R\bar{u}_i^2(t)\right]dt + H\hat{x}_i^2(T)\right\} \tag{4.40}$$

where $(\bar{x}(t)_{\hat{x}_0}, k(t)_{\hat{x}_0})$ satisfies (4.17). We have

Lemma 4.4.

$$\sup_{0\leqslant t\leqslant T} \mathbb{E}\Big|\tilde{x}^{(N)}(t) - \bar{x}(t)_{\hat{x}_0}\Big|^2 = O\Big(\frac{1}{N}\Big), \tag{4.41}$$

$$\Big|\mathcal{J}_0(\tilde{u}_0, \tilde{u}_{-0}) - J_0(\bar{u}_0)\Big| = O\Big(\frac{1}{\sqrt{N}}\Big). \tag{4.42}$$

Proof. By (4.37), we have

$$\begin{cases} d\tilde{x}^{(N)}(t) = \Big[\big(A+D-B^2R^{-1}P(t)\big)\tilde{x}^{(N)}(t) - B^2R^{-1}k(t)_{\tilde{x}_0} + \alpha\tilde{x}_0(t)\Big]dt + \dfrac{1}{N}\displaystyle\sum_{i=1}^N \sigma dW_i(t), \\ \tilde{x}^{(N)}(0) = x_0^{(N)} \end{cases}$$

where $x_0^{(N)} = \frac{1}{N}\sum\limits_{i=1}^N x_{i0}$. Noting that

$$\mathbb{E}\Big|x_0^{(N)} - x\Big|^2 \sim \mathbb{E}\left|\int_0^t \frac{1}{N}\sum_{i=1}^N \sigma dW_i(s)\right|^2 = O\Big(\frac{1}{N}\Big),$$

by (4.17) and Gronwall's inequality, we obtain (4.41).

It is easily got that $\sup\limits_{0\leqslant t\leqslant T} \mathbb{E}\big|\hat{x}_0(t) - \bar{x}(t)_{\hat{x}_0}\big|^2 < +\infty$. Applying Cauchy-Schwarz inequality, we have

$$\sup_{0\leqslant t\leqslant T} \mathbb{E}\Big|\big|\tilde{x}_0(t) - \tilde{x}^{(N)}(t)\big|^2 - \big|\hat{x}_0(t) - \bar{x}(t)_{\hat{x}_0}\big|^2\Big| = O\Big(\frac{1}{\sqrt{N}}\Big). \tag{4.43}$$

In addition, by (4.10) and (4.33), we have $\tilde{u}_0(\cdot) = \hat{u}_0(\cdot)$. Thus (4.42) is obtained. □

For minor players, we have

Lemma 4.5.

$$\sup_{1\leqslant i\leqslant N}\left[\sup_{0\leqslant t\leqslant T}\mathbb{E}\Big|\tilde{x}_i(t)-\hat{x}_i(t)\Big|^2\right]=O\Big(\frac{1}{N}\Big), \tag{4.44}$$

$$\sup_{1\leqslant i\leqslant N}\left[\sup_{0\leqslant t\leqslant T}\mathbb{E}\Big|\tilde{u}_i(t)-\bar{u}_i(t)\Big|^2\right]=O\Big(\frac{1}{N}\Big), \tag{4.45}$$

$$\Big|\mathcal{J}_i(\tilde{u}_i,\tilde{u}_{-i})-J_i(\bar{u}_i)\Big|=O\Big(\frac{1}{\sqrt{N}}\Big),\quad 1\leqslant i\leqslant N. \tag{4.46}$$

Proof. For $\forall\ 1\leqslant i\leqslant N$, applying Gronwall's inequality, we get (4.44) from (4.41). (4.45) follows from (4.44) obviously. Using the same technique as (4.43) and noting $\sup\limits_{0\leqslant t\leqslant T}\mathbb{E}\big|\hat{x}_i(t)-\bar{x}(t)_{\hat{x}_0}\big|^2<+\infty$, $\sup\limits_{0\leqslant t\leqslant T}\mathbb{E}\big|\bar{u}_i(t)\big|^2<+\infty$, $\sup\limits_{0\leqslant t\leqslant T}\mathbb{E}\big|\hat{x}_i(t)\big|^2<+\infty$, we obtain (4.46). $\square$

Until now, we have addressed some estimates of states and costs corresponding to control $\tilde{u}_i$ and $\bar{u}_i$,$0\leqslant i\leqslant N$. Next we will focus on the ϵ-Nash equilibrium for **(BF-MM)**. Consider a perturbed control $u_0\in\mathcal{U}_0$ for $\mathcal{A}_0$ and introduce the dynamics

$$\begin{cases} dl_0(t)=\Big[A_0l_0(t)+B_0u_0(t)+C_0q_0(t)\Big]dt+q_0(t)dW_0(t),\\ x_0(T)=\xi \end{cases} \tag{4.47}$$

whereas minor players keep the control $\tilde{u}_i, 1\leqslant i\leqslant N$, i.e.,

$$\begin{cases} dl_i(t)=\Big[(A-B^2R^{-1}P(t))l_i(t)-B^2R^{-1}k(t)_{l_0}+Dl^{(N)}(t)+\alpha l_0(t)\Big]dt+\sigma dW_i(t),\\ l_i(0)=x_{i0} \end{cases} \tag{4.48}$$

where $l^{(N)}(t)=\frac{1}{N}\sum\limits_{k=1}^{N}l_k(t)$; $k(t)_{l_0}$ associated with l_0 satisfies

$$\begin{cases} dk(t)_{l_0}=\Big[\big(-A+B^2R^{-1}P(t)\big)k(t)_{l_0}+\big(Q-DP(t)\big)\bar{x}(t)_{l_0}-\alpha P(t)l_0(t)\Big]dt\\ \qquad\qquad +\theta_0(t)_{l_0}dW_0(t),\\ d\bar{x}(t)_{l_0}=\Big[\big(A+D-B^2R^{-1}P(t)\big)\bar{x}(t)_{l_0}-B^2R^{-1}k(t)_{l_0}+\alpha l_0(t)\Big]dt,\\ k(T)_{l_0}=0,\ \bar{x}(0)_{l_0}=x. \end{cases} \tag{4.49}$$

And for any fixed i, $1 \leqslant i \leqslant N$, consider a perturbed control $u_i \in \mathcal{U}_i$ for $\mathcal{A}_i$, whereas the major and other minor players keep the control $\tilde{u}_j, 0 \leqslant j \leqslant N, j \neq i$. Introduce the dynamics

$$\begin{cases} dm_i(t) = \Big[Am_i(t) + Bu_i(t) + Dm^{(N)}(t) + \alpha\tilde{x}_0(t)\Big]dt + \sigma dW_i(t), \\ m_i(0) = x_{i0} \end{cases} \tag{4.50}$$

and for $1 \leqslant j \leqslant N,\ j \neq i$,

$$\begin{cases} dm_j(t) = \Big[(A - B^2R^{-1}P(t))m_j(t) - B^2R^{-1}k(t)_{\tilde{x}_0} + Dm^{(N)}(t) + \alpha\tilde{x}_0(t)\Big]dt + \sigma dW_i(t), \\ m_j(0) = x_{j0} \end{cases} \tag{4.51}$$

where $m^{(N)}(t) = \frac{1}{N}\sum_{k=1}^{N} m_k(t)$; $k(t)_{\tilde{x}_0}$ satisfies (4.17) due to $\tilde{x}_0(\cdot) = \hat{x}_0(\cdot)$.

If $\tilde{u}_j,\ 0 \leqslant j \leqslant N$ is an ϵ-Nash equilibrium with respect to cost $\mathcal{J}_j$, it holds that

$$\mathcal{J}_j(\tilde{u}_j, \tilde{u}_{-j}) \geqslant \inf_{u_j \in \mathcal{U}_j} \mathcal{J}_j(u_j, \tilde{u}_{-j}) \geqslant \mathcal{J}_j(\tilde{u}_j, \tilde{u}_{-j}) - \epsilon.$$

Then, when making the perturbation, we just need to consider $u_j \in \mathcal{U}_j$ such that $\mathcal{J}_j(u_j, \tilde{u}_{-j}) \leqslant \mathcal{J}_j(\tilde{u}_j, \tilde{u}_{-j})$, which implies

$$\frac{1}{2}\mathbb{E}\int_0^T Ru_j^2(t)dt \leqslant \mathcal{J}_j(u_i, \tilde{u}_{-j}) \leqslant \mathcal{J}_j(\tilde{u}_j, \tilde{u}_{-j}) = J_j(\bar{u}_j) + O\Big(\frac{1}{\sqrt{N}}\Big).$$

In the limiting cost functional J_j, by the optimality of $(\bar{x}_j, \bar{u}_j)$, we get that $(\bar{x}_j, \bar{u}_j)$ is L^2-bounded. Then we obtain the boundedness of $J_j(\bar{u}_j)$, i.e.,

$$\mathbb{E}\int_0^T u_j^2(t)dt \leqslant C_3, \quad 0 \leqslant j \leqslant N \tag{4.52}$$

where C_3 is a positive constant and independent of N. Then we have the following proposition.

Proposition 4.1. $\sup\limits_{0\leqslant t\leqslant T}\mathbb{E}\big|l_0(t)\big|^2$, $\sup\limits_{1\leqslant k\leqslant N}\Big[\sup\limits_{0\leqslant t\leqslant T}\mathbb{E}\big|l_k(t)\big|^2\Big]$, $\sup\limits_{1\leqslant k\leqslant N}\Big[\sup\limits_{0\leqslant t\leqslant T}\mathbb{E}\big|m_k(t)\big|^2\Big]$ *are bounded.*

Proof. By (4.52), applying the usual technique of BSDE, we get the boundedness of $\sup\limits_{0\leqslant t\leqslant T}\mathbb{E}\big|l_0(t)\big|^2$. It follows from (4.48) that

$$\mathbb{E}\left[\sum_{k=1}^{N}|l_k(t)|^2\right]\leqslant C_4\Bigg\{\mathbb{E}\left[\sum_{k=1}^{N}|x_{k0}|^2\right]+\mathbb{E}\int_0^t\left[\sum_{k=1}^{N}|l_k(s)|^2+N|k(s)_{l_0}|^2+N|l_0(s)|^2\right]ds$$
$$+\sum_{k=1}^{N}\mathbb{E}\Big|\int_0^t\sigma dW_k(s)\Big|^2\Bigg\}.$$

From (4.50) and (4.51), it holds that

$$\mathbb{E}\left[\sum_{k=1}^{N}|m_k(t)|^2\right]\leqslant C_5\Bigg\{\mathbb{E}\left[\sum_{k=1}^{N}|x_{k0}|^2\right]+\mathbb{E}\int_0^t\Bigg[\sum_{k=1}^{N}|m_k(s)|^2+|u_i(s)|^2+\sum_{k=1,k\neq i}^{N}|\tilde{u}_k(s)|^2$$
$$+N|\tilde{x}_0(s)|^2\Bigg]ds+\sum_{k=1}^{N}\mathbb{E}\Big|\int_0^t\sigma dW_k(s)\Big|^2\Bigg\}.$$

Here, C_4 and C_5 are both positive constants. Since $\sup\limits_{0\leqslant t\leqslant T}\mathbb{E}\big|l_0(t)\big|^2$ is bounded, we get the boundedness of $\sup\limits_{0\leqslant t\leqslant T}\mathbb{E}\big|k(t)_{l_0}\big|^2$ by (4.49). It follows from (4.52) that $\mathbb{E}|u_i(\cdot)|^2$ is bounded. Besides, the optimal controls $\tilde{u}_k(\cdot), k\neq i$ is L^2-bounded. Then by Gronwall's inequality, it follows that

$$\sup_{0\leqslant t\leqslant T}\mathbb{E}\left[\sum_{k=1}^{N}|l_k(t)|^2\right]\sim\sup_{0\leqslant t\leqslant T}\mathbb{E}\left[\sum_{k=1}^{N}|m_k(t)|^2\right]=O(N).$$

Thus, for any $1\leqslant k\leqslant N$, $\sup\limits_{0\leqslant t\leqslant T}\mathbb{E}|l_k(t)|^2$ and $\sup\limits_{0\leqslant t\leqslant T}\mathbb{E}|m_k(t)|^2$ are bounded. Hence the result. $\square$

Correspondingly, the dynamics for agent $\mathcal{A}_0$ under control u_0 for **(L-BF-MM)** is as follows

$$\begin{cases} dl_0'(t) = \big[A_0 l_0'(t) + B_0 u_0(t) + C_0 q_0'(t)\big]dt + q_0'(t)dW_0(t), \\ x_0'(T) = \xi \end{cases} \tag{4.53}$$

and for agent $\mathcal{A}_i, 1 \leqslant i \leqslant N$,

$$\begin{cases} d\hat{l}_i(t) = \big[(A - B^2R^{-1}P(t))\hat{l}_i(t) - B^2R^{-1}k(t)_{l_0'} + D\bar{x}(t)_{l_0'} + \alpha l_0'(t)\big]dt + \sigma dW_i(t), \\ \hat{l}_i(0) = x_{i0}. \end{cases} \tag{4.54}$$

where $(k(t)_{l_0'}, \bar{x}(t)_{l_0'})$ associated with l_0' satisfying

$$\begin{cases} dk(t)_{l_0'} = \big[\big(-A + B^2R^{-1}P(t)\big)k(t)_{l_0'} + \big(Q - DP(t)\big)\bar{x}(t)_{l_0'} - \alpha P(t)l_0'(t)\big]dt \\ \qquad\qquad + \theta_0(t)_{l_0'}dW_0(t), \\ d\bar{x}(t)_{l_0'} = \big[\big(A + D - B^2R^{-1}P(t)\big)\bar{x}(t)_{l_0'} - B^2R^{-1}k(t)_{l_0'} + \alpha l_0'(t)\big]dt, \\ k(T)_{l_0'} = 0,\ \bar{x}(0)_{l_0'} = x. \end{cases} \tag{4.55}$$

Then we have

Lemma 4.6.

$$\sup_{0\leqslant t\leqslant T} \mathbb{E}\Big|l^{(N)}(t) - \bar{x}(t)_{l_0'}\Big|^2 = O\Big(\frac{1}{N}\Big), \tag{4.56}$$

$$\Big|\mathcal{J}_0(u_0, \tilde{u}_{-0}) - J_0(u_0)\Big| = O\Big(\frac{1}{\sqrt{N}}\Big). \tag{4.57}$$

Proof. From (4.47) and (4.53), by the existence and uniqueness of BSDE, for the same perturbed control $u_0(\cdot)$ we have $(l_0', q_0') = (l_0, q_0)$. Further, noting FBSDE (4.49) and (4.55), we get $(k(t)_{l_0'}, \bar{x}(t)_{l_0'}) = (k(t)_{l_0}, \bar{x}(t)_{l_0})$.

It follows from (4.48) that

$$\begin{cases} dl^{(N)}(t) = \Big[\big(A + D - B^2R^{-1}P(t)\big)l^{(N)}(t) - B^2R^{-1}k(t)_{l_0} + \alpha l_0(t)\Big]dt + \dfrac{1}{N}\sum_{i=1}^{N}\sigma dW_i(t), \\ l^{(N)}(0) = x_0^{(N)}. \end{cases}$$

Noting (4.55) and

$$\mathbb{E}\left|x_0^{(N)}-x_0\right|^2 \sim \mathbb{E}\left|\int_0^t \frac{1}{N}\sum_{i=1}^N \sigma dW_i(s)\right|^2 = O\Big(\frac{1}{N}\Big),$$

and applying Gronwall's inequality, we get (4.56). Using the same technique as Lemma 4.4 and noting $\sup\limits_{0\leqslant t\leqslant T}\mathbb{E}\left|l_0'(t)-\bar{x}(t)_{l_0'}\right|^2<+\infty$, we obtain (4.57). □

Now, we will focus on the difference of states and cost functionals for the perturbed control and optimal control of minor players. Given the system of $\mathcal{A}_i$ under control u_i for **(L-BF-MM)**

$$\begin{cases} dm_i'(t) = \big[Am_i'(t)+Bu_i(t)+D\bar{x}(t)_{\hat{x}_0}+\alpha\hat{x}_0(t)\big]dt+\sigma dW_i(t),\\ m_i'(0) = x_{i0}\end{cases} \tag{4.58}$$

and for agent $\mathcal{A}_j,\ 1\leqslant j\leqslant N,\ j\neq i$,

$$\begin{cases} d\hat{m}_j(t) = \big[(A-B^2R^{-1}P(t))\hat{m}_j(t)-B^2R^{-1}k(t)_{\hat{x}_0}+D\bar{x}(t)_{\hat{x}_0}+\alpha\hat{x}_0(t)\big]dt+\sigma dW_i(t),\\ \hat{m}_j(0) = x_{j0}\end{cases} \tag{4.59}$$

where $(\bar{x}(t)_{\hat{x}_0}, k(t)_{\hat{x}_0})$ satisfies (4.17).

In order to give necessary estimates in **(BF-MM)** and **(L-BF-MM)**, we need to introduce some intermediate states as

$$\begin{cases} d\check{m}_i(t) = \left[A\check{m}_i(t)+Bu_i(t)+\frac{N-1}{N}D\check{m}^{(N-1)}(t)+\alpha\tilde{x}_0(t)\right]dt+\sigma dW_i(t),\\ \check{m}_i(0) = x_{i0}\end{cases} \tag{4.60}$$

and for $1\leqslant j\leqslant N,\ j\neq i$,

$$\begin{cases} d\check{m}_j(t) = \Big[\Big(A-B^2R^{-1}P(t)\Big)\check{m}_j(t)-B^2R^{-1}k(t)_{\tilde{x}_0}+\frac{N-1}{N}D\check{m}^{(N-1)}(t)\\ \qquad\qquad +\alpha\tilde{x}_0(t)\Big]dt+\sigma dW_j(t),\\ \check{m}_j(0) = x_{j0}\end{cases} \tag{4.61}$$

where $\check{m}^{(N-1)}(t) = \frac{1}{N-1}\sum_{j=1,j\neq i}^{N}\check{m}_j(t)$.

Define $m^{(N-1)}(t) := \frac{1}{N-1}\sum_{j=1,j\neq i}^{N} m_j(t),\ x_0^{(N-1)} := \frac{1}{N-1}\sum_{j=1,j\neq i}^{N} x_{j0}$. By (4.51) and (4.61), we get

$$
\begin{cases}
dm^{(N-1)}(t) =& \Big[\Big(A - B^2R^{-1}P(t) + \dfrac{N-1}{N}D\Big)m^{(N-1)}(t) - B^2R^{-1}k(t)_{\tilde{x}_0} \\
& + \alpha\tilde{x}_0(t) + \dfrac{D}{N}m_i(t)\Big]dt + \dfrac{1}{N-1}\displaystyle\sum_{j=1,j\neq i}^{N}\sigma dW_j(t), \\
m^{(N-1)}(0) =& x_0^{(N-1)}
\end{cases}
\tag{4.62}
$$

and

$$
\begin{cases}
d\check{m}^{(N-1)}(t) =& \Big[\Big(A - B^2R^{-1}P(t) + \dfrac{N-1}{N}D\Big)\check{m}^{(N-1)}(t) - B^2R^{-1}k(t)_{\tilde{x}_0} \\
& + \alpha\tilde{x}_0(t)\Big]dt + \dfrac{1}{N-1}\displaystyle\sum_{j=1,j\neq i}^{N}\sigma dW_j(t), \\
\check{m}^{(N-1)}(0) =& x_0^{(N-1)}.
\end{cases}
\tag{4.63}
$$

Then we have the following proposition.

Proposition 4.2.

$$\sup_{0\leqslant t\leqslant T}\mathbb{E}\Big|m^{(N-1)}(t) - \check{m}^{(N-1)}(t)\Big|^2 = O\Big(\frac{1}{N^2}\Big), \tag{4.64}$$

$$\sup_{0\leqslant t\leqslant T}\mathbb{E}\Big|m^{(N)}(t) - m^{(N-1)}(t)\Big|^2 = O\Big(\frac{1}{N}\Big), \tag{4.65}$$

$$\sup_{0\leqslant t\leqslant T}\mathbb{E}\Big|\check{m}^{(N-1)}(t) - \bar{x}(t)_{\hat{x}_0}\Big|^2 = O\Big(\frac{1}{N}\Big). \tag{4.66}$$

Proof. From (4.62)-(4.63), applying Proposition 4.1 and Gronwall's inequality, the assertion (4.64) holds. (4.65) follows from (H4.1) and the L^2-boundness of controls $u_i(\cdot)$ and $\tilde{u}_j(\cdot), j \neq i$. From (4.63) and (4.17), noting $(\bar{x}(t)_{\tilde{x}_0}, k(t)_{\tilde{x}_0}, \tilde{x}_0) =$

$(\bar{x}(t)_{\hat{x}_0}, k(t)_{\hat{x}_0}, \hat{x}_0)$, we get

$$\begin{cases} d\Big(\check{m}^{(N-1)}(t) - \bar{x}(t)_{\hat{x}_0}\Big) = \Big[\frac{N-1}{N} D\big(\check{m}^{(N-1)}(t) - \bar{x}(t)_{\hat{x}_0}\big) - \frac{D}{N}\bar{x}(t)_{\hat{x}_0}\Big] dt + \frac{1}{N-1}\sum_{j=1, j\neq i}^{N} \sigma dW_j(t), \\ \check{m}^{(N-1)}(0) - \bar{x}(0)_{\hat{x}_0} = x_0^{(N-1)} - x. \end{cases}$$

Therefore (4.66) is obtained. □

Based on Proposition 4.2, we obtain more direct estimates to prove Theorem 4.2.

Lemma 4.7. *For fixed $i, 1 \leqslant i \leqslant N$, we have*

$$\sup_{0\leqslant t\leqslant T} \mathbb{E}\Big|m^{(N)}(t) - \bar{x}(t)_{\hat{x}_0}\Big|^2 = O\Big(\frac{1}{N}\Big), \tag{4.67}$$

$$\sup_{0\leqslant t\leqslant T} \mathbb{E}\Big|m_i(t) - m_i'(t)\Big|^2 = O\Big(\frac{1}{N}\Big), \tag{4.68}$$

$$\Big|\mathcal{J}_i(u_i, \tilde{u}_{-i}) - J_i(u_i)\Big| = O\Big(\frac{1}{\sqrt{N}}\Big). \tag{4.69}$$

Proof. (4.67) follows from Proposition 4.2 directly. From (4.50) and (4.58), we get (4.68) by applying (4.67). Further, $\sup_{0\leqslant t\leqslant T} \mathbb{E}\Big||m_i(t)|^2 - |m_i'(t)|^2\Big| = O\Big(\frac{1}{\sqrt{N}}\Big)$. In addition, $\sup_{0\leqslant t\leqslant T} \mathbb{E}\Big|\big(m_i(t) - m^{(N)}(t)\big)^2 - \big(m_i'(t) - \bar{x}(t)_{\hat{x}_0}\big)^2\Big| = O\Big(\frac{1}{\sqrt{N}}\Big)$. Then (4.69) follows. □

Proof of Theorem 4.2: Combining (4.42) and (4.57), we have

$$\begin{aligned} \mathcal{J}_0(\tilde{u}_0, \tilde{u}_{-0}) = J_0(\bar{u}_0) + O\Big(\frac{1}{\sqrt{N}}\Big) &\leqslant J_0(u_0) + O\Big(\frac{1}{\sqrt{N}}\Big) \\ &= \mathcal{J}_0(u_0, \tilde{u}_{-0}) + O\Big(\frac{1}{\sqrt{N}}\Big). \end{aligned}$$

It follows from (4.46) and (4.69) that

$$\begin{aligned} \mathcal{J}_i(\tilde{u}_i, \tilde{u}_{-i}) = J_i(\bar{u}_i) + O\Big(\frac{1}{\sqrt{N}}\Big) &\leqslant J_i(u_i) + O\Big(\frac{1}{\sqrt{N}}\Big) \\ &= \mathcal{J}_i(u_i, \tilde{u}_{-i}) + O\Big(\frac{1}{\sqrt{N}}\Big). \end{aligned}$$

Thus, Theorem 4.2 follows by taking $\epsilon = O\Big(\frac{1}{\sqrt{N}}\Big)$. □

Chapter 5

Leader-Follower LQMFGs Involving Major and Minor Agents

This chapter investigates the combination problems of leader-follower and major-minor systems, where the large scale population is also under consideration. In the entire system, the major and minor agents are together regarded as the leaders, which are called major-leader and minor-leaders respectively. This chapter is devoted to giving the frameworks and processing methods of three topics. In the first topic ("Serial-Parallel Coupling"–Case I), the optimization problems of followers are solved firstly, and the left is a classic major-minor problem and solved in the way of [51]. The major-leader imposes some direct impacts to the followers in the second topic ("Serial-Parallel Coupling"–Case II). The processing way is similar to the first topic, but the corresponding variation mode is different. In the third topic ("Serial Coupling"), motivated by [55], the problem, which is seemed as "major leader—minor leader—follower" model, is investigated in the "anticipating" manner and solved from back to front. In all three topics, the agents track different convex combinations of the centroid and dynamics of agents. In the end, three consistency condition systems are obtained.

5.1 “Serial-Parallel Coupling”–Case I

$(\Omega, \mathcal{F}, P)$ is a complete probability space on which a standard $(1 + N_L + N_F)$-dimensional Brownian motion $\{W_0(t), W_i(t), \widetilde{W}_j(t),\ 1 \leqslant i \leqslant N_L,\ 1 \leqslant j \leqslant N_F\}_{0 \leqslant t \leqslant T}$ is defined. Here, N_L and N_F stands for the population size of minor-leaders and followers respectively. We define the filtration $\mathcal{F}_t^0 := \sigma\{W_0(s), 0 \leqslant s \leqslant t\}$, $\mathcal{F}_t^i := \sigma\{W_0(s), W_i(s), 0 \leqslant s \leqslant t\}$, for $1 \leqslant i \leqslant N_L$, $\mathcal{G}_t^j := \sigma\{W_0(s), \widetilde{W}_j(s), 0 \leqslant s \leqslant t\}$, for $1 \leqslant j \leqslant N_F$, $\mathcal{F}_t := \sigma\{W_0(s), W_i(s), \widetilde{W}_j(s), 0 \leqslant s \leqslant t;\ 1 \leqslant i \leqslant N_L,\ 1 \leqslant j \leqslant N_F\}$.

We consider a large-population system with $(1 + N_L + N_F)$ individual agents, including the major-leader (the government or supervisory, denoted by $\mathcal{A}_0$), the minor-leaders (firms, denoted by $\mathcal{A}_i, 1 \leqslant i \leqslant N_L$) and the followers (suppliers of raw material or manufacturers of primary commodity, denoted by $\mathcal{B}_j, 1 \leqslant j \leqslant N_F$). The dynamics of $\mathcal{A}_0, \mathcal{A}_i, \mathcal{B}_j$ are given as follows:

$$\begin{cases} dx_0(t) = [A_0(t)x_0(t) + B_0(t)u_0(t)]dt + D_0(t)dW_0(t), \\ x_0(0) = x_0, \end{cases} \tag{5.1}$$

$$\begin{cases} dx_i(t) = [A(t)x_i(t) + B(t)u_i(t)]dt + D(t)dW_i(t) + C(t)dW_0(t), \\ x_i(0) = x,\ i = 1, 2, \dots, N_L, \end{cases} \tag{5.2}$$

and

$$\begin{cases} dy_j(t) = [\tilde{A}(t)y_j(t) + \tilde{B}(t)v_j(t)]dt + \tilde{D}(t)d\widetilde{W}_j(t) + \tilde{C}(t)dW_0(t), \\ y_j(0) = y,\ j = 1, 2, \dots, N_F. \end{cases} \tag{5.3}$$

Here, $W_i, \widetilde{W}_j$ denote the individual random noise while W_0 denotes the random noise of the major-leader. The admissible controls $u_0 \in \mathcal{U}_0,\ u_i \in \mathcal{U}_i,\ v_j \in \mathcal{V}_j$ where the admissible control set $\mathcal{U}_0, \mathcal{U}_i$ and $\mathcal{V}_j$ are defined as

$$\mathcal{U}_0 := \Big\{u_0 | u_0(\cdot) \in L^2_{\mathcal{F}_t^0}(0, T; \mathbb{R})\Big\}, \quad \mathcal{U}_i := \Big\{u_i | u_i(\cdot) \in L^2_{\mathcal{F}_t^i}(0, T; \mathbb{R})\Big\},\ 1 \leqslant i \leqslant N_L,$$

$$\mathcal{V}_j := \Big\{v_j | v_j(\cdot) \in L^2_{\mathcal{G}_t^j}(0, T; \mathbb{R})\Big\},\ 1 \leqslant j \leqslant N_F.$$

Let $(u_0,u,v) = (u_0,u_1,\dots,u_{N_L},v_1,\dots,v_{N_F})$ denote the set of control strategies of all $(1+N_L+N_F)$ agents; $u = (u_1,\dots,u_i,\dots,u_{N_L})$ the set of control strategies of all N_L major-leader agents; $v = (v_1,\dots,v_j,\dots,v_{N_F})$ the set of control strategies of all N_F follower agents; $u_{-i} = (u_1,\dots,u_{i-1},\ u_{i+1},\dots,u_{N_L})$ the control strategy set of major-leader agents except the i^{th} one $\mathcal{A}_i$; $v_{-j} = (v_1,\dots,v_{j-1},\ v_{j+1},\dots,v_{N_F})$ the control strategy set of follower agents except the j^{th} follower agent $\mathcal{B}_j$. Introduce the following cost functional

$$\begin{aligned}\mathcal{J}_0(u_0,u,v) =& \frac{1}{2}\mathbb{E}\Big\{\int_0^T \Big[Q_0(t)\Big(x_0(t)-\big(\lambda_0 x^{(N_L)}(t)+(1-\lambda_0)y^{(N_F)}(t)\big)\Big)^2\\ &+R_0(t)u_0^2(t)\Big]dt + H_0x_0^2(T)\Big\},\end{aligned} \tag{5.4}$$

for $\mathcal{A}_0$;

$$\begin{aligned}\mathcal{J}_i^L(u_0,u_i,u_{-i}) =& \frac{1}{2}\mathbb{E}\Big\{\int_0^T \Big[Q(t)\Big(x_i(t)-\big(\lambda x^{(N_L)}(t)+(1-\lambda)x_0(t)\big)\Big)^2\\ &+R(t)u_i^2(t)\Big]dt + Hx_i^2(T)\Big\},\end{aligned} \tag{5.5}$$

for $\mathcal{A}_i,\ 1\leqslant i\leqslant N_L$; and

$$\begin{aligned}\mathcal{J}_j^F(u_0,u,v_j,v_{-j}) =& \frac{1}{2}\mathbb{E}\Big\{\int_0^T \Big[\tilde{Q}(t)\Big(y_j(t)-\big(\tilde{\lambda} x^{(N_L)}(t)+(1-\tilde{\lambda})y^{(N_F)}(t)\big)\Big)^2\\ &+\tilde{R}(t)v_j^2(t)\Big]dt + \tilde{H}y_j^2(T)\Big\},\end{aligned} \tag{5.6}$$

for $\mathcal{B}_j,\ 1\leqslant j\leqslant N_F$. Here, $x^{(N_L)}(t) = \frac{1}{N}\sum_{i=1}^{N_L} x_i(t)$ and $y^{(N_F)}(t) = \frac{1}{N}\sum_{j=1}^{N_F} y_j(t)$ are state-average.

For the coefficients of (5.1)-(5.6), we set the following assumptions:

(H5.1) $A_0(\cdot),B_0(\cdot),D_0(\cdot)\in L^\infty(0,T;\mathbb{R}), x_0\in\mathbb{R}, Q_0(\cdot),R_0(\cdot)\in L^\infty(0,T;\mathbb{R})$, $Q_0(\cdot)\geqslant 0, R_0(\cdot)\geqslant\delta_0$, for $\delta_0>0,\ \lambda_0,H_0\geqslant 0$.

(H5.2) $A(\cdot),B(\cdot),C(\cdot),D(\cdot)\in L^\infty(0,T;\mathbb{R}), x\in\mathbb{R}, Q(\cdot),R(\cdot)\in L^\infty(0,T;\mathbb{R})$, $Q(\cdot)\geqslant 0, R(\cdot)\geqslant\delta$, for $\delta>0,\ \lambda,H\geqslant 0$.

(H5.3) $\tilde{A}(\cdot), \tilde{B}(\cdot), \tilde{C}(\cdot), \tilde{D}(\cdot) \in L^\infty(0,T;\mathbb{R}), y \in \mathbb{R}, \tilde{Q}(\cdot), \tilde{R}(\cdot) \in L^\infty(0,T;\mathbb{R})$,

$\tilde{Q}(\cdot) \geqslant 0, \tilde{R}(\cdot) \geqslant \tilde{\delta}$, for $\tilde{\delta} > 0$, $\tilde{\lambda}, \tilde{H} \geqslant 0$.

Now, we formulate the large-population LQ games with leader-followers and major-minors (LF-MFG) as follows.

Problem (LF-MFG). Find a control strategy set $(\bar{u}_0, \bar{u}, \bar{v}) = (\bar{u}_0, \bar{u}_1, \ldots, \bar{u}_{N_L}, \bar{v}_1, \ldots, \bar{v}_{N_F})$ which satisfies

$$\begin{aligned}
&\mathcal{J}_0(\bar{u}_0, \bar{u}, \bar{v}) = \inf_{u_0 \in \mathcal{U}_0} \mathcal{J}_0(u_0, \bar{u}, \bar{v}),\\
&\mathcal{J}_i^L(\bar{u}_0, \bar{u}_i, \bar{u}_{-i}) = \inf_{u_i \in \mathcal{U}_i} \mathcal{J}_i^L(\bar{u}_0, u_i, \bar{u}_{-i}),\ 1 \leqslant i \leqslant N_L,\\
&\mathcal{J}_j^F(\bar{u}_0, \bar{u}, \bar{v}_j, \bar{v}_{-j}) = \inf_{v_j \in \mathcal{V}_j} \mathcal{J}_j^F(\bar{u}_0, \bar{u}, v_j, \bar{v}_{-j}),\ 1 \leqslant j \leqslant N_F
\end{aligned}$$

where $\bar{u}_{-i}$ represents $(\bar{u}_1, \ldots, \bar{u}_{i-1}, \bar{u}_{i+1}, \ldots, \bar{u}_{N_L})$, and $\bar{v}_{-j}$ represents $(\bar{v}_1, \ldots, \bar{v}_{j-1}, \bar{v}_{j+1}, \ldots, \bar{v}_{N_F})$.

To study (**LF-MFG**), one efficient approach is to discuss the associated MFGs via limiting problem when the agent number N_L and N_F tends to infinity. As $N_L, N_F \to +\infty$, suppose $x^{(N_L)}$ and $y^{(N_F)}$ can be approximated by $\mathcal{F}_t^0$-measurable functions $\bar{x}$ and $\bar{y}$, respectively.

Introduce the following auxiliary cost functionals as

$$\begin{aligned}
J_0(u_0) =& \frac{1}{2}\mathbb{E}\Big\{ \int_0^T \Big[Q_0(t)\Big(x_0(t) - \big(\lambda_0 \bar{x}(t) + (1-\lambda_0)\bar{y}(t)\big)\Big)^2 + R_0(t)u_0^2(t)\Big]dt \\
&+ H_0 x_0^2(T)\Big\},
\end{aligned} \tag{5.7}$$

for $\mathcal{A}_0$;

$$\begin{aligned}
J_i^L(u_0, u_i) =& \frac{1}{2}\mathbb{E}\Big\{ \int_0^T \Big[Q(t)\Big(x_i(t) - \big(\lambda \bar{x}(t) + (1-\lambda)x_0(t)\big)\Big)^2 + R(t)u_i^2(t)\Big]dt \\
&+ H x_i^2(T)\Big\},
\end{aligned} \tag{5.8}$$

for $\mathcal{A}_i,\ 1 \leqslant i \leqslant N_L$; and

$$\begin{aligned} J_j^F(v_j) =& \frac{1}{2}\mathbb{E}\Big\{ \int_0^T \Big[\tilde{Q}(t)\Big(y_j(t) - \big(\tilde{\lambda}\bar{x}(t) + (1-\tilde{\lambda})\bar{y}(t)\big)\Big)^2 + \tilde{R}(t)v_j^2(t)\Big]dt \\ & + \tilde{H}y_j^2(T)\Big\}, \end{aligned} \tag{5.9}$$

for $\mathcal{B}_j,\ 1 \leqslant j \leqslant N_F$. Now we formulate the following limiting LQ games.

Problem (L-LF-MFG). For $\mathcal{A}_0$, $\mathcal{A}_i, i = 1, 2, \ldots, N_L$, and $\mathcal{B}_j, j = 1, 2, \ldots, N_F$, find $(\bar{u}_0, \bar{u}_i, \bar{v}_j) \in \mathcal{U}_0 \times \mathcal{U}_i \times \mathcal{V}_j$ satisfying

$$\begin{aligned} & J_0(\bar{u}_0) = \inf_{u_0 \in \mathcal{U}_0} J_0(u_0), \\ & J_i^L(\bar{u}_0, \bar{u}_i) = \inf_{u_i \in \mathcal{U}_i} J_i^L(\bar{u}_0, u_i),\ 1 \leqslant i \leqslant N_L, \\ & J_j^F(\bar{v}_j) = \inf_{v_j \in \mathcal{V}_j} J_j^F(v_j),\ 1 \leqslant j \leqslant N_F. \end{aligned}$$

Then $(\bar{u}_0, \bar{u}_i, \bar{v}_j)$ is called an optimal control for Problem (**L-LF-MFG**). In this topic, we use three steps to solve Problem (**L-LF-MFG**). The entire system is seemed as the "leaders-followers" manner and $(\bar{x}(\cdot), \bar{y}(\cdot))$ are supposed two fixed stochastic process. Firstly, the optimization problems of the followers are solved, and the left is a classic major-minor problem. With the help of ideas in [51], the MF problem of the major-leader is processed in the second step. And then in the last step, the optimization problems of minor-leaders are considered.

Step 1. Mean-field games of followers.

Applying the standard variational method, we have:

Lemma 5.1. *Under* (H5.3), *the optimal control for the follower of Problem* (**L-LF-MFG**) *is given by*

$$\bar{v}_j(t) = -\tilde{R}^{-1}(t)\tilde{B}(t)p_j(t) \tag{5.10}$$

where the optimal trajectory $\bar{y}_j(t)$ and the adjoint process $p_j(t)$ satisfy the FBSDE

$$
\begin{cases}
d\bar{y}_j(t) = \Big[\tilde{A}(t)\bar{y}_j(t) - \tilde{B}^2(t)\tilde{R}^{-1}(t)p_j(t)\Big]dt + \tilde{D}(t)d\widetilde{W}_j(t) + \tilde{C}dW_0(t), \\
dp_j(t) = -\Big[\tilde{A}(t)p_j(t) + \tilde{Q}(t)\Big(\bar{y}_j(t) - \big(\tilde{\lambda}\bar{x}(t) + (1-\tilde{\lambda})\bar{y}(t)\big)\Big)\Big]dt \\
\qquad\qquad + q_j(t)d\widetilde{W}_j(t) + q_0(t)dW_0(t), \\
\bar{y}_j(0) = y, \ \ p_j(T) = \tilde{H}\bar{y}_j(T), \ \ j = 1, 2, \ldots, N_F.
\end{cases}
\tag{5.11}
$$

By the terminal condition of (5.11), we suppose $p_j(t) = P(t)\bar{y}_j(t) + \Phi(t)$, for some $P(\cdot) \in L^\infty(0,T;\mathbb{R})$ and $\Phi(\cdot) \in L^2_{\mathcal{F}^0_t}(0,T;\mathbb{R})$ with terminal conditions

$$P(T) = \tilde{H}, \ \ \Phi(T) = 0.$$

Applying Itô's formula to $P(t)\bar{y}_j(t) + \Phi(t)$, noting (5.11) and comparing coefficients, we obtain

$$
\begin{cases}
\dot{P}(t) + 2\tilde{A}(t)P(t) - \tilde{B}^2(t)\tilde{R}^{-1}(t)P^2(t) + \tilde{Q}(t) = 0, \\
P(T) = \tilde{H}.
\end{cases}
\tag{5.12}
$$

and

$$
\begin{cases}
d\Phi(t) = \Big[-\big(\tilde{A}(t) - \tilde{B}^2(t)\tilde{R}^{-1}(t)P(t)\big)\Phi(t) + \tilde{Q}(t)\big(\tilde{\lambda}\bar{x}(t) + (1-\tilde{\lambda})\bar{y}(t)\big)\Big]dt + k(t)dW_0(t), \\
\Phi(T) = 0.
\end{cases}
\tag{5.13}
$$

Note that the optimal state $\bar{y}_j(t)$ can be represented by

$$d\bar{y}_j(t) = \Big[\tilde{A}(t)\bar{y}_j(t) - \tilde{B}^2(t)\tilde{R}^{-1}(t)(P(t)\bar{y}_j(t) + \Phi(t))\Big]dt + \tilde{D}(t)d\widetilde{W}_j(t) + \tilde{C}(t)dW_0(t).$$

Therefore the state-average satisfies:

$$
\begin{aligned}
d\bar{y}^{(N_F)}(t) =& \Big[\big(\tilde{A}(t) - \tilde{B}^2(t)\tilde{R}^{-1}(t)P(t)\big)\bar{y}^{(N)}(t) - \tilde{B}^2(t)\tilde{R}^{-1}(t)\Phi(t)\Big]dt \\
&+ \tilde{D}(t)\frac{1}{N}\sum_{j=1}^{N} d\widetilde{W}_j(t) + \tilde{C}(t)dW_0(t).
\end{aligned}
$$

Let $N_F \to +\infty$, the limiting process $\bar{y}(t)$ is given by

$$d\bar{y}(t)=\Big[\big(\tilde{A}(t)-\tilde{B}^2(t)\tilde{R}^{-1}(t)P(t)\big)\bar{y}(t)-\tilde{B}^2(t)\tilde{R}^{-1}(t)\Phi(t)\Big]dt+\tilde{C}(t)dW_0(t). \tag{5.14}$$

Noting (5.12), $P(t)$ can be computed off-line. Then it follows from (5.13) and (5.14) that

$$\begin{cases} d\bar{y}(t)=\Big[\big(\tilde{A}(t)-\tilde{B}^2(t)\tilde{R}^{-1}(t)P(t)\big)\bar{y}(t)-\tilde{B}^2(t)\tilde{R}^{-1}(t)\Phi(t)\Big]dt+\tilde{C}(t)dW_0(t), \\ d\Phi(t)=\Big[-\big(\tilde{A}(t)-\tilde{B}^2(t)\tilde{R}^{-1}(t)P(t)\big)\Phi(t)+\tilde{Q}(t)\big(\tilde{\lambda}\bar{x}(t)+(1-\tilde{\lambda})\bar{y}(t)\big)\Big]dt \\ \qquad\qquad +k(t)dW_0(t), \\ \bar{y}(0)=y,\ \Phi(T)=0 \end{cases} \tag{5.15}$$

where $\bar{x}(\cdot)$ is to be determined.

Step 2. Mean-field games of major-leader.

Similar to Step 1, applying the standard variational method, we have:

Lemma 5.2. *Under* (H5.1)*, the optimal control for the major-leader of Problem* (**L-LF-MFG**) *is*

$$\bar{u}_0(t)=-R_0^{-1}(t)B_0(t)(P_0(t)\bar{x}_0(t)+\Phi_0(t)) \tag{5.16}$$

and the decoupled system implies

$$\begin{cases} d\bar{x}_0(t)=\Big[\big(A_0(t)-B_0^2(t)R_0^{-1}(t)P_0(t)\big)\bar{x}_0(t)-B_0^2(t)R_0^{-1}(t)\Phi_0(t)\Big]dt+D_0(t)dW_0(t), \\ d\Phi_0(t)=\Big[-\big(A_0(t)-B_0^2(t)R_0^{-1}(t)P_0(t)\big)\Phi_0(t)+Q_0(t)\big(\lambda_0\bar{x}(t)+(1-\lambda_0)\bar{y}(t)\big)\Big]dt \\ \qquad\qquad +k_0(t)dW_0(t), \\ \bar{x}_0(0)=x_0,\ \Phi_0(T)=0 \end{cases} \tag{5.17}$$

where $P_0(\cdot)$ satisfies

$$\begin{cases} \dot{P}_0(t)+2A_0(t)P_0(t)-B_0^2(t)R_0^{-1}(t)P_0^2(t)+Q_0(t)=0, \\ P_0(T)=H_0. \end{cases} \tag{5.18}$$

Step 3. Mean-field games of minor-leaders.

In the same way, we get

Lemma 5.3. *Under* (H5.2)*, the optimal control for the minor-leader of Problem* (**L-LF-MFG**) *has the following form*

$$\bar{u}_i(t) = -R^{-1}(t)B(t)(K(t)\bar{x}_i(t) + \Psi(t)) \tag{5.19}$$

where the optimal trajectory $\bar{x}_i(t)$ satisfies

$$d\bar{x}_i(t) = \Big[A(t)\bar{x}_i(t) - B^2(t)R^{-1}(t)(K(t)\bar{x}_i(t) + \Psi(t))\Big]dt + D(t)dW_i(t) + C(t)dW_0(t)$$

and the decoupled system is

$$\begin{cases} d\bar{x}(t) = \Big[\big(A(t) - B^2(t)R^{-1}(t)K(t)\big)\bar{x}(t) - B^2(t)R^{-1}(t)\Psi(t)\Big]dt + C(t)dW_0(t), \\ d\Psi(t) = \Big[-\big(A(t) - B^2(t)R^{-1}(t)K(t)\big)\Psi(t) + Q(t)\big(\lambda\bar{x}(t) + (1-\lambda)\bar{x}_0(t)\big)\Big]dt \\ \qquad\qquad + k'(t)dW_0(t), \\ \bar{x}(0) = x, \ \Psi(T) = 0. \end{cases} \tag{5.20}$$

Here, $K(\cdot)$ satisfies the following Riccati equation

$$\begin{cases} \dot{K}(t) + 2A(t)K(t) - B^2(t)R^{-1}(t)K^2(t) + Q(t) = 0, \\ K(T) = H. \end{cases} \tag{5.21}$$

Combining (5.15), (5.17) and (5.20), we derive the following consistency condition system

$$
\begin{cases}
d\bar{y}(t) = \Big[\big(\tilde{A}(t) - \tilde{B}^2(t)\tilde{R}^{-1}(t)P(t)\big)\bar{y}(t) - \tilde{B}^2(t)\tilde{R}^{-1}(t)\Phi(t)\Big]dt + \tilde{C}(t)dW_0(t), \\
d\Phi(t) = \Big[- \big(\tilde{A}(t) - \tilde{B}^2(t)\tilde{R}^{-1}(t)P(t)\big)\Phi(t) + \tilde{Q}(t)\big(\tilde{\lambda}\bar{x}(t) + (1-\tilde{\lambda})\bar{y}(t)\big)\Big]dt \\
\qquad\qquad + k(t)dW_0(t), \\
d\bar{x}_0(t) = \Big[\big(A_0(t) - B_0^2(t)R_0^{-1}(t)P_0(t)\big)\bar{x}_0(t) - B_0^2(t)R_0^{-1}(t)\Phi_0(t)\Big]dt \\
\qquad\qquad + D_0(t)dW_0(t), \\
d\Phi_0(t) = \Big[- \big(A_0(t) - B_0^2(t)R_0^{-1}(t)P_0(t)\big)\Phi_0(t) + Q_0(t)\big(\lambda_0\bar{x}(t) \\
\qquad\qquad + (1-\lambda_0)\bar{y}(t)\big)\Big]dt + k_0(t)dW_0(t), \\
d\bar{x}(t) = \Big[\big(A(t) - B^2(t)R^{-1}(t)K(t)\big)\bar{x}(t) - B^2(t)R^{-1}(t)\Psi(t)\Big]dt + C(t)dW_0(t), \\
d\Psi(t) = \Big[- \big(A(t) - B^2(t)R^{-1}(t)K(t)\big)\Psi(t) + Q(t)\big(\lambda\bar{x}(t) + (1-\lambda)\bar{x}_0(t)\big)\Big]dt \\
\qquad\qquad + k'(t)dW_0(t), \\
\bar{y}(0) = y,\ \Phi(T) = 0,\ \bar{x}_0(0) = x_0,\ \Phi_0(T) = 0,\ \bar{x}(0) = x,\ \Psi(T) = 0.
\end{cases}
\tag{5.22}
$$

5.2 "Serial-Parallel Coupling"–Case II

In this topic, a case that the major-leader imposes some direct impacts to the followers is under considerable. The impacts are reflected in the coupling in the followers' cost functionals. The processing sequence coincides with "Serial-Parallel Coupling"–Case I, but the corresponding variational technique is different.

Because the states of all agents and the cost functionals of major-leader and minor-leaders stay the same with that in "Serial-Parallel Coupling"–Case I, the cost functionals of followers are only given out. Introduce the following cost functional

$$
\begin{aligned}
\mathcal{J}_j^F(u_0, u, v_j) =& \frac{1}{2}\mathbb{E}\Big\{\int_0^T \Big[\tilde{Q}(t)\Big(y_j(t) - \big(\tilde{\lambda}x^{(N_L)}(t) + (1-\tilde{\lambda})x_0(t)\big)\Big)^2 \\
&+ \tilde{R}(t)v_j^2(t)\Big]dt + \tilde{H}y_j^2(T)\Big\},
\end{aligned}
\tag{5.23}
$$

for $\mathcal{B}_j,\ 1 \leqslant j \leqslant N_F$. Here, $x^{(N_L)}(t) = \frac{1}{N}\sum_{i=1}^{N_L} x_i(t)$ is the state-average of minor-leaders.

Now, we formulate the large-population LQ games with leader-followers and major-minors as follows.

Problem (LF-MFG'). Find a control strategy set $(\bar{u}_0, \bar{u}, \bar{v}) = (\bar{u}_0, \bar{u}_1, \ldots, \bar{u}_{N_L}, \bar{v}_1, \ldots, \bar{v}_{N_F})$ which satisfies

$$
\begin{aligned}
&\mathcal{J}_0(\bar{u}_0, \bar{u}, \bar{v}) = \inf_{u_0 \in \mathcal{U}_0} \mathcal{J}_0(u_0, \bar{u}, \bar{v}),\\
&\mathcal{J}_i^L(\bar{u}_0, \bar{u}_i, \bar{u}_{-i}) = \inf_{u_i \in \mathcal{U}_i} \mathcal{J}_i^L(\bar{u}_0, u_i, \bar{u}_{-i}),\ 1 \leqslant i \leqslant N_L,\\
&\mathcal{J}_j^F(\bar{u}_0, \bar{u}, \bar{v}_j) = \inf_{v_j \in \mathcal{V}_j} \mathcal{J}_j^F(\bar{u}_0, \bar{u}, v_j),\ 1 \leqslant j \leqslant N_F
\end{aligned}
$$

where $\mathcal{J}_0$ and $\mathcal{J}_i^L$ are given by (5.4) and (5.5), respectively. $x^{(N_L)}$ and $y^{(N_F)}$ are still supposed to be approximated by $\mathcal{F}_t^0$-measurable functions $\bar{x}$ and $\bar{y}$, as $N_L, N_F \to +\infty$.

Introduce the following auxiliary cost functional

$$
\begin{aligned}
J_j^F(u_0, v_j) = &\frac{1}{2}\mathbb{E}\Big\{\int_0^T \Big[\tilde{Q}(t)\Big(y_j(t) - \big(\tilde{\lambda}\bar{x}(t) + (1-\tilde{\lambda})x_0(t)\big)\Big)^2 + \tilde{R}(t)v_j^2(t)\Big]dt\\
&\qquad + \tilde{H}y_j^2(T)\Big\},
\end{aligned}
\tag{5.24}
$$

for $\mathcal{B}_j,\ 1 \leqslant j \leqslant N_F$. Now we formulate the following limiting LQ games.

Problem (L-LF-MFG'). For $\mathcal{A}_0$, $\mathcal{A}_i, i = 1, 2, \ldots, N_L$, and $\mathcal{B}_j, j = 1, 2, \ldots, N_F$, find $(\bar{u}_0, \bar{u}_i, \bar{v}_j) \in \mathcal{U}_0 \times \mathcal{U}_i \times \mathcal{V}_j$ satisfying

$$
\begin{aligned}
&J_0(\bar{u}_0) = \inf_{u_0 \in \mathcal{U}_0} J_0(u_0),\\
&J_i^L(\bar{u}_0, \bar{u}_i) = \inf_{u_i \in \mathcal{U}_i} J_i^L(\bar{u}_0, u_i),\ 1 \leqslant i \leqslant N_L,\\
&J_j^F(\bar{u}_0, \bar{v}_j) = \inf_{v_j \in \mathcal{V}_j} J_j^F(\bar{u}_0, v_j),\ 1 \leqslant j \leqslant N_F.
\end{aligned}
$$

Then $(\bar{u}_0, \bar{u}_i, \bar{v}_j)$ is called an optimal control for Problem (**L-LF-MFG'**).

Step 1. Mean-field games of followers.

Similar to Step 1 of "Serial-Parallel Coupling"–Case I, applying the standard variational method, we get

$$
\begin{cases}
d\bar{y}(t) = \Big[\big(\tilde{A}(t) - \tilde{B}^2(t)\tilde{R}^{-1}(t)P(t)\big)\bar{y}(t) - \tilde{B}^2(t)\tilde{R}^{-1}(t)\Phi(t)\Big]dt + \tilde{C}(t)dW_0(t),\\
d\Phi(t) = \Big[-\big(\tilde{A}(t) - \tilde{B}^2(t)\tilde{R}^{-1}(t)P(t)\big)\Phi(t) + \tilde{Q}(t)\big(\tilde{\lambda}\bar{x}(t) + (1-\tilde{\lambda})\bar{x}_0(t)\big)\Big]dt\\
\qquad\qquad + k(t)dW_0(t),\\
\bar{y}(0) = y,\ \Phi(T) = 0
\end{cases}
\tag{5.25}
$$

where $P(\cdot)$ also satisfies (5.12), and $\bar{x}(\cdot)$, $\bar{x}_0(\cdot)$ are to be determined.

Step 2. Mean-field games of major-leader.

In the following, we define

$$
\check{A}(t) := \tilde{A}(t) - \tilde{B}^2(t)\tilde{R}^{-1}(t)P(t), \quad \check{B}(t) := \tilde{B}^2(t)\tilde{R}^{-1}(t). \tag{5.26}
$$

Then we have the following state equation system

$$
\begin{cases}
d\bar{y}(t) = \Big[\check{A}(t)\bar{y}(t) - \check{B}(t)\Phi(t)\Big]dt + \tilde{C}(t)dW_0(t),\\
d\Phi(t) = \Big[-\check{A}(t)\Phi(t) + \tilde{Q}(t)\big(\tilde{\lambda}\bar{x}(t) + (1-\tilde{\lambda})\bar{x}_0(t)\big)\Big]dt + k(t)dW_0(t),\\
d\bar{x}_0(t) = [A_0(t)\bar{x}_0(t) + B_0(t)\bar{u}_0(t)]dt + D_0(t)dW_0(t),\\
\bar{y}(0) = y,\ \Phi(T) = 0,\ x_0(0) = x_0
\end{cases}
\tag{5.27}
$$

and variational equation system

$$
\begin{cases}
d\eta(t) = \Big[\check{A}(t)\eta(t) - \check{B}(t)\xi(t)\Big]dt,\\
d\xi(t) = \Big[-\check{A}(t)\xi(t) + \tilde{Q}(t)(1-\tilde{\lambda})\eta_0(t)\Big]dt + \delta k(t)dW_0(t),\\
d\eta_0(t) = [A_0(t)\eta_0(t) + B_0(t)\delta u_0(t)]dt,\\
\eta(0) = 0,\ \xi(T) = 0,\ \eta_0(0) = 0.
\end{cases}
\tag{5.28}
$$

Then we obtain

Lemma 5.4. *Under* (H5.1), *the optimal control for the major-leader of Problem* (**L-LF-MFG'**) *is given by*

$$\bar{u}_0(t) = -R_0^{-1}(t)B_0(t)p_0(t) \tag{5.29}$$

and the adjoint equation is given by

$$\begin{cases} dp_0(t) = -\Big[A_0(t)p_0(t) + Q_0(t)\Big(\bar{x}_0(t) - \big(\lambda_0\bar{x}(t) + (1-\lambda_0)\bar{y}(t)\big)\Big) \\ \qquad\qquad + \tilde{Q}(t)(1-\tilde{\lambda})S(t)\Big]dt + q_0(t)dW_0(t), \\ dp(t) = \Big[-\check{A}(t)p(t) + (1-\lambda_0)Q_0(t)\Big(\bar{x}_0(t) - \big(\lambda_0\bar{x}(t) + (1-\lambda_0)\bar{y}(t)\big)\Big)\Big]dt \\ \qquad\qquad + q(t)dW_0(t), \\ dS(t) = \Big[\check{A}(t)S(t) + \check{B}(t)p(t)\Big]dt, \\ p_0(T) = H_0\bar{x}_0(T),\ p(T) = 0,\ S(0) = 0. \end{cases} \tag{5.30}$$

Proof. The variation of cost functional is

$$\begin{aligned} 0 = \frac{\delta J_0(u_0)}{\delta u_0} = \mathbb{E}\Big\{\int_0^T \Big[Q_0(t)\Big(\bar{x}_0(t) - \big(\lambda_0\bar{x}(t) + (1-\lambda_0)\bar{y}(t)\big)\Big) \\ \cdot\big(\eta_0(t) - (1-\lambda_0)\eta(t)\big) + R_0(t)\bar{u}_0(t)\delta u_0(t)\Big]dt + H_0\bar{x}_0(T)\eta_0(T)\Big\}. \end{aligned} \tag{5.31}$$

Applying Itô's formula to $p_0(t)\eta_0(t) + p(t)\eta(t) + S(t)\xi(t)$, the results are easily obtained. □

Then the following coupled system follows

$$
\begin{cases}
d\bar{x}_0(t) = \Big[A_0(t)\bar{x}_0(t) - B_0^2(t)R_0^{-1}(t)p_0(t)\Big]dt + D_0(t)dW_0(t), \\
d\bar{y}(t) = \Big[\breve{A}(t)\bar{y}(t) - \breve{B}(t)\Phi(t)\Big]dt + \tilde{C}(t)dW_0(t), \\
d\Phi(t) = \Big[-\breve{A}(t)\Phi(t) + \tilde{Q}(t)\big(\tilde{\lambda}\bar{x}(t) + (1-\tilde{\lambda})\bar{x}_0(t)\big)\Big]dt + k(t)dW_0(t), \\
dp_0(t) = -\Big[A_0(t)p_0(t) + Q_0(t)\Big(\bar{x}_0(t) - \big(\lambda_0\bar{x}(t) + (1-\lambda_0)\bar{y}(t)\big)\Big) \\
\qquad\qquad + \tilde{Q}(t)(1-\tilde{\lambda})S(t)\Big]dt + q_0(t)dW_0(t), \\
dp(t) = \Big[-\breve{A}(t)p(t) + (1-\lambda_0)Q_0(t)\Big(\bar{x}_0(t) - \big(\lambda_0\bar{x}(t) + (1-\lambda_0)\bar{y}(t)\big)\Big)\Big]dt \\
\qquad\qquad + q(t)dW_0(t), \\
dS(t) = \Big[\breve{A}(t)S(t) + \breve{B}(t)p(t)\Big]dt, \\
\bar{x}_0(0) = x_0,\ \bar{y}(0) = y,\ \Phi(T) = 0,\ p_0(T) = H_0\bar{x}_0(T),\ p(T) = 0,\ S(0) = 0
\end{cases}
\tag{5.32}
$$

where $\breve{A}(t)$ and $\breve{B}(t)$ are given by (5.26), and $\bar{x}(\cdot)$ is to be determined.

Step 3. Mean-field games of minor-leaders.

Same to Step 3 of "Serial-Parallel Coupling"–Case I, the decoupled system is derived (same to (5.20))

$$
\begin{cases}
d\bar{x}(t) = \Big[\big(A(t) - B^2(t)R^{-1}(t)K(t)\big)\bar{x}(t) - B^2(t)R^{-1}(t)\Psi(t)\Big]dt + C(t)dW_0(t), \\
d\Psi(t) = \Big[-\big(A(t) - B^2(t)R^{-1}(t)K(t)\big)\Psi(t) + Q(t)\big(\lambda\bar{x}(t) + (1-\lambda)\bar{x}_0(t)\big)\Big]dt \\
\qquad\qquad + k'(t)dW_0(t), \\
\bar{x}(0) = x,\ \Psi(T) = 0.
\end{cases}
\tag{5.33}
$$

where $K(\cdot)$ satisfies (5.21).

Combining (5.32) and (5.33) implies the following consistency condition system

$$
\begin{cases}
d\bar{x}_0(t) = \Big[A_0(t)\bar{x}_0(t) - B_0^2(t)R_0^{-1}(t)p_0(t)\Big]dt + D_0(t)dW_0(t),\\
d\bar{y}(t) = \Big[\check{A}(t)\bar{y}(t) - \check{B}(t)\Phi(t)\Big]dt + \tilde{C}(t)dW_0(t),\\
d\Phi(t) = \Big[-\check{A}(t)\Phi(t) + \tilde{Q}(t)\big(\tilde{\lambda}\bar{x}(t) + (1-\tilde{\lambda})\bar{x}_0(t)\big)\Big]dt + k(t)dW_0(t),\\
dp_0(t) = -\Big[A_0(t)p_0(t) + Q_0(t)\Big(\bar{x}_0(t) - \big(\lambda_0\bar{x}(t) + (1-\lambda_0)\bar{y}(t)\big)\Big)\\
\qquad\qquad + \tilde{Q}(t)(1-\tilde{\lambda})S(t)\Big]dt + q_0(t)dW_0(t),\\
dp(t) = \Big[-\check{A}(t)p(t) + (1-\lambda_0)Q_0(t)\Big(\bar{x}_0(t) - \big(\lambda_0\bar{x}(t) + (1-\lambda_0)\bar{y}(t)\big)\Big)\Big]dt\\
\qquad\qquad + q(t)dW_0(t),\\
dS(t) = \Big[\check{A}(t)S(t) + \check{B}(t)p(t)\Big]dt,\\
d\bar{x}(t) = \Big[\big(A(t) - B^2(t)R^{-1}(t)K(t)\big)\bar{x}(t) - B^2(t)R^{-1}(t)\Psi(t)\Big]dt + C(t)dW_0(t),\\
d\Psi(t) = \Big[-\big(A(t) - B^2(t)R^{-1}(t)K(t)\big)\Psi(t) + Q(t)\big(\lambda\bar{x}(t) + (1-\lambda)\bar{x}_0(t)\big)\Big]dt\\
\qquad\qquad + k'(t)dW_0(t),\\
\bar{x}_0(0) = x_0,\ \bar{y}(0) = y,\ \Phi(T) = 0,\ p_0(T) = H_0\bar{x}_0(T),\ p(T) = 0,\ S(0) = 0,\\
\bar{x}(0) = x,\ \Psi(T) = 0
\end{cases}
\tag{5.34}
$$

where $\check{A}(t)$ and $\check{B}(t)$ are given by (5.26).

5.3 "Serial Coupling"

The model implied in this topic is the same to "Serial-Parallel Coupling"–Case I. However, in this topic, another method is under considerable. Motivated by [55], this problem is seemed as "major leader-minor leader-follower", which can be view as a "Serial Coupling". It is solved from back to front and the variational technique is used widely. We also suppose $x^{(N_L)}$ and $y^{(N_F)}$ can be approximated by $\mathcal{F}_t^0$-measurable functions $\bar{x}$ and $\bar{y}$, as $N_L, N_F \to +\infty$.

Step 1. Mean-field games of followers.

This step is same to Step 1 of "Serial-Parallel Coupling"–Case I. And the following system is obtained (same to (5.15))

$$\begin{cases} d\bar{y}(t) = \Big[\big(\tilde{A}(t) - \tilde{B}^2(t)\tilde{R}^{-1}(t)P(t)\big)\bar{y}(t) - \tilde{B}^2(t)\tilde{R}^{-1}(t)\Phi(t)\Big]dt + \tilde{C}(t)dW_0(t), \\ d\Phi(t) = \Big[-\big(\tilde{A}(t) - \tilde{B}^2(t)\tilde{R}^{-1}(t)P(t)\big)\Phi(t) + \tilde{Q}(t)\big(\tilde{\lambda}\bar{x}(t) + (1-\tilde{\lambda})\bar{y}(t)\big)\Big]dt \\ \qquad\qquad + k(t)dW_0(t), \\ \bar{y}(0) = y,\ \Phi(T) = 0. \end{cases} \tag{5.35}$$

Here, $P(\cdot)$ satisfies (5.12) and $\bar{x}(\cdot)$ is to be determined.

Step 2. Mean-field games of minor-leaders.

Because the minor-leaders do not "dominate" the behaviors of the major-leader, the variational technique is out of use for $x_0(\cdot)$. Therefore, this step is same to Step 3 of "Serial-Parallel Coupling"–Case I and the decoupled system is derived (same to (5.20))

$$\begin{cases} d\bar{x}(t) = \Big[\big(A(t) - B^2(t)R^{-1}(t)K(t)\big)\bar{x}(t) - B^2(t)R^{-1}(t)\Psi(t)\Big]dt + C(t)dW_0(t), \\ d\Psi(t) = \Big[-\big(A(t) - B^2(t)R^{-1}(t)K(t)\big)\Psi(t) + Q(t)\big(\lambda\bar{x}(t) + (1-\lambda)\bar{x}_0(t)\big)\Big]dt \\ \qquad\qquad + k'(t)dW_0(t), \\ \bar{x}(0) = x,\ \Psi(T) = 0. \end{cases} \tag{5.36}$$

where $K(\cdot)$ satisfies (5.21) and $\bar{x}_0(\cdot)$ is to be determined.

Step 3. Mean-field games of major-leader.

In the following, we define

$$
\begin{cases}
\check{A}(t) := A(t) - B^2(t)R^{-1}(t)K(t),\\
\check{B}(t) := B^2(t)R^{-1}(t),\\
\check{C}(t) := \tilde{A}(t) - \tilde{B}^2(t)\tilde{R}^{-1}(t)P(t),\\
\check{D}(t) := \tilde{B}^2(t)\tilde{R}^{-1}(t).
\end{cases}
\tag{5.37}
$$

Introduce the following variational equation system

$$
\begin{cases}
d\eta_0(t) = [A_0(t)\eta_0(t) + B_0(t)\delta u_0(t)]dt,\\
d\xi(t) = \Big[\check{A}(t)\xi(t) - \check{B}(t)\theta(t)\Big]dt,\\
d\theta(t) = \Big[- \check{A}(t)\theta(t) + Q(t)\big(\lambda\xi(t) + (1-\lambda)\eta_0(t)\big)\Big]dt + \delta k'(t)dW_0(t),\\
d\zeta(t) = \Big[\check{C}(t)\zeta(t) - \check{D}(t)\beta(t)\Big]dt,\\
d\beta(t) = \Big[- \check{C}(t)\beta(t) + \tilde{Q}(t)\big(\tilde{\lambda}\xi(t) + (1-\tilde{\lambda})\zeta(t)\big)\Big]dt + \delta k(t)dW_0(t),\\
\eta_0(0) = 0,\ \xi(0) = 0,\ \theta(T) = 0,\ \zeta(0) = 0,\ \beta(T) = 0.
\end{cases}
\tag{5.38}
$$

Then we have

Lemma 5.5. *Under* (H5.1)*, the optimal control for the major-leader of Problem* (**L-LF-MFG**) *is given by*

$$
\bar{u}_0(t) = -R_0^{-1}(t)B_0(t)p_0(t) \tag{5.39}
$$

and the adjoint equation satisfies

$$
\begin{cases}
dp_0(t) = -\Big[A_0(t)p_0(t) + Q(t)m(t)(1-\lambda) + Q_0(t)\Big(\bar{x}_0(t) - \big(\lambda_0\bar{x}(t) \\
\qquad\qquad + (1-\lambda_0)\bar{y}(t)\big)\Big)\Big]dt + q_0(t)dW_0(t), \\
dp(t) = \Big[-\check{A}(t)p(t) - \tilde{\lambda}\tilde{Q}(t)l(t) - \lambda Q(t)m(t) + \lambda_0 Q_0(t)\Big(\bar{x}_0(t) - \big(\lambda_0\bar{x}(t) \\
\qquad\qquad + (1-\lambda_0)\bar{y}(t)\big)\Big)\Big]dt + q(t)dW_0(t), \\
dm(t) = \Big[\check{A}(t)m(t) + \check{B}(t)p(t)\Big]dt, \\
dn(t) = \Big[-\check{C}(t)n(t) - (1-\tilde{\lambda})\tilde{Q}(t)l(t) + (1-\lambda_0)Q_0(t)\Big(\bar{x}_0(t) - \big(\lambda_0\bar{x}(t) \\
\qquad\qquad + (1-\lambda_0)\bar{y}(t)\big)\Big)\Big]dt + r(t)dW_0(t), \\
dl(t) = \Big[\check{C}(t)l(t) + \check{D}(t)n(t)\Big]dt, \\
p_0(T) = H_0\bar{x}_0(T),\ p(T) = 0,\ m(0) = 0,\ n(T) = 0,\ l(0) = 0.
\end{cases}
\tag{5.40}
$$

Proof. The variation of cost functional is

$$
\begin{aligned}
0 = \frac{\delta J_0(u_0)}{\delta u_0} = \mathbb{E}\Big\{\int_0^T \Big[Q_0(t)\Big(\bar{x}_0(t) - \big(\lambda_0\bar{x}(t) + (1-\lambda_0)\bar{y}(t)\big)\Big) \\
\cdot\Big(\eta_0(t) - \big(\lambda_0\xi(t) + (1-\lambda_0)\zeta(t)\big)\Big) + R_0(t)\bar{u}_0(t)\delta u_0(t)\Big]dt + H_0\bar{x}_0(T)\eta_0(T)\Big\}.
\end{aligned}
\tag{5.41}
$$

Applying Itô's formula to $p_0(t)\eta_0(t) + p(t)\xi(t) + m(t)\theta(t) + n(t)\zeta(t) + l(t)\beta(t)$, hence the results. $\square$

From (5.35), (5.36) and (5.40), the consistency condition system is given as follows

$$
\begin{cases}
d\bar{x}_0(t) = \Big[A_0(t)\bar{x}_0(t) - B_0^2(t)R_0^{-1}(t)p_0(t)\Big]dt + D_0(t)dW_0(t),\\[2mm]
d\bar{x}(t) = \Big[\breve{A}(t)\bar{x}(t) - \breve{B}(t)\Psi(t)\Big]dt + C(t)dW_0(t),\\[2mm]
d\Psi(t) = \Big[-\breve{A}(t)\Psi(t) + Q(t)\big(\lambda\bar{x}(t) + (1-\lambda)\bar{x}_0(t)\big)\Big]dt + k'(t)dW_0(t),\\[2mm]
d\bar{y}(t) = \Big[\breve{C}(t)\bar{y}(t) - \breve{D}(t)\Phi(t)\Big]dt + \tilde{C}(t)dW_0(t),\\[2mm]
d\Phi(t) = \Big[-\breve{C}(t)\Phi(t) + \tilde{Q}(t)\big(\tilde{\lambda}\bar{x}(t) + (1-\tilde{\lambda})\bar{y}(t)\big)\Big]dt + k(t)dW_0(t),\\[2mm]
dp_0(t) = -\Big[A_0(t)p_0(t) + Q(t)m(t)(1-\lambda) + Q_0(t)\Big(\bar{x}_0(t) - \big(\lambda_0\bar{x}(t)\\[2mm]
\qquad\qquad + (1-\lambda_0)\bar{y}(t)\big)\Big)\Big]dt + q_0(t)dW_0(t),\\[2mm]
dp(t) = \Big[-\breve{A}(t)p(t) - \tilde{\lambda}\tilde{Q}(t)l(t) - \lambda Q(t)m(t) + \lambda_0 Q_0(t)\Big(\bar{x}_0(t) - \big(\lambda_0\bar{x}(t)\\[2mm]
\qquad\qquad + (1-\lambda_0)\bar{y}(t)\big)\Big)\Big]dt + q(t)dW_0(t),\\[2mm]
dm(t) = \Big[\breve{A}(t)m(t) + \breve{B}(t)p(t)\Big]dt,\\[2mm]
dn(t) = \Big[-\breve{C}(t)n(t) - (1-\tilde{\lambda})\tilde{Q}(t)l(t) + (1-\lambda_0)Q_0(t)\Big(\bar{x}_0(t) - \big(\lambda_0\bar{x}(t)\\[2mm]
\qquad\qquad + (1-\lambda_0)\bar{y}(t)\big)\Big)\Big]dt + r(t)dW_0(t),\\[2mm]
dl(t) = \Big[\breve{C}(t)l(t) + \breve{D}(t)n(t)\Big]dt,\\[2mm]
\bar{x}_0(0) = x_0,\ \bar{x}(0) = x,\ \Psi(T) = 0,\ \bar{y}(0) = y,\ \Phi(T) = 0,\ p_0(T) = H_0\bar{x}_0(T),\\[2mm]
p(T) = 0,\ m(0) = 0,\ n(T) = 0,\ l(0) = 0
\end{cases}
\tag{5.42}
$$

where $\breve{A}(t), \breve{B}(t), \breve{C}(t)$ and $\breve{D}(t)$ are given by (5.37).

Remark 5.1. *To conclude this chapter, we give some remarks concerning the solutions of Riccati equations and consistency condition systems.*

(1) The sufficient conditions for the existence and uniqueness of $P(\cdot), P_0(\cdot), K(\cdot)$ (solutions of (5.12), (5.18), (5.21)*) are given in [95]. In addition, with the help*

of the sufficient condtions, the explicit forms of $P(\cdot), P_0(\cdot), K(\cdot)$ *can be obtained as nonnegative functions, respectively.*

(2) Based on $P(\cdot), P_0(\cdot), K(\cdot)$, *the following work is to seek the wellposedness of the consistency condition systems* (5.22), (5.34) *and* (5.42), *which are all coupled FBSDEs. Though there is some classical literature for wellposedness of FBSDE (see [88, 90, 92, 96], etc.), it is still challenging due to the complicated coupled structures of* (5.22), (5.34) *and* (5.42). *If the wellposedness of the consistency condition systems is obtained, the decentralized strategies and the corresponding* ϵ*-Nash equilibrium properties will be studied.*

Chapter 6

LQMFGs with Partial Information — An FBSDE Representation

The purpose of this chapter is to consider the dynamic optimization of large-population system with partial information structure. In this framework, the individual agents can only access the filtration generated by one observable component of the underlying Brownian motion. The most significant feature is that the limiting state average in this setup turns out to be some stochastic process driven by the common Brownian motion.

Two classes of large-population systems are proposed in this chapter: one class is characterized by forward dynamics, and the other class is governed by backward one. In the first class, the LQ system is proposed, the limiting state average is represented by a MF SDE and its consistency condition is equivalent to the wellposedness of some common Riccati equation system. This case differs from [81] because in [81] an infinite-time horizon was defined, as a result the algebra Riccati equations were involved. Moreover, the limiting state average in [81] was deterministic as there was no common noise. In the backward class, the explicit forms of the decentralized strategies and some BSDE satisfied by the limiting process are obtained. In both cases, with the help of estimates to SDE and BSDE, the ϵ-Nash equilibrium properties are presented.

6.1 Problem Formulation

The information structure of our large-population system can be described as follows. First, introduce $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leqslant t\leqslant T}, P)$ as the complete probability space on which a standard $(1+N)$-dimensional Brownian motion $\{W(t), W_i(t),\ 1\leqslant i\leqslant N\}_{0\leqslant t\leqslant T}$ is defined. Depending on which problems to be addressed, we have different setup to the information structure. In case of forward partial information problem, we denote by $\{\mathcal{F}_t^{w_i}\}_{0\leqslant t\leqslant T}$ the filtration generated by the component W_i; $\{\mathcal{F}_t^{w}\}_{0\leqslant t\leqslant T}$ the filtration generated by the component W. Here, $\{\mathcal{F}_t^{w_i}\}_{0\leqslant t\leqslant T}$ stands for the individual information owning by the i^{th} agent; $\{\mathcal{F}_t^{w}\}_{0\leqslant t\leqslant T}$ the information of some macro process imposing on all agents (firms) due to the common external economic factors which can't be directly observed by our agents (say, some latent marco-economic process, or hidden action process). $\mathcal{F}_t^i := \mathcal{F}_t^{w_i}\bigcup\mathcal{F}_t^{w}$. $\mathcal{G}_t := \bigcup_{i=1}^N \mathcal{F}_t^i$ denotes the complete information of system. In case of backward partial information problem, we let $\mathcal{F}_t := \bigcup_{i=1}^N \mathcal{F}_t^{w_i}$ denote the information accessible to all agents. Actually, in this case $\mathcal{G}_t = \mathcal{F}_t\bigcup\mathcal{F}_t^{w}$ denotes the complete information of large-population system.

6.1.1 Forward LQMFGs with partial information

Now, we first consider the forward large-population system with N individual agents $\{\mathcal{A}_i\}_{1\leqslant i\leqslant N}$ in partial information structure. The state x_i for $\mathcal{A}_i$ satisfies the following controlled linear stochastic system:

$$\begin{cases} dx_i(t) =\Big[A(t)x_i(t) + B(t)u_i(t) + \alpha x^{(N)}(t) + m(t)\Big]dt + \sigma(t)dW_i(t) \\ \qquad\qquad + \tilde{\sigma}(t)dW(t), \\ x_i(0) = x \end{cases} \tag{6.1}$$

where $x^{(N)}(t) = \frac{1}{N}\sum\limits_{i=1}^{N} x_i(t)$ is the state-average, $\alpha\in\mathbb{R}$ denotes the coupling constant which maybe positive or negative. In (6.1), W_i denotes the individual random noise while W denotes the common random noise. Other work discussing the large-population system with common noise W includes [43]. Thus, the admissible control $u_i \in \mathcal{U}_i$ where the admissible control set $\mathcal{U}_i$ is defined by

$$\mathcal{U}_i := \Big\{ u_i \Big| u_i(\cdot) \in L^2_{\mathcal{F}_t^{w_i}}(0,T;\mathbb{R}) \Big\}, \ 1 \leqslant i \leqslant N.$$

Denote $u_{-i} = (u_1, \ldots, u_{i-1}, \ u_{i+1}, \ldots u_N)$ the strategies of all agents except $\mathcal{A}_i$. The cost functional of $\mathcal{A}_i$ is

$$\mathcal{J}_i(u_i(\cdot), u_{-i}(\cdot)) = \mathbb{E}\left[\int_0^T \big(Q(t)(x_i(t) - x^{(N)}(t))^2 + R(t)u_i^2(t) \big)\, dt + Gx_i^2(T) \right]. \quad (6.2)$$

Moreover, we have the following assumption:

(H6.1) $A(\cdot), B(\cdot), m(\cdot), \sigma(\cdot), \tilde{\sigma}(\cdot), Q(\cdot), R(\cdot) \in L^\infty(0,T;\mathbb{R}), \alpha \in \mathbb{R}, Q(\cdot) \geqslant 0,$
$R(\cdot) \geqslant \delta$, for $\delta > 0$, $G \geqslant 0$.

Now, we formulate the forward large-population LQ games with partial filtration (F-PI).

Problem (F-PI). Find a control strategies set $\bar{u} = (\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_N)$ which satisfies

$$\mathcal{J}_i(\bar{u}_i(\cdot), \bar{u}_{-i}(\cdot)) = \inf_{u_i(\cdot) \in \mathcal{U}_i} \mathcal{J}_i(u_i(\cdot), \bar{u}_{-i}(\cdot))$$

where $\bar{u}_{-i}$ represents $(\bar{u}_1, \ldots, \bar{u}_{i-1}, \bar{u}_{i+1}, \ldots, \bar{u}_N)$.

6.1.2 Backward MFGs with partial information

In some case, it is very natural to consider the backward MFGs. To this end, we formulate the following backward MFGs in which the large-population system is weakly-coupled in the cost functional :

$$\textbf{(B-PI)} \begin{cases} \text{state} : \begin{cases} -dy_i(t) = \big[Ay_i(t) + Bu_i(t) \big] dt - z_i(t)dW_i(t) - \tilde{z}_i(t)dW(t), \\ \quad y_i(T) = \eta_i, \end{cases} \\ \text{cost functional} : \mathcal{J}_i\big(u_i(\cdot), u_{-i}(\cdot)\big) \\ \qquad\qquad = \mathbb{E}\Big[\int_0^T Ru_i^2(t)dt + 2y_i(0)\big(\alpha - \beta y^{(N)}(0)\big) \Big]. \end{cases} \quad (6.3)$$

Here, we assume A, B are scalar constants, $R > 0, \alpha \geqslant 0, \beta \geqslant 0$; $\eta_i \in \mathcal{F}_T^i$, $i = 1, 2, \cdots, N$, are the terminal conditions for individual agents; $y^{(N)}(t) = \frac{1}{N}\sum\limits_{i=1}^{N} y_i(t)$ is the state average, $y^{(N)}(0)$ is its initial value. The admissible control $u_i \in \mathcal{V}_i$ is defined as

$$\mathcal{V}_i := \Big\{u_i \big| u_i(\cdot) \in L^2_{\mathcal{F}_t^{w_i}}(0, T; \mathbb{R})\Big\},\ 1 \leqslant i \leqslant N.$$

In partial information structure, we make the following assumption:

(H6.2) $\{\eta_i\}_{i=1}^N$ are identically conditional distributed w.r.t. $\mathcal{F}_T^w$ with $\mathbb{E}|\eta_i|^2 < +\infty$. Moreover, the distribution of each η_i is not depending on i and N.

It follows that under (H6.2), the state equation in (6.3) admits a unique solution $(y_i, z_i, \tilde{z}_i) \in L^2_{\mathcal{F}^i}(0, T; \mathbb{R}) \times L^2_{\mathcal{F}^i}(0, T; \mathbb{R}) \times L^2_{\mathcal{F}^i}(0, T; \mathbb{R})$ for all $u_i \in \mathcal{V}_i$. In fact, the uniqueness is obtained by [84] directly in partial information framework. Noting the identically conditional distributions of $\{\eta_i\}_{i=1}^N$ in (H6.2), it is easy to obtain that $\mathbb{E}\big(\eta_1|\mathcal{F}_T^w\big) = \cdots = \mathbb{E}\big(\eta_N|\mathcal{F}_T^w\big)$, which is denoted by $\eta \in \mathcal{F}_T^w$. Then applying the results of [114], we get that conditionally on $\mathcal{F}_T^w$, $\frac{1}{N}\sum\limits_{i=1}^{N}\eta_i \to \eta,\ a.s.$, as $N \to +\infty$. It is worth pointing out that if η_i has the following linear or nonlinear structure, $\{\eta_i\}_{i=1}^N$ satisfy (H6.2) easily: $\eta_i = \alpha_i + \beta$ or $\eta_i = \phi(\alpha_i, \beta)$, where $\alpha_i \in \mathcal{F}_T^{w_i}$ with identical distribution, $\beta \in \mathcal{F}_T^w$, $\mathbb{E}|\alpha_i|^2 < +\infty$, $\mathbb{E}|\beta|^2 < +\infty$, $i = 1, \cdots, N$, and $\phi(\cdot)$ is a measurable function. And $\frac{1}{N}\sum\limits_{i=1}^{N}\eta_i \to \mathbb{E}\alpha_1 + \beta$ or $\mathbb{E}\big(\phi(\alpha_1, \beta)|\mathcal{F}_T^w\big)$ $a.s.$, as $N \to +\infty$.

Remark 6.1. *(1) We now present some remarks to the real meaning of system* (6.3)*. In reality, the LQ BSDE system stands for the benchmark tracking problem with portfolio selection in financial market. If a given portfolio strategy emphasizes one aspect or one product, it will be adjusted by considering the whole behaviors throughout the market.*

(2) In this system, the state average is not coupled in dynamics. There are two reasons. The first reason is from practical point: the coupling in cost functional arise

naturally when we consider the relative (investment) performance (see e.g., [113]). In particular, the penalty over the initial average or states enables us to consider the relative or comparable criteria based on the average performance of all other peers through the whole sector (industry). The second reason is more technical: in partial information structure, the optimal control involves filtering equations and this always leads to considerable interrelated and complicated filter estimations. It is difficult to get similar estimated results as in the full information problem. Thus, we consider the coupled cost functional in (6.3) *due to its financial meanings.*

6.2 (F-PI): Forward LQMFGs with Partial Information

To study **(F-PI)**, one efficient protocol is the LQMFGs which bridges the "centralized" LQ problems via the limiting state-average, as the number of agents tends to infinity.

6.2.1 The limiting control of (L-F-PI)

Due to partial filtration structure, it is natural to set the following feedback control on filters

$$u_i(t) = -a(t)\mathbb{E}(x_i(t)|\mathcal{F}_t^{w_i}) + \sum_{j=1, j\neq i}^{N} \tilde{a}(t)\mathbb{E}(x_j(t)|\mathcal{F}_t^{w_i}) + b(t) \tag{6.4}$$

where the coefficients $a(\cdot), \tilde{a}(\cdot)$ and $b(\cdot)$ are deterministic functions and $\tilde{a}(\cdot) = O(\frac{1}{N})$. Inserting (6.4) into state equation (6.1), we get the following realized state dynamics

$$\begin{aligned} dx_i(t) =& \Big[A(t)x_i(t) - B(t)a(t)\mathbb{E}(x_i(t)|\mathcal{F}_t^{w_i}) + B(t)\tilde{a}(t)\sum_{j=1, j\neq i}^{N}\mathbb{E}(x_j(t)|\mathcal{F}_t^{w_i}) + B(t)b(t) \\ &+ \alpha x^{(N)}(t) + m(t)\Big]dt + \sigma(t)dW_i(t) + \tilde{\sigma}(t)dW(t),\ 1 \leqslant i \leqslant N. \end{aligned} \tag{6.5}$$

Take summation of the above N equations and divide by N,

$$
\begin{aligned}
d\Big(\frac{1}{N}\sum_{i=1}^{N} x_i(t)\Big) =& \Big[A(t)\frac{1}{N}\sum_{i=1}^{N} x_i(t) - B(t)a(t)\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}(x_i(t)|\mathcal{F}_t^{w_i}) + B(t)b(t) + \alpha x^{(N)}(t) + m(t) \\
&+ B(t)\tilde{a}(t)\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1, j\neq i}^{N}\mathbb{E}(x_j(t)|\mathcal{F}_t^{w_i})\Big]dt + \sigma(t)\frac{1}{N}\sum_{i=1}^{N} dW_i(t) + \tilde{\sigma}(t)dW(t).
\end{aligned}
$$

Letting $N \to +\infty$, we obtain the following limiting process which is a MF SDE:

$$
\begin{cases}
dx_0(t) = \Big[(A(t)+\alpha)x_0(t) - \tilde{\alpha}(t)\mathbb{E}x_0(t) + \tilde{b}(t)\Big]dt + \tilde{\sigma}(t)dW(t), \\
x_0(0) = x
\end{cases}
\tag{6.6}
$$

where the functions $\tilde{\alpha}(\cdot), \tilde{b}(\cdot)$ are to be determined. Now, we introduce an auxiliary state:

$$
\begin{cases}
dx_i(t) = \Big[A(t)x_i(t) + B(t)u_i(t) + \alpha x_0(t) + m(t)\Big]dt + \sigma(t)dW_i(t) + \tilde{\sigma}(t)dW(t), \\
x_i(0) = x
\end{cases}
\tag{6.7}
$$

with the auxiliary cost functional

$$
J_i(u_i(\cdot)) = \mathbb{E}\left[\int_0^T \big(Q(t)(x_i(t) - x_0(t))^2 + R(t)u_i^2(t)\big)\,dt + Gx_i^2(T)\right]
\tag{6.8}
$$

where $x_0(\cdot)$ is given by (6.6). Note that (6.7) and (6.8) are obtained from (6.1) and (6.2) with $x^{(N)}(\cdot)$ replaced by $x_0(\cdot)$. Thus, we formulate the following limiting forward partial information (L-F-PI) LQ game.

Problem (L-F-PI). For the i^{th} agent, $i = 1, 2, \ldots, N$, find $\bar{u}_i(\cdot) \in \mathcal{U}_i$ satisfying

$$
J_i(\bar{u}_i(\cdot)) = \inf_{u_i(\cdot)\in\mathcal{U}_i} J_i(u_i(\cdot)).
$$

Then $\bar{u}_i(\cdot)$ is called an optimal control for Problem **(L-F-PI)**. Applying the variational method (similar to Proposition 3.1), we have the following result to the optimal control of **(L-F-PI)**.

Proposition 6.1. *Let* (H6.1) *hold. Suppose there exists an optimal control* $\bar{u}_i(\cdot)$ *of Problem* **(L-F-PI)** *and* $\bar{x}_i(\cdot)$ *is the corresponding optimal state, then there exists an adjoint process* $p_i(\cdot) \in L^2_{\mathcal{F}^i_t}(0,T;\mathbb{R})$ *satisfying the following BSDE for some* $\beta(\cdot)$ *and* $\tilde{\beta}(\cdot)$*:*

$$\begin{cases} dp_i(t) = \Big[-A(t)p_i(t) - Q(t)(\bar{x}_i(t) - x_0(t))\Big]dt + \beta(t)dW_i(t) + \tilde{\beta}(t)dW(t), \\ p_i(T) = G\bar{x}_i(T), \ i = 1, 2, \ldots, N \end{cases} \tag{6.9}$$

such that

$$\bar{u}_i(t) = -R^{-1}(t)B(t)\mathbb{E}(p_i(t)|\mathcal{F}^{w_i}_t)$$

where the conditional expectation is defined in its optional projection version.

6.2.2 The consistency condition

With the results above, consequently, we get the following Hamiltonian system for $\mathcal{A}_i$:

$$\begin{cases} dx_0(t) = \Big[(A(t) + \alpha)x_0(t) - \tilde{\alpha}(t)\mathbb{E}x_0(t) + \tilde{b}(t)\Big]dt + \tilde{\sigma}(t)dW(t), \\ d\bar{x}_i(t) = \Big[A(t)\bar{x}_i(t) - B^2(t)R^{-1}(t)\mathbb{E}(p_i(t)|\mathcal{F}^{w_i}_t) + \alpha x_0(t) + m(t)\Big]dt \\ \qquad\qquad + \sigma(t)dW_i(t) + \tilde{\sigma}(t)dW(t), \\ dp_i(t) = \Big[-A(t)p_i(t) - Q(t)(\bar{x}_i(t) - x_0(t))\Big]dt + \beta(t)dW_i(t) + \tilde{\beta}(t)dW(t), \\ x_0(0) = \bar{x}_i(0) = x, \ p_i(T) = G\bar{x}_i(T), \ i = 1, 2, \ldots, N. \end{cases} \tag{6.10}$$

After obtaining $\tilde{\alpha}(\cdot), \tilde{b}(\cdot)$ in Theorem 6.1 (see below), by the monotonic conditions of FBSDE (see [92]), it is easy to see that (6.10) admits a unique solution $(x_0(\cdot), \bar{x}_i(\cdot), p_i(\cdot)) \in L^2_{\mathcal{F}^w_t}(0,T;\mathbb{R}) \times L^2_{\mathcal{F}^i_t}(0,T;\mathbb{R}) \times L^2_{\mathcal{F}^i_t}(0,T;\mathbb{R})$. Note that in system (6.10), the forward optimal state $\bar{x}_i(\cdot)$ depends on the backward adjoint process $p_i(\cdot)$ through its filtering state $\mathbb{E}(p_i(t)|\mathcal{F}^{w_i}_t)$. In this sense, (6.10) becomes a filtered FBSDE system and its decoupling should be proceeded through some FBSDE that involves the filtering state only. To this end, we introduce the following filter notations

$$\hat{\bar{x}}_i(t) = \mathbb{E}[\bar{x}_i(t)|\mathcal{F}_t^{w_i}], \quad \hat{p}_i(t) = \mathbb{E}[p_i(t)|\mathcal{F}_t^{w_i}]$$

where the conditional expectations to the partial filtration $\mathcal{F}_t^{w_i}$ should be understood in the version of optional projection. Then we reach a FBSDE system involving the state filters only:

$$\begin{cases} d\hat{\bar{x}}_i(t) = \Big[A(t)\hat{\bar{x}}_i(t) - B^2(t)R^{-1}(t)\hat{p}_i(t) + \alpha\mathbb{E}x_0(t) + m(t)\Big]dt + \sigma(t)dW_i(t), \\ \hat{\bar{x}}_i(0) = x, \\ d\hat{p}_i(t) = \Big[-A(t)\hat{p}_i(t) - Q(t)(\hat{\bar{x}}_i(t) - \mathbb{E}x_0(t))\Big]dt + \beta(t)dW_i(t), \\ \hat{p}_i(T) = G\hat{\bar{x}}_i(T), \ i = 1, 2, \ldots, N. \end{cases} \tag{6.11}$$

Note that system (6.11) is driven by W_i only so it becomes observable to agent $\mathcal{A}_i$. It can be viewed a filtering system of (6.10) that is unobservable as driven by W_i and W both. Taking expectation on (6.6),

$$\begin{cases} d\mathbb{E}x_0(t) = \Big[(A(t) + \alpha - \tilde{\alpha}(t))\mathbb{E}x_0(t) + \tilde{b}(t)\Big]dt, \\ \mathbb{E}x_0(0) = x \end{cases} \tag{6.12}$$

where $\tilde{\alpha}(\cdot)$, $\tilde{b}(\cdot)$ are functions to be determined. One key step in MFG is to analyze the related consistency condition (which is also called Nash certainty equivalence (NCE) principle, see [81], [18], etc).

Remark 6.2. *To intuitively explain the consistency condition, we give some remarks.*

(1) Unlike most literature on MFGs, there is no fixed-point argument involved here (e.g., some contraction mapping based on the datum of our problem) to characterize the consistency condition. Instead, our consistency condition is transformed into the wellposedness of Riccati equation system (6.13) *(see below). Actually,* $(\hat{P}(\cdot), \Phi(\cdot))$

depend on $(\tilde{\alpha}(\cdot), \tilde{b}(\cdot))$, *thus* (6.18) *(see below) can be rewritten by*

$$\begin{cases} \tilde{\alpha} = \mathcal{T}_1(\tilde{\alpha}) := B^2 R^{-1}(P + \hat{P}(\tilde{\alpha})), \\ \tilde{b} = \mathcal{T}_2(\tilde{b}) := -B^2 R^{-1}\Phi(\tilde{\alpha}, \tilde{b}) + m. \end{cases}$$

In this sense, (6.13) *can be understood as the consistency condition of* **(L-F-PI)**.

(2) The advantages of handling the consistency condition of $(\tilde{\alpha}(\cdot), \tilde{b}(\cdot))$ *are as follows. The consistency condition imposed on* $(\tilde{\alpha}(\cdot), \tilde{b}(\cdot))$ *is equivalent to the well-posedness of Riccati equation* (6.13) *(see below) which can be ensured in an arbitrary time interval. On the other hand, as addressed in [37], the fixed-point analysis on* x *will preferably lead to the consistency condition only on a small time interval.*

Now we first state the following result.

Theorem 6.1. *Suppose* (H6.1) *hold true and the following Riccati equation system*

$$\begin{cases} \dot{\Pi}(t) + (2A(t) + \alpha)\Pi(t) - B^2(t)R^{-1}(t)\Pi^2(t) = 0, \\ \dot{\Phi}(t) + [A(t) - B^2(t)R^{-1}(t)\Pi(t)]\Phi(t) + m(t)\Pi(t) = 0, \\ \Pi(T) = G, \ \Phi(T) = 0 \end{cases} \tag{6.13}$$

admits unique solution $(\Pi(\cdot), \Phi(\cdot))$, *then* $(\tilde{\alpha}(\cdot), \ \tilde{b}(\cdot))$ *can be uniquely determined by*

$$\begin{cases} \tilde{\alpha}(t) = B^2(t)R^{-1}(t)\Pi(t), \\ \tilde{b}(t) = -B^2(t)R^{-1}(t)\Phi(t) + m(t). \end{cases} \tag{6.14}$$

Proof. By the terminal condition of (6.10) or (6.11), we suppose

$$\hat{p}_i(t) = P(t)\hat{\bar{x}}_i(t) + \hat{P}(t)\mathbb{E}x_0(t) + \Phi(t) \tag{6.15}$$

for some $P(\cdot), \hat{P}(\cdot) \in L^\infty(0, T; \mathbb{R})$ and $\Phi(t) \in L^\infty(0, T; \mathbb{R})$ with terminal conditions

$$P(T) = G, \ \hat{P}(T) = \Phi(T) = 0.$$

Applying Itô's formula to (6.15) and noting (6.10), we have

$$\begin{aligned} d\hat{p}_i(t) = & \Big(\dot{P}(t) + P(t)A(t) - B^2(t)R^{-1}(t)P^2(t)\Big)\hat{\bar{x}}_i(t)dt \\ & + \Big(\dot{\hat{P}}(t) + \hat{P}(t)(A(t) + \alpha - \tilde{\alpha}(t)) - P(t)B^2(t)R^{-1}(t)\hat{P}(t) + \alpha P(t)\Big)\mathbb{E}x_0(t)dt \\ & + \Big(\dot{\Phi}(t) - P(t)B^2(t)R^{-1}(t)\Phi(t) + P(t)m(t) + \hat{P}(t)\tilde{b}(t)\Big)dt + P(t)\sigma(t)dW_i(t) \\ = & \Big[(-Q(t) - A(t)P(t))\,\hat{\bar{x}}_i(t) + (Q(t) - A(t)\hat{P}(t))\mathbb{E}x_0(t) - A(t)\Phi(t)\Big]dt + \beta(t)dW_i(t). \end{aligned}$$

Comparing coefficients, we obtain

$$\begin{cases} \dot{P}(t) + P(t)A(t) - B^2(t)R^{-1}(t)P^2(t) = -Q(t) - A(t)P(t), \\ \dot{\hat{P}}(t) + \hat{P}(t)(A(t) + \alpha - \tilde{\alpha}(t)) - P(t)B^2(t)R^{-1}(t)\hat{P}(t) + \alpha P(t) = Q(t) - A(t)\hat{P}(t), \\ \dot{\Phi}(t) - P(t)B^2(t)R^{-1}(t)\Phi(t) + P(t)m(t) + \hat{P}(t)\tilde{b}(t) = -A(t)\Phi(t), \\ \beta(t) = P(t)\sigma(t). \end{cases} \tag{6.16}$$

Note that the above Riccati equations are parameterized by the undetermined functions $(\tilde{\alpha}(t), \tilde{b}(t))$ which are to be specified below. To this end, note that the optimal state $\bar{x}_i(t)$ can be represented by

$$\begin{aligned} d\bar{x}_i(t) =& [A(t)\bar{x}_i(t) - B^2(t)R^{-1}(t)(P(t)\hat{\bar{x}}_i(t) + \hat{P}(t)\mathbb{E}x_0(t) + \Phi(t)) + \alpha x_0(t) + m(t)]dt \\ & + \sigma(t)dW_i(t) + \tilde{\sigma}(t)dW(t). \end{aligned}$$

Therefore the state-average satisfies:

$$\begin{aligned} d\bar{x}^{(N)}(t) =& \Big[A(t)\bar{x}^{(N)}(t) - B^2(t)R^{-1}(t)(P(t)\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}(\bar{x}_i(t)|\mathcal{F}_t^{w_i}) + \hat{P}(t) \\ & \cdot \mathbb{E}x_0(t) + \Phi(t)) + \alpha x_0(t) + m(t)\Big]dt + \sigma(t)\frac{1}{N}\sum_{i=1}^{N}dW_i(t) + \tilde{\sigma}(t)dW(t). \end{aligned}$$

Let $N \to +\infty$, the limiting process x_0 is given by

$$\begin{aligned} dx_0(t) =& \Big[(A(t) + \alpha)x_0(t) - B^2(t)R^{-1}(t)(P(t) + \hat{P}(t))\mathbb{E}x_0(t) \\ & - B^2(t)R^{-1}(t)\Phi(t) + m(t)\Big]dt + \tilde{\sigma}(t)dW(t). \end{aligned} \tag{6.17}$$

Comparing the coefficients with (6.10), we have

$$\begin{cases} \tilde{\alpha}(t) = B^2(t)R^{-1}(t)(P(t) + \hat{P}(t)), \\ \tilde{b}(t) = -B^2(t)R^{-1}(t)\Phi(t) + m(t). \end{cases} \tag{6.18}$$

Thus we rewrite (6.16) as

$$\begin{cases} \dot{P}(t) + 2A(t)P(t) - B^2(t)R^{-1}(t)P^2(t) + Q(t) = 0, \\ \dot{\hat{P}}(t) + \hat{P}(t)[2A(t) + \alpha - B^2(t)R^{-1}(t)(P(t) + \hat{P}(t)) - B^2(t)R^{-1}(t)P(t)] + \alpha P(t) - Q(t) = 0, \\ \dot{\Phi}(t) + [A(t) - (P(t) + \hat{P}(t))B^2(t)R^{-1}(t)]\Phi(t) + (P(t) + \hat{P}(t))m(t) = 0, \\ P(T) = G,\ \hat{P}(T) = \Phi(T) = 0. \end{cases}$$

Letting $\Pi(t) = P(t) + \hat{P}(t)$, we get

$$\begin{cases} \dot{\Pi}(t) + (2A(t) + \alpha)\Pi(t) - B^2(t)R^{-1}(t)\Pi^2(t) = 0, \\ \Pi(T) = G. \end{cases} \tag{6.19}$$

This completes the proof. $\square$

Moreover, the filtering system (6.11) can be decoupled as

$$\begin{cases} d\hat{\check{x}}_i(t) = \Big[\Big(A(t) - B^2(t)R^{-1}(t)P(t)\Big)\hat{\check{x}}_i(t) + \Big(\alpha - B^2(t)R^{-1}(t)(\Pi(t) - P(t))\Big) \\ \qquad\qquad \cdot \mathbb{E}x_0(t) - B^2(t)R^{-1}(t)\Phi(t) + m(t)\Big]dt + \sigma(t)dW_i(t), \\ \hat{p}_i(t) = P(t)\hat{\check{x}}_i(t) + (\Pi(t) - P(t))\mathbb{E}x_0(t) + \Phi(t), \\ \hat{\check{x}}_i(0) = x_i(0),\ \hat{p}_i(T) = G\hat{\check{x}}_i(T). \end{cases} \tag{6.20}$$

Taking average of all and sending $N \to +\infty$, we regenerate

$$\begin{cases} d\mathbb{E}x_0(t) = \Big[\big(A(t) + \alpha - B^2(t)R^{-1}(t)\Pi(t)\big)\mathbb{E}x_0(t) - B^2(t)R^{-1}(t)\Phi(t) + m(t)\Big]dt, \\ \mathbb{E}x_0(0) = x. \end{cases} \tag{6.21}$$

Remark 6.3. *To conclude this section, we give some remarks concerning Theorem 6.1.*

(1) The sufficient conditions for the existence and uniqueness of $P(\cdot)$ and $\Pi(\cdot)$ can be found in [95] hence the solvability of $\hat{P}(\cdot)$ follows directly by noting $\Pi(t) = P(t) + \hat{P}(t)$. In addition, the solvability of $\Phi(\cdot)$ follows from that of $\Pi(\cdot)$.

(2) As referred in Remark 6.2, in [37] the fixed-point analysis on x preferably leads to the consistency condition defined only on a small time interval. This finding also corresponds to the standard result in FBSDE theory: as discussed in [88], the usual contraction mapping on forward-backward system will always lead to its existence and uniqueness in a very small time interval.

6.2.3 ϵ-Nash equilibrium for (F-PI)

Now we show that $(\bar{u}_1, \bar{u}_2, \dots, \bar{u}_N)$ satisfies the ϵ-Nash equilibrium for (**F-PI**).

Theorem 6.2. *Let* (H6.1) *hold and* (6.13) *admit a solution* (Π, Φ), *then* $(\bar{u}_1, \bar{u}_2, \dots, \bar{u}_N)$ *satisfies the ϵ-Nash equilibrium of Problem* **(F-PI)**. *Here, for $1 \leqslant i \leqslant N$, $\bar{u}_i$ is given by*

$$\bar{u}_i(t) = -R^{-1}(t)B(t)\Big[P(t)\hat{\bar{x}}_i(t) + (\Pi(t) - P(t))\mathbb{E}x_0(t) + \Phi(t)\Big] \tag{6.22}$$

where $\hat{\bar{x}}_i$ and $\mathbb{E}x_0$ satisfy (6.20) *and* (6.21) *respectively.*

As preliminaries of proving the theorem, several lemmas are presented to produce some estimates on the state and cost difference between Problem **(F-PI)**) and **(L-F-PI)** and the proofs are available upon request. Recall that

$$\begin{cases} d\bar{x}_i(t) =\Big[A(t)\bar{x}_i(t) - B^2(t)R^{-1}(t)\Big(P(t)\hat{\bar{x}}_i(t) + \hat{P}(t)\mathbb{E}x_0(t) + \Phi(t)\Big) + \alpha x_0(t) \\ \qquad\qquad + m(t)\Big]dt + \sigma(t)dW_i(t) + \tilde{\sigma}(t)dW(t), \\ d\hat{\bar{x}}_i(t) =\Big[\Big(A(t) - B^2(t)R^{-1}(t)P(t)\Big)\hat{\bar{x}}_i(t) + \Big(\alpha - B^2(t)R^{-1}(t)\hat{P}(t)\Big)\mathbb{E}x_0(t) \\ \qquad\qquad - B^2(t)R^{-1}(t)\Phi(t) + m(t)\Big]dt + \sigma(t)dW_i(t), \\ \bar{x}_i(0) =\hat{\bar{x}}_i(t) = x_i(0), \end{cases} \tag{6.23}$$

and denote

$$\bar{x}^{(N)}(t) = \frac{1}{N}\sum_{i=1}^{N}\bar{x}_i(t), \qquad \hat{\bar{x}}^{(N)}(t) = \frac{1}{N}\sum_{i=1}^{N}\hat{\bar{x}}_i(t).$$

Here, $\bar{x}^{(N)}(t)$ denotes the average of state [in (L-F-PI)] while $\hat{\bar{x}}^{(N)}$ denotes the average of filtered states. Note that $\hat{\bar{x}}_i(t)$ is driven by W_i only thus it is observable to the individual agent $\mathcal{A}_i$. It enters the state dynamics (6.23) as an input process when applying the optimal strategy. Some estimates are as follows.

Lemma 6.1.

$$\sup_{0\leqslant t\leqslant T}\mathbb{E}\Big|\hat{\bar{x}}^{(N)}(t) - \mathbb{E}x_0(t)\Big|^2 = O\Big(\frac{1}{N}\Big), \tag{6.24}$$

$$\sup_{0\leqslant t\leqslant T}\mathbb{E}\Big|\bar{x}^{(N)}(t) - x_0(t)\Big|^2 = O\Big(\frac{1}{N}\Big). \tag{6.25}$$

Proof. By (6.20) and (6.23), we have

$$\begin{cases} d\Big(\hat{\bar{x}}^{(N)}(t) - \mathbb{E}x_0(t)\Big) = \Big(A(t) - B^2(t)R^{-1}(t)P(t)\Big)\Big(\hat{\bar{x}}^{(N)}(t) - \mathbb{E}x_0(t)\Big)dt \\ \qquad\qquad\qquad\qquad + \dfrac{1}{N}\sigma(t)\displaystyle\sum_{i=1}^{N}dW_i(t), \\ \hat{\bar{x}}^{(N)}(0) - \mathbb{E}x_0(0) = x^{(N)}(0) - x. \end{cases}$$

Thus

$$\begin{aligned}\Big|\hat{\bar{x}}^{(N)}(t) - \mathbb{E}x_0(t)\Big|^2 \leqslant{}& 3\Big|x^{(N)}(0) - x\Big|^2 + 3\int_0^t \Big|A(s) - B^2(s)R^{-1}(s)P(s)\Big|^2 \\ &\cdot\Big|\hat{\bar{x}}^{(N)}(s) - \mathbb{E}x_0(s)\Big|^2 ds + 3\Big|\int_0^t \frac{1}{N}\sigma(s)\sum_{i=1}^{N}dW_i(s)\Big|^2.\end{aligned}$$

By the independence of $\{W_i(t)\}_{t\geqslant 0}, 1\leqslant i\leqslant N$, we have

$$\mathbb{E}\Big|\int_0^t \frac{1}{N}\sigma(s)\sum_{i=1}^{N}dW_i(s)\Big|^2 = O\Big(\frac{1}{N}\Big).$$

So (6.24) follows by Gronwall's inequality. Combining with (6.24), the assertion (6.25) can be proved in a similar way. $\square$

Denote $y_i, 1 \leqslant i \leqslant N$, the state of $\mathcal{A}_i$ to the control $\bar{u}_i, 1 \leqslant i \leqslant N$ in Problem **(F-PI)**, namely,

$$\begin{cases} dy_i(t) =\Big[A(t)y_i(t) - B^2(t)R^{-1}(t)\Big(P(t)\hat{\bar{x}}_i(t) + \hat{P}(t)\mathbb{E}x_0(t) + \Phi(t)\Big) \\ \qquad\qquad + \alpha y^{(N)}(t) + m(t)\Big]dt + \sigma(t)dW_i(t) + \tilde{\sigma}(t)dW(t), \\ y_i(0) =x_i(0) \end{cases} \tag{6.26}$$

where $y^{(N)}(t) = \frac{1}{N}\sum\limits_{j=1}^{N} y_j(t)$. By the difference of states related to $\bar{u}_i$ in **(F-PI)** and **(L-F-PI)**, we have the following estimates:

Lemma 6.2.

$$\sup_{0\leqslant t\leqslant T} \mathbb{E}\Big|y^{(N)}(t) - x_0(t)\Big|^2 = O\Big(\frac{1}{N}\Big), \tag{6.27}$$

$$\sup_{1\leqslant i\leqslant N}\Big[\sup_{0\leqslant t\leqslant T} \mathbb{E}\Big|y_i(t) - \bar{x}_i(t)\Big|^2\Big] = O\Big(\frac{1}{N}\Big), \tag{6.28}$$

$$\sup_{1\leqslant i\leqslant N}\Big[\sup_{0\leqslant t\leqslant T} \mathbb{E}\Big||y_i(t)|^2 - |\bar{x}_i(t)|^2\Big|\Big] = O\Big(\frac{1}{\sqrt{N}}\Big). \tag{6.29}$$

Proof. By (6.26) and (6.20), the estimate (6.27) can be verified by the same method in Lemma 6.1. According to (6.26) and (6.23), we have

$$\begin{cases} d\Big(y_i(t) - \bar{x}_i(t)\Big) = \Big[A(t)(y_i(t) - \bar{x}_i(t)) + \alpha(y^{(N)}(t) - x_0(t))\Big]dt, \\ y_i(0) - \bar{x}_i(0) = 0. \end{cases}$$

Thus, (6.28) follows from (6.27). Since $\sup\limits_{0\leqslant t\leqslant T} \mathbb{E}|\bar{x}_i(t)|^2 < +\infty$, applying Cauchy-Schwarz inequality, it follows

$$\sup_{0\leqslant t\leqslant T} \mathbb{E}\Big||y_i(t)|^2 - |\bar{x}_i(t)|^2\Big| = \ O\Big(\frac{1}{\sqrt{N}}\Big)$$

which completes the proof. $\square$

As to the difference of cost functionals, it holds

Lemma 6.3. *For* $\forall\ 1 \leqslant i \leqslant N$,

$$\Big|\mathcal{J}_i(\bar{u}_i, \bar{u}_{-i}) - J_i(\bar{u}_i)\Big| = O\Big(\frac{1}{\sqrt{N}}\Big). \tag{6.30}$$

Proof. Similar to the proof of Lemma 6.2, combining with the fact that $\sup_{0 \leqslant t \leqslant T} \mathbb{E}|\bar{x}_i(t) - x_0(t)|^2 < +\infty$, we obtain

$$\sup_{0 \leqslant t \leqslant T} \mathbb{E}\Big||y_i(t) - y^{(N)}(t)|^2 - |\bar{x}_i(t) - x_0(t)|^2\Big| = O\Big(\frac{1}{\sqrt{N}}\Big).$$

Thus,

$$\begin{aligned}
&\Big|\mathcal{J}_i(\bar{u}_i, \bar{u}_{-i}) - J_i(\bar{u}_i)\Big| \\
\leqslant&\mathbb{E}\int_0^T \Big|Q(t)(y_i(t) - y^{(N)}(t))^2 - Q(t)(\bar{x}_i(t) - x_0(t))^2\Big|dt + \mathbb{E}\Big|Gy_i^2(T) - G\bar{x}_i^2(T)\Big| \\
=&O\Big(\frac{1}{\sqrt{N}}\Big).
\end{aligned}$$

The assertion (6.30) follows. $\square$

After addressing the above estimates of states and costs corresponding to control $\bar{u}_i, 1 \leqslant i \leqslant N$, given by (6.22), our goal is to prove that the control strategies set $(\bar{u}_1, \dots, \bar{u}_N)$ is an ϵ-Nash equilibrium for Problem **(F-PI)**. For any fixed i, $1 \leqslant i \leqslant N$, consider an admissible control $u_i \in \mathcal{U}_i$ for $\mathcal{A}_i$ and denote z_i the corresponding state process in Problem **(F-PI)**, that is

$$\begin{cases}
dz_i(t) =\Big[A(t)z_i(t) + B(t)u_i(t) + \alpha z^{(N)}(t) + m(t)\Big]dt + \sigma(t)dW_i(t) \\
\qquad\qquad + \tilde{\sigma}(t)dW(t), \\
z_i(0) =x_i(0)
\end{cases} \tag{6.31}$$

whereas other agents keep the control $\bar{u}_j, 1 \leqslant j \leqslant N, j \neq i$, i.e.,

$$\begin{cases} dz_j(t) =\Big[A(t)z_j(t) - B^2(t)R^{-1}(t)\Big(P(t)\hat{\bar{x}}_j(t) + \hat{P}(t)\mathbb{E}x_0(t) + \Phi(t)\Big) \\ \qquad\qquad + \alpha z^{(N)}(t) + m(t)\Big]dt + \sigma(t)dW_j(t) + \tilde{\sigma}(t)dW(t), \\ z_j(0) =x_j(0) \end{cases} \tag{6.32}$$

where $z^{(N)}(t) = \frac{1}{N}\sum\limits_{j=1}^{N} z_j(t)$ and $\hat{\bar{x}}_j(t)$ is given by (6.23). If $\bar{u}_i,\ 1 \leqslant i \leqslant N$ is an ϵ-Nash equilibrium with respect to cost $\mathcal{J}_i$, it holds that

$$\mathcal{J}_i(\bar{u}_i, \bar{u}_{-i}) \geqslant \inf_{u_i \in \mathcal{U}_i} \mathcal{J}_i(u_i, \bar{u}_{-i}) \geqslant \mathcal{J}_i(\bar{u}_i, \bar{u}_{-i}) - \epsilon.$$

Then, when making the perturbation, we just need to consider $u_i \in \mathcal{U}_i$ such that $\mathcal{J}_i(u_i, \bar{u}_{-i}) \leqslant \mathcal{J}_i(\bar{u}_i, \bar{u}_{-i})$, which implies

$$\mathbb{E}\int_0^T R(t)u_i^2(t)dt \leqslant \mathcal{J}_i(u_i, \bar{u}_{-i}) \leqslant \mathcal{J}_i(\bar{u}_i, \bar{u}_{-i}) = J_i(\bar{u}_i) + O\Big(\frac{1}{\sqrt{N}}\Big).$$

In the limiting cost functional, by the optimality of $(\bar{x}_i, \bar{u}_i)$, we get that $(\bar{x}_i, \bar{u}_i)$ is L^2-bounded. Then we obtain the boundedness of $J_i(\bar{u}_i)$, i.e.,

$$\mathbb{E}\int_0^T R(t)u_i^2(t)dt \leqslant C_1 \tag{6.33}$$

where C_1 is a positive constant, independent of N. Thus we have

Proposition 6.2. *For any fixed $i, 1 \leqslant i \leqslant N$, $\sup\limits_{0\leqslant t\leqslant T} \mathbb{E}|z_i(t)|^2$ is bounded.*

Proof. By (6.31) and (6.32), it holds that

$$\begin{aligned} \mathbb{E}\Big[\sum_{k=1}^{N}|z_k(t)|^2\Big] \leqslant& 4\mathbb{E}\Big[\sum_{k=1}^{N}|x_k(0)|^2\Big] + 4C_2\mathbb{E}\int_0^t \Big[2\sum_{k=1}^{N}|z_k(s)|^2 + |u_i(s)|^2 + \sum_{k=1,k\neq i}^{N}|\bar{u}_k(s)|^2 \\ &+ N|m(s)|^2\Big]ds + 4\sum_{k=1}^{N}\mathbb{E}\Big|\int_0^t \sigma(s)dW_k(s)\Big|^2 + 4N\mathbb{E}\Big|\int_0^t \tilde{\sigma}(s)dW(s)\Big|^2 \end{aligned}$$

where $C_2 := \max\limits_{0\leqslant t\leqslant T}\Big(A^2(t)+B^2(t)\Big)+\alpha^2$. By (6.33), we can see that $\mathbb{E}|u_i(t)|^2$ is bounded. Besides, the optimal controls $\bar{u}_k(t), k\neq i$ are L^2-bounded. Then by Gronwall's inequality, it follows that $\sup\limits_{0\leqslant t\leqslant T}\mathbb{E}\Big[\sum\limits_{k=1}^{N}|z_k(t)|^2\Big]=O(N)$, and $\sup\limits_{0\leqslant t\leqslant T}\mathbb{E}|z_i(t)|^2$ is bounded. $\square$

Correspondingly, the state process $\bar{x}_i^0$ for agent $\mathcal{A}_i$ under control u_i in Problem **(L-F-PI)** satisfies

$$\begin{cases} d\bar{x}_i^0(t) =\Big[A(t)\bar{x}_i^0(t)+B(t)u_i(t)+\alpha x_0(t)+m(t)\Big]dt+\sigma(t)dW_i(t)+\tilde{\sigma}(t)dW(t),\\ \bar{x}_i^0(0) =x_i(0) \end{cases} \tag{6.34}$$

and for agent $\mathcal{A}_j$, $j\neq i$,

$$\begin{cases} d\bar{x}_j(t) =\Big[A(t)\bar{x}_j(t)-B^2(t)R^{-1}(t)\Big(P(t)\hat{\bar{x}}_j(t)+\hat{P}(t)\mathbb{E}x_0(t)+\Phi(t)\Big)+\alpha x_0(t)\\ \qquad\qquad +m(t)\Big]dt+\sigma(t)dW_j(t)+\tilde{\sigma}(t)dW(t),\\ \bar{x}_j(0) =x_j(0) \end{cases} \tag{6.35}$$

where $\hat{\bar{x}}_j$ and x_0 are given in (6.20).

In order to give necessary estimates of perturbed states and costs in Problem **(F-PI)** and **(L-F-PI)**, we introduce some intermediate states and present some of their properties. Denote

$$z^{(N-1)}(t)=\frac{1}{N-1}\sum_{j=1,j\neq i}^{N}z_j(t),\quad \hat{\bar{x}}^{(N-1)}(t)=\frac{1}{N-1}\sum_{j=1,j\neq i}^{N}\hat{\bar{x}}_j(t).$$

Then by (6.32), we have

$$\begin{cases} dz^{(N-1)}(t) =\Big[(A(t)+\frac{N-1}{N}\alpha)z^{(N-1)}(t)-B^2(t)R^{-1}(t)\Big(P(t)\hat{\bar{x}}^{(N-1)}(t)+\hat{P}(t)\mathbb{E}x_0(t)\\ \qquad\qquad +\Phi(t)\Big)+\frac{\alpha}{N}z_i(t)+m(t)\Big]dt+\frac{1}{N-1}\sum\limits_{j=1,j\neq i}^{N}\sigma(t)dW_j(t)+\tilde{\sigma}(t)dW(t),\\ z^{(N-1)}(0) =x^{(N-1)}(0) \end{cases}$$

where $x^{(N-1)}(0) = \frac{1}{N-1}\sum_{j=1,j\neq i}^{N} x_j(0)$. Besides, we introduce

$$
\begin{cases}
d\check{z}_i(t) = & \Big[A(t)\check{z}_i(t) + B(t)u_i(t) + \dfrac{N-1}{N}\alpha\check{z}^{(N-1)}(t) + m(t)\Big]dt + \sigma(t)dW_i(t) \\
& + \tilde{\sigma}(t)dW(t), \\
\check{z}_i(0) = & x_i(0)
\end{cases}
\tag{6.36}
$$

and for $j \neq i$,

$$
\begin{cases}
d\check{z}_j(t) = & \Big[A(t)\check{z}_j(t) - B^2(t)R^{-1}(t)\Big(P(t)\hat{\check{x}}_j(t) + \hat{P}(t)\mathbb{E}x_0(t) + \Phi(t)\Big) \\
& + \dfrac{N-1}{N}\alpha\check{z}^{(N-1)}(t) + m(t)\Big]dt + \sigma(t)dW_j(t) + \tilde{\sigma}(t)dW(t), \\
\check{z}_j(0) = & x_j(0)
\end{cases}
\tag{6.37}
$$

where $\check{z}^{(N-1)}(t) = \frac{1}{N-1}\sum\limits_{j=1,j\neq i}^{N} \check{z}_j(t)$.

We have the following estimates on these states.

Proposition 6.3.

$$\sup_{0\leqslant t\leqslant T} \mathbb{E}\Big|\hat{\check{x}}^{(N-1)}(t) - \mathbb{E}x_0(t)\Big|^2 = O\Big(\frac{1}{N}\Big), \tag{6.38}$$

$$\sup_{0\leqslant t\leqslant T} \mathbb{E}\Big|z^{(N)}(t) - z^{(N-1)}(t)\Big|^2 = O\Big(\frac{1}{N}\Big), \tag{6.39}$$

$$\sup_{0\leqslant t\leqslant T} \mathbb{E}\Big|\check{z}^{(N-1)}(t) - z^{(N-1)}(t)\Big|^2 = O\Big(\frac{1}{N^2}\Big), \tag{6.40}$$

$$\sup_{0\leqslant t\leqslant T} \mathbb{E}\Big|\check{z}^{(N-1)}(t) - x_0(t)\Big|^2 = O\Big(\frac{1}{N}\Big), \tag{6.41}$$

$$\sup_{0\leqslant t\leqslant T} \mathbb{E}\Big|z_i(t) - \check{z}_i(t)\Big|^2 = O\Big(\frac{1}{N^2}\Big), \tag{6.42}$$

$$\sup_{0\leqslant t\leqslant T} \mathbb{E}\Big|\check{z}_i(t) - \bar{x}_i^0(t)\Big|^2 = O\Big(\frac{1}{N}\Big). \tag{6.43}$$

Proof. From (6.37), it follows that

$$\begin{cases} d\check{z}^{(N-1)}(t) = \Big[(A(t) + \dfrac{N-1}{N}\alpha)\check{z}^{(N-1)}(t) - B^2(t)R^{-1}(t)\Big(P(t)\hat{\check{x}}^{(N-1)}(t) \\ \qquad + \hat{P}(t)\mathbb{E}x_0(t) + \Phi(t)\Big) + m(t)\Big]dt + \dfrac{1}{N-1}\displaystyle\sum_{j=1,j\neq i}^{N} \sigma(t)dW_j(t) + \tilde{\sigma}(t)dW(t), \\ \check{z}^{(N-1)}(0) = x^{(N-1)}(0). \end{cases} \tag{6.44}$$

Then we have

$$\begin{cases} d\Big(\check{z}^{(N-1)}(t) - z^{(N-1)}(t)\Big) = \Big[(A(t) + \dfrac{N-1}{N}\alpha)\Big(\check{z}^{(N-1)}(t) - z^{(N-1)}(t)\Big) - \dfrac{\alpha}{N}z_i(t)\Big]dt, \\ \check{z}^{(N-1)}(0) - z^{(N-1)}(0) = 0. \end{cases}$$

By the L^2-boundness of $z_i(t)$ and Gronwall's inequality, the assertions (6.39) and (6.40) hold. And by (6.24), we can get (6.38). Besides, it follows that

$$\sup_{0\leqslant t\leqslant T} \mathbb{E}\Big|\check{z}^{(N-1)}(t) - x_0(t)\Big|^2 = O\Big(\frac{1}{N}\Big)$$

from (6.44), (6.20) and (6.38). Then (6.34), (6.36) and (6.41) imply that

$$\sup_{0\leqslant t\leqslant T} \mathbb{E}\Big|\check{z}_i(t) - \bar{x}_i^0(t)\Big|^2 = O\Big(\frac{1}{N}\Big).$$

Finally, by Proposition 6.2, we easily get (6.42). □

Further, more direct estimates about states and costs of Problem **(F-PI)** and **(L-F-PI)** under perturbed controls can be obtained, which enable us to prove Theorem 6.2.

Lemma 6.4.

$$\sup_{0\leqslant t\leqslant T}\mathbb{E}\Big|z_i(t)-\bar{x}_i^0(t)\Big|^2=O\Big(\frac{1}{N}\Big), \tag{6.45}$$

$$\sup_{0\leqslant t\leqslant T}\mathbb{E}\Big|z^{(N)}(t)-x_0(t)\Big|^2=O\Big(\frac{1}{N}\Big), \tag{6.46}$$

$$\sup_{0\leqslant t\leqslant T}\mathbb{E}\Big||z_i(t)|^2-|\bar{x}_i^0(t)|^2\Big|=O\Big(\frac{1}{\sqrt{N}}\Big), \tag{6.47}$$

$$\Big|\mathcal{J}_i(u_i,\bar{u}_{-i})-J_i(u_i)\Big|=O\Big(\frac{1}{\sqrt{N}}\Big). \tag{6.48}$$

Proof. (6.45) and (6.46) follow from Proposition 6.3 directly. By Proposition 6.2, we get that both $\sup_{0\leqslant t\leqslant T}\mathbb{E}|\bar{x}_i^0(t)|^2$ and $\sup_{0\leqslant t\leqslant T}\mathbb{E}|\bar{x}_i^0(t)-x_0(t)|^2$ are bounded. Similar to the proof of Lemma 6.2, (6.47) holds. Besides,

$$\sup_{0\leqslant t\leqslant T}\mathbb{E}\Big||z_i(t)-z^{(N)}(t)|^2-|\bar{x}_i^0(t)-x_0(t)|^2\Big|=\quad O\Big(\frac{1}{\sqrt{N}}\Big),$$

then

$$\begin{aligned}&\Big|\mathcal{J}_i(u_i,\bar{u}_{-i})-J_i(u_i)\Big|\\ \leqslant&\mathbb{E}\int_0^T\Big|Q(t)(z_i(t)-z^{(N)}(t))^2-Q(t)(\bar{x}_i^0(t)-x_0(t))^2\Big|dt+\mathbb{E}\Big|Gz_i^2(T)-G(\bar{x}_i^0(T))^2\Big|\\ =&O\Big(\frac{1}{\sqrt{N}}\Big),\end{aligned}$$

which implies (6.48). □

Proof of Theorem 6.2: Consider the ϵ-Nash equilibrium for $\mathcal{A}_i$. Combining Lemma 6.3 and 6.4, we have

$$\begin{aligned}\mathcal{J}_i(\bar{u}_i,\bar{u}_{-i})&=J_i(\bar{u}_i)+O\Big(\frac{1}{\sqrt{N}}\Big)\\ &\leqslant J_i(u_i)+O\Big(\frac{1}{\sqrt{N}}\Big)\\ &=\mathcal{J}_i(u_i,\bar{u}_{-i})+O\Big(\frac{1}{\sqrt{N}}\Big).\end{aligned}$$

Thus, Theorem 6.2 follows by taking $\epsilon = O\Big(\frac{1}{\sqrt{N}}\Big)$. $\Box$

6.3 (B-PI): Backward MFGs with Partial Information

Now, we turn to study the backward MFGs with partial information **(B-PI)**. Similar to above, we need also introduce and study the associated MFGs via limiting state average.

6.3.1 The limiting control of (L-B-PI)

Considering the large-population system with partial information structure, suppose the feedback control for $\mathcal{A}_i$ takes the following feedback form on the state filters

$$u_i(t) = -\,a(t)\mathbb{E}\big(y_i(t)|\mathcal{F}_t^{w_i}\big) + \sum_{j=1,j\neq i}^{N} \tilde{a}(t)\mathbb{E}\big(y_j(t)|\mathcal{F}_t^{w_i}\big) + b(t) \tag{6.49}$$

where the regulator coefficients $a(\cdot), \tilde{a}(\cdot), b(\cdot) \in L^2(0,T;\mathbb{R})$ and $\tilde{a}(\cdot) = O(\frac{1}{N})$. Inserting (6.49) into the state equation in (6.3), we have

$$\begin{aligned}-dy_i(t) =& \Big[Ay_i(t) - Ba(t)\mathbb{E}\big(y_i(t)|\mathcal{F}_t^{w_i}\big) + B\tilde{a}(t)\sum_{j=1,j\neq i}^{N}\mathbb{E}\big(y_j(t)|\mathcal{F}_t^{w_i}\big) + Bb(t)\Big]dt\\ &- z_i(t)dW_i(t) - \tilde{z}_i(t)dW(t),\ 1 \leqslant i \leqslant N.\end{aligned} \tag{6.50}$$

Then consider the state average, we get

$$\begin{aligned}-\,d\bigg(\frac{1}{N}\sum_{i=1}^{N}y_i(t)\bigg) =& \bigg[A\frac{1}{N}\sum_{i=1}^{N}y_i(t) - Ba(t)\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}(y_i(t)|\mathcal{F}_t^{w_i}) + Bb(t)\\ &+ B\tilde{a}(t)\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}\mathbb{E}(y_j(t)|\mathcal{F}_t^{w_i})\bigg]dt - \frac{1}{N}\sum_{i=1}^{N}z_i(t)dW_i(t) - \frac{1}{N}\sum_{i=1}^{N}\tilde{z}_i(t)dW(t).\end{aligned}$$

Thus, we assume there exists a limiting process $(y^*(t), z^*(t))$, which satisfies the following BSDE

$$\begin{cases} -dy^*(t) = \big[Ay^*(t) + \tilde{B}(t)\mathbb{E}y^*(t) + r(t)\big]dt - z^*(t)dW(t), \\ y^*(T) = \eta \end{cases} \tag{6.51}$$

where $\eta \in \mathcal{F}_T^w$ is obtained by (H6.2), $\tilde{B}(\cdot)$ and $r(\cdot) \in L^2(0,T;\mathbb{R})$ are to be determined.

Now, we introduce the limiting partial-information system

$$\begin{cases} -dy_i(t) =[Ay_i(t) + Bu_i(t)]dt - z_i(t)dW_i(t) - \tilde{z}_i(t)dW(t), \\ y_i(T) =\eta_i \end{cases} \tag{6.52}$$

with the cost functional

$$J_i(u_i) = \mathbb{E}\left[\int_0^T Ru_i^2(t)dt + 2y_i(0)\big(\alpha - \beta y^*(0)\big)\right] \tag{6.53}$$

where $y^*(\cdot)$ is given by (6.51).

Now, we formulate the limiting backward partial information (L-B-PI) games.

Problem (L-B-PI). For the i^{th} agent, $i = 1, 2, \ldots, N$, find $\hat{u}_i \in \mathcal{V}_i$ satisfying

$$J_i(\hat{u}_i) = \inf_{u_i \in \mathcal{V}_i} J_i(u_i).$$

Then $\hat{u}_i$ is called an optimal control of problem **(L-B-PI)**. Further we have

Proposition 6.4. *Let* (H6.2) *hold. Then the optimal control of* **(L-B-PI)** *is*

$$\hat{u}_i(t) = -R^{-1}Bh_i(t)$$

where $h_i(t) \in L^2(0,T;\mathbb{R})$ *satisfies the following ODE:*

$$\begin{cases} dh_i(t) = Ah_i(t)dt, \\ h_i(0) = \alpha - \beta y^*(0), \ i = 1, 2, \ldots, N. \end{cases} \tag{6.54}$$

6.3.2 The explicit representation

For $\forall\ 1 \leqslant i \leqslant N$, solving ODE (6.54) directly, we have

$$h_i(t) = \big(\alpha - \beta y^*(0)\big)e^{At}.$$

Thus, the optimal control $\hat{u}_i(t)$ is given by

$$\hat{u}_i(t) = -R^{-1}B\big(\alpha - \beta y^*(0)\big)e^{At}. \tag{6.55}$$

Applying the control law (6.55) for i^{th} agent $\mathcal{A}_i$, the closed-loop system (6.3) becomes

$$\begin{cases} -dy_i(t) = \Big[Ay_i(t) - B^2R^{-1}\big(\alpha - \beta y^*(0)\big)e^{At}\Big]dt - z_i(t)dW_i(t) - \tilde{z}_i(t)dW(t), \\ y_i(T) = \eta_i. \end{cases} \tag{6.56}$$

Summing the above N equations of (6.56) and dividing by N, we get

$$\begin{cases} -dy^{(N)}(t) = \Big[Ay^{(N)}(t) - B^2R^{-1}\big(\alpha - \beta y^*(0)\big)e^{At}\Big]dt - \dfrac{1}{N}\displaystyle\sum_{i=1}^{N} z_i(t)dW_i(t) \\ \qquad\qquad - \dfrac{1}{N}\displaystyle\sum_{i=1}^{N} \tilde{z}_i(t)dW(t), \\ y^{(N)}(T) = \eta^{(N)} \end{cases} \tag{6.57}$$

where $\eta^{(N)} = \frac{1}{N}\sum\limits_{i=1}^{N}\eta_i$. Taking $N \to +\infty$ and noting (6.51), we have $\tilde{B}(t) \equiv 0$ and

$$r(t) = -B^2R^{-1}\big(\alpha - \beta y^*(0)\big)e^{At}. \tag{6.58}$$

Then we rewrite (6.51) as

$$\begin{cases} -dy^*(t) = \big[Ay^*(t) + r(t)\big]dt - z^*(t)dW(t), \\ y^*(T) = \eta. \end{cases} \tag{6.59}$$

Taking expectation and solving the corresponding backward ODE, we get

$$\mathbb{E}y^*(t) = \eta_0 e^{A(T-t)} + \int_t^T r(s)e^{A(s-t)}ds$$

where $\eta_0 := \mathbb{E}\eta$. Thus,

$$y^*(0) = \mathbb{E}y^*(0) = \eta_0 e^{AT} + \int_0^T r(s)e^{As}ds.$$

Further we have

$$r(t) = -B^2R^{-1}e^{At}\bigg\{\alpha - \beta\Big[\eta_0 e^{AT} + \int_0^T r(s)e^{As}ds\Big]\bigg\}.$$

Then we have the following proposition.

Proposition 6.5. *$r(\cdot)$ can be explicitly solved as*

$$r(t) = \begin{cases} -\frac{2AB^2e^{At}\left(\alpha-\beta\eta_0e^{AT}\right)}{2AR-B^2\beta\left(e^{2AT}-1\right)}, & \textit{if } A \neq 0,\ 2AR - B^2\beta\big(e^{2AT}-1\big) \neq 0; \\ 0, & \textit{if } A \neq 0,\ 2AR - B^2\beta\big(e^{2AT}-1\big) = 0; \\ -\frac{B^2(\alpha-\beta\eta_0)}{R-B^2\beta T}, & \textit{if } A = 0,\ R - B^2\beta T \neq 0; \\ 0, & \textit{if } A = 0,\ R - B^2\beta T = 0. \end{cases} \tag{6.60}$$

Moreover, $y^*(\cdot)$ in (6.59) *can be determined based on $r(\cdot)$.*

Proof. The proof is similar to that of Proposition 3.2 and omitted. □

Remark 6.4. *By Proposition 6.5 it follows that there exists a unique bounded continuous function $r(\cdot)$. Then* (6.59) *admits a unique solution $\big(y^*(\cdot), z^*(\cdot)\big)$, in which $y^*(\cdot)$ is approximated by the state average $y^{(N)}$. Applying $y^*(\cdot)$, we get the optimal control for* **(L-B-PI)***, which is important to analyze the properties of ϵ-Nash equilibrium.*

6.3.3 ϵ-Nash equilibrium for (B-PI)

In this section, we analyze the asymptotic property of the decentralized control strategies and verify the ϵ-Nash equilibrium property for **(B-PI)**. To begin with, we state the main result.

Theorem 6.3. *Let* (H6.2) *hold. Then the strategy set $(\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_N)$ satisfies the ϵ-Nash equilibrium of* **(B-PI)***, with ϵ is of order $1/\sqrt{N}$.*

Let y_i denote the state process corresponding to $\hat{u}_i$ for **(B-PI)**, $\hat{y}_i$ denote the state process corresponding to $\hat{u}_i$ for **(L-B-PI)**. Note that in partial information structure, state average is coupled in cost only therefore applying $\hat{u}_i$, y_i is same to $\hat{y}_i,\ i=1,2,\ldots,N$.

Lemma 6.5.

$$\sup_{0\leqslant t\leqslant T}\mathbb{E}\Big|y^{(N)}(t)-y^*(t)\Big|^2=O\Big(\frac{1}{N}\Big),\tag{6.61}$$

$$\Big|\mathcal{J}_i(\hat{u}_i,\hat{u}_{-i})-J_i(\hat{u}_i)\Big|=O\Big(\frac{1}{\sqrt{N}}\Big),\ \forall\ 1\leqslant i\leqslant N.\tag{6.62}$$

Proof. By (6.57) and (6.59), we have

$$\begin{cases}-d\big(y^{(N)}(t)-y^*(t)\big)=A\big[y^{(N)}(t)-y^*(t)\big]dt-\dfrac{1}{N}\displaystyle\sum_{i=1}^{N}z_i(t)dW_i(t)+\left[z^*(t)-\dfrac{1}{N}\displaystyle\sum_{i=1}^{N}\tilde{z}_i(t)\right]dW(t),\\[2ex] y^{(N)}(T)-y^*(T)=\eta^{(N)}-\eta.\end{cases}\tag{6.63}$$

Introducing a 1-dimensional dual process $X(s,t)$, which satisfies

$$\begin{cases}dX(s,t)=AX(s,t)ds,\\ X(t,t)=1,\ t\leqslant s\leqslant T,\end{cases}$$

and applying Itô's formula, we get

$$y^{(N)}(t)-y^*(t)=X(T,t)\mathbb{E}\big(\eta^{(N)}-\eta\big|\mathcal{G}_t\big).$$

It is easy to obtain that

$$\mathbb{E}\Big|\eta^{(N)}-\eta\Big|^2=\frac{1}{N^2}\sum_{i=1}^{N}\mathbb{E}\Big|\eta_i-\eta\Big|^2+\frac{2}{N^2}\sum_{i<j}\mathbb{E}\big(\eta_i-\eta\big)\big(\eta_j-\eta\big).$$

Since $\mathbb{E}\Big|\eta_i-\eta\Big|^2<+\infty$, we have $\frac{1}{N^2}\sum_{i=1}^{N}\mathbb{E}\Big|\eta_i-\eta\Big|^2=O\Big(\frac{1}{N}\Big)$. Besides, it follows that

$$\mathbb{E}\big(\eta_i-\eta\big)\big(\eta_j-\eta\big)=\mathbb{E}\Big[\mathbb{E}\big[\big(\eta_i-\eta\big)\big(\eta_j-\eta\big)\big|\mathcal{F}_T^w\big]\Big]=\mathbb{E}\Big[\mathbb{E}\big[\eta_i\eta_j\big|\mathcal{F}_T^w\big]-\eta^2\Big].$$

Under (H6.2), applying the results of [114, 115, 116, 117], we can derive that

$$\mathbb{E}\big[\eta_i\eta_j\big|\mathcal{F}_T^w\big] = \mathbb{E}\big[\eta_i\big|\mathcal{F}_T^w\big]\mathbb{E}\big[\eta_j\big|\mathcal{F}_T^w\big] = \eta^2.$$

Thus, $\mathbb{E}\Big|\eta^{(N)}-\eta\Big|^2 = O\Big(\frac{1}{N}\Big)$ and (6.61) follows. In addition, note that the state equation of **(B-PI)** coincides with its limiting equation (6.52), since the state equation in (6.3) does not contain the state-average term $y^{(N)}$. Therefore, after applying the optimal control $\hat{u}_i$ in (6.55), we get that $y_i = \hat{y}_i$ P-$a.s.$. Thus, we have

$$\Big|\mathcal{J}_i(\hat{u}_i,\hat{u}_{-i}) - J_i(\hat{u}_i)\Big| \leqslant 2\beta\mathbb{E}\Big[\big|\hat{y}_i(0)\big|\big|y^{(N)}(0) - y^*(0)\big|\Big] = O\Big(\frac{1}{\sqrt{N}}\Big)$$

where the last equality follows by Hölder's inequality and (6.61). $\square$

For any fixed i, $1 \leqslant i \leqslant N$, consider an admissible alternative control $u_i \in \mathcal{V}_i$ for the i^{th} agent $\mathcal{A}_i$ and denote the corresponding state as

$$\begin{cases} -dk_i(t) = \big[Ak_i(t) + Bu_i(t)\big]dt - n_i(t)dW_i(t) - \tilde{n}_i(t)dW(t), \\ \quad k_i(T) = \eta_i \end{cases} \tag{6.64}$$

while all other agents keep the control $\hat{u}_j, 1 \leqslant j \leqslant N, j \neq i$, i.e.,

$$\begin{cases} -dk_j(t) =\Big[Ak_j(t) - B^2R^{-1}\big(\alpha - \beta y^*(0)\big)e^{At}\Big]dt - n_j(t)dW_j(t) - \tilde{n}_j(t)dW(t), \\ \quad k_j(T) =\eta_j \end{cases} \tag{6.65}$$

with the cost functional

$$\mathcal{J}_i\big(u_i(\cdot),\hat{u}_{-i}(\cdot)\big) = \mathbb{E}\left[\int_0^T Ru_i^2(t)dt + 2k_i(0)\big(\alpha - \beta k^{(N)}(0)\big)\right] \tag{6.66}$$

where $k^{(N)}(t) = \frac{1}{N}\sum_{j=1}^{N} k_j(t)$, $k^{(N)}(0) = \frac{1}{N}\sum_{j=1}^{N} k_j(0)$ is its initial value.

If $\hat{u}_i,\ 1 \leqslant i \leqslant N$ is an ϵ-Nash equilibrium with respect to the cost $\mathcal{J}_i$, we have

$$\mathcal{J}_i(\hat{u}_i,\hat{u}_{-i}) \geqslant \inf_{u_i\in\mathcal{V}_i} \mathcal{J}_i(u_i,\hat{u}_{-i}) \geqslant \mathcal{J}_i(\hat{u}_i,\hat{u}_{-i}) - \epsilon.$$

Then, when making the perturbation, we just need to consider $u_i \in \mathcal{V}_i$ such that $\mathcal{J}_i(u_i, \hat{u}_{-i}) \leqslant \mathcal{J}_i(\hat{u}_i, \hat{u}_{-i})$. Besides, by (6.64) (6.65) and applying the estimates of BSDE, we obtain the L^2 boundness of $k_j, j \neq i$ and the following inequality

$$\sup_{0\leqslant t\leqslant T} \mathbb{E}|k_i(t)|^2 \leqslant C_3\Big[1 + \mathbb{E}\int_0^T |u_i(s)|^2 ds\Big]$$

where C_3 is a positive constant. For the cost functional (6.66), we have

Proposition 6.6. *$\mathcal{J}_i\big(u_i(\cdot), \hat{u}_{-i}(\cdot)\big)$ is strictly convex and coercive with respect to $u_i(\cdot)$, if N is large enough. Specially, a bound is given as $N > \frac{4\beta T}{R} e^{2|A|T}$.*

Proof. (6.66) can be rewritten as

$$\mathcal{J}_i\big(u_i(\cdot), \hat{u}_{-i}(\cdot)\big) = \mathbb{E}\left[\int_0^T R u_i^2(t)dt + k_i(0)\Big(2\alpha - \frac{2\beta}{N}\sum_{j=1, j\neq i}^{N} k_j(0)\Big) - \frac{2\beta}{N} k_i^2(0)\right].$$

Applying the dual method of BSDE, (6.64) satisfies

$$k_i(0) = e^{AT}\eta_0 + \mathbb{E}\int_0^T e^{As} u_i(s)ds.$$

Plugging $k_i(0)$ into $\mathcal{J}_i\big(u_i(\cdot), \hat{u}_{-i}(\cdot)\big)$, we obtain

$$\begin{aligned}
\mathcal{J}_i\big(u_i(\cdot), \hat{u}_{-i}(\cdot)\big) =& \mathbb{E}\int_0^T R u_i^2(t)dt - \frac{2\beta}{N}\Big(\mathbb{E}\int_0^T e^{At} u_i(t)dt\Big)^2 \\
&+ \Big(2\alpha - \frac{2\beta}{N}\sum_{j=1, j\neq i}^{N} k_j(0) - \frac{4\beta}{N} e^{AT}\eta_0\Big)\mathbb{E}\int_0^T e^{At} u_i(t)dt \\
&+ \Big(2\alpha - \frac{2\beta}{N}\sum_{j=1, j\neq i}^{N} k_j(0)\Big) e^{AT}\eta_0 - \frac{2\beta}{N} e^{2AT}\eta_0^2.
\end{aligned}$$

To prove the strict convexity of $\mathcal{J}_i$, we consider $u_i(\cdot), v_i(\cdot) \in \mathcal{V}_i$ such that $(\texttt{Leb} \otimes P)(\Omega_0) > 0$ and $\lambda \in (0,1)$. Here, Ω_0 is defined as $\Omega_0 = \big\{(t,\omega) \in [0,T] \times \Omega \big| u_i(t,\omega) \neq$

$v_i(t,\omega)\big\}$. Then by Hölder's inequality, we have

$$
\begin{aligned}
&\mathcal{J}_i\big(\lambda u_i + (1-\lambda)v_i\big) - \lambda \mathcal{J}_i(u_i) - (1-\lambda)\mathcal{J}_i(v_i)\\
=&\lambda(1-\lambda)\Big\{ - R\,\mathbb{E}\int_0^T \big(u_i(t)-v_i(t)\big)^2 dt + \frac{2\beta}{N}\Big(\mathbb{E}\int_0^T e^{At}\big(u_i(t)-v_i(t)\big)dt\Big)^2\Big\}\\
\leqslant&\lambda(1-\lambda)\Big(- R + \frac{2\beta T}{N}e^{2|A|T}\Big)\mathbb{E}\int_0^T \big(u_i(t)-v_i(t)\big)^2 dt.
\end{aligned}
$$

If N is large enough, specially $N > \frac{4\beta T}{R}e^{2|A|T}$, $\mathcal{J}_i\big(\lambda u_i + (1-\lambda)v_i\big) < \lambda\mathcal{J}_i(u_i) + (1-\lambda)\mathcal{J}_i(v_i)$. The strict convexity of $\mathcal{J}_i$ is obtained. Similarly, if $N > \frac{4\beta T}{R}e^{2|A|T}$, we get

$$
\begin{aligned}
\mathcal{J}_i\big(u_i(\cdot),\hat{u}_{-i}(\cdot)\big) \geqslant&\frac{R}{2}\mathbb{E}\int_0^T u_i^2(t)dt - \Big|2\alpha - \frac{2\beta}{N}\sum_{j=1,j\neq i}^N k_j(0) - \frac{4\beta}{N}e^{AT}\eta_0\Big|e^{|A|T}T^{\frac{1}{2}}\\
&\cdot\Big(\mathbb{E}\int_0^T u_i^2(t)dt\Big)^{\frac{1}{2}} + \Big(2\alpha - \frac{2\beta}{N}\sum_{j=1,j\neq i}^N k_j(0)\Big)e^{AT}\eta_0 - \frac{2\beta}{N}e^{2AT}\eta_0^2.
\end{aligned}
$$

Then $\mathcal{J}_i\big(u_i(\cdot),\hat{u}_{-i}(\cdot)\big)$ tends to $+\infty$ as $\mathbb{E}\int_0^T u_i^2(t)dt \to +\infty$. Hence the coercive property. $\square$

It follows from Lemma 6.5 that $\mathcal{J}_i(\hat{u}_i,\hat{u}_{-i})(= J_i(\hat{u}_i) + O\big(\frac{1}{\sqrt{N}}\big))$ is bounded by noting $\hat{u}_i$ is already optimal for J_i. Therefore, by Proposition 6.6, when making the perturbation we need only consider the control u_i which is L^2 bounded, otherwise $\mathcal{J}_i\big(u_i,\hat{u}_{-i}\big)$ will tend to $+\infty$. Thus we have

$$
\mathbb{E}\int_0^T u_i^2(t)dt \leqslant C_4 \tag{6.67}
$$

where C_4 is a positive constant which is independent of N. Further, we can get the boundness of $\sup\limits_{0\leqslant t\leqslant T}\mathbb{E}|k_i(t)|^2$.

Remark 6.5. *Note that C_4 is independent of N. Actually, we should first get $\mathbb{E}\int_0^T u_i^2(t)dt \leqslant \tilde{C}_4$ for some $\tilde{C}_4$ containing the terms $\frac{1}{N}$ and $\frac{1}{\sqrt{N}}$ due to the terms*

$k^{(N)}$ and $O\Big(\frac{1}{\sqrt{N}}\Big)$. However, they will vanish in asymptotic sense as N is large enough. Thus, we can obtain some C_4, which is independent of N.

For the i^{th} agent $\mathcal{A}_i$, consider the perturbation in **(L-B-PI)** and introduce some auxiliary system

$$\begin{cases} -dk_i^0(t) = \big[Ak_i^0(t) + Bu_i(t)\big]dt - n_i^0(t)dW_i(t) - \tilde{n}_i^0(t)dW(t), \\ \quad k_i^0(T) = \eta_i \end{cases} \tag{6.68}$$

and for $j \neq i$,

$$\begin{cases} -d\hat{k}_j(t) =\Big[A\hat{k}_j(t) - B^2R^{-1}\big(\alpha - \beta y^*(0)\big)e^{At}\Big]dt - \hat{n}_j(t)dW_j(t) - \hat{\tilde{n}}_j(t)dW(t), \\ \quad \hat{k}_j(T) =\eta_j \end{cases} \tag{6.69}$$

with the cost functional

$$J_i\big(u_i(\cdot)\big) = \mathbb{E}\left[\int_0^T Ru_i^2(t)dt + 2k_i^0(0)\big(\alpha - \beta y^*(0)\big)\right]. \tag{6.70}$$

Noting (6.64) and (6.68), we can see that $(k_i, n_i, \tilde{n}_i)$ is same to $(k_i^0, n_i^0, \tilde{n}_i^0)$. Besides, by (6.64) and (6.65), we have

$$\begin{cases} -dk^{(N)}(t) = \left[Ak^{(N)}(t) + \dfrac{B}{N}\Big(u_i(t) + \displaystyle\sum_{j=1,j\neq i}^N \hat{u}_j(t)\Big)\right]dt - \dfrac{1}{N}\displaystyle\sum_{j=1}^N n_j(t)dW_j(t) \\ \qquad\qquad\qquad - \dfrac{1}{N}\displaystyle\sum_{j=1}^N \tilde{n}_j(t)dW(t), \\ \quad k^{(N)}(T) =\eta^{(N)} \end{cases} \tag{6.71}$$

where $\hat{u}_j(t) = -BR^{-1}\big(\alpha - \beta y^*(0)\big)e^{At},\ j \neq i$. Then we have the following lemma.

Lemma 6.6.

$$\sup_{0\leqslant t\leqslant T} \mathbb{E}\Big|k^{(N)}(t) - y^*(t)\Big|^2 = O\Big(\frac{1}{N}\Big), \tag{6.72}$$

$$\Big|\mathcal{J}_i(u_i, \hat{u}_{-i}) - J_i(u_i)\Big| = O\Big(\frac{1}{\sqrt{N}}\Big). \tag{6.73}$$

Proof. By (6.59) and (6.71), we have

$$\begin{cases} -d\big(k^{(N)}(t)-y^*(t)\big)=\Big[A\big(k^{(N)}(t)-y^*(t)\big)+\dfrac{1}{N}\big(Bu_i(t)-r(t)\big)\Big]-\dfrac{1}{N}\displaystyle\sum_{j=1}^{N}n_j(t)dW_j(t) \\ \qquad\qquad\qquad\qquad +\Big[z^*(t)-\dfrac{1}{N}\displaystyle\sum_{j=1}^{N}\tilde{n}_j(t)\Big]dW(t), \\ k^{(N)}(T)-y^*(T)=\eta^{(N)}-\eta. \end{cases}$$

Noting (6.67) (6.58) and applying the estimates of BSDE, we obtain (6.72). Thus, we have

$$\Big|\mathcal{J}_i(u_i,\hat{u}_{-i})-J_i(u_i)\Big|\leqslant 2\beta\mathbb{E}\Big[\big|k_i(0)\big|\big|k^{(N)}(0)-y^*(0)\big|\Big]=O\Big(\frac{1}{\sqrt{N}}\Big)$$

which completes the proof. □

Proof of Theorem 6.3: Consider the ϵ-Nash equilibrium of $\mathcal{A}_i$ for **(B-PI)**. Combining Lemma 6.5 and 6.6, we have

$$\begin{aligned}\mathcal{J}_i(\hat{u}_i,\hat{u}_{-i})=J_i(\hat{u}_i)+O\Big(\frac{1}{\sqrt{N}}\Big)\leqslant& J_i(u_i)+O\Big(\frac{1}{\sqrt{N}}\Big)\\ =&\mathcal{J}_i(u_i,\hat{u}_{-i})+O\Big(\frac{1}{\sqrt{N}}\Big).\end{aligned}$$

Thus, Theorem 6.3 follows by taking $\epsilon=O\Big(\frac{1}{\sqrt{N}}\Big)$. □

6.3.4 Extensions

Now, we present some possible extensions based on our previous analysis. The first extension is to consider the following cost functional

$$\mathcal{J}_i^1\big(u_i,u_{-i}\big)=\mathbb{E}\left[\int_0^T Ru_i^2(t)dt+2y_i(0)\Big(\alpha+\frac{\beta}{y^{(N)}(0)}\Big)\right] \tag{6.74}$$

where α,β are nonnegative constants. Such cost functional characterizes the so-called bench-mark performance criteria in investment. To be more precise, suppose there has a large-population system which consists of considerable small investors who aim to achieve (or, hedge) some terminal targets η_i by portfolio selection. The term $y^{(N)}(0)$ denotes the average hedging cost for all investors while $\frac{y_i(0)}{y^{(N)}(0)}$ denotes the relative hedging costs for i^{th} investor, and β denotes its weight. In case $\beta = 0$, it is reduced to the classical individual own performance. In case $\beta > 0$, the investor should get some balance between its own individual performance and the average population performance. In other words, the investor aims to minimize its initial hedging cost by taking account of the average cost of the whole market participants. In this case, we aim to minimize the weighted cost functional $\mathcal{J}_i^1$.

Another extension is to consider the so-called convex portfolio selection. In this case, the given individual investor will take into account their relative performance by comparison to their peers in convex combination. In accordance with [113], in which the security writers aim to maximize the utility function of terminal wealth. Here, we aim to minimize the following initial hedging cost

$$\mathcal{J}_i^2\big(u_i, u_{-i}\big) = \mathbb{E}\left[\frac{1}{2}\int_0^T Ru_i^2(t)dt + (1-\lambda)y_i(0) + \lambda\Big(y_i(0) - y^{(N)}(0)\Big)\right] \tag{6.75}$$

where $\lambda \in [0,1]$ is the parameter of relative interest.

For above two extensions, following the similar arguments to our previous analysis, we can get the corresponding optimal decentralized controls as

$$\begin{cases} \bar{u}_i^1(t) = -R^{-1}B\Big(\alpha + \dfrac{\beta}{y^1(0)}\Big)e^{At}, \\ \bar{u}_i^2(t) = -R^{-1}Be^{At} \end{cases} \tag{6.76}$$

where $y^1(0)$ is the initial value of the limiting process of state average. Besides, the fixed points principle and the ϵ-Nash equilibrium properties for $\mathcal{J}_i^1, \mathcal{J}_i^2$ are obtained respectively. Since there are some other financial models in the form of large-population with partial information structure, our theoretical results may have potential applications in finance and economics.

Chapter 7

Conclusions and Future Work

This chapter draws conclusions on the thesis, and points out some possible research directions related to the work done in this thesis.

7.1 Conclusions

The focus of the thesis has been placed on the LQMFGs of FBSDE sysems. Specifically, five research problems have been investigated in detail.

1. The large-population LQ games with forward-backward structure are discussed. Unlike the forward case, the consistency conditions of the forward-backward MFGs involve six Riccati and force rate equations. The decentralized control is derived based on the consistency conditions. The ϵ-Nash equilibrium property is also verified with the help of the estimates of forward-backward stochastic systems.

2. The backward LQMFGs are introduced. Different to the well-studied forward LQMFGs, the terminal conditions of individual players are specified here as a priori and as a result, the decentralized control and consistency condition are determined in backward manner. The ϵ-Nash equilibrium is verified using the estimates of BSDE and its limiting equation.

3. The LQMFGs with major and minor agents but in backward-forward setup are studied. The state dynamics of major agent satisfies some BSDE while the minor agents are modeled by some SDEs. To derive the decentralized strategies, the MFG is formulated in backward-forward and major-minor framework. An auxiliary MF SDE and a mixed BFSDE are thus introduced and analyzed. The consistency condition is not directly analyzed via the fixed-point analysis and contraction mapping. Instead, it is connected to the well-posedness of the mixed BFSDE system and is obtained under some weak monotonic conditions. The decentralized strategies are also verified to satisfy the ϵ-Nash equilibrium property. For this purpose, some estimates to BFSDE is applied.

4. The combination problems of leader-follower and major-minor systems involving large-population are investigated. The frameworks and processing methods are mainly presented in three different manners. For "Serial-Parallel Coupling"–Case I, the optimization problems of followers are solved firstly, and then a classic major-minor problem. For "Serial-Parallel Coupling"–Case II, the major-leader imposes some direct impacts to the followers, and the corresponding variation method is different to the first one. As to "Serial Coupling", the problem is investigated in the "anticipating" manner and solved from back to front. In all the topics, the agents track different convex combinations of the centroid and dynamics of agents, and three consistency condition systems are obtained.

5. The dynamic optimization of large-population systems with partial information is considered. Due to the information structure, the state-average limit in this setup turns out to be some stochastic process driven by the common Brownian motion. The large-population systems are driven by SDEs and BSDEs. The associated MFGs are formulated and studied. In addition, the decentralized

strategies and the ϵ-Nash equilibrium properties are presented.

7.2 Future Work

Related topics for the future research work are listed below.

1. The large-population dynamic optimization problems investigated in Chapter 2 are driven by partially-coupled FBSDEs. One possible direction is to investigate the *fully-coupled* forward-backward LQMFGs for more theoretical results where the forward dynamics also involves the backward one. In the future, seeking for the auxiliary systems and decentralized strategies is also challenging if the *backward* state-average is involved.

2. In Chapter 3, the individual agents of large-population system are only weakly coupled in their state dynamics. It suggests to include the first solution component $y_i(\cdot)$ and its average $y^{(N)}(\cdot)$ into the running cost to be minimized. This brings additional technical difficulty as the decoupling method via Riccati equation is not workable for backward setup and the explicit solution can't be obtained (because the adjoint equation becomes a SDE). Another direction is to introduce the second component $z_i(\cdot)$ into the state or cost functional. It is worth discussing them in the future work.

3. As to major-minor problem, in the future, one possible direction is that state-average appears in dynamics of major player, which may bring lots of trouble in proving the ϵ-Nash equilibrium property. Wellposedness of the corresponding 3×2 mixed FBSDE system is also worth considering. Another direction is that dynamics of minor players are formulated by BSDEs. In this case, the consistent condition analysis may be more complicated and some technical difficulties may arise.

4. In Chapter 5, three consistency condition systems are obtained. Actually, it is challenging to seek the wellposedness of the systems due to the complicated coupled structures. After getting it, the decentralized strategies and the corresponding ϵ-Nash equilibrium properties will be studied.

5. The dynamic optimization of large-population systems with partial information is considered in Chapter 6, in which the individual agents can only access the partial filtration. One possible research direction is to study a more complicated–partial observed case. More filter theory should be applied to derive the optimal strategies. As to the backward formulation, the second point above can be also extended to partial information or partial observed structures.

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