POSITIVE SEMI-DEFINITENESS AND SUM-OF-SQUARES PROPERTY OF HANKEL TENSORS AND CIRCULANT TENSORS

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POSITIVE SEMI-DEFINITENESS AND SUM-OF-SQUARES PROPERTY OF HANKEL TENSORS AND CIRCULANT TENSORS

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________________________________(Signed)

WANG Qun (Name of student)

Dedicate to my parents.

Abstract

Problems in many applications, such as physics, computing sciences, economic and engineering, can be formulated as tensor problems that usually have special structures. In recently years, there are some classes structured tensors including Toeplitz tensors, Hankel tensors, Hilbert tensors, Vandermonde tensors, Cauchy tensors, M-tensors, P-tensors and others have been generalized from matrices and studied.

Hankel tensors and Toeplitz tensors arise from signal processing and some other applications. The positive semi-definiteness of Hankel tensors is a condition that guarantee the existence of solution for a multidimensional moment problem. To identify a general tensor is positive semi-definite (PSD) or not is NP-hard but it is easier to check for structured tensors. The aim of this work is to identify the positive semi-definiteness of Hankel tensors and circulant tensors.

A symmetric tensor is uniquely corresponding to a homogeneous polynomial. SOS (sum-of-squares) tensors are connected with SOS polynomials, which is easily to check by solving a semi-definite linear programming problem. SOS tensors are PSD tensors, but not vice versa. Based on these facts, we study the existence problem of PSD non SOS Hankel tensor in the following cases: sixth order three dimensional Hankel tensors, fourth order four dimensional Hankel tensors and generalized anti-circulant tensors. There are no PSD non SOS Hankel tensors to be found in these cases.

We also study the three dimensional strongly symmetric circulant tensors, which

are special Toeplitz tensors. We give a sufficient and necessary condition for an even order three dimensional strongly symmetric circulant tensors to be positive semi-definite in some cases.

For a given even order symmetric tensor, it is positive semi-definite (positive definite) if and only if all of its H- or Z-eigenvalues are nonnegative (positive). In other words, it is positive semi-definite if and only if the smallest H- or Z-eigenvalue is nonnegative. We propose an algorithm to compute extreme eigenvalues of large scale Hankel tensors, which can be used to not only identify positive semi-definiteness but also solve many problems in other applications, such as automatic control, medical imaging, quantum information, and spectral graph theory. Numerical examples show the efficiency of the proposed method.

Underlying papers

This thesis is based on the following papers written by the author during the period of stay at the Department of Applied Mathematics, The Hong Kong Polytechnic University as a graduate student:

1. G. Li, L. Qi, and Q. Wang. Are there sixth order three dimensional Hankel tensors? arXiv:1411.2368, November 2014, submitted.

2. Y. Chen, L. Qi, and Q. Wang. Positive semi-definiteness and sum-of-squares property of fourth order four dimensional Hankel tensors. Journal of Computational and Applied Mathematics, 302 (2016) 356-368.

3. G. Li, L. Qi and Q. Wang. Positive semi-definiteness of generalized anti-circulant tensors. Communications in Mathematical Sciences, 14 (2016) 941-952.

4. L. Qi, Q. Wang and Y. Chen. Three dimensional strongly symmetric circulant tensors. Linear Algebra and Its Applications, 482 (2015) 207-220.

5. Y. Chen, L. Qi, and Q. Wang. Computing extreme eigenvalues of large scale Hankel tensors. Journal of Scientific Computing, DOI:10.1007/s10915-015-0155-8 (2016).

In addition, the following is a list of other papers written by the author during the period of her Ph.D study.

1. L. Qi, J. Shao and Q. Wang. Regular uniform hypergraphs, s-cycles, s-paths and their largest Laplacian H-eigenvalues. Linear Algebra and Its Applications, 443 (2014) 215-227.

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$\Re$	set of real numbers
$\Re^n$	set of n-dimensional real vectors
$\Re^{m\times n}$	set of $m \times n$ real matrices
x^T	transpose of matrix/vector x
$\mathcal{I}$	the unit tensor
$\mathcal{O}$	the zero tensor
$S_{m,n}$	set of all the real symmetric tensors of order m and dimension n.
$\mathbf{e}_i$	the i-th unit vector in $\Re^n$.
x_i	the ith component of $\mathbf{x}$
$\mathbf{0}$	zero vector in $\Re^n$.
$\|\cdot\|$	the Euclidean norm.
$[n]$	the index set $\{1, \cdots, n\}$.
$GCD(m, r)$	the greatest common divisor of the two nonnegative integers m and r.
F_l	$l \times l$ Fourier matrix.
$\lceil x \rceil$	the smallest integer that is not less than x.

Chapter 1

Introduction

1.1 Background

Just as linear operators and their coordinate representations i.e., matrices, are the main objects of interest in linear algebra. Tensors and their coordinate representations i.e., hypermatrices, are the main objects of interest in multilinear algebra. Tensors can be represented as multidimensional array and the order of a tensor is the number of indices. Without misleading, we use the tensor instead of multidimensional arrays or hypermatrices.

Tensors provide a framework for solving the problems in many applications, such as physics, engineering, medical science and fluid mechanics. Tensors were first introducted by Tullio Levi-Civita and Gregorio Ricci-Curbastro, who continued the earlier work of Bernhard Riemann and Elwin Bruno Christoffel and others, as part of the absolute differential calculus [48].

Many problems in computing sciences, economic and engineering can be reduced to tensor problems that usually have special structures. In recent years, there are some classes structured tensors such as Toeplitz tensors [19], Hankel tensors [74, 30, 58, 92], Hilbert tensors [86], Vandermonde tensors [74, 92], Cauchy tensors [14, 13], M-tensors [93, 29], P-tensors [87] and others that have been extended from the corresponding matrices and their properties have been studied.

Most tensor problems are NP-hard [41]. The problems for determining a general even order symmetric tensor is positive semi-definite (PSD) or not is NP-hard and is also important both theoretically and practically [1, 40, 73]. However, to determining the tensors with some special structure is PSD or not may not be NP-hard. Recently, it was discovered that several easily checkable classes of special even order symmetric tensors are PSD or there are easily checkable conditions to identify, including even order symmetric diagonally dominated tensors [73], even order symmetric B_0 tensors [77], even order Hilbert tensors [86], even order symmetric M tensors [93], even order symmetric double B_0 tensors [55], even order symmetric strong H tensors [56, 47], even order strong Hankel tensors [74], even order positive Cauchy tensors [14], etc. However, some kinds of structured tensors are not easy to identify their positive semi-definiteness, such as Hankel tensors and Toeplitz tensors.

Hankel tensors arise from signal processing and some other applications [74, 30, 4, 72]. They are symmetric tensors. If a multidimensional sequence generates Hankel tensors and all the Hankel matrices, generated by this sequence, are positive semi-definite, then this sequence is a multidimensional moment sequence [75].

Moment problems are important topic in mathematics [7, 8, 81, 90]. The theory of one dimensional moment problems have received wide attention. For a sequence of real numbers $S = \{s_k\}$ with integers $k \geq 0$, the moment problem [81, 90] is to find sufficient and necessary conditions on S that guarantee the existence of a positive measure μ such that $\int t^k d\mu = s_k$. If such a measure μ exists, the sequence S is called a moment sequence.

There are three classical moment problems: the Hamburger moment problem in which the support interval of μ is the whole real line $\Re$, the Stieltjes moment problem for interval $[0, +\infty)$, and the Hausdorff moment problem for a bounded interval $[0, 1]$. It is well know that the Hamburger moment problem is solvable if and only if the Hankel matrix $H = H(S) := (s_{i+j})_{i,j}$ is positive semi-definite [81].

The multidimensional moment problem generalizes the moment problem to higher dimensions. According to [8, 81, 75], a multidimensional sequence

$$S = \{b_{j_1 \cdots j_{n-1}} : j_1, \cdots, j_{n-1} \geq 0\} \tag{1.1}$$

is called a multidimensional moment sequence if there is a nonnegative measure μ on $\Re^{n-1}$ satisfying:

$$b_{j_1 \cdots j_{n-1}} = \int_{\Re^{n-1}} t_1^{j_1} \cdots t_{n-1}^{j_{n-1}} d\mu, \quad \text{for} \quad j_1, \cdots, j_{n-1} \geq 0, \tag{1.2}$$

are all finite. For a given multidimensional sequence S defined by (1.1), is it a multidimensional moment sequence? i.e., Is there a nonnegative measure such that (1.2) holds? This problem is called the multidimensional moment problem [7, 8, 81].

For any $m > 0$, we may define a homogeneous polynomial of n variables and degree m:

$$\begin{aligned} f(\mathbf{x}) = \ & \sum\{b_{j_1 \cdots j_{n-1}} \tfrac{m!}{j_1! \cdots j_{n-1}!(m-j_1-\cdots-j_{n-1})!} x_1^{j_1} x_{n-1}^{j_{n-1}} x_0^{m-j_1-\cdots-j_{n-1}} \\ & : j_1, \cdots, j_{n-1} \geq 0, j_1 + \cdots + j_{n-1} \leq m\}. \end{aligned} \tag{1.3}$$

According to [81], S is a multidimensional moment sequence if and only if for all m, $f(\mathbf{x})$ has a sum of mth power (SOM) form. A homogeneous polynomial $f(\mathbf{x})$ of n variables and degree m is corresponding to an mth order n-dimensional symmetric tensor $\mathcal{A} = (a_{i_1 \cdots i_m})$, where

$$a_{i_1 \cdots i_m} = b_{j_1 \cdots j_{n-1}} \tag{1.4}$$

for $j_{n-1} \geq 0, j_1 + \cdots + j_{n-1} \leq m$, if in $\{i_1, \cdots, i_m\}$, the frequency of k is exactly j_k, $k = 1, \cdots, n-1$. Then $f(\mathbf{x})$ is an SOM form if and only if there are vectors $u_k \in \Re^n$ for $k = 1, \cdots, r$ such that

$$\mathcal{A} = \sum_{k=1}^{r} u_k^m, \tag{1.5}$$

where for a vector $\mathbf{v} \in \Re^n$, $\mathbf{v}^m = (v_{i_1} \cdots v_{i_m})$ denotes a symmetric rank-one tensor. Such a symmetric tensor is called a completely decomposable tensor in [58].

Thus, a given multidimensional sequence S defined by (1.1), is a multidimensional moment sequence if and only if all the symmetric tensors $\mathcal{A}$ generated by it are completely decomposable tensors for all m. Note that when m is odd, a symmetric tensor is always completely decomposable [58].

Suppose now that for $j_1, \cdots, j_{n-1}$, $l_1, \cdots, l_{n-1} \geq 0$, we have

$$b_{j_1 \cdots j_{n-1}} = b_{l_1 \cdots l_{n-1}}$$

if

$$j_1 + 2j_2 + \cdots + (n-1)j_{n-1} = l_1 + 2l_2 + \cdots + (n-1)l_{n-1}. \tag{1.6}$$

By (1.4), for $i_1, \cdots, i_n, k_1, \cdots, k_n \geq 0$, we have

$$a_{i_1 \cdots i_m} = a_{k_1 \cdots k_m} \tag{1.7}$$

as long as

$$i_1 + \cdots + i_m = k_1 + \cdots + k_m. \tag{1.8}$$

By [74], such a tensor is called a Hankel tensor. Thus, we call a multidimensional sequence S satisfying (1.8) a Hankel multidimensional sequence.

By [58], a strong Hankel tensor is completely decomposable. An explicit decomposition expression of a strong Hankel tensor is given in [31]. Furthermore, by (1.6), we see that

$$v_{j_1+2j_2+\cdots+(n-1)j_{n-1}} = b_{j_1 \cdots j_{n-1}}, \tag{1.9}$$

for $j_1, \cdots, j_{n-1} \geq 0$, i.e., the components of $\mathbf{v}$ are independent from m. Thus, (1.9) defines an infinite sequence $V = \{v_k : k \geq 0\}$. This infinite sequence V generates a sequence of Hankel matrices $H_p = (h_{ij})$, with $i, j = 0, \cdots, p-1, p > 0$, and

$$h_{ij} = v_{i+j} \tag{1.10}$$

for $i, j \geq 0$. We have the following theorem.

Theorem 1.1. *Suppose that a given multidimensional sequence S defined by (1.1), satisfies (1.6), i.e., it is a Hankel multidimensional sequence. If all the Hankel tensors generated by V are strong Hankel tensors, i.e., all the Hankel matrices H_p generated by the sequence V are positive semi-definite, then S is a multidimensional moment sequence.*

This links the classical result for the Hamburger moment problem [81], and gives an application of the results in [58, 74, 31].

The positive semi-definiteness of Hankel tensors is a condition that guarantees the existence of solution for a multidimensional moment problem.

It is not easy to identify the positive semi-definiteness of Hankel tensors. In [43], sum of squares (SOS) tensors were introduced and SOS tensors are connected with SOS polynomials. SOS tensors are PSD, but not vice versa. This result is from [40]. In 1888, Hilbert proved that only in the following three cases, a PSD homogeneous polynomial of degree m in n variables is an SOS polynomial: 1) $m = 2$; 2) $n = 2$; 3) $m = 4$ and $n = 3$. Hilbert proved that in all the other possible combinations of n and even m, there are PSD non-SOS (PNS) homogeneous polynomials. Chesi [20] used the abbreviation PNS for PSD non-SOS in 2007. However, Hilbert did not give an explicit example for PNS homogeneous polynomials. The first explicit example for PNS homogeneous polynomials was given by Motzkin [66] in 1967. More examples of PNS homogeneous polynomials can be found in [22, 82].

In [13], there are some classes of structured tensors that have been proved to be SOS tensors, including positive Cauchy tensors, weakly diagonally dominated tensors, B_0-tensors, double B-tensors, quasi-double B_0-tensors, MB_0-tensors, H-tensors, absolute tensors of positive semi-definite Z-tensors and extended Z-tensors.

The question raised in [58] is the Hilbert's seventeenth problem under the Hankel constraint. It can be stated as:

Does there exist a PNS Hankel tensor?

If there are no PNS Hankel tensors, then the problem for determining a given even order Hankel tensor is PSD or not is polynomial time solvable [58].

In [74], two classes of positive semi-definite Hankel tensors were identified. They are even order strong Hankel tensors and even order complete Hankel tensors. It was proved that complete Hankel tensors are strong Hankel tensors, and even order strong Hankel tensors are SOS Hankel tensors in [58]. Some other PSD Hankel tensors were identified in [58]. They are not strong Hankel tensors. But they are still SOS Hankel tensors.

According to Hilbert [40, 82], the cases with low values of m and n, in which there are PNS homogeneous polynomials, are that $m = 6$ and $n = 3$ and $m = n = 4$. We explore the conditions for positive semi-definiteness of Hankel tensors with order six and dimension three and order four and dimension four in Chapter 3 . If there are PNS Hankel tensors in these two cases, the answer of the above problem is no.

Anti-circulant tensors were introduced in [30] and have applications in exponential data fitting. They are extensions of anti-circulant matrices in matrix theory [27, 94]. Anti-circulant tensors are Hankel tensors that arise from signal processing and some other applications [74, 4, 72]. An anti-circulant tensor with order six and dimension three has been studied as a special case in [57]. We extend anti-circulant tensors to generalized anti-circulant tensors, which are still Hankel tensors, and study the conditions for positive semi-definiteness of generalized anti-circulant tensors in Chapter 4.

Toeplitz tensors are special classes of even order symmetric tensors, whose positive semi-definiteness is also not easily checkable. Are they PNS? A good candidate for such PNS tensors is the class of even order strongly symmetric circulant tensors.

Strongly symmetric tensors were introduced in [79]. Circulant tensor has applications in stochastic process and spectral hypergraph theory [19, 76] and is a special class of Toeplitz tensor. An even order circulant B_0 tensor is positive semi-definite. An even order circulant B tensor is positive definite [19]. This shows that the Laplacian tensor and the signless Laplacian tensor of a directed circulant even-uniform hypergraph are positive semi-definite [19]. If a stochastic process is mth order stationary, where m is even, then its mth order moment, which is a circulant tensor, must be positive semi-definite [19]. We study even order three dimensional strongly symmetric circulant tensors in Chapter 5.

In many applications, large scale tensors are important tools. For a given even order symmetric tensor, it is positive semi-definite (positive definite) if and only if all of its H- or Z-eigenvalues are nonnegative (positive) [73]. In other words, it is positive semi-definite if and only if the smallest H- or Z-eigenvalue is nonnegative. Qi [74] and Xu [92] studied the spectra of Hankel tensors and gave some upper bounds and lower bounds for the smallest and the largest eigenvalues. In [30], Ding et al. proposed a fast computational framework for products of a Hankel tensor and vectors. In Chapter 6, we propose a method to compute the smallest and the largest eigenvalues of relatively large Hankel tensors. The algorithms to compute eigenvalues of large scale Hankel tensors can be used to not only identify the positive semi-definiteness but also solve many problems in other applications, such as automatic control [68], medical imaging [84, 80, 16], quantum information [67], and spectral graph theory [24].

1.2 Summary of contributions of the thesis

The original contributions of this thesis are as follows:

- We study the existence problem of several classes of PNS Hankel tensors, including sixth order three dimensional Hankel tensors, fourth order four dimensional Hankel tensors, generalized anti-circulant tensors. We examine various important classes of sixth order three dimensional Hankel tensors and there are no PNS Hankel tensors are found in these cases. We show that there are no fourth order four dimensional PNS hankel tensor in a 45-degree planar closed convex cone, a segment, a ray and an additional point. Numerical tests check various grid points and find that there are no PNS Hankel tensors found. For some cases, we give necessary and sufficient conditions for even order PSD generalized anti-circulant tensors and show that in these cases, they are SOS tensors.

- We give a necessary and sufficient condition for an even order three dimensional strongly symmetric circulant tensor to be positive semi-definite and this condition can be a sufficient condition for such a tensor to be SOS in some cases. There are no PNS strongly symmetric circulant tensors found in numerical tests.

- We propose an algorithm to get the largest and the smallest H- (or Z-)eigenvalues of Hankel tensors which can be used to not only identify the positive semi-definiteness but also solve many problems in other applications, such as automatic control, medical imaging, quantum information, and spectral graph theory.

1.3 Organization of the thesis

The thesis is structured as follows.

- Chapter 2 reviews the preliminary knowledge, including some definitions and some preliminary results which are useful in the following chapters.

- Chapter 3 focuses on the existence problem of low order low dimensional PNS Hankel tensors. One case is sixth order three dimensional Hankel tensors, and we study four special classes Hankel tensors: truncated Hankel tensors, quasi-truncated Hankel tensors, anti-circulant tensors and alternatively anti-circulant tensors.

  Another case is fourth order four dimensional Hankel tensor. Under the assumption that the generating vector is symmetric, we show that there are no fourth order four dimensional PNS Hankel tensors in a 45-degree planar closed convex cone, a segment, a ray and an additional point. Numerical tests also show that no PNS Hankel tensor is found.

- Chapter 4 is devoted to an special subclasses of Hankel tensors, generalized anti-circulant tensors, which is extended from the definition of anti-circulant tensors by using a circulant index r such that the entries of generating vector of a Hankel tensor are circulant with module r. For the cases that $GCD(m, r) = 1$, $GCD(m, r) = 2$ and some other cases, we give the conditions for positive semi-definiteness of even order generalized anti-circulant tensors and they also are SOS tensors in these cases.

- Chapter 5 shows that the sufficient and necessary condition for positive semi-definiteness of an even order three dimensional strongly symmetric circulant tensor, and this condition can be a sufficient condition for such a tensor to be SOS in some cases. Numerical tests indicate that this is also true in the other cases.

- Chapter 6 proposes an inexact curvilinear search optimization method to compute the extreme H- (or Z-) eigenvalues of large scale Hankel tensors. The sequence generated by the new algorithm converges to a unique critical point, which is an eigen-pair of Hankel tensor. We analyze the linear convergence rate of iterate sequence by the Kurdyka-Łojasiewicz property. The numerical experiments are reported to show the efficiency for computing the extreme H- (or Z-) eigenvalues of large scale Hankel tensors by the new method.

- Chapter 7 concludes the whole thesis and plans for the future work.

Chapter 2

Preliminaries

2.1 Structured tensors

Denote that $[n] := \{1, \cdots, n\}$, m, n and k are integers and $m, n \geq 2$. A tensor $\mathcal{A} = (a_{i_1 \cdots i_m})$ of order m and dimension n has entries $a_{i_1 \cdots i_m}$ with $i_j \in [n]$ for $j \in [m]$. Tensor $\mathcal{A}$ is said to be a symmetric tensor if its entries $a_{i_1 \cdots i_m}$ is invariant under any index permutation. Denote the set of all the real symmetric tensors of order m and dimension n by $S_{m,n}$. Then $S_{m,n}$ is a linear space. Throughout this thesis, we only discuss real symmetric tensors. We use $\|\mathcal{A}\|$ to denote the Frobenius norm of tensor $\mathcal{A} = (a_{i_1 \cdots i_m})$, i.e., $\|\mathcal{A}\| = \sum_{i_1 \cdots i_m \in [n]} a_{i_1 \cdots i_m}^2$.

Let $\mathbf{v} = (v_0, \cdots, v_{(n-1)m})^\top$. Define $\mathcal{A} = (a_{i_1 \cdots i_m}) \in S_{m,n}$ by

$$a_{i_1 \cdots i_m} = v_{i_1 + \cdots + i_m - m}, \tag{2.1}$$

for $i_1, \cdots, i_m \in [n]$. Then $\mathcal{A}$ is a **Hankel tensor** [74, 58, 18, 17, 31, 57] and $\mathbf{v}$ is called the **generating vector** of $\mathcal{A}$. We see that a sufficient and necessary condition for $\mathcal{A} = (a_{i_1 \cdots i_m}) \in S_{m,n}$ to be a Hankel tensor is that whenever $i_1 + \cdots + i_m = j_1 + \cdots + j_m$,

$$a_{i_1 \cdots i_m} = a_{j_1 \cdots j_m}. \tag{2.2}$$

If the entries of the generating vector of a Hankel tensor satisfy

$$v_i = v_{i+n},$$

for $i = 0, \cdots, (n-1)m - n$, then $\mathcal{A}$ is called an anti-circulant tensor.

A tensor $\mathcal{A}$ is called a **Hilbert tensor** [86] if

$$a_{i_1 \cdots i_m} = \frac{1}{i_1 + i_2 + \cdots + i_m - m + 1}$$

for $i_1, \cdots, i_m \in [n]$. An mth order n dimensional Hilbert tensor is a Hankel tensor with $v = \left(1, \frac{1}{2}, \frac{1}{3}, \cdots \frac{1}{mn}\right)$.

If for $i_1, \cdots, i_m \in [n-1]$, we have

$$a_{i_1 \cdots i_m} = a_{i_1+1 \cdots i_m+1},$$

then we say that $\mathcal{A}$ is an mth order **Toeplitz tensor** [19]. By the definition, all the diagonal entries of a Toeplitz tensor are the same.

An mth order n dimensional tensor $\mathcal{A} = (a_{i_1 \cdots i_m})$ is called a **circulant tensor** [19] if

$$a_{i_1 \cdots i_m} \equiv a_{j_1 \cdots j_m}$$

as long as $j_l \equiv i_l + 1, (\text{mod } n)$ for $l = 1, \cdots, m$. Clearly, a circulant tensor is a Toeplitz tensor. Circulant tensors have applications in stochastic process and spectral hypergraph theory [19].

Strongly symmetric tensors were introduced in [79]. An mth order n dimensional tensor $\mathcal{A} = (a_{i_1 \cdots i_m})$ is called a **strongly symmetric tensor** if

$$a_{i_1 \cdots i_m} \equiv a_{j_1 \cdots j_m}$$

as long as $\{i_1, \cdots, i_m\} = \{j_1, \cdots, j_m\}$. Note that a symmetric matrix is a strongly symmetric tensor of order 2. Hence, strongly symmetric tensors are also extensions of symmetric matrices.

2.2 Positive semi-definite and positive definite tensors

Let $\mathbf{x} \in \Re^n$. Then $\mathbf{x}^m$ is a rank-one symmetric tensor with entries $x_{i_1} \cdots x_{i_m}$. For $\mathcal{A} \in S_{m,n}$ and $\mathbf{x} \in \Re^n$, we have a homogeneous polynomial $f(\mathbf{x})$ of n variables and degree m,

$$f(\mathbf{x}) = \mathcal{A}\mathbf{x}^{\otimes m} \equiv \sum_{i_1,\cdots,i_m \in [n]} a_{i_1 \cdots i_m} x_{i_1} \cdots x_{i_m}. \tag{2.3}$$

Note that there is a one to one relation between homogeneous polynomials and symmetric tensors. If $f(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \Re^n$, then homogeneous polynomial $f(\mathbf{x})$ and symmetric tensor $\mathcal{A}$ are called **positive semi-definite**(PSD). If $f(\mathbf{x}) > 0$ for all $\mathbf{x} \in \Re^n$, $\mathbf{x} \neq 0$, then $f(x)$ and $\mathcal{A}$ are called **positive definite** (PD). In (2.3), if $\mathcal{A}$ is a Hankel tensor, then $f(\mathbf{x})$ is called a Hankel polynomial. Clearly, if m is odd, there is no positive definite symmetric tensor and there is only one positive semi-definite tensor $\mathcal{O}$. Thus, we assume that $m = 2k$ when we discuss positive definite and semi-definite tensors (polynomials). $\mathcal{A}\mathbf{x}^{m-1}$ is a column vector

$$(\mathcal{A}\mathbf{x}^{m-1})_i = \sum_{i_2,i_3,\cdots,i_m=1}^{n} a_{i,i_2,\cdots,i_m} x_{i_2} \cdots x_{i_m}, \quad \text{for } i = 1, \cdots, n.$$

For a vector $x \in \Re^n$, we use x_i to denote its components, and $x^{[m]}$ to denote a vector in $\Re^n$ such that

$$x_i^{[m]} = x_i^m$$

for all i. In [73], Qi introduced that the definition of the H-eigenvalue (eigenvalue) and the Z-eigenvalue (E-eigenvalue) of a tensor $\mathcal{A} \in \Re_n^m$.

A real number $\lambda \in \Re$ is called an H-eigenvalue of $\mathcal{A}$, iff $\exists x \in \Re^n$ satisfies

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]},$$

x is called the H-eigenvector corresponding to λ.

A real number $\lambda \in \Re$ is called an Z-eigenvalue of $\mathcal{A}$, iff $\exists x \in \Re^n$ satisfies

$$\begin{cases} \mathcal{A}x^{m-1} = \lambda x \\ x^\top x = 1, \end{cases} \tag{2.4}$$

x is called the Z-eigenvector corresponding to λ.

Theorem 2.1. *[73] Assume that m is even. The following conclusions hold for $\mathcal{A} \in S_{m,n}$:*

(1) A always has H-eigenvalues. $\mathcal{A}$ is positive definite (positive semi-definite) if and only if all of its H-eigenvalues are positive (nonnegative).

(2) A always has Z-eigenvalues. $\mathcal{A}$ is positive definite (positive semi-definite) if and only if all of its Z-eigenvalues are positive (nonnegative).

This theorem shows that a tensor $\mathcal{A}$ is positive semi-definite if and only if the smallest H- or Z-eigenvalue of $\mathcal{A}$ is nonnegative.

2.3 PSD Hankel tensors and SOS Hankel tensors

If $f(x)$ can be decomposed to the sum of squares of polynomials of degree k, then $f(x)$ is called a sum-of-squares polynomial, and the corresponding symmetric tensor $\mathcal{A}$ is called an **SOS tensor** [43]. SOS polynomials play a central role in the modern theory of polynomial optimization [53, 54]. Clearly, an SOS polynomial (tensor) is a PSD polynomial, but not vice versa. Actually, this was shown by young Hilbert [40, 66, 22, 82] that for homogeneous polynomial, only in the following three cases, a PSD polynomial definitely is an SOS polynomial: 1)$n = 2$; 2)$m = 2$; 3)$m = 4$ and $n = 3$. For tensors, the second case corresponds to the symmetric matrices, i.e., a PSD symmetric matrix is always an SOS matrix. Hilbert proved that in all the other possible combinations of $m = 2k$ and n, there are PNS homogeneous polynomials. The most well-known PNS homogeneous polynomial is the Motzkin polynomial [66]

$$f_M(\mathbf{x}) = x_3^6 + x_1^2x_2^4 + x_1^4x_2^2 - 3x_1^2x_2^2x_3^2.$$

By the Arithmetic-Geometric inequality, we see that it is a PSD polynomial. But it is not an SOS polynomial [82]. The other two PNS homogeneous polynomials with small m and n are given by Choi and Lam [22]

$$f_{CL1}(\mathbf{x}) = x_4^4 + x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2 - 4x_1x_2x_3x_4$$

and

$$f_{CL2}(\mathbf{x}) = x_1^4x_2^2 + x_2^4x_3^2 + x_3^4x_1^2 - 3x_1^2x_2^2x_3^2.$$

Denote the set of all SOS tensors in $S_{m,n}$ by $SOS_{m,n}$. Then it is also a closed convex cone [43].

By (2.2), the three PNS polynomials $f_M(\mathbf{x})$, $f_{CL1}(\mathbf{x})$ and $f_{CL2}(\mathbf{x})$ are not Hankel polynomials. These three polynomials are still non-SOS PSD polynomials if we switch the indices of their variables.

Suppose that $\mathcal{A}$ is a Hankel tensor defined by (2.1). Let $A = (a_{ij})$ be an $\lceil \frac{(n-1)m+2}{2} \rceil \times \lceil \frac{(n-1)m+2}{2} \rceil$ matrix with $a_{ij} \equiv v_{i+j-2}$, where $v_{2\lceil \frac{(n-1)m}{2} \rceil}$ is an additional number when $(n-1)m$ is odd. Then A is a Hankel matrix, associated with the Hankel tensor $\mathcal{A}$. Clearly, when m is even, such an associated Hankel matrix is unique. Recall from [74] that $\mathcal{A}$ is called a **strong Hankel tensor** if there exists an associated Hankel matrix A is positive semi-definite. Thus, whether a tensor is a strong Hankel tensor or not can be verified by using tools from matrix analysis. It has also been shown in [74] that $\mathcal{A}$ is a strong Hankel tensor if and only if it is a Hankel tensor and there exists an absolutely integrable real valued function $h : (-\infty, +\infty) \to [0, +\infty)$ such that its generating vector $\mathbf{v} = (v_0, v_1, \cdots, v_{(n-1)m})^\top$ satisfies

$$v_k = \int_{-\infty}^{\infty} t^k h(t)dt, \quad k = 0, 1, \cdots, (n-1)m. \tag{2.5}$$

Such a real valued function h is called the generating function of the strong Hankel tensor $\mathcal{A}$. A vector $\mathbf{u} = (1, \gamma, \gamma^2, \cdots, \gamma^{n-1})^\top$ for some $\gamma \in \Re$ is called a Vandermonde vector [74]. If tensor $\mathcal{A}$ has the form

$$\mathcal{A} = \sum_{i \in [r]} \alpha_i (\mathbf{u}_i)^m, \tag{2.6}$$

where $\mathbf{u}_i$ for $i = 1, \cdots, r$, are all Vandermonde vectors, then we say that $\mathcal{A}$ has a Vandermonde decomposition. It was shown in [74] that a symmetric tensor is a Hankel tensor if and only if it has a Vandermonde decomposition. If the coefficients α_i for $i = 1, \cdots, r$, are all nonnegative, then $\mathcal{A}$ is called a **complete Hankel tensor** [74].

Let $\mathcal{A} \in S_{m,n}$. If there are vectors $x_j \in \Re^n$ for $j \in [r]$ such that

$$A = \sum_{j \in [r]} x_j^{\otimes m},$$

then we say that $\mathcal{A}$ is a **completely r-decomposable tensor**, or a completely decomposable tensor. If $x_j \in \Re^n_+$ for all $j \in [r]$, then $\mathcal{A}$ is called a completely positive tensor [79].

Clearly, a complete Hankel tensor is a completely decomposable tensor. Unlike strong Hankel tensors, there is no clear method to check whether a Hankel tensor is a complete Hankel tensor or not, as the Vandermonde decompositions of a Hankel tensor are not unique.

It was proved that even order strong or complete Hankel tensors are positive semi-definite in [74], complete Hankel tensors are strong Hankel tensors and all of them are SOS Hankel tensors in [58]. A even order strong Hankel tensor is a completely decomposable tensor and a completely decomposable tensor is a SOS Hankel tensor. There is a even order strong Hankel tensor which is not a complete Hankel tensor, whenever m is a positive even number and $n \geq 2$.

Chapter 3

Low Order Low Dimensional Hankel Tensor

If there are no PNS Hankel tensor, then the problem for determining an even order Hankel tensor is PSD or not can be solved in polynomial-time. By Hilbert, the cases of low order (degree) and dimension (number of variables), in which PNS symmetric tensors (homogeneous polynomials) exists, is of order six and dimension three and order four and dimension four.

In this chapter, we study the existence problem of sixth order three dimensional and fourth order four dimensional PNS Hankel tensors. We examine various important classes of sixth order three dimensional Hankel tensors. No PNS Hankel tensors are found in these cases. We also show that there are no fourth order four dimensional PNS Hankel tensors to be found in the following cases: a 45-degree planar closed convex cone, a segment, a ray and an additional point. Numerical tests check various grid points, and no PNS Hankel tensors are found.

3.1 Sixth order three dimensional Hankel tensors

3.1.1 Introduction

Let $\mathbf{v} = (v_0, v_1, \cdots, v_{12})^\top \in \Re^{13}$. A sixth order three dimensional Hankel tensor $\mathcal{A} = (a_{i_1 \cdots i_6})$ is defined by

$$a_{i_1 \cdots i_6} = v_{i_1 + \cdots + i_6 - 6},$$

for $i_1, \cdots, i_6 = 1, 2, 3$. The corresponding vector $\mathbf{v}$ that defines the Hankel tensor $\mathcal{A}$ is called the generating vector of $\mathcal{A}$. For $\mathbf{x} = (x_1, x_2, x_3)^\top \in \Re^3$, $\mathcal{A}$ uniquely defines a homogeneous polynomial (a ternary sextic)

$$f(\mathbf{x}) \equiv \mathcal{A}\mathbf{x}^{\otimes 6} = \sum_{i_1, \cdots, i_6 = 1}^{3} a_{i_1 \cdots i_6} x_{i_1} \cdots x_{i_6} = \sum_{i_1, \cdots, i_6 = 1}^{3} v_{i_1 + \cdots + i_6 - 6} x_{i_1} \cdots x_{i_6}. \tag{3.1}$$

We call such a polynomial a **(ternary sextic) Hankel polynomial**.

We study several special classes of sixth order three dimensional Hankel tensors.

The first class of Hankel tensors we examined is called truncated Hankel tensors. The generating vector $\mathbf{v}$ of a sixth order three dimensional truncated Hankel tensor $\mathcal{A}$ has only three nonzero entries: v_0, v_6 and v_{12}. We provide a sufficient and necessary condition that a sixth order three dimensional truncated Hankel tensor to be PSD. We show that such truncated Hankel tensors are PSD if and only if they are SOS. We also show that such SOS Hankel tensors are not strong Hankel tensors unless $v_6 = 0$.

The second class of Hankel tensors is called quasi-truncated Hankel tensors. The generating vector $\mathbf{v}$ of a sixth order three dimensional quasi-truncated Hankel tensor $\mathcal{A}$ has five nonzero entries: v_0, v_1, v_6, v_{11} and v_{12}. It is still true that such SOS Hankel tensors are not strong Hankel tensors unless $v_1 = v_6 = v_{11} = 0$. In this case, still no PNS Hankel tensors are found.

To motivate the third class of Hankel tensors, we recall that, beside the Motzkin polynomial, there is another well-known PNS homogeneous polynomial for $m = 6$ and $n = 3$. This is the Choi-Lam polynomial [22, 82]:

$$f_{CL}(\mathbf{x}) = x_1^4 x_2^2 + x_2^4 x_3^2 + x_3^4 x_1^2 - 3x_1^2 x_2^2 x_3^2.$$

An important property of the Choi-Lam polynomial is that

$$f(x_1, x_2, x_3) = f(x_2, x_3, x_1) = f(x_3, x_1, x_2)$$

for any $\mathbf{x} \in \Re^3$. The generating vector $\mathbf{v}$ of a sixth order three dimensional Hankel tensor $\mathcal{A}$, associated with such a ternary sextic has the property

$$v_i = v_{i+3}, \tag{3.2}$$

for $i = 0, \cdots, 9$. By [30], a Hankel tensor satisfying (3.2) is called an anti-circulant tensor. The name "anti-circulant tensor" is an extension of the name "anti-circulant matrix" [27]. We show that a sixth order three dimensional anti-circulant tensor is PSD if and only if it is a nonnegative multiple of the all one tensor, which is an SOS Hankel tensor. Thus, no PNS Hankel tensors are found in this case.

The fourth class of Hankel tensors is defined that the generating vectors $\mathbf{v}$ of such Hankel tensors satisfy

$$v_i = v_{i+2},$$

for $i = 0, \cdots, 10$. We call such Hankel tensors alternatively anti-circulant tensors. We give a sufficient and necessary condition for a sixth order three dimensional alternatively anti-circulant tensor to be PSD, and show that a sixth order three dimensional PSD alternatively anti-circulant tensor is a strong Hankel tensor, hence an SOS Hankel tensor. Thus, still no PNS Hankel tensors are found.

Since we cannot find sixth order three dimensional PNS Hankel tensors in all the above four special cases, we turn our search to numerical tests. To conduct the numerical tests, we randomly generate several thousands of sixth order three dimensional Hankel tensors and make them PSD but not positive definite by adding adequate multiple of a fixed sixth order three dimensional positive definite Hankel tensor. Again, still no PNS Hankel tensors are found. Thus, we make a conjecture that there are no sixth order three dimensional PNS Hankel tensors. If this conjecture is true, then the problem for determining a given sixth order three dimensional Hankel tensor is PSD or not can be solved by a semi-definite linear programming problem.

In (3.1), if $f(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \Re^3$, then f is called a PSD Hankel polynomial and $\mathcal{A}$ is called a PSD Hankel tensor [73]. Denote $\mathbf{0} = (0,0,0)^\top \in \Re^3$. If $f(\mathbf{x}) > 0$ for all $\mathbf{x} \in \Re^3, \mathbf{x} \neq \mathbf{0}$, then f and $\mathcal{A}$ are called positive definite. If f can be decomposed to the sum of squares of polynomials of degree three, then f is called an SOS Hankel polynomial and $\mathcal{A}$ is called an SOS Hankel tensor [44, 43, 58, 63]. Clearly, an SOS Hankel tensor is a PSD Hankel tensor but not vice versa. By [74], a necessary condition for $\mathcal{A}$ to be PSD is that

$$v_0 \geq 0, \ v_6 \geq 0, \ v_{12} \geq 0. \tag{3.3}$$

Let $\mathbf{e}_1 = (1,0,0)^\top, \mathbf{e}_2 = (0,1,0)^\top$ and $\mathbf{e}_3 = (0,0,1)^\top$. Substitute them to (3.1). Then we get (3.3) directly. The generating vector $\mathbf{v}$ may also generate a 7×7 Hankel matrix $A = (a_{ij})$ by

$$a_{ij} = v_{i+j-2},$$

for $i, j = 1, \cdots, 7$. If the associated Hankel matrix A is PSD, then the Hankel tensor $\mathcal{A}$ is called a strong Hankel tensor [74]. In [58], it was proved that an even order strong Hankel tensor is an SOS Hankel tensor. On the other hand, the converse is not true in general [74, 58]. A necessary condition for $\mathcal{A}$ to be a strong Hankel tensor is that

$$v_0 \geq 0, \ v_2 \geq 0, \ v_4 \geq 0, \ v_6 \geq 0, \ v_8 \geq 0, \ v_{10} \geq 0, \ v_{12} \geq 0. \tag{3.4}$$

A simple example of Hankel tensor is the Hilbert tensor. The sixth order three dimensional Hilbert tensor $\mathcal{H}$ has the form $\mathcal{H} = (\frac{1}{i_1+\cdots+i_6-5})$. Its generating vector is

$\mathbf{v} = (1, \frac{1}{2}, \cdots, \frac{1}{13})^\top$. It was shown in [86] that $\mathcal{H}$ is positive definite. It is easy to see that the associated Hankel matrix of the Hilbert tensor is a Hilbert matrix, which is positive definite. Thus, the sixth order three dimensional Hilbert tensor $\mathcal{H}$ is a strong Hankel tensor, and hence is an SOS tensor.

We may write out (3.1) explicitly in terms of the coordinates of its generating vector $\mathbf{v}$. Then we have

$$\begin{aligned} f(\mathbf{x}) \;=\; & v_0 x_1^6 + 6v_1 x_1^5 x_2 + v_2(15x_1^4x_2^2 + 6x_1^5x_3) + v_3(20x_1^3x_2^3 + 30x_1^4x_2x_3) \\ & + v_4(15x_1^2x_2^4 + 60x_1^3x_2^2x_3 + 15x_1^4x_3^2) + v_5(6x_1x_2^5 + 60x_1^2x_2^3x_3 + 60x_1^3x_2x_3^2) \\ & + v_6(x_2^6 + 30x_1x_2^4x_3 + 90x_1^2x_2^2x_3^2 + 20x_1^3x_3^3) \\ & + v_7(6x_2^5x_3 + 60x_1x_2^3x_3^2 + 60x_1^2x_2x_3^3) + v_8(15x_2^4x_3^2 + 60x_1x_2^2x_3^3 + 15x_1^2x_3^4) \\ & + v_9(20x_2^3x_3^3 + 30x_1x_2x_3^4) + v_{10}(15x_2^2x_3^4 + 6x_1x_3^5) + 6v_{11}x_2x_3^5 + v_{12}x_3^6. \end{aligned} \tag{3.5}$$

Let $g(\mathbf{y}) = \mathbf{y}^\top A\mathbf{y}$, where $\mathbf{y} = (y_1, \cdots, y_7)^\top \in \Re^7$ and A is the associated Hankel matrix of $\mathcal{A}$. Then

$$\begin{aligned} g(\mathbf{y}) \;=\; & v_0y_1^2 + 2v_1y_1y_2 + v_2(y_2^2 + 2y_1y_3) + v_3(2y_1y_4 + 2y_2y_3) \\ & + v_4(y_3^2 + 2y_1y_5 + 2y_2y_3) + v_5(2y_1y_6 + 2y_2y_5 + 2y_3y_4) \\ & + v_6(y_4^2 + 2y_1y_7 + 2y_2y_6 + 2y_3y_5) \\ & + v_7(2y_2y_7 + 2y_3y_6 + 2y_4y_5) + v_8(y_5^2 + 2y_3y_7 + 2y_4y_6) \\ & + v_9(2y_4y_7 + 2y_5y_6) + v_{10}(y_6^2 + 2y_5y_7) + 2v_{11}y_6y_7 + v_{12}y_7^2. \end{aligned} \tag{3.6}$$

Thus, $\mathcal{A}$ is a strong Hankel tensor if and only if g is PSD.

These will be helpful for our further discussion.

If $\mathbf{v} = (1, 1, \cdots, 1)^\top$, then $\mathcal{A}$ is the all one tensor. By (3.5), in this case, $f(\mathbf{x}) = (x_1 + x_2 + x_3)^6$. Thus, the all one tensor is an SOS Hankel tensor, but not a positive definite tensor. By (3.6), it is a strong Hankel tensor.

Now we may have some simple properties of sixth order three dimensional Hankel tensors.

Theorem 3.1. *Suppose that* $\mathcal{A} = (a_{i_1\cdots i_6})$ *is a Hankel tensor generated by its generating vector* $\mathbf{v} = (v_0, v_1, \cdots, v_{12})^\top \in \Re^{13}$. *If* $\mathcal{A}$ *is a PSD (or positive definite, or SOS, or strong) Hankel tensor, then the Hankel tensors* $\mathcal{B}, \mathcal{C}, \mathcal{D}$*, generated by* $(v_{12}, v_{11}, \cdots, v_0)^\top, (v_0, -v_1, v_2, -v_3, \cdots, v_{12})^\top$, $(v_0, 0, v_2, 0, \cdots, v_{12})^\top$ *are also a PSD (or positive definite, or SOS, or strong) Hankel tensor.*

Proof. In (3.5) and (3.6), changing $\mathbf{x} = (x_1, x_2, x_3)^\top$ and $\mathbf{y} = (y_1, \cdots, y_7)^\top$ to $(x_3, x_2, x_1)^\top$ and $(y_7, \cdots, y_1)^\top$ respectively, we see that the conclusions on $\mathcal{B}$ hold.

In (3.5) and (3.6), changing $\mathbf{x} = (x_1, x_2, x_3)^\top$ and $\mathbf{y} = (y_1, \cdots, y_7)^\top$ to $(x_1, -x_2, x_3)^\top$ and $(y_1, -y_2, y_3, -y_4, \cdots, y_7)^\top$ respectively, we get the conclusions on $\mathcal{C}$.

Since $\mathcal{D} = \frac{\mathcal{A}+\mathcal{C}}{2}$, the conclusions on $\mathcal{D}$ follow. □

3.1.2 Sixth order three dimensional truncated Hankel tensors

In this section, we consider the case that the Hankel tensor $\mathcal{A}$ is generated by $\mathbf{v} = (v_0, 0, 0, 0, 0, 0, v_6, 0, 0, 0, 0, 0, v_{12})^\top$. Now, (3.5) and (3.6) have the simple form

$$f(\mathbf{x}) = v_0 x_1^6 + v_6(x_2^6 + 30x_1x_2^4x_3 + 90x_1^2x_2^2x_3^2 + 20x_1^3x_3^3) + v_{12}x_3^6 \tag{3.7}$$

and

$$g(\mathbf{y}) = v_0 y_1^2 + v_6(y_4^2 + 2y_1y_7 + 2y_2y_6 + 2y_3y_5) + v_{12}y_7^2. \tag{3.8}$$

We call such a Hankel tensor a **truncated Hankel tensor**. Since we are only concerned about PSD Hankel tensors, we may assume that (3.3) holds. From (3.7) and (3.8), we have the following proposition.

Proposition 3.1. *Suppose that (3.3) holds. If* $v_6 = 0$*, then the truncated Hankel tensor* $\mathcal{A}$ *is a strong Hankel tensor and an SOS Hankel tensor. If* $v_6 > 0$*, then* $\mathcal{A}$ *is not a strong Hankel tensor.*

Proof. When $v_6 = 0$, from (3.7) and (3.8), we see that the truncated Hankel tensor $\mathcal{A}$ is a strong Hankel tensor and an SOS Hankel tensor. If $v_6 > 0$, consider $\bar{\mathbf{y}} = (0, 0, 1, 0, -1, 0, 0)^\top$. We see that $g(\bar{\mathbf{y}}) = -2v_6 < 0$. Hence $\mathcal{A}$ is not a strong Hankel tensor in this case. □

We now give the main result of this section.

Theorem 3.2. *The following statements are equivalent:*

(i) *The truncated Hankel tensor $\mathcal{A}$ is a PSD Hankel tensor;*

(ii) *The truncated Hankel tensor $\mathcal{A}$ is an SOS Hankel tensor;*

(iii) *The relation (3.3) holds and*

$$\sqrt{v_0 v_{12}} \geq (560 + 70\sqrt{70}) v_6. \tag{3.9}$$

Furthermore, the truncated Hankel tensor $\mathcal{A}$ is positive definite if and only if $v_0, v_6, v_{12} > 0$ and strict inequality holds in (3.9).

Proof. [(i) ⇒ (iii)] Suppose that $\mathcal{A}$ is PSD, then clearly (3.3) holds. To see (iii), we only need to show (3.9) holds. Let $t \geq 0$ and let $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2, \bar{x}_3)^\top$, where

$$\bar{x}_1 = v_{12}^{\frac{1}{6}}, \ \bar{x}_2 = \sqrt{t}(v_0 v_{12})^{\frac{1}{12}}, \ \bar{x}_3 = -v_0^{\frac{1}{6}}.$$

Substitute them to (3.7). If $\mathcal{A}$ is PSD, then $f(\bar{\mathbf{x}}) \geq 0$. It follows from (3.7) that

$$v_0 v_{12} + v_6(t^3 - 30t^2 + 90t - 20)\sqrt{v_0 v_{12}} + v_0 v_{12} \geq 0.$$

From this, we have

$$\sqrt{v_0 v_{12}} \geq \frac{-t^3 + 30t^2 - 90t + 20}{2} v_6.$$

Substituting $t = 10 + \sqrt{70}$ to it, we have (3.9).

[(iii) $\Rightarrow$ (ii)] We now assume that (3.3) and (3.9) hold. We will show that $\mathcal{A}$ is SOS. If $v_6 = 0$, then by Proposition 3.1, $\mathcal{A}$ is an SOS Hankel tensor. Assume that $v_6 > 0$. By (3.9), $v_0 > 0$ and $v_{12} > 0$. We now have

$$\begin{aligned} f(\mathbf{x}) &= f_1(\mathbf{x}) + 10v_6\left(\left(\frac{v_0}{v_{12}}\right)^{\frac{1}{4}} x_1^3 + \left(\frac{v_{12}}{v_0}\right)^{\frac{1}{4}} x_3^3\right)^2 \\ &\quad + v_6\left(\sqrt{\frac{10-\sqrt{70}}{2}}x_2^3 + \sqrt{150+15\sqrt{70}}x_1x_2x_3\right)^2, \end{aligned}$$

where

$$\begin{aligned} f_1(\mathbf{x}) &= \left(v_0 - 10v_6\left(\frac{v_0}{v_{12}}\right)^{\frac{1}{2}}\right)x_1^6 + \frac{\sqrt{70}-8}{2}v_6x_2^6 + \left(v_{12} - 10v_6\left(\frac{v_{12}}{v_0}\right)^{\frac{1}{2}}\right)x_3^6 \\ &\quad -(60+15\sqrt{70})v_6x_1^2x_2^2x_3^2. \end{aligned} \tag{3.10}$$

We see that $f_1(\mathbf{x})$ is a diagonal minus tail form [33]. By the arithmetic-geometric inequality, we have

$$\begin{aligned} &\left(v_0 - 10v_6\left(\frac{v_0}{v_{12}}\right)^{\frac{1}{2}}\right)x_1^6 + \frac{\sqrt{70}-8}{2}v_6x_2^6 + \left(v_{12} - 10v_6\left(\frac{v_{12}}{v_0}\right)^{\frac{1}{2}}\right)x_3^6 \\ &\geq 3\left(\frac{\sqrt{70}-8}{2}v_6(\sqrt{v_0v_{12}} - 10v_6)^2\right)^{\frac{1}{3}} x_1^2x_2^2x_3^2. \end{aligned}$$

By (3.9),

$$3\left(\frac{\sqrt{70}-8}{2}v_6(\sqrt{v_0v_{12}} - 10v_6)^2\right)^{\frac{1}{3}} x_1^2x_2^2x_3^2 \geq (60+15\sqrt{70})v_6x_1^2x_2^2x_3^2. \tag{3.11}$$

Thus, f_1 is a PSD diagonal minus tail form. By [33], f_1 is an SOS polynomial. Hence, f is also an SOS polynomial if (3.3) and (3.9) hold.

[(ii) $\Rightarrow$ (i)] This implication is direct by the definition.

We now prove the last conclusion of this theorem. First, we assume that $\mathcal{A}$ is positive definite. Then, $v_6 = f(\mathbf{e}_2) > 0$ as $\mathbf{e}_2 \neq \mathbf{0}$. Similarly, $v_0 = f(\mathbf{e}_1) > 0$ and $v_{12} = f(\mathbf{e}_3) > 0$. Note that in the above [(i) $\Rightarrow$ (iii)] part, $f(\bar{\mathbf{x}}) > 0$ as $\bar{\mathbf{x}} \neq \mathbf{0}$. Then strict inequality holds for the last two inequalities in the above [(i) $\Rightarrow$ (iii)] part. This implies that strict inequality holds in (3.9).

On the other hand, assume that $v_0, v_6, v_{12} > 0$ and strict inequality holds in (3.9). Let $\mathbf{x} = (x_1, x_2, x_3)^\top \neq \mathbf{0}$. If $x_1 \neq 0, x_2 \neq 0$ and $x_3 \neq 0$, then strict inequality holds in (3.11) as $v_6 > 0$ and strict inequality holds in (3.9). Then $f_1(\mathbf{x}) > 0$. If $x_2 \neq 0$ but $x_1 x_3 = 0$, then from (3.10), we still have $f_1(\mathbf{x}) > 0$. If $x_2 = 0$ and one of x_1 and x_3 are nonzero, then we still have $f_1(\mathbf{x}) > 0$ by (3.10). Thus, we always have $f_1(\mathbf{x}) > 0$ as long as $\mathbf{x} \neq \mathbf{0}$. This implies $f(\mathbf{x}) > 0$ as long as $\mathbf{x} \neq \mathbf{0}$. Hence, $\mathcal{A}$ is positive definite. $\square$

3.1.3 Sixth order three dimensional quasi-truncated Hankel tensors

In this section, we consider the case that the Hankel tensor $\mathcal{A}$ is generated by $\mathbf{v} = (v_0, v_1, 0, 0, 0, 0, v_6, 0, 0, 0, 0, v_{11}, v_{12})^\top \in \Re^{13}$. Adding v_1 and v_{11} to the case in the last section, we get this case. We call such a Hankel tensor a **quasi-truncated Hankel tensor**. Hence, truncated Hankel tensors are quasi-truncated Hankel tensors.

Since we are only concerned about PSD Hankel tensors, we may assume that (3.3) holds. Now, (3.5) and (3.6) have the simple form

$$f(\mathbf{x}) = v_0 x_1^6 + 6v_1 x_1^5 x_2 + v_6(x_2^6 + 30x_1 x_2^4 x_3 + 90x_1^2 x_2^2 x_3^2 + 20x_1^3 x_3^3) + 6v_{11} x_2 x_3^5 + v_{12} x_3^6, \tag{3.12}$$

and

$$g(\mathbf{y}) = v_0 y_1^2 + 2v_1 y_1 y_2 + v_6(y_4^2 + 2y_1 y_7 + 2y_2 y_6 + 2y_3 y_5) + 2v_{11} y_6 y_7 + v_{12} y_7^2. \tag{3.13}$$

We first show that a result with the form of Proposition 3.1 continues to hold in this case.

Proposition 3.2. *Suppose that (3.3) holds. If $v_6 = 0$, then the quasi-truncated Hankel tensor $\mathcal{A}$ is PSD if and only if $v_1 = v_{11} = 0$. In this case, $\mathcal{A}$ is a strong Hankel tensor and an SOS Hankel tensor. If $v_6 > 0$, then $\mathcal{A}$ is not a strong Hankel tensor.*

Proof. Suppose that $v_6 = 0$. Assume that $v_1 \neq 0$. If $v_0 = 0$, consider $\hat{\mathbf{x}} = (1, -v_1, 0)^\top$. Then $f(\hat{\mathbf{x}}) < 0$. If $v_0 > 0$, consider $\tilde{\mathbf{x}} = (1, -\frac{v_0}{v_1}, 0)^\top$. Then $f(\tilde{\mathbf{x}}) < 0$. Thus, $\mathcal{A}$ is not PSD in these two cases. Similar discussion holds for the case that $v_{11} = 0$. Assume now that $v_1 = v_{11} = 0$. By Proposition 3.1, we see that the truncated Hankel tensor $\mathcal{A}$ is a strong Hankel tensor and an SOS Hankel tensor in this case. This proves the first part of this proposition.

Suppose that $v_6 > 0$. Consider $\bar{\mathbf{y}} = (0, 0, 1, 0, -1, 0, 0)^\top \in \Re^7$. We see that $g(\bar{\mathbf{y}}) = -2v_6 < 0$. Hence $\mathcal{A}$ is not a strong Hankel tensor in this case. $\square$

To present a necessary condition for a sixth order three dimensional quasi-truncated Hankel tensor to be PSD, we first prove the following lemma.

Lemma 3.1. *Consider*

$$\hat{f}(x_1, x_2) = v_0 x_1^6 + 6v_1 x_1^5 x_2 + v_6 x_2^6.$$

Then $\hat{f}$ is PSD if and only if $v_0 \geq 0$, $v_6 \geq 0$ and

$$|v_1| \leq \left(\frac{v_0}{5}\right)^{\frac{5}{6}} v_6^{\frac{1}{6}}. \tag{3.14}$$

Proof. Suppose that $v_0 \geq 0$, $v_6 \geq 0$ and (3.14) holds. Then, by the arithmetic-geometric inequality, one has

$$\begin{aligned} v_0x_1^6 + v_6x_2^6 &= \frac{1}{5}v_0x_1^6 + \frac{1}{5}v_0x_1^6 + \frac{1}{5}v_0x_1^6 + \frac{1}{5}v_0x_1^6 + \frac{1}{5}v_0x_1^6 + v_6x_2^6 \\ &\geq 6\left(\left(\frac{v_0}{5}\right)^5 x_1^{30}v_6x_2^6\right)^{\frac{1}{6}} \\ &\geq 6|v_1x_1^5x_2|. \end{aligned}$$

This implies that $\hat{f}(x_1, x_2) \geq 0$ for any $(x_1, x_2)^\top \in \Re^2$, i.e., $\hat{f}(x_1, x_2)$ is PSD.

Suppose that $\hat{f}(x_1, x_2)$ is PSD. It is easy to see that $v_0 \geq 0$ and $v_6 \geq 0$. Assume now that (3.14) does not hold, i.e.,

$$|v_1| > \left(\frac{v_0}{5}\right)^{\frac{5}{6}} v_6^{\frac{1}{6}}. \tag{3.15}$$

If $v_0 = v_6 = 0$, let $x_1 = 1$ and $x_2 = -v_1$. Then $\hat{f}(x_1, x_2) < 0$. We get a contradiction. If $v_0 = 0$ and $v_6 \neq 0$, let $x_1 = v_6^{\frac{1}{5}}$ and $x_2 = -v_1^{\frac{1}{5}}$. Again, $\hat{f}(x_1, x_2) < 0$. We get a contradiction. Similarly, if $v_0 \neq 0$ and $v_6 = 0$, we may get a contradiction. If $v_0 \neq 0$ and $v_6 \neq 0$, let $x_1 = (5v_6)^{\frac{1}{6}}$ and $x_2 = -\frac{v_1}{|v_1|}v_0^{\frac{1}{6}}$. Then by (3.15),

$$\hat{f}(x_1, x_2) = 6v_0v_6 - 6|v_1|(5v_6)^{\frac{5}{6}}v_0^{\frac{1}{6}} < 0.$$

We still get a contradiction. This completes the proof. □

We now present a necessary condition for a sixth order three dimensional quasi-truncated Hankel tensor to be PSD.

Proposition 3.3. *Suppose that (3.3) holds. If $\mathcal{A}$ is a PSD quasi-truncated Hankel tensor, then (3.14) and the following inequalities*

$$|v_1| \leq \left(\frac{v_{12}}{5}\right)^{\frac{5}{6}} v_6^{\frac{1}{6}} \tag{3.16}$$

and

$$\sqrt{v_0 v_{12}} \geq 10 v_6 \tag{3.17}$$

hold. If furthermore

$$v_1 v_{12}^{\frac{5}{6}} = v_{11} v_0^{\frac{5}{6}}, \tag{3.18}$$

then (3.9) also holds.

Proof. Suppose that $\mathcal{A}$ is PSD. In (3.12), let $x_3 = 0$. By Lemma 3.1, (3.14) holds. In (3.12), let $x_1 = 0$. By an argument similar to Lemma 3.1, (3.16) holds. In (3.12), let $x_2 = 0$. Since $\mathcal{A}$ is PSD, we may easily get (3.17).

Suppose further that (3.18) holds. As in [(i) $\Rightarrow$ (iii)] part of the proof of Theorem 3.2, we let $t \geq 0$ and let $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2, \bar{x}_3)^\top$, where $\bar{x}_1 = v_{12}^{\frac{1}{6}}$, $\bar{x}_2 = \sqrt{t}(v_0 v_{12})^{\frac{1}{12}}$, $\bar{x}_3 = -v_0^{\frac{1}{6}}$. It follows from (3.18) that

$$6 v_1 \bar{x}_1^5 \bar{x}_2 + 6 v_{11} \bar{x}_2 \bar{x}_3^5 = 0. \tag{3.19}$$

This together with (3.12) implies that

$$f(\bar{\mathbf{x}}) = v_0 v_{12} + v_6 (t^3 - 30t^2 + 90t - 20)\sqrt{v_0 v_{12}} + v_0 v_{12} \geq 0.$$

Proceed as in [(i) $\Rightarrow$ (iii)] part of the proof of Theorem 3.2, we see that (3.9) holds in this case. This completes the proof. $\square$

We may also present a sufficient condition for a sixth order three dimensional quasi-truncated Hankel tensor to be SOS.

Proposition 3.4. *Let $\mathcal{A}$ be a quasi-truncated Hankel tensor. Suppose that $v_0, v_6, v_{12} > 0$. Let $t_1, t_2 > 0$. If*

$$|v_1| \leq \frac{1}{t_1} - \frac{10 v_6}{t_1 \sqrt{v_0 v_{12}}} \tag{3.20}$$

$$|v_{11}| \leq \frac{1}{t_2} - \frac{10 v_6}{t_2 \sqrt{v_0 v_{12}}} \tag{3.21}$$

$$|v_1|\left(\frac{5}{t_1 v_0}\right)^5 + |v_{11}|\left(\frac{5}{t_2 v_{12}}\right)^5 \le \frac{\sqrt{70}-8}{2} v_6 \tag{3.22}$$

and

$$\left(v_0 - 10v_6\left(\frac{v_0}{v_{12}}\right)^{\frac{1}{2}} - |v_1|t_1v_0\right)\left(v_{12} - 10v_6\left(\frac{v_{12}}{v_0}\right)^{\frac{1}{2}} - |v_{11}|t_2v_{12}\right)$$

$$\times\left(\frac{\sqrt{70}-8}{2}v_6 - |v_1|\left(\frac{5}{t_1v_0}\right)^5 - |v_{11}|\left(\frac{5}{t_2v_{12}}\right)^5\right)$$

$$\ge \frac{1}{27}v_6^3(60+15\sqrt{70})^3 \tag{3.23}$$

hold, then $\mathcal{A}$ is SOS.

Proof. We write $f(\mathbf{x}) = \sum_{i=1}^5 f_i(\mathbf{x})$, where

$$f_2(\mathbf{x}) = 10v_6\left(\left(\frac{v_0}{v_{12}}\right)^{\frac{1}{4}} x_1^3 + \left(\frac{v_{12}}{v_0}\right)^{\frac{1}{4}} x_3^3\right)^2,$$

$$f_3(\mathbf{x}) = |v_1|t_1v_0x_1^6 + 6v_1x_1^5x_2 + |v_1|\left(\frac{5}{t_1v_0}\right)^5 x_2^6,$$

$$f_4(\mathbf{x}) = |v_{11}|t_2v_{12}x_3^6 + 6v_{11}x_3^5x_2 + |v_{11}|\left(\frac{5}{t_2v_{12}}\right)^5 x_2^6,$$

$$f_5(\mathbf{x}) = v_6\left(\sqrt{\frac{10-\sqrt{70}}{2}}x_2^3 + \sqrt{150+15\sqrt{70}}x_1x_2x_3\right)^2$$

and

$$\begin{aligned}
&f_1(\mathbf{x})\\
&= \left(v_0 - 10v_6\left(\frac{v_0}{v_{12}}\right)^{\frac{1}{2}} - |v_1|t_1v_0\right)x_1^6 - \left(v_{12} - 10v_6\left(\frac{v_{12}}{v_0}\right)^{\frac{1}{2}} - |v_{11}|t_2v_{12}\right)x_3^6\\
&+\left(\frac{\sqrt{70}-8}{2}v_6 - |v_1|\left(\frac{5}{t_1v_0}\right)^5 - |v_{11}|\left(\frac{5}{t_2v_{12}}\right)^5\right)x_2^6\\
&-v_6(60+15\sqrt{70})x_1^2x_2^2x_3^2.
\end{aligned}$$

Clearly, f_2 and f_5 are squares. From Lemma 1, we may show that f_3 and f_4 are PSD. Since each of f_3 and f_4 has only two variables, they are SOS. If (3.20-3.23) hold, by the arithmetic-geometric inequality, f_1 is PSD. In this case, f_1 is a PSD diagonal minus tail form. By [33], f_1 is SOS. Thus, if (3.20-3.23) hold, then f, hence $\mathcal{A}$, is SOS. $\square$

To get more insights for quasi-truncated Hankel tensors, we conduct some numerical tests for sixth order three dimensional quasi-truncated Hankel tensors. For simplicity purpose, we let $v_0 = v_{12}$ and $v_1 = v_{11}$. Note that in this case (3.18) holds. Thus, by Proposition 3.3, (3.9) holds, i.e., a necessary condition for $\mathcal{A}$ to be PSD is that $v_0 \geq 560 + 70\sqrt{70}$. Numerically, we observe that there is a function $\phi(\theta) \geq 0$, defined for $\theta \geq 560 + 70\sqrt{70}$ such that in this case, $\mathcal{A}$ is PSD if and only if $v_0 \geq 560 + 70\sqrt{70}$ and $|v_1| \leq \phi(v_0)$. In this case, $\mathcal{A}$ is also SOS. In the following, we give a table and a figure to sketch the graph of the function ϕ.

We get the value of ϕ by using the toolbox (Gloptipoly3 and SeDuMi) to confirm whether a sixth order three dimensional Hankel tensor is PSD or not and use the toolbox (YALMIP) [61] to test whether a sixth order three dimensional Hankel tensor is SOS or not. We tested ten different values of v_0 and the corresponding values of ϕ are in Table 3.1 and Figure3.1. Note that approximately

$$560 + 70\sqrt{70} \approx 1145.7.$$

Hence, no PNS Hankel tensors are found in this case.

3.1.4 Sixth order three dimensional anti-circulant tensors

In this section, we consider sixth order three dimensional Hankel tensors $\mathcal{A}$, satisfying

$$f(\mathbf{x}) \equiv \mathcal{A}\mathbf{x}^{\otimes 6} \equiv f(x_1, x_2, x_3) = f(x_2, x_3, x_1) = f(x_3, x_1, x_2). \tag{3.24}$$

Table 3.1: The values of ϕ for different v_0

v_0	ϕ	v_0	ϕ
1146	1.3034	1160	8.4925
1147	2.5853	1170	11.0947
1148	3.4183	1180	13.2144
1149	4.0855	1190	15.0563
1150	4.6585	1200	16.7130

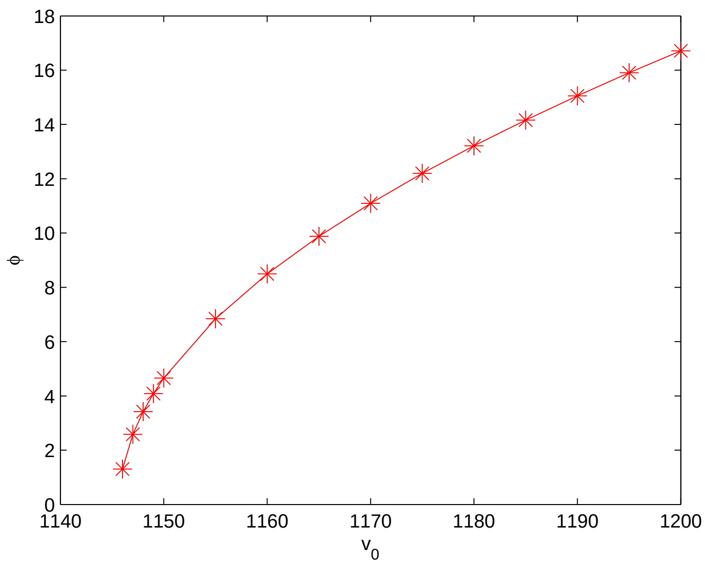

Figure 3.1: The value of ϕ

Notably, the all one tensor satisfies (3.24). Comparing (3.24) with (3.5), we find that the entries of the generating vector of a sixth order three dimensional anti-circulant tensor $\mathcal{A}$ satisfy

$$v_i = v_{i+3},$$

for $i = 0, \cdots, 9$. By [30], such a Hankel tensor is called an **anti-circulant tensor**. Thus, the generating vector of such a Hankel tensor has the following form

$$\mathbf{v} = (v_0, v_1, v_2, v_0, v_1, v_2, v_0, v_1, v_2, v_0, v_1, v_2, v_0)^\top \in \Re^{13}.$$

There are only three independent entries v_0, v_1 and v_2. Now, (3.5) has the simple form:

$$
\begin{aligned}
&f(\mathbf{x})\\
&= v_0\left[x_1^6 + x_2^6 + x_3^6 + 20(x_1^3x_2^3 + x_2^3x_3^3 + x_1^3x_3^3) + 30(x_1^4x_2x_3 + x_1x_2^4x_3 + x_1x_2x_3^4) + 90x_1^2x_2^2x_3^2\right]\\
&+v_1\left[6(x_1^5x_2 + x_2^5x_3 + x_1x_3^5) + 15(x_1^2x_2^4 + x_2^2x_3^4 + x_1^4x_3^2) + 60(x_1^3x_2^2x_3 + x_1x_2^3x_3^2 + x_1^2x_2x_3^3)\right]\\
&+v_2\left[6(x_1x_2^5 + x_2x_3^5 + x_1^5x_3) + 15(x_1^4x_2^2 + x_2^4x_3^2 + x_1^2x_3^4) + 60(x_1^2x_2^3x_3 + x_1^3x_2x_3^2 + x_1x_2^2x_3^3).\right]
\end{aligned}
\tag{3.25}
$$

Since we are only concerned about PSD Hankel tensors, we may assume that (3.3) holds which, in this case, means $v_0 \geq 0$.

Let us write

$$f(\mathbf{x}) = v_0 f_0(\mathbf{x}) + v_1 f_1(\mathbf{x}) + v_2 f_2(\mathbf{x}),$$

where f_0, f_1 and f_2 are given by

$$f_0(\mathbf{x}) = x_1^6 + x_2^6 + x_3^6 + 20(x_1^3x_2^3 + x_2^3x_3^3 + x_1^3x_3^3) + 30(x_1^4x_2x_3 + x_1x_2^4x_3 + x_1x_2x_3^4) + 90x_1^2x_2^2x_3^2, \tag{3.26}$$

$$f_1(\mathbf{x}) = 6(x_1^5x_2 + x_2^5x_3 + x_1x_3^5) + 15(x_1^2x_2^4 + x_2^2x_3^4 + x_1^4x_3^2) + 60(x_1^3x_2^2x_3 + x_1x_2^3x_3^2 + x_1^2x_2x_3^3), \tag{3.27}$$

$$f_2(\mathbf{x}) = 6(x_1x_2^5 + x_2x_3^5 + x_1^5x_3) + 15(x_1^4x_2^2 + x_2^4x_3^2 + x_1^2x_3^4) + 60(x_1^2x_2^3x_3 + x_1^3x_2x_3^2 + x_1x_2^2x_3^3). \tag{3.28}$$

Next, we provide a characterization for a sixth order three dimensional anti-circulant tensor $\mathcal{A}$ to be PSD.

Theorem 3.3. *Suppose that $\mathcal{A}$ is a sixth order three dimensional anti-circulant tensor. Then $\mathcal{A}$ is PSD if and only $v_0 = v_1 = v_2 \geq 0$. In this case,*

$$f(\mathbf{x}) = v_0(x_1 + x_2 + x_3)^6. \tag{3.29}$$

This implies that $\mathcal{A}$ is SOS if only if it is PSD.

Proof. Suppose that $\mathcal{A}$ is PSD. Then $f(1,-1,0) \geq 0$ and $f(1,1,-2) \geq 0$. From (3.25), we derive that $v_1 + v_2 \geq 2v_0$ and $v_1 + v_2 \leq 2v_0$ respectively. So, $v_1 + v_2 = 2v_0$. Let $v_1 = v_0(1+\alpha)$ with $\alpha \in \Re$. Then $v_2 = v_0(1-\alpha)$ and

$$f(\mathbf{x}) = v_0(x_1 + x_2 + x_3)^6 + v_0\alpha(f_1(\mathbf{x}) - f_2(\mathbf{x})),$$

where f_1 and f_2 are defined as in (3.27) and (3.28) respectively. From this and $f(1,2,-3) \geq 0$, we have $\alpha \geq 0$. From this and $f(1,-3,2) \geq 0$, we have $\alpha \leq 0$. Thus $\alpha = 0$ and (3.29) follows. $\square$

Thus, there are no sixth order three dimensional PNS anti-circulant tensors.

3.1.5 Sixth order three dimensional alternatively anti-circulant tensors

In this section, we consider sixth order three dimensional Hankel tensors $\mathcal{A}$, whose generating vector has the form $\mathbf{v} = (v_0, v_1, v_0, v_1, v_0, v_1, v_0, v_1, v_0, v_1, v_0, v_1, v_0)^\top$. We call such a Hankel tensor an **alternatively anti-circulant tensor**. Since we are only concerned about PSD Hankel tensors, we may assume that (3.3) holds, i.e., $v_0 \geq 0$. Now, (3.5) and (3.6) have the simple form

$$\begin{aligned} f(\mathbf{x}) \;=\;& v_0 \left[x_1^6 + x_2^6 + x_3^6 + 6(x_1^5x_3 + x_1x_3^5) + 20x_1^3x_3^3 + 30x_1x_2^4x_3 + 90x_1^2x_2^2x_3^2 \right.\\ & \left. +15(x_1^4x_2^2 + x_1^2x_2^4 + x_1^4x_3^2 + x_2^4x_3^2 + x_1^2x_3^4 + x_2^2x_3^4) + 60(x_1^3x_2^2x_3 + x_1x_2^2x_3^3)\right] \\ & +v_1 \left[6(x_1^5x_2 + x_1x_2^5 + x_2^5x_3 + x_2x_3^5) + 20(x_1^3x_2^3 + x_2^3x_3^3) \right.\\ & \left. +60(x_1^2x_2^3x_3 + x_1^3x_2x_3^2 + x_1x_2^3x_3^2 + x_1^2x_2x_3^3) + 30(x_1^4x_2x_3 + x_1x_2x_3^4)\right] \end{aligned} \tag{3.30}$$

and

$$
\begin{aligned}
g(\mathbf{y}) &= v_0(y_1^2+y_2^2+y_3^2+y_4^2+y_5^2+y_6^2+y_7^2) \\
&\quad +2v_0(y_1y_3+y_1y_5+y_2y_3+y_1y_7+y_2y_6+y_3y_5+y_3y_7+y_4y_6+y_5y_7) \\
&\quad +2v_1(y_1y_2+y_1y_4+y_2y_3+y_1y_6+y_2y_5+y_3y_4+y_2y_7+y_3y_6+y_4y_5 \\
&\quad +y_4y_7+y_5y_6+y_6y_7). \qquad (3.31)
\end{aligned}
$$

We have the following theorem which provides a characterization for a sixth order three dimensional alternatively anti-circulant tensor $\mathcal{A}$ to be PSD.

Theorem 3.4. *Suppose that $\mathcal{A}$ is a sixth order three dimensional alternatively anti-circulant tensor defined above. Then $\mathcal{A}$ is PSD if and only if $|v_1| \le v_0$. In this case, $\mathcal{A}$ is a strong Hankel tensor, and thus an SOS Hankel tensor.*

Proof. Suppose that $\mathcal{A}$ is PSD. From $f(1,1,0) \ge 0$ and (3.30), we have $v_0+v_1 \ge 0$. From $f(1,-1,0) \ge 0$ and (3.30), we have $v_0 - v_1 \ge 0$. This implies that $v_0 \ge |v_1|$. On the other hand, suppose that $v_0 \ge |v_1|$. We may write $v_1 = v_0(2t-1)$, where $t \in [0,1]$. Write $f(\mathbf{x}) = v_0 f_0(\mathbf{x}) + v_1 f_1(\mathbf{x})$ where

$$
\begin{aligned}
f_0(\mathbf{x}) &= x_1^6+x_2^6+x_3^6+6(x_1^5x_3+x_1x_3^5)+20x_1^3x_3^3+30x_1x_2^4x_3+90x_1^2x_2^2x_3^2 \\
&\quad +15(x_1^4x_2^2+x_1^2x_2^4+x_1^4x_3^2+x_2^4x_3^2+x_1^2x_3^4+x_2^2x_3^4)+60(x_1^3x_2^2x_3+x_1x_2^2x_3^3)
\end{aligned}
$$

and

$$
\begin{aligned}
f_1(\mathbf{x}) &= 6(x_1^5x_2+x_1x_2^5+x_2^5x_3+x_2x_3^5)+20(x_1^3x_2^3+x_2^3x_3^3)+30(x_1^4x_2x_3+x_1x_2x_3^4) \\
&\quad +60(x_1^2x_2^3x_3+x_1^3x_2x_3^2+x_1x_2^3x_3^2+x_1^2x_2x_3^3).
\end{aligned}
$$

It can be verfied that $f_0(\mathbf{x})+f_1(\mathbf{x}) = (x_1+x_2+x_3)^6$ and $f_0(\mathbf{x})-f_1(\mathbf{x}) = (x_1-x_2+x_3)^6$

for all $\mathbf{x} = (x_1, x_2, x_3)^\top \in \Re^{13}$. It then follows from (3.30) that

$$\begin{aligned} f(\mathbf{x}) = v_0 f_0(\mathbf{x}) + v_1 f_1(\mathbf{x}) &= v_0 f_0(\mathbf{x}) + (2t-1) v_0 f_1(\mathbf{x}) \\ &= t v_0 (f_0(\mathbf{x}) + f_1(\mathbf{x})) + (1-t) v_0 (f_0(\mathbf{x}) - f_1(\mathbf{x})) \\ &= t v_0 (x_1 + x_2 + x_3)^6 + (1-t) v_0 (x_1 - x_2 + x_3)^6. \end{aligned}$$

Similarly, we have

$$g(\mathbf{y}) = t v_0 (y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)^2 + (1-t) v_0 (y_1 - y_2 + y_3 - y_4 + y_5 - y_6 + y_7)^2.$$

The conclusions now follow from the definitions of PSD, SOS and strong Hankel tensors. □

Thus, there are no sixth order three dimensional PNS alternatively anti-circulant tensors. We also note that the above theorem can be easily extended to general even order alternatively anti-circulant tensors.

3.1.6 Numerical tests and a conjecture

In this section, we conduct numerical experiments to search sixth order three dimensional PNS Hankel tensors. We first explain how to generate a positive semi-definite Hankel tensor $\mathcal{A}_\alpha$ with a parameter α randomly, and determine a value α_0 such that $\mathcal{A}_\alpha$ is PSD if and only if $\alpha \geq \alpha_0$.

We first generate a vector $\mathbf{v} \in \Re^{13}$ randomly. We form a Hankel tensor $\mathcal{A}_0$ by using $\mathbf{v}$ as its generating vector. Then, we consider a parameterized tensor $\mathcal{A}_\alpha = \mathcal{A}_0 + \frac{\alpha}{2}(\mathcal{H} + \tilde{\mathcal{H}})$ where $\alpha \in \Re$, $\mathcal{H}$ is the sixth order three dimensional Hilbert tensor and $\tilde{\mathcal{H}}$ is the Hankel tensor generating by $\tilde{\mathbf{v}} = (\frac{1}{13}, \frac{1}{12}, \cdots, 1)^\top \in \mathbb{R}^{13}$. As $\mathcal{H}$ is a positive definite Hankel tensor, $\tilde{\mathcal{H}}$ is also positive definite Hankel tensor by Theorem 3.1. So, $\mathcal{A}_\alpha$ is also a Hankel tensor and $\mathcal{A}_\alpha$ is positive definite if α is large enough. We then find α_0 to be the smallest number α such that $\mathcal{A}_\alpha$ is positive semi-definite.

Here, α_0 can be negative if $\mathcal{A}_0$ is positive definite. Now we test if $\mathcal{A}_{\alpha_0}$ is SOS or not. If $\mathcal{A}_{\alpha_0}$ is not SOS, then we find a sixth order three dimensional PNS Hankel tensor. If $\mathcal{A}_{\alpha_0}$ is SOS, then we see that $\mathcal{A}_\alpha$ is also an SOS Hankel tensor if $\alpha > \alpha_0$, as

$$\mathcal{A}_\alpha = \mathcal{A}_{\alpha_0} + \frac{\alpha - \alpha_0}{2}(\mathcal{H} + \tilde{\mathcal{H}})$$

and $\frac{\alpha-\alpha_0}{2}(\mathcal{H} + \tilde{\mathcal{H}})$ is also an SOS Hankel tensor. Thus, if $\mathcal{A}_{\alpha_0}$ is SOS, then there is no α with $\alpha \geq \alpha_0$ such that $\mathcal{A}_\alpha$ is a PNS Hankel tensor.

We use $\frac{1}{2}(\mathcal{H}+\tilde{\mathcal{H}})$ as the reference positive definite Hankel tensor instead of using $\mathcal{H}$, as the entries of the generating vector of $\frac{1}{2}(\mathcal{H}+\tilde{\mathcal{H}})$ is distributed somewhat evenly. This makes our numerical tests more efficient in terms of finding $\mathcal{A}_{\alpha_0}$.

Due to numerical inaccuracy, instead of finding α_0, we find α_1 such that $\alpha_1 \geq \alpha_0$ and $\alpha_1 - \alpha_0 \leq \epsilon$, where ϵ is a given very small positive number. Then we test if $\mathcal{A}_{\alpha_1}$ is SOS or not.

Here, we use the toolbox (Gloptipoly3 and SeDuMi) to confirm whether a sixth order three dimensional Hankel tensor is PSD or not and use the toolbox (YALMIP) to test whether a sixth order three dimensional Hankel tensor is SOS or not. All codes were written by MATLAB 2014a and run on a Lenovo desktop computer with Core processor 2.83 GHz and 4 GB memory. We have not found any PNS Hankel tensor in six thousand tests.

Taking into account of the four special classes we examined and our numerical experiment, we now make the following conjecture:

There are no sixth order three dimensional PNS Hankel tensors.

If this conjecture turns out to be true, then determining a given sixth order three dimensional Hankel tensor is PSD or not can be solved by a semi-definite linear programming problem.

3.2 Fourth order four dimensional Hankel tensors

3.2.1 Introduction

Let $\mathbf{v} = (v_0, v_1, \dots, v_{12})^\top \in \Re^{13}$. A fourth order four dimensional Hankel tensor $\mathcal{A} = (a_{i_1 i_2 i_3 i_4})$ is defined by

$$a_{i_1 i_2 i_3 i_4} = v_{i_1+i_2+i_3+i_4-4},$$

for $i_1, i_2, i_3, i_4 = 1, 2, 3, 4$. The corresponding vector $\mathbf{v}$ that defines the Hankel tensor $\mathcal{A}$ is called the generating vector of $\mathcal{A}$. For $\mathbf{x} = (x_1, x_2, x_3, x_4)^\top \in \Re^4$, a Hankel tensor $\mathcal{A}$ uniquely defines a Hankel polynomial

$$f(\mathbf{x}) \equiv \mathcal{A}\mathbf{x}^{\otimes 4} = \sum_{i_1,i_2,i_3,i_4=1}^{4} a_{i_1 i_2 i_3 i_4} x_{i_1} x_{i_2} x_{i_3} x_{i_4} = \sum_{i_1,i_2,i_3,i_4=1}^{4} v_{i_1+i_2+i_3+i_4-4} x_{i_1} x_{i_2} x_{i_3} x_{i_4}. \tag{3.32}$$

We may see that the role of v_j is symmetric in $f(\mathbf{x})$. We assume that

$$v_j = v_{12-j} \tag{3.33}$$

for $j = 0, \dots, 5$. Under this assumption, if $\mathcal{A}$ is PSD, we have $v_0 = v_{12} \geq 0$ and $v_4 = v_8 \geq 0$. Moreover, if $v_4 = v_8 = 0$ and $\mathcal{A}$ is PSD, $\mathcal{A}$ is SOS. Thus, we may only consider the case that $v_4 = v_8 > 0$. Since $\mathcal{A}$ is PSD or SOS or PNS if and only if $\alpha\mathcal{A}$ is PSD or SOS or PNS respectively, where α is an arbitrary positive number, we may simply assume that

$$v_4 = v_8 = 1. \tag{3.34}$$

Next, we show that there is a function $\eta(v_5, v_6)$ such that $\eta(v_5, v_6) \leq 1$ if $\mathcal{A}$ is PSD. We propose that there are two functions $M_0(v_2, v_6, v_1, v_3, v_5) \geq N_0(v_2, v_6, v_1, v_3, v_5)$, defined for $\eta(v_5, v_6) < 1$, such that $\mathcal{A}$ is SOS if and only if $v_0 \geq M_0$, and $\mathcal{A}$ is PSD if and only if $v_0 \geq N_0$. If $M_0 = N_0$ for some v_2, v_6, v_1, v_3, v_5, then there are no fourth order four dimensional PNS Hankel tensors for such v_2, v_6, v_1, v_3, v_5 under the symmetric assumption (3.33). We call such a point $P = (v_2, v_6, v_1, v_3, v_5)^\top \in \Re^5$ a **PNS-free point** of fourth order four dimensional Hankel tensors, or simply a PNS-free point. We call the set of points in $\Re^5$, satisfying $\eta(v_5, v_6) < 1$, the **effective domain** of fourth order four dimensional Hankel tensors, or simply the effective domain, and denote it by S. We show that if all the points in S are PNS-free, then there are no fourth order four dimensional PNS Hankel tensors with symmetric generating vectors.

We show that a point P in S is PNS-free if there is a value M, such that when $v_0 = M$, $f_0(\mathbf{x}) \equiv f(\mathbf{x})$ has an SOS decomposition, and $f_0(\bar{\mathbf{x}}) = 0$ for $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4)^\top \in \Re^4$ with $\bar{x}_1^2 + \bar{x}_4^2 \neq 0$. We call such a value M, such an SOS decomposition of $f_0(\mathbf{x})$, and such a vector $\bar{\mathbf{x}}$ the **critical value**, the **critical SOS decomposition** and the **critical minimizer** of $\mathcal{A}$ at P, respectively. Then, we show that the segment $L = \{(v_2, v_6, v_1, v_3, v_5)^\top = (1, 1, t, t, t)^\top : t \in [-1, 1]\}$ is PNS-free. We conjecture that this segment is the minimizer set of both M_0 and N_0. Then, we show that the 45-degree planar closed convex cone $C = \{(v_2, v_6, v_1, v_3, v_5)^\top = (a, b, 0, 0, 0)^\top : a \geq b \geq 1\}$, the ray $R = \{(v_2, v_6, v_1, v_3, v_5)^\top = (a, 0, 0, 0, 0)^\top : a \leq 0\}$ and the point $A = (1, 0, 0, 0, 0)^\top$ are also PNS-free. We illustrate L, C, R and A in Figure 3.2.

Numerical tests check various grid points, and find that $M_0 = N_0$ there. Thus, they are also PNS-free. Therefore, numerical tests indicate that there are no fourth order four dimensional PNS Hankel tensors with symmetric generating vectors.

We write out (3.32) explicitly in terms of the coordinates of its generating vector

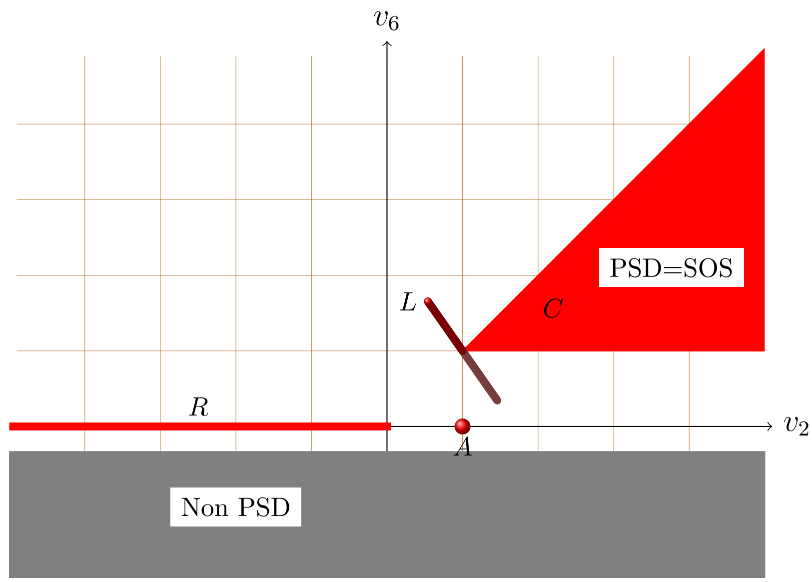

Figure 3.2: The segment L, the planar closed convex cone C, the ray R and the point A.

$\mathbf{v}$:

$$
\begin{aligned}
f(\mathbf{x}) \quad = \quad & v_0x_1^4 + 4v_1x_1^3x_2 + v_2(4x_1^3x_3 + 6x_1^2x_2^2) + v_3(4x_1x_2^3 + 4x_1^3x_4 + 12x_1^2x_2x_3) \\
& + v_4(x_2^4 + 6x_1^2x_3^2 + 12x_1x_2^2x_3 + 12x_1^2x_2x_4) \\
& + v_5(4x_2^3x_3 + 12x_1x_2x_3^2 + 12x_1x_2^2x_4 + 12x_1^2x_3x_4) \\
& + v_6(4x_1x_3^3 + 4x_2^3x_4 + 6x_1^2x_4^2 + 6x_2^2x_3^2 + 24x_1x_2x_3x_4) \\
& + v_7(4x_2x_3^3 + 12x_2^2x_3x_4 + 12x_1x_3^2x_4 + 12x_1x_2x_4^2) \\
& + v_8(x_3^4 + 6x_2^2x_4^2 + 12x_2x_3^2x_4 + 12x_1x_3x_4^2) \\
& + v_9(4x_3^3x_4 + 4x_1x_4^3 + 12x_2x_3x_4^2) + v_{10}(4x_2x_4^3 + 6x_3^2x_4^2) + 4v_{11}x_3x_4^3 + v_{12}x_4^4.
\end{aligned} \tag{3.35}
$$

The following theorem gives some necessary conditions for fourth order four dimensional Hankel tensors being PSD. Particularly, we note that four key elements of its generating vector v_0, v_4, v_8, v_{12} must be nonnegative.

Theorem 3.5. *Suppose that $\mathcal{A} = (a_{i_1i_2i_3i_4})$ is a Hankel tensor generated by its generating vector $\mathbf{v} = (v_0, v_1, \ldots, v_{12})^\top \in \Re^{13}$. If $\mathcal{A}$ is a PSD (or positive definite, or*

SOS, or strong) Hankel tensor, then we have

$$v_i \geq 0, \tag{3.36}$$

for $i = 0, 4, 8, 12$,

$$v_i + 6v_{i+2} + v_{i+4} \geq 4|v_{i+1} + v_{i+3}|, \tag{3.37}$$

for $i = 0, 4, 8$,

$$v_i + 6v_{i+4} + v_{i+8} \geq 4|v_{i+2} + v_{i+6}|, \tag{3.38}$$

for $i = 0, 4$, *and*

$$v_0 + 6v_6 + v_{12} \geq 4|v_3 + v_9|. \tag{3.39}$$

Proof. Let $\mathbf{e}_k$ be the kth column of a 4-by-4 identity matrix, for $k = 1, 2, 3, 4$. Substituting $\mathbf{x} = \mathbf{e}_k$ to (3.35) for $k = 1, 2, 3, 4$, by $f(\mathbf{e}_k) \geq 0$, we have (3.36) for $i = 0, 4, 8, 12$.

Substituting $\mathbf{x} = \mathbf{e}_k + \mathbf{e}_{k+1}$ to (3.35) for $k = 1, 2, 3$, by $f(\mathbf{e}_k + \mathbf{e}_{k+1}) \geq 0$, we have

$$v_i + 4v_{i+1} + 6v_{i+2} + 4v_{i+3} + v_{i+4} \geq 0,$$

for $i = 0, 4, 8$. Substituting $\mathbf{x} = \mathbf{e}_k - \mathbf{e}_{k+1}$ to (3.35) for $k = 1, 2, 3$, by $f(\mathbf{e}_k - \mathbf{e}_{k+1}) \geq 0$, we have

$$v_i - 4v_{i+1} + 6v_{i+2} - 4v_{i+3} + v_{i+4} \geq 0,$$

for $i = 0, 4, 8$. Combining these two inequalities, we have (3.37) for $i = 0, 4, 8$.

Similarly, by $f(\mathbf{e}_k + \mathbf{e}_{k+2}) \geq 0$ and $f(\mathbf{e}_k - \mathbf{e}_{k+2}) \geq 0$ for $k = 1, 2$, we have (3.38) for $i = 0, 4$. By $f(\mathbf{e}_1 + \mathbf{e}_4) \geq 0$ and $f(\mathbf{e}_1 - \mathbf{e}_4) \geq 0$, we have (3.39). The theorem is proved. $\square$

Whereafter, we say that a PSD Henkel tensor is SOS if there is a key element of its generating vector v_0, v_4, v_8, v_{12} vanishes. Before we show this, the following lemma is useful.

Lemma 3.2. *If a polynomial in one variable is always nonnegative:*

$$p(t) = a_0 t^{2k+1} + a_1 t^{2k} + \cdots + a_{2k+1} \geq 0, \qquad \forall\, t \in \Re.$$

Then $a_0 = 0$.

Proof. If $a_0 > 0$, we let $t \to -\infty$ and get $p(t) \to -\infty$, which contradicts that $p(t)$ is nonnegative.

If $a_0 < 0$, we let $t \to +\infty$ and get $p(t) \to -\infty$, which also contradicts that $p(t)$ is nonnegative.

Hence, there must be $a_0 = 0$. $\square$

Theorem 3.6. *Suppose the fourth order four dimensional Hankel tensor $\mathcal{A}$ is PSD and its generating vector is $\mathbf{v}$. If $v_0 v_{12} = 0$, then $v_j = 0$, for $j = 1, \ldots, 11$, and $\mathcal{A}$ is SOS.*

Proof. Without loss of generality, we assume that $v_0 = 0$.

To prove $v_1 = 0$, we take $\mathbf{x}_1 = (t, 1, 0, 0)^\top$. Then, the homogeneous polynomial (3.35) reduces to

$$f(\mathbf{x}_1) = 4v_1 t^3 + 6v_2 t^2 + 4v_3 t + v_4.$$

From Lemma 3.2, we have $v_1 = 0$ since $f(\mathbf{x}_1)$ is nonnegative. Similarly, we can prove $v_2 = v_3 = 0$ if we take $\mathbf{x}_2 = (t, 0, 1, 0)^\top$ and $\mathbf{x}_3 = (t, 0, 0, 1)^\top$ respectively.

From Theorem 3.5, we know $v_4 \geq 0$. When we take $\mathbf{x}_4 = (t^2, t, -\frac{1}{\sqrt{6}}, 0)^\top$, the homogeneous polynomial (3.35) reduces to

$$f(\mathbf{x}_4) = -(2\sqrt{6} - 2)v_4 t^4 + \mathcal{O}(t^3).$$

Let $t \to \infty$. Since $f(\mathbf{x}_4)$ is always nonnegative, we have $v_4 \leq 0$. Hence, there must be $v_4 = 0$.

If we take $\mathbf{x}_5 = (t^3, 0, t, 1)^\top$, the homogeneous polynomial (3.35) is

$$f(\mathbf{x}_5) = 12v_5 t^7 + \mathcal{O}(t^6).$$

From Lemma 3.2, we have $v_5 = 0$ since $f(\mathbf{x}_5)$ is nonnegative.

We take $\mathbf{x}_6 = (t, 0, 1, 0)^\top$. Then, the homogeneous polynomial (3.35) is

$$f(\mathbf{x}_6) = 4v_6 t + v_8.$$

From Lemma 3.2, we have $v_6 = 0$ since $f(\mathbf{x}_6)$ is nonnegative. Similarly, we can prove $v_7 = 0$ when we take $\mathbf{x}_7 = (0, t, 1, 0)^\top$.

We take $\mathbf{x}_8 = (t^4, 0, t, 1)^\top$. Then we have

$$f(\mathbf{x}_8) = 12v_8 t^5 + \mathcal{O}(t^4).$$

From Lemma 3.2, we have $v_8 = 0$ since the polynomial $f(\mathbf{x}_8)$ is nonnegative.

We could prove $v_9 = 0$, $v_{10} = 0$ and $v_{11} = 0$ if we takes $\mathbf{x}_9 = (t, 0, 0, 1)^\top$, $\mathbf{x}_{10} = (0, t, 0, 1)^\top$ and $\mathbf{x}_{11} = (0, 0, t, 1)^\top$, respectively.

Finally, since $v_0 = v_1 = \cdots = v_{11} = 0$, we have

$$f(\mathbf{x}) = v_{12} x_4^4.$$

By Theorem 3.5, we get $v_{12} \geq 0$. Hence, the Hankel tensor $\mathcal{A}$ is obviously SOS. $\square$

Theorem 3.7. *Suppose the fourth order four dimensional Hankel tensor $\mathcal{A}$ is PSD and its generating vector is $\mathbf{v}$. If $v_4 v_8 = 0$, then $v_j = 0$ for $j = 1, 2, \ldots, 11$, and $\mathcal{A}$ is SOS.*

Proof. By symmetry, we only need to prove this theorem under the condition $v_4 = 0$.

If we take $\mathbf{x}_1 = (1, t, 0, 0)^\top$, the homogeneous polynomial (3.35) reduces to

$$f(\mathbf{x}_1) = 4v_3 t^3 + 6v_2 t^2 + 4v_1 t + v_0.$$

From Lemma 3.2, we have $v_3 = 0$ since $f(\mathbf{x}_1)$ is nonnegative. Similarly, we can prove $v_5 = v_6 = 0$ if we take $\mathbf{x}_2 = (0, t, 1, 0)^\top$ and $\mathbf{x}_3 = (0, t, 0, 1)^\top$ respectively.

To prove $v_7 = 0$, we take $\mathbf{x}_4 = (0, t^2, t, 1)^\top$. Then, the homogeneous polynomial (3.35) reduces to

$$f(\mathbf{x}_4) = 16v_7t^5 + \mathcal{O}(t^4).$$

From Lemma 3.2, we have $v_7 = 0$ since $f(\mathbf{x}_4)$ is nonnegative.

From Theorem 3.5, we know $v_8 \geq 0$. When we take $\mathbf{x}_5 = (0, -t^2, t, 1)^\top$, the homogeneous polynomial (3.35) reduces to

$$f(\mathbf{x}_5) = -5v_8t^4 + \mathcal{O}(t^3).$$

Let $t \to \infty$. Since $f(\mathbf{x}_5)$ is always nonnegative, we have $v_8 \leq 0$. Hence, there must be $v_8 = 0$.

If we take $\mathbf{x}_6 = (0, 0, t, 1)^\top$, the homogeneous polynomial (3.35) is

$$f(\mathbf{x}_6) = 4v_9t^3 + \mathcal{O}(t^2).$$

From Lemma 3.2, we have $v_9 = 0$ since $f(\mathbf{x}_6)$ is nonnegative. Similarly, we could prove $v_{10} = 0$ and $v_{11} = 0$ if we takes $\mathbf{x}_7 = (0, t, 0, 1)^\top$ and $\mathbf{x}_8 = (0, 0, t, 1)^\top$, respectively.

The prove of $v_1 = 0$ and $v_2 = 0$ could be similarly obtained if we take $\mathbf{x}_9 = (1, t, 0, 0)^\top$ and $\mathbf{x}_{10} = (1, 0, t, 0)^\top$ respectively.

Finally, since $v_j = 0$ for $j = 1, \ldots, 11$, we have

$$f(\mathbf{x}) = v_0x_1^4 + v_{12}x_4^4.$$

By Theorem 3.5, we get $v_0 \geq 0$ and $v_{12} \geq 0$. Hence, the Hankel tensor $\mathcal{A}$ is obviously SOS. □

Now, we make assumptions (3.33) and (3.34). At the beginning, we consider a mini problem which is the Hankel polynomial with $x_1 = x_4 = 0$. This problem helps us to analyze the effective domain of two important surfaces M_0 and N_0.

We consider a two variable quartic polynomial

$$g(y_1,y_2)=\alpha y_1^4+4\beta y_1^3y_2+6\gamma y_1^2y_2^2+4\beta y_1y_2^3+\alpha y_2^4.$$

Its PSD property is completely characterized by the following theorem.

Theorem 3.8. *The quartic polynomial $g(y_1,y_2)$ is PSD if and only if*

$$\alpha\geq\eta(\beta,\gamma):=\begin{cases}4|\beta|-3\gamma & \text{if } \gamma\leq|\beta|,\\[2mm] \dfrac{3\gamma-\sqrt{9\gamma^2-8\beta^2}}{2} & \text{if } \gamma>|\beta|.\end{cases}$$

Proof. First, if $g(y_1,y_2)$ is PSD, from $g(1,-1)\geq 0$ and $g(1,1)\geq 0$, we have $\alpha\geq 4|\beta|-3\gamma$. Thus, in any case, $\eta(\beta,\gamma)\geq 4|\beta|-3\gamma$.

Second, suppose that $\alpha\geq 4|\beta|-3\gamma$. If $\gamma\leq 0$, we get

$$g(y_1,y_2)=(\alpha-4|\beta|+3\gamma)(y_1^4+y_2^4)+4|\beta|(y_1+y_2)^2(y_1^2-y_1y_2+y_2^2)-3\gamma(y_1^2-y_2^2)^2\geq 0.$$

If $0<\gamma\leq|\beta|$, we rewrite $g(y_1,y_2)$ as follows

$$g(y_1,y_2)=(\alpha-4|\beta|+3\gamma)(y_1^4+y_2^4)+(y_1+y_2)^2\left[(4|\beta|-3\gamma)(y_1^2+y_2^2)-(4|\beta|-6\gamma)y_1y_2\right].$$

Since $(4|\beta|-6\gamma)^2-4(4|\beta|-3\gamma)^2=-48|\beta|(|\beta|-\gamma)\leq 0$, it yields that $g(y_1,y_2)\geq 0$.

Finally, we consider the case $\gamma>|\beta|$. Let $\bar{\alpha}=\frac{3\gamma-\sqrt{9\gamma^2-8\beta^2}}{2}>0$. Then, we have

$$g(y_1,y_2)=(\alpha-\bar{\alpha})(y_1^4+y_2^4)+\bar{\alpha}\left(y_1^2+\frac{2\beta}{\bar{\alpha}}y_1y_2+y_2^2\right)^2.$$

Obviously, if $\alpha\geq\bar{\alpha}$, $g(y_1,y_2)$ is SOS and PSD.

Next, we show that $y_1^2+\frac{2\beta}{\bar{\alpha}}y_1y_2+y_2^2=0$ has nonzero real roots. For the convenience, we denote $t=\frac{y_1}{y_2}$ and prove that $t^2+\frac{2\beta}{\bar{\alpha}}t+1=0$ has real roots. It is easy to see that $t=0$ is not its root. Since $\gamma>|\beta|$, we have

$$\frac{|\beta|}{\bar{\alpha}}=\frac{2|\beta|}{3\gamma-\sqrt{9\gamma^2-8\beta^2}}=\frac{2|\beta|(3\gamma+\sqrt{9\gamma^2-8\beta^2})}{8\beta^2}\geq\frac{8|\beta|\gamma}{8\beta^2}\geq 1.$$

Hence, $|\beta| \geq \bar{\alpha}$. The discriminant of the quadratic in t is

$$\left(\frac{2\beta}{\bar{\alpha}}\right)^2 - 4 = 4\frac{\beta^2 - \bar{\alpha}^2}{\bar{\alpha}^2} \geq 0.$$

Therefore, there are nonzero (y_1, y_2) such that $g(y_1, y_2) = (\alpha - \bar{\alpha})(y_1^4 + y_2^4)$. Obviously, if $g(y_1, y_2)$ is PSD, we have $\alpha \geq \bar{\alpha}$. Thus, we say $\eta(\beta, \gamma) = \bar{\alpha}$ if $\gamma > |\beta|$. $\square$

Then we have another necessary condition for a fourth order four dimensional Hankel tensor $\mathcal{A}$ to be PSD under assumptions (3.33) and (3.34).

Corollary 3.1. *Under assumptions (3.33) and (3.34), if $\mathcal{A}$ is PSD, then $\eta(v_5, v_6) \leq 1$.*

Proof. Let $x_1 = x_4 = 0$, $x_2 = y_1$ and $x_3 = y_2$. By Theorem 3.8, we have the conclusion. $\square$

We now establish two surface M_0 and N_0, in the following theorem.

Theorem 3.9. *Suppose that assumptions (3.33) and (3.34) hold. Then, there are two functions $M_0(v_2, v_6, v_1, v_3, v_5) \geq N_0(v_2, v_6, v_1, v_3, v_5) > 0$ defined for*

$$\eta(v_5, v_6) < 1, \tag{3.40}$$

such that $\mathcal{A}$ is SOS if and only if $v_0 \geq M_0(v_2, v_6, v_1, v_3, v_5)$, and $\mathcal{A}$ is PSD if and only if $v_0 \geq N_0(v_2, v_6, v_1, v_3, v_5)$. If for all v_5 and v_6 satisfying (3.40), we have $M_0(v_2, v_6, v_1, v_3, v_5) = N_0(v_2, v_6, v_1, v_3, v_5)$, then there are no fourth order four dimensional PNS Hankel tensors under assumption (3.33).

Proof. Using assumptions (3.33) and (3.34), we rewrite (3.35) as

$$f(\mathbf{x}) = v_0(x_1^4 + x_4^4) + \bar{v}_4(x_2^4 + x_3^4) + f_1(\mathbf{x}) + f_2(\mathbf{x}),$$

where

$$f_1(\mathbf{x}) = \eta(v_5, v_6)(x_2^4 + x_3^4) + 4v_5(x_2^3 x_2 + x_2 x_3^3) + 6v_6 x_2^2 x_3^2$$

and

$$\bar{v}_4 = 1 - \eta(v_5, v_6).$$

Then $\bar{v}_4 > 0$ by (3.40). By Theorem 3.8, $f_1(\mathbf{x})$ is PSD. Since $f_1(\mathbf{x})$ has only two variables, it is also SOS by Hilbert [40, 82].

We now consider terms in $f_2(\mathbf{x})$. Each monomial in $f_2(\mathbf{x})$ has at least one factor as a power of x_1 or x_4. We may order the monomials of $f_2(\mathbf{x})$. For example, consider $12v_5x_1x_2x_3^2$. Assume that it is ordered as the kth monomial of $f_2(\mathbf{x})$. Then by the arithmetic-geometric inequality, we may see that

$$-12v_5x_1x_2x_3^2 \leq 3|v_5| \left(\frac{1}{\epsilon_k^3} x_1^4 + \epsilon_k x_2^4 + 2\epsilon_k x_3^4 \right),$$

where ϵ_k is a small positive number. We may let ϵ_k be small enough such that the sum of the coefficients for x_2^4 on the right hand side of the above inequality for all possible k is less than $\bar{v}_4$. By symmetry, the sum of the coefficients for x_3^4 on the right hand side of the above inequality for all possible k is less than $\bar{v}_4$. We see that

$$12v_5x_1x_2x_3^2 + 3|v_5| \left(\frac{1}{\epsilon_k^3} x_1^4 + \epsilon_k x_2^4 + 2\epsilon_k x_3^4 \right)$$

is a PSD diagonal minus tail form. By [33], it is SOS. Thus, as long as v_0 is big enough, when (3.40) is satisfied, $f(\mathbf{x})$ is SOS. From this, we see that M_0 and N_0 exist, such that they are defined as long as (3.40) is satisfied, $M_0 \geq N_0$, $\mathcal{A}$ is SOS if and only if $v_0 \geq M_0$, and $\mathcal{A}$ is PSD if and only if $v_0 \geq N_0$.

By Theorem 3.8, we now only need to consider the case that $\eta(v_5, v_6) = 1$. Suppose that for all v_5 and v_6 satisfying (3.40), we have $M_0(v_2, v_6, v_1, v_3, v_5) = N_0(v_2, v_6, v_1, v_3, v_5)$. Since the sets for PSD Hankel tensors and SOS Hankel tensors are closed [58], this implies that for all v_5 and v_6 satisfying $\eta(v_5, v_6) = 1$, we also have $M_0(v_2, v_6, v_1, v_3, v_5) = N_0(v_2, v_6, v_1, v_3, v_5)$, as long as N_0 is defined there. Thus, in this case, by Theorem 3.7, there are no fourth order four dimensional PNS Hankel tensors under assumption (3.33). $\square$

For the variables of M_0 and N_0, we put v_2 and v_6 before v_1, v_3 and v_5, as v_2, v_6 play a more important role in the PSD and SOS properties of $\mathcal{A}$, comparing with v_1, v_3 and v_5.

We now regard $P = (v_2, v_6, v_1, v_3, v_5)^\top$ as a point in $\Re^5$. If $M_0(P) = N_0(P)$, P is called a **PNS-free point**. We call

$$S = \{(v_2, v_6, v_1, v_3, v_5)^\top \in \Re^5 : \eta(v_5, v_6) < 1\}$$

the **effective domain**. Theorem 3.9 says that if all the points in the effective domain are PNS-free, then there are no fourth order four dimensional PNS Hankel tensors with symmetric generating vectors. In the next sections, we will study more on PNS-free points.

For the convenience, we present formally three ingredients used in theoretical proofs. If a point belongs to the effective domain and enjoys these ingredients, it is PNS-free.

Definition 3.1. *Suppose that assumptions (3.33) and (3.34) hold and* $P = (v_2, v_6, v_1, v_3, v_5)^\top \in S$. *Suppose that there is a number* M *such that* $\mathcal{A}$ *is SOS if* $v_0 = M$, *and a point* $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4)^\top \in \Re^4$ *such that* $\bar{x}_1^2 + \bar{x}_4^2 > 0$ *and* $f_0(\bar{\mathbf{x}}) = 0$, *where* $f_0(\mathbf{x}) \equiv f(\mathbf{x})$ *with* $v_0 = M$. *Then we call* M *the* **critical value** *of* $\mathcal{A}$ *at* P, *the SOS decomposition* $f_0(\mathbf{x})$ *the* **critical SOS decomposition** *of* $\mathcal{A}$ *at* P, *and* $\bar{\mathbf{x}}$ *the* **critical minimizer** *of* $\mathcal{A}$ *at* P.

Theorem 3.10. *Let* $P \in S$. *Then* P *is PNS-free if* $\mathcal{A}$ *has a critical value* M, *a critical SOS decomposition* $f_0(\mathbf{x})$ *and a critical minimizer* $\bar{\mathbf{x}}$ *at* P.

Proof. Suppose that $\mathcal{A}$ has a critical value M, a critical SOS decomposition $f_0(\mathbf{x})$ and a critical minimizer $\bar{\mathbf{x}}$ at P. Then we have $M \geq M_0(P)$ by the definition of M_0.

If $v_0 < M$, then

$$f(\bar{\mathbf{x}}) = (v_0 - M)(\bar{x}_1^4 + \bar{x}_4^4) + f_0(\bar{\mathbf{x}}) < 0.$$

This implies that $N_0(P) \geq M$ by the definition of N_0. But $N_0(P) \leq M_0(P)$. Thus, $M_0(P) = N_0(P) = M$, i.e., P is PNS-free. $\square$

We believe that all the effective domain S is PNS-free. In the next four subsections, we theoretically prove that some regions of S are PNS-free.

3.2.2 A PNS-free segment

We have the following theorem.

Theorem 3.11. *Suppose that* $P = (v_2, v_6, v_1, v_3, v_5)^\top = (1, 1, t, t, t)^\top$, *where* $t \in [-1, 1]$. *Then,* P *is PNS-free, with the critical value* 1 *and the critical minimizer* $(1, 0, -1, 0)^\top$.

Proof. For $P = (v_2, v_6, v_1, v_3, v_5)^\top = (1, 1, t, t, t)^\top$, where $t \in [-1, 1]$, and $M = 1$, we have

$$f_0(\mathbf{x}) = \frac{1+t}{2}(x_1 + x_2 + x_3 + x_4)^4 + \frac{1-t}{2}(x_1 - x_2 + x_3 - x_4)^4$$

is SOS, and

$$f_0(1, 0, -1, 0) = 0.$$

Hence, P is PNS-free. $\square$

By numerical experiments, we have the following conjecture.

Conjecture 1. *The segment* $L = \{(v_2, v_6, v_1, v_3, v_5)^\top = (1, 1, t, t, t)^\top : t \in [-1, 1]\}$, *is the minimizer set of both* M_0 *and* N_0.

3.2.3 A PNS-free planar cone

Theorem 3.12. *Suppose that* $P = (v_2, v_6, v_1, v_3, v_5)^\top = (v_2, v_6, 0, 0, 0)^\top$ *with* $v_2 \geq v_6 \geq 1$. *Then,* P *is PNS-free.*

If we parameterize $v_6 = b$ *and* $v_2 = (\theta + 3b - 1)(\theta^2 + (3b - 2)\theta - 3b + 4)$. *Then, the critical value at* P *is*

$$M = (\theta + 3b - 1)^2(3\theta^2 + (10b - 6)\theta + 3b^2 - 10b + 9)$$

and the critical minimizer is $\bar{\mathbf{x}} = (1, 0, -(\theta + 3b - 1), 0)^\top$.

Proof. Note that for $v_2 \geq v_6 \geq 1$, we may let $v_6 = b$ and $v_2 = (\theta + 3b - 1)(\theta^2 + (3b - 2)\theta - 3b + 4)$, where the parameter

$$\theta \geq \bar{\theta} = (b-1)^{\frac{1}{3}}(b+1)^{\frac{2}{3}} + (b-1)^{\frac{2}{3}}(b+1)^{\frac{1}{3}} - 2b + 1.$$

In fact, $\bar{\theta}$ is the largest real root of the cubic equation $v_2 - v_6 = 0$.

With the critical value as $M = (\theta + 3b - 1)^2(3\theta^2 + (10b - 6)\theta + 3b^2 - 10b + 9)$, the critical SOS decomposition at P is as follows

$$\begin{aligned} f_0(\mathbf{x}) &= \frac{1}{v_0}(v_0 x_1^2 + 2v_2 x_1 x_3 + \alpha_1 x_3^2)^2 + \frac{1}{v_0}(v_0 x_4^2 + 2v_2 x_2 x_4 + \alpha_1 x_2^2)^2 \\ &\quad + \alpha_2((\theta + 3b - 1)x_1 x_3 + x_3^2)^2 + \alpha_2((\theta + 3b - 1)x_2 x_4 + x_2^2)^2 \\ &\quad + \frac{6}{b}(x_1 x_2 + x_3 x_4 + b x_2 x_3 + b x_1 x_4)^2 + \frac{6(b^2 - 1)}{b}(x_1 x_2 + x_3 x_4)^2 \\ &\quad + 6(v_2 - b)[x_1^2 x_2^2 + x_3^2 x_4^2], \end{aligned}$$

where the involved parameters are as follows:

$$\begin{aligned} \alpha_1 &= -(\theta^2 + (4b - 2)\theta + 3b^2 - 4b + 1), \\ \alpha_2 &= \frac{2(\theta^2 + (4b - 2)\theta + b^2 - 4b + 4)}{3\theta^2 + (10b - 6)\theta + 3b^2 - 10b + 9}. \end{aligned}$$

Since $f_0(1, 0, -(\theta + 3b - 1), 0) = 0$, the corresponding critical minimizer is $\bar{\mathbf{x}} = (1, 0, -(\theta + 3b - 1), 0)^\top$. Hence, $P = (v_2, v_6, 0, 0, 0)^\top$ with $v_2 \geq v_6 \geq 1$ is PNS-free. $\square$

The cone $C = \{(v_2, v_6, v_1, v_3, v_5)^\top = (a, b, 0, 0, 0)^\top : a \geq b \geq 1\}$ is a 45-degree planar closed convex cone. Its end point is just the mid point of the segment $L = \{(v_2, v_6, v_1, v_3, v_5)^\top = (1, 1, t, t, t)^\top : t \in [-1, 1]\}$, discussed in the last subsection.

3.2.4 A PNS-free ray

In this subsection, we show that the ray $R = \{(v_2, v_6, v_1, v_3, v_5)^\top = (a, 0, 0, 0, 0)^\top : a \leq 0\}$ is PNS-free. Let $a = -\rho$, where $\rho \geq 0$ is a constant. We report that, at a point $P = (-\rho, 0, 0, 0, 0)^\top$, $\mathcal{A}$ has the critical value

$$M = 3\sqrt[3]{\theta_1 + 32\sqrt{\theta_2}} + \frac{\theta_3}{3\sqrt[3]{\theta_1 + 32\sqrt{\theta_2}}} + 6\rho^2 + 138\rho + 609,$$

where

$$\begin{aligned}
\theta_1 &:= -\rho^6 + 272\rho^5 + 12608\rho^4 + 204032\rho^3 + 1558528\rho^2 + 5750784\rho + 8290304, \\
\theta_2 &:= -(\rho + 6)^2(\rho + 4)^3(\rho^2 + 4\rho - 16)^3, \\
\theta_3 &:= 9(\rho + 8)(\rho^3 + 152\rho^2 + 1728\rho + 5120).
\end{aligned}$$

The function $f_0(\mathbf{x})$ enjoys a critical SOS decomposition:

$$f_0(\mathbf{x}) = \sum_{k=1}^{5} q_k^2(\mathbf{x}),$$

where

$$\begin{aligned}
q_1(\mathbf{x}) &= x_3^2 + 6x_2x_4 + \alpha_1 x_1^2 + \alpha_2 x_4^2, \\
q_2(\mathbf{x}) &= x_2^2 + 6x_1x_3 + \alpha_2 x_1^2 + \alpha_1 x_4^2, \\
q_3(\mathbf{x}) &= \alpha_3 x_2x_4 + \alpha_4 x_1^2 + \alpha_5 x_4^2, \\
q_4(\mathbf{x}) &= \alpha_3 x_1x_3 + \alpha_5 x_1^2 + \alpha_4 x_4^2, \\
q_5(\mathbf{x}) &= \alpha_6 x_1^2 - \alpha_6 x_4^2.
\end{aligned}$$

The involved parameters are listed as follows:

$$\begin{aligned}
\alpha_1 &= -\frac{(\rho+23)M_1(-\rho)-9\rho^3-21\rho^2+105\rho+9}{M_1(-\rho)+3\rho^2+6\rho-33}, \\
\alpha_2 &= -3\rho, \\
\alpha_3 &= \sqrt{-30-2\alpha_{15}}, \\
\alpha_4 &= \frac{6(1-\alpha_{15})}{\alpha_{33}}, \\
\alpha_5 &= \frac{16\rho}{\alpha_{33}}, \\
\alpha_6 &= \sqrt{-6\rho\alpha_{15}-\frac{192\rho(\alpha_{15}-1)}{\alpha_{33}^2}}.
\end{aligned}$$

Theorem 3.13. *Suppose that assumptions (3.33) and (3.34) hold. Then, for any constant $\rho \geq 0$, $P = (-\rho, 0, 0, 0, 0)^\top$ is PNS-free.*

Proof. We only need to prove that there is a critical minimizer. Let

$$\bar{\mathbf{x}} = (\alpha_{33}, \alpha_{35}+\alpha_{36}, -\alpha_{35}-\alpha_{36}, -\alpha_{33})^\top.$$

Then, we get $q_3(\bar{\mathbf{x}}) = q_4(\bar{\mathbf{x}}) = q_5(\bar{\mathbf{x}}) = 0$ immediately. Moreover, we have

$$q_1(\bar{\mathbf{x}}) = q_2(\bar{\mathbf{x}}) = (\alpha_{35}+\alpha_{36})^2 - 6(\alpha_{35}+\alpha_{36})\alpha_{33} + \alpha_{15}\alpha_{33}^2 - 3\rho\alpha_{33}^2 = 0.$$

We check the validation of the last equality by a mathematical software Maple. Hence, $f_0(\bar{\mathbf{x}}) = 0$ and $\bar{\mathbf{x}}$ is a critical minimizer at P. Hence, we get the conclusion by Theorem 3.10. $\square$

3.2.5 A PNS-free point

We now show that the point $A = (1, 0, 0, 0, 0)^\top$ is PNS-free. In fact, the critical value at A is

$$M = 477 + 3\sqrt[3]{3906351+9120\sqrt{57}} + \frac{74403}{\sqrt[3]{3906351+9120\sqrt{57}}}.$$

The critical SOS decomposition of $f_0(\mathbf{x})$ is as follows

$$f_0(\mathbf{x}) = \sum_{k=1}^{7} q_k(\mathbf{x})^2,$$

where

$$\begin{aligned}
q_1(\mathbf{x}) &= x_3^2 + 6x_2x_4 - 21x_1^2 + \alpha_1 x_4^2,\\
q_2(\mathbf{x}) &= x_2^2 + 6x_1x_3 - 21x_4^2 + \alpha_1 x_1^2,\\
q_3(\mathbf{x}) &= 2\sqrt{3}x_2x_4 + \alpha_2 x_1^2 + \alpha_3 x_4^2,\\
q_4(\mathbf{x}) &= 2\sqrt{3}x_1x_3 + \alpha_2 x_4^2 + \alpha_3 x_1^2,\\
q_5(\mathbf{x}) &= \alpha_4 x_1^2 - \alpha_4 x_4^2,\\
q_6(\mathbf{x}) &= \beta_1 x_1 x_2 + \beta_2 x_1 x_4,\\
q_7(\mathbf{x}) &= \beta_1 x_3 x_4 + \beta_2 x_1 x_4.
\end{aligned}$$

Some involved parameters are listed as follows:

$$\begin{aligned}
\beta_1 &= \frac{\sqrt{-6(M_2-36)(3M_2-4336)}}{\sqrt{M_2^2 - 1302M_2 + 25056}},\\
\beta_2 &= \frac{\beta_1(3\beta_1^2+116)}{\beta_1^2+12},\\
\alpha_1 &= 3 - \frac{1}{2}\beta_1^2,\\
\alpha_2 &= 22\sqrt{3} - \frac{\sqrt{3}}{6}\beta_1\beta_2,\\
\alpha_3 &= -\frac{8\sqrt{3}}{3} + \frac{\sqrt{3}}{2}\beta_1^2,\\
\alpha_4 &= \sqrt{-42\alpha_1 + 2\alpha_2\alpha_3 + \beta_2^2}.
\end{aligned}$$

Theorem 3.14. *Suppose that assumptions (3.33) and (3.34) hold. Then,* $A = (1,0,0,0,0)^\top$ *is PNS-free.*

$v_2 \setminus v_6$	$-.2$	$-.1$	0	.5	1	1.5	2	4
-4.0	3.54e4	8.74e3	3.76e3	4.78e2	3.12e2	3.92e2	6.23e2	6.37e3
-2.0	2.98e4	6.77e3	2.73e3	2.75e2	1.25e2	1.70e2	3.57e2	6.11e3
-1.0	2.72e4	5.85e3	2.26e3	1.91e2	6.15e1	9.26e1	2.73e2	6.06e3
-0.5	2.59e4	5.42e3	2.04e3	1.53e2	3.78e1	6.41e1	2.48e2	6.06e3
0.0	2.46e4	4.99e3	1.82e3	1.20e2	1.96e1	4.50e1	2.39e2	6.07e3
0.5	2.34e4	4.57e3	1.62e3	8.90e1	7.058	4.18e1	2.45e2	6.09e3
1.0	2.21e4	4.17e3	1.42e3	6.21e1	1.000	4.93e1	2.56e2	6.11e3
1.5	2.09e4	3.78e3	1.23e3	3.90e1	4.191	5.69e1	2.67e2	6.14e3
2.0	1.98e4	3.41e3	1.06e3	2.02e1	8.00e0	6.46e1	2.78e2	6.16e3
3.0	1.75e4	2.70e3	7.28e2	7.16e0	1.66e1	8.01e1	3.01e2	6.21e3
4.0	1.53e4	2.04e3	4.41e2	1.23e1	2.60e1	9.60e1	3.23e2	6.25e3

Table 3.2: The values of $M_0(v_2, v_6, 0, 0, 0) = N_0(v_2, v_6, 0, 0, 0)$ on some grid points.

Proof. Using the mathematical software Maple, we calculate

$$f(\mathbf{x}) - \sum_{k=1}^{7} q_k^2(\mathbf{x}) = \frac{-\beta_1^6 - 120\beta_1^4 + (4v_0 - 4944)\beta_1^2 + 48v_0 - 69376}{4(\beta_1^2 + 12)}(x_1^4 + x_4^4).$$

Substituting the value of $v_0 = M$ and β_1, we get $f_0(\mathbf{x}) - \sum_{k=1}^{7} q_k^2(\mathbf{x}) = 0$.

Let $\bar{\mathbf{x}} = (\beta_1, \beta_2, -\beta_2, -\beta_1)^\top$. Obviously, we obtain $q_5(\bar{\mathbf{x}}) = q_6(\bar{\mathbf{x}}) = q_7(\bar{\mathbf{x}}) = 0$. We find that $q_3(\bar{\mathbf{x}})$ and $q_4(\bar{\mathbf{x}})$ vanishes if we rewrite all the parameters using β_1. Using the value of each parameter, we find that $q_1(\bar{\mathbf{x}}) = q_2(\bar{\mathbf{x}}) = 0$. Since $\bar{x}_1 = \beta_1 \approx 1.73$, $\bar{\mathbf{x}}$ is the critical minimizer. Therefore, this theorem is valid according to Theorem 3.10. □

3.2.6 Numerical experiments

We have proved that some regions are PNS-free. What about the other cases? We try to answer this problem by a numerical approach. We use the YALMIP software with an SOS module [61, 60] to compute $M_0(v_2, v_6, v_1, v_3, v_5)$, which is the smallest value of v_0 such that the fourth order four dimensional Hankel tensor $\mathcal{A}$ with the generating vector $(v_0, v_1, v_2, v_3, 1, v_5, v_6, v_5, 1, v_3, v_2, v_1, v_0)^\top$ is SOS. Gloptipoly [25] and SeDuMi [88] are employed to compute $N_0(v_2, v_6, v_1, v_3, v_5)$, which is the smallest

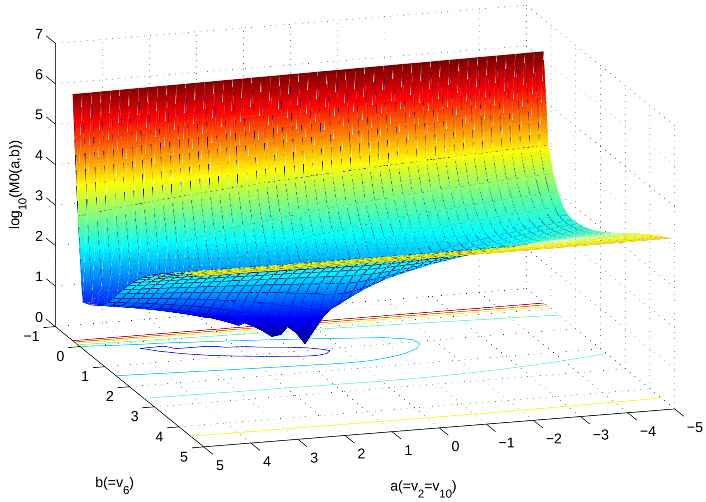

Figure 3.3: The contour profile of $M_0(v_2, v_6, 0, 0, 0) = N_0(v_2, v_6, 0, 0, 0)$.

value of v_0 such that the Hankel tensor $\mathcal{A}$ is PSD.

First, we focus on two elements v_2 and v_6 of generating vectors and set $v_1 = v_3 = v_5 = 0$. By Theorem 3.8, owing to the effective domain, we have $b > -\frac{1}{3}$. We choose $v_2 = -4, -2, -1, -0.5, 0, 0.5, 1, 1.5, 2, 3, 4$ and $v_6 = -0.2, -0.1, 0, 0.5, 1, 1.5, 2, 4$ and compute M_0 and N_0 in these grid points respectively. By our experiments, we found

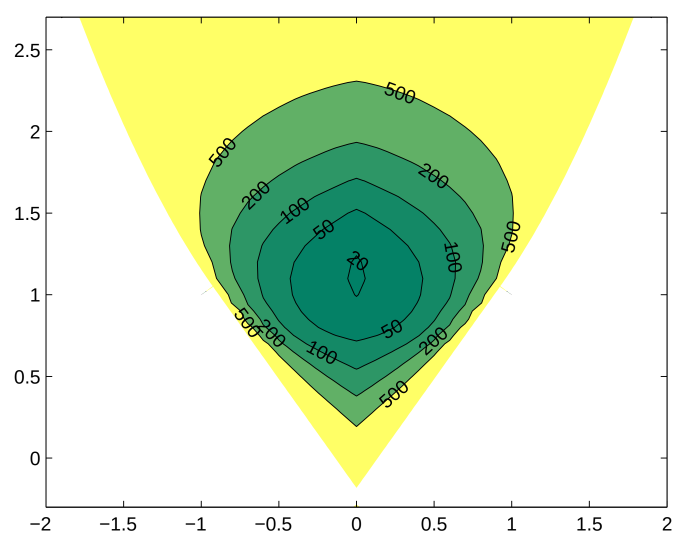

Figure 3.4: The contour profile of $M_0(0, v_6, 0, 0, v_5)$.

that these two functions are equivalent on all of the grid points. Thus, no PNS tensors are detected here. The detailed value of M_0 and N_0 are reported in Table 3.2.

A more intuitional profile of $M_0 = N_0$ is illustrated in Figure 3.3. It is easy to see that $(v_2, v_6) = (1, 1)$ is the minimizer of both M_0 and N_0 when we set $v_1 = v_3 = v_5 = 0$.

We consider the case that the generating vector of a fourth order four dimensional Hankel tensor has nonzero odd elements. According to Theorem 3.9, we say that v_5 and v_6 must satisfy $\eta(v_5, v_6) < 1$. So we study them first and set $v_1 = v_2 = v_3 = 0$. We compute a plenty of grid points with different v_5 and v_6. The function $M_0(0, v_6, 0, 0, v_5)$ is still equivalent to the function $N_0(0, v_6, 0, 0, v_5)$. That is to say, no PNS tensors are found.

The contour of $M_0(0, v_6, 0, 0, v_5) = M_0(0, v_6, 0, 0, v_5)$ is shown in Figure 3.4. We could see that the nonlinear contour of $M_0 = N_0 = 500$ looks like a fire balloon.

Finally, we consider all of the elements of symmetric generating vectors of fourth order four dimensional Hankel tensors. The contours of $M_0(v_2, v_6, v_1, v_3, v_5)$ and $N_0(v_2, v_6, v_1, v_3, v_5)$ for various combinations of v_2, v_6, v_1, v_3 and v_5 are reported in Figure 3.5. In all of our tests, values of the function $M_0(v_2, v_6, v_1, v_3, v_5)$ in grid points are always equivalent to the corresponding values of the function $N_0(v_2, v_6, v_1, v_3, v_5)$. So, no fourth order four dimensional PNS Hankel tensors with symmetric generating vectors are detected.

From Figures 3.4 and 3.5, we could say that the second element v_1 of the generating vector of a Hankel tensor affect functions $M_0(v_2, v_6, v_1, v_3, v_5)$ and $N_0(v_2, v_6, v_1, v_3, v_5)$ slightly. When we fix $v_4 = 1$, the middle element v_6 of the generating vector $\mathbf{v}$ plays a more important role since it has direct impact on the effective domain.

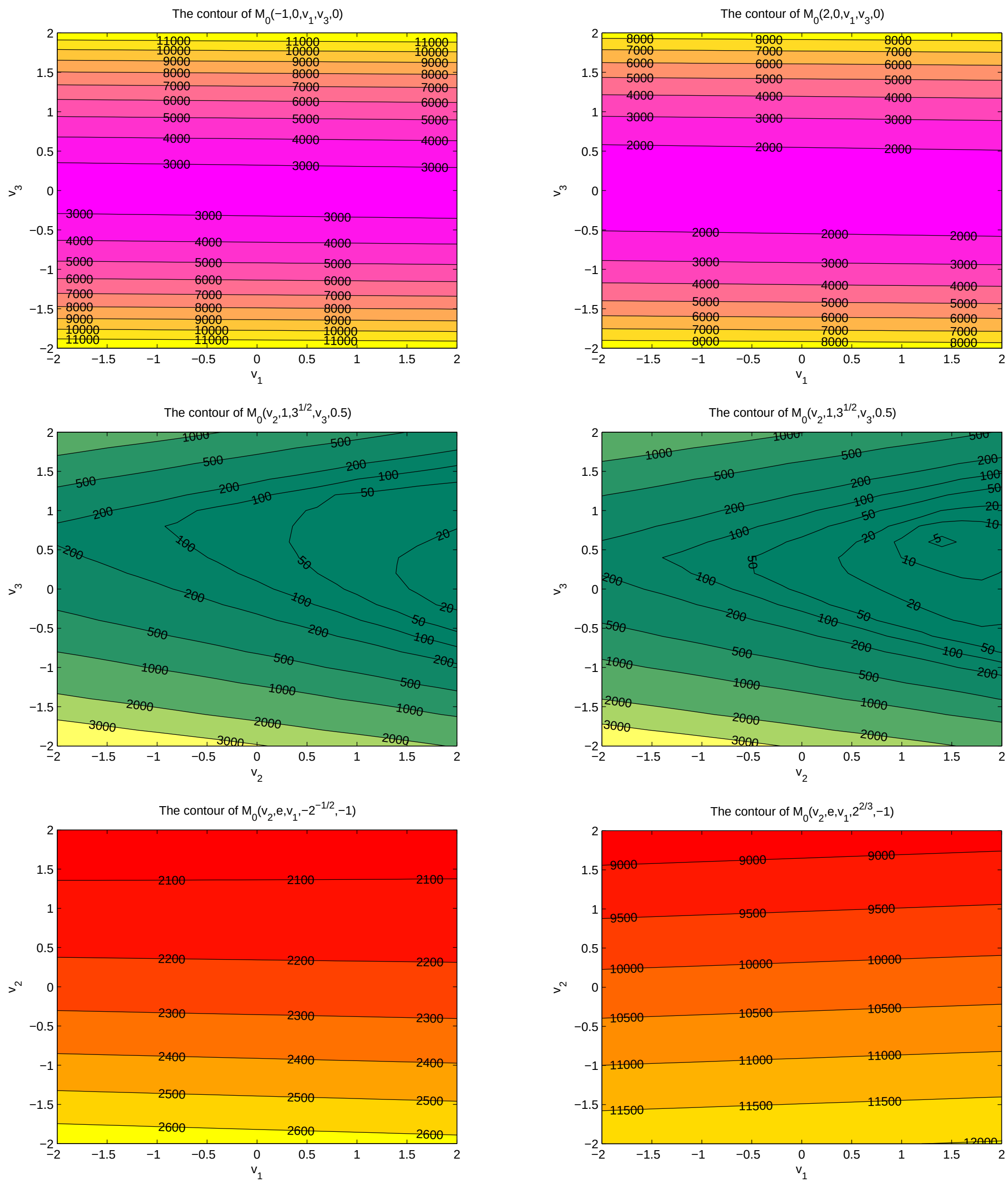

Figure 3.5: The contour profiles of $M_0(v_2, v_6, v_1, v_3, v_5)$ which are equivalent to $N_0(v_2, v_6, v_1, v_3, v_5)$.

Chapter 4

Generalized Anti-circulant Tensors

Anti-circulant tensors have applications in exponential data fitting. They are special Hankel tensors. In this chapter, we extend the definition of anti-circulant tensors to generalized anti-circulant tensors by introducing a circulant index r such that the entries of the generating vector of a Hankel tensor are circulant with module r. In the special case when $r = n$, where n is the dimension of the Hankel tensor, the generalized anticirculant tensor reduces to the anti-circulant tensor. Hence, generalized anti-circulant tensors are still special Hankel tensors. For the cases that $GCD(m, r) = 1$, $GCD(m, r) = 2$ and some other cases, including the matrix case that $m = 2$, we give necessary and sufficient conditions for positive semi-definiteness of even order generalized anti-circulant tensors, and show that in these cases, they are sum of squares tensors. This shows that, in these cases, there are no PNS (positive semidefinite tensors which are not sum of squares) Hankel tensors.

4.1 Introduction

Anti-circulant tensors were introduced in [30]. They are extensions of anti-circulant matrices in matrix theory [27, 94]. They have applications in exponential data fitting [30]. Anti-circulant tensors are Hankel tensors. Hankel tensors arise from signal processing and some other applications [4, 72, 74].

Let $\mathbf{v} = (v_0, \cdots, v_{(n-1)m})^\top \in \Re^{(n-1)m+1}$, where $m, n \geq 2$. An mth order n dimensional Hankel tensor $\mathcal{A} = (a_{i_1\cdots i_m})$ is defined by

$$a_{i_1\cdots i_m} = v_{i_1+\cdots+i_m-m},$$

for $i_1, \cdots, i_m = 1, \cdots, n$. If

$$v_i = v_{i+r}, \tag{4.1}$$

for $i = 0, \cdots, (n-1)m - r$, where $1 \leq r \leq n$, then $\mathcal{A}$ is called a **generalized anti-circulant tensor** with **circulant index** r. If $r = n$, then $\mathcal{A}$ is an anti-circulant tensor according to [30].

For $\mathbf{x} = (x_1, \cdots, x_n)^\top \in \Re^n$, $\mathcal{A}$ uniquely define a homogeneous polynomial

$$f(\mathbf{x}) \equiv \mathcal{A}\mathbf{x}^{\otimes m} = \sum_{i_1,\cdots,i_m=1}^{n} a_{i_1\cdots i_m} x_{i_1} \cdots x_{i_m} = \sum_{i_1,\cdots,i_m=1}^{n} v_{i_1+\cdots+i_m-m} x_{i_1} \cdots x_{i_m}. \tag{4.2}$$

We call such a polynomial a **Hankel polynomial**. By [74], a necessary condition for $\mathcal{A}$ to be PSD is that

$$v_{jm} \geq 0, \tag{4.3}$$

for $j = 0, \cdots, n-1$. In [58], it was proved that an even order strong Hankel tensor is an SOS Hankel tensor. Then, a necessary condition for $\mathcal{A}$ to be a strong Hankel tensor is that

$$v_{2j} \geq 0, \tag{4.4}$$

for $j = 0, \cdots, (n-1)k$.

When r is odd, for the case that $m = 2k, k \geq 1, GCD(m, r) = 1$ and $n \geq r$, we show that $\mathcal{A}$ is PSD if and only if $v_0 = \cdots = v_{r-1} \geq 0$. In this case, we show that

$$f(\mathbf{x}) = v_0(x_1 + \cdots + x_n)^m,$$

and $\mathcal{A}$ is a strong Hankel tensor. We show that this result is still true for $r = 3, n \geq r$ and $m = 6, 12, 18, 30, 42$.

When r is even, for the case that $m = 2k, k \geq 1, GCD(m, r) = 2$ and $n \geq r$, we show that $\mathcal{A}$ is PSD if and only if $v_0 = v_2 = \cdots = v_{r-2}$, $v_1 = v_3 = \cdots = v_{r-1}$, and $v_0 \geq |v_1|$. In these cases, we may write $v_1 = v_0(2t - 1)$, where $t \in [0, 1]$. We show that

$$f(\mathbf{x}) = tv_0(x_1 + \cdots x_n)^m + (1 - t)v_0 \left(x_1 - x_2 + x_3 - \cdots + (-1)^{n-1}x_n\right)^m,$$

and $\mathcal{A}$ is a strong Hankel tensor. We show that this result is still true in the case that $m = 4$, $r = 4$ and $n \geq 4$.

Note that these two results are true in the matrix case for all $r \geq 1$. In fact, in the matrix case, for even r, we show the result is true as long as $2 \leq r \equiv 2p \leq 2n - 4$. We believe that our results are new even in the matrix case.

4.2 A theorem on circulant numbers

We have the following theorem.

Theorem 4.1. *Let $M \geq 1$ and $p \geq 2$. Suppose that we have a sequence $\{u_j : j = 0, 1, \cdots\}$, satisfying*

$$u_{j+p} = u_j,$$

for $j = 0, 1, \cdots$. If

$$\sum_{j=0}^{M} \binom{M}{j} (-1)^j u_{i+j} \geq 0, \tag{4.5}$$

for $i = 0, \cdots, p - 1$, or

$$\sum_{j=0}^{M} \binom{M}{j} (-1)^j u_{i+j} \leq 0, \tag{4.6}$$

for $i = 0, \cdots, p - 1$, then $u_0 = u_1 = \cdots = u_{p-1}$.

Proof. We may prove this theorem by induction on M. Obviously, it is true for $M = 1, 2$. Suppose that it is true for $M = 2, \cdots, k$. We now prove that it is true for $M = k + 1$. Define

$$w_i = \sum_{j=0}^{k} \binom{k}{j} (-1)^j u_{i+j}$$

for $i = 0, \cdots, p-1$. Then $w_{i+p} = w_i$ for $i = 0, \cdots, p-1$. Suppose that (4.5) holds for $M = k + 1$. Note that

$$\begin{aligned}
&w_i - w_{i+1} \\
&= \left(u_0 + \sum_{j=1}^{k} \binom{k}{j} (-1)^j u_{i+j} \right) - \left(\sum_{j=0}^{k-1} \binom{k}{j} (-1)^j u_{i+j+1} + (-1)^k u_{i+k+1} \right) \\
&= u_0 + \left(\sum_{j=1}^{k} \left(\binom{k}{j} + \binom{k}{j-1} \right) (-1)^j u_{i+j} \right) + (-1)^{k+1} u_{i+k+1} \\
&= u_0 + \left(\sum_{j=1}^{k} \binom{k+1}{j} (-1)^j u_{i+j} \right) + (-1)^{k+1} u_{i+k+1} \\
&= \sum_{j=0}^{k+1} \binom{k+1}{j} (-1)^j u_{i+j}.
\end{aligned}$$

Then (4.5) is equivalent to

$$w_i - w_{i+1} \geq 0$$

for $i = 0, \cdots, p-1$. This implies that $w_0 = w_1 \cdots = w_{p-1}$. Thus, either (4.5) or (4.6) holds for $M = k$. By our induction assumption, we have $u_0 = u_1 = \cdots = u_{p-1}$. Similarly, if (4.6) holds for $M = k+1$, we may show that $u_0 = u_1 = \cdots = u_{p-1}$. This proves the theorem. $\square$

4.3 The case that r is odd

4.3.1 The case that $r = 1$

This case is trivial. However, we present the statement of the result for the reader's convenience here, as it covers the sufficiency part of the results for the cases that r is an odd number with $r \geq 3$.

Proposition 4.1. *Suppose that $\mathcal{A}$ is an mth order n dimensional generalized anti-circulant tensor with circulant index 1, where $m = 2k \geq 2$ and $n \geq 2$. Then $\mathcal{A}$ is PSD if and only if $v_0 \geq 0$. In this case, we have*

$$f(\mathbf{x}) = v_0(x_1 + \cdots + x_n)^m. \tag{4.7}$$

and

$$\mathbf{y}^\top A\mathbf{y} = v_0 \left(y_1 + \cdots + y_{(nk-k+1)}\right)^2, \tag{4.8}$$

where A is the associated Hankel matrix, which implies that $\mathcal{A}$ is a strong Hankel tensor and hence an SOS Hankel tensor.

The proof is trivial and we omit it here.

4.3.2 The case that $GCD(m, r) = 1$

We have the following theorem.

Theorem 4.2. *Suppose that $\mathcal{A}$ is an mth order n dimensional generalized anti-circulant tensor with $m = 2k$, $GCD(m, r) = 1$, $1 \leq r \leq n$ and $k \geq 1$. Then $\mathcal{A}$ is PSD if and only if $v_0 = \cdots = v_{r-1} \geq 0$. In this case, we still have (4.7) and (4.8), which implies that $\mathcal{A}$ is a strong Hankel tensor and hence an SOS Hankel tensor.*

Proof. Suppose that $\mathcal{A}$ is PSD. Let $\mathbf{x} = \mathbf{e}_q - \mathbf{e}_{q+1}$ for $q = 1, \cdots, n$, with $\mathbf{e}_{n+1} \equiv \mathbf{e}_1$. From $f(\mathbf{x}) \geq 0$, we have

$$\sum_{j=0}^{m} \binom{m}{j} (-1)^j v_{(q-1)m+j} \geq 0, \tag{4.9}$$

for $q = 1, \cdots, n$. Since $GCD(m, r) = 1$, for each $i = 0, \cdots, r-1$, there is an integer q, $1 \le q \le n$ such that $(q-1)m = i$, $\mathrm{mod}(r)$. Then $v_{i+j} = v_{(q-1)m+j}$ for such i, q and $j = 0, \cdots, m$. Thus, (4.9) implies that

$$\sum_{j=0}^{m} \binom{m}{j} (-1)^j v_{i+j} \ge 0,$$

for $i = 0, \cdots, r-1$. Applying Theorem 4.1 with $M = m, u_j = v_j$ and $p = r$, we have $v_0 = \cdots = v_{r-1}$. By (4.3), $v_0 \ge 0$. Thus, we have $v_0 = \cdots = v_{r-1} \ge 0$.

The "if" part follows from Proposition 4.1. $\square$

4.3.3 The case that $GCD(m, r) \neq 1$

The case that $GCD(m, r) \neq 1$ and r is odd includes the case that $r = 3, m = 6l$ for $l \ge 1$, the case that $r = 5, m = 10l$ for $l \ge 1$, etc. By [57], Theorem 4.2 still holds for the case that $m = 6$ and $r = 3$. We may see that Theorem 4.2 still holds for more cases that $GCD(m, r) \neq 1$ and r is odd.

We now assume that $m = 6l$, $r = 3$ for $l \ge 1$.

In this case, (4.1) and (4.2) have the following form:

$$v_i = v_{i+3} \tag{4.10}$$

for $i = 0, \cdots, (n-1)m - 3$, and for $\mathbf{x} = (x_1, \cdots, x_n)^\top \in \Re^n$,

$$f(\mathbf{x}) \equiv \mathcal{A}\mathbf{x}^{\otimes m} = \sum_{i_1, \cdots, i_m = 1}^{n} v_{i_1 + \cdots + i_m} x_{i_1} \cdots x_{i_m} = v_0 f_0(\mathbf{x}) + v_1 f_1(\mathbf{x}) + v_2 f_2(\mathbf{x}), \tag{4.11}$$

where

$$f_j(\mathbf{x}) = \sum \left\{ x_{i_1} \cdots x_{i_m} : i_1 + \cdots + i_m = j, \ \mathrm{mod}(3), i_1, \cdots, i_m = 1, \cdots, n \right\}, \tag{4.12}$$

for $j = 0, 1, 2$. We may see that

$$f_0(\mathbf{x}) + f_1(\mathbf{x}) + f_2(\mathbf{x}) = (x_1 + \cdots + x_n)^m. \tag{4.13}$$

Since we are concerned about PSD generalized anti-circulant tensors, we may assume that (4.3) holds, i.e., $v_0 \geq 0$.

Proposition 4.2. *Suppose that $\mathcal{A}$ is an mth order n dimensional generalized anti-circulant tensor with circulant index* 3*, where $m = 6, 12, 18, 30, 42$ and $n \geq 3$. Assume that $v_0 \geq 0$. If $\mathcal{A}$ is PSD, then*

$$v_1 + v_2 = 2v_0. \tag{4.14}$$

Proof. Suppose that $\mathcal{A}$ is PSD and $v_0 \geq 0$. Then we have $f(1, -1, 0, \cdots, 0) \geq 0$. Note that

$$\begin{aligned}
&f_0(1, -1, 0, \cdots, 0) \\
&= \sum \{x_{i_1} \cdots x_{i_m} : i_1 + \cdots + i_m = 0,\ \mathrm{mod}(3),\ i_1, \cdots, i_m = 1, 2,\ x_1 = 1,\ x_2 = -1\} \\
&= \sum \{x_{i_1} \cdots x_{i_m} : \text{the number of } i_j = 1 \text{ is } m - p,\ \text{the number of } i_j = 2 \text{ is } p, \\
&\qquad\qquad p = 0, 3, \cdots, m,\ x_1 = 1,\ x_2 = -1\} \\
&= \sum \left\{ (-1)^p \binom{m}{p} : p = 0, 3, \cdots, m \right\}.
\end{aligned}$$

Similarly, we can prove that

$$f_1(1, -1, 0, \cdots, 0) = \sum \left\{ (-1)^p \binom{m}{p} : p = 1, 4, \cdots, m - 2 \right\}$$

and

$$f_2(1, -1, 0, \cdots, 0) = \sum \left\{ (-1)^p \binom{m}{p} : p = 2, 5, \cdots, m - 1 \right\}.$$

By direct calculation, for $m = 6, 18, 30, 42$, we have

$$f_0(1, -1, 0, \cdots, 0) < 0. \tag{4.15}$$

Since $\binom{m}{p} \equiv \binom{m}{m-p}$, we have

$$f_1(1, -1, 0, \cdots, 0) = f_2(1, -1, 0, \cdots, 0). \tag{4.16}$$

By (4.13),

$$f_0(1,-1,0,\cdots,0)+f_1(1,-1,0,\cdots,0)+f_2(1,-1,0,\cdots,0)=0. \tag{4.17}$$

By (4.11), (4.15-4.17) and $f(1,-1,0,\cdots,0)\geq 0$, we have $v_1+v_2-2v_0\geq 0$.

On the other hand, for $m=6,18,30,42$,

$$\begin{aligned}
&f_0(1,1,-2,0,\cdots,0)\\
&=\sum\{x_{i_1}\cdots x_{i_m}: i_1+\cdots+i_m=0,\ \mathrm{mod}(3),\ i_1,\cdots,i_m=1,2,3,\ x_1=1,\\
&\qquad x_2=1,\ x_3=-2\}\\
&=\sum\{x_{i_1}\cdots x_{i_m}: \text{the number of } i_j=1 \text{ is } m-p-q,\ \text{the number of } i_j=2\\
&\qquad \text{is } q, \text{the number of } i_j=3 \text{ is } p,\ 2p+q=0,\ \mathrm{mod}(3),\ 0\leq p,\ q\leq m,\\
&\qquad x_1=1,\ x_2=1,\ x_3=-2\}\\
&=\sum_{p=0}^{m}\sum\{x_{i_1}\cdots x_{i_m}: \text{the number of } i_j=1 \text{ is } m-p-q,\ \text{the number of } i_j=2\\
&\qquad \text{is } q,\ \text{the number of } i_j=3 \text{ is } p,\ 2p+q=0,\ \mathrm{mod}(3),\ 0\leq q\leq m,\\
&\qquad x_1=1,\ x_2=1,\ x_3=-2\}\\
&=\sum_{p=0}^{m}\sum\left\{(-2)^p\binom{m}{p}\binom{m-p}{q}: 2p+q=0,\ \mathrm{mod}(3),\ 0\leq q\leq m\right\}.
\end{aligned}$$

Similarly, we can prove that

$$\begin{aligned}
&f_1(1,1,-2,0,\cdots,0)\\
&=\sum_{p=0}^{m}\sum\left\{(-2)^p\binom{m}{p}\binom{m-p}{q}: 2p+q=1,\ \mathrm{mod}(3),\ 0\leq q\leq m\right\}
\end{aligned}$$

and

$$\begin{aligned}
&f_2(1,1,-2,0,\cdots,0)\\
&=\sum_{p=0}^{m}\sum\left\{(-2)^p\binom{m}{p}\binom{m-p}{q}: 2p+q=2,\ \mathrm{mod}(3),\ 0\leq q\leq m\right\}.
\end{aligned}$$

By direct calculation, for $m = 6, 18, 30, 42$, we have

$$f_0(1, 1, -2, 0, \cdots, 0) > 0. \tag{4.18}$$

Note that $\binom{m-p}{q} \equiv \binom{m-p}{m-p-q}$. Also, $2p + q = 2$, mod(3) is equivalent to $2p + m - p - q = 1$, mod(3). Thus,

$$f_1(1, 1, -2, 0, \cdots, 0) = f_2(1, 1, -2, 0, \cdots, 0). \tag{4.19}$$

By (4.13),

$$f_0(1, 1, -2, 0, \cdots, 0) + f_1(1, 1, -2, 0, \cdots, 0) + f_2(1, 1, -2, 0, \cdots, 0) = 0. \tag{4.20}$$

By $f(1, 1, -2, 0, \cdots, 0) \geq 0$, (4.11) and (4.18-4.20) we can derive $v_1 + v_2 - 2v_0 \leq 0$. This proves (4.14).

For the case that $m = 12$, (4.16) still holds. By direct computation, we have

$$f_0(1, -1, 0, 0, \cdots, 0) > 0. \tag{4.21}$$

By (4.11), (4.16-4.17) and (4.21), we have $v_1 + v_2 - 2v_0 \leq 0$. On the other hand, consider $f(1, -3, 2, 0, \cdots, 0)$ and $f(1, 2, -3, 0, \cdots, 0)$. By direct computation, we have

$$f_0(1, -3, 2, 0, \cdots, 0) < 0. \tag{4.22}$$

We have that

$$\begin{aligned} & f_0(1, -3, 2, 0, \cdots, 0) \\ & = \sum_{p=0}^{m} \sum \left\{ 2^p (-3)^q \binom{m}{p} \binom{m-p}{q} : 2p + q = 0, \ \mathrm{mod}(3), \ 0 \leq q \leq m \right\} \end{aligned}$$

and

$$\begin{aligned} & f_0(1, 2, -3, 0, \cdots, 0) \\ & = \sum_{p=0}^{m} \sum \left\{ 2^q (-3)^p \binom{m}{p} \binom{m-p}{q} : 2p + q = 0, \ \mathrm{mod}(3), \ 0 \leq q \leq m \right\} \end{aligned}$$

We may see that $2p + q = 0$, mod(3) if and only if $p + 2q = 0$, mod(3). Thus,

$$f_0(1, 2, -3, 0, \cdots, 0) = f_0(1, -3, 2, 0, \cdots, 0) < 0. \tag{4.23}$$

Similarly, we may show that

$$\begin{aligned} & f_1(1, 2, -3, 0, \cdots, 0) - f_2(1, 2, -3, 0, \cdots, 0) \\ = & f_2(1, -3, 2, 0, \cdots, 0) - f_1(1, -3, 2, 0, \cdots, 0). \end{aligned} \tag{4.24}$$

By $f(1, -3, 2, 0, \cdots, 0) + f(1, 2, -3, 0, \cdots, 0) \geq 0$, (4.11) and (4.23-4.24) we can derive $v_1 + v_2 - 2v_0 \geq 0$. This proves that (4.14) still holds for $m = 12$. □

We now have the following theorem.

Theorem 4.3. *Suppose that $\mathcal{A}$ is an mth order n dimensional generalized anti-circulant tensor with $m = 6, 12, 18, 30, 42$, $r = 3$ and $n \geq r$. Then $\mathcal{A}$ is PSD if and only if $v_0 = v_1 = v_2 \geq 0$. In this case, we still have (4.7) and (4.8), which implies that $\mathcal{A}$ is a strong Hankel tensor and hence an SOS Hankel tensor.*

Proof. Suppose that $\mathcal{A}$ is PSD. Then $v_0 \geq 0$. Without loss of generality, assume that $v_0 > 0$. By Proposition 4.2, $v_1 + v_2 = 2v_0$. Let $v_1 = v_0(1 + \alpha)$. Then $v_2 = v_0(1 - \alpha)$ and

$$f(\mathbf{x}) = v_0(x_1 + \cdots + x_n)^m + v_0\alpha(f_1(\mathbf{x}) - f_2(\mathbf{x})), \tag{4.25}$$

where f_1 and f_2 are defined as in (4.12). We may see that

$$\begin{aligned} & f_1(1, 2, -3, 0, \cdots, 0) - f_2(1, 2, -3, 0, \cdots, 0) \\ = & f_2(1, -3, 2, 0, \cdots, 0) - f_1(1, -3, 2, 0, \cdots, 0) \neq 0. \end{aligned}$$

Then from this, (4.25), $f(1, 2, -3, 0, \cdots, 0) \geq 0$ and $f(1, -3, 2, 0, \cdots, 0) \geq 0$, we have $\alpha = 0$. This proves that $v_0 = v_1 = v_2 \geq 0$. The remaining conclusions now follow from Proposition 4.1. □

In the proof of Proposition 4.2, we use direct calculation to show (4.15), (4.18), (4.21) and (4.22). Are (4.15) and (4.18) still true for $m = 12l + 6$ with $l \geq 4$? Are (4.21) and (4.22) still true for $m = 12l$ with $l \geq 2$? How can we prove these by some analytical technique? The case that $m = 2k, r = 2p + 1, GCD(m, r) \neq 1$ for $k \geq 2$ and $p \geq 2$ also remains unknown. These are some further research topics.

4.4 The case that r is even

4.4.1 The case that $r = 2$

We see that the results in [57] for $m = 6$ and $n = 3$ can be extended to this case.

In this case, (4.1) and (4.2) have the following form:

$$v_i = v_{i+2} \tag{4.26}$$

for $i = 0, \cdots, (n-1)m - 2$, and for $\mathbf{x} = (x_1, \cdots, x_n)^\top \in \Re^n$,

$$f(\mathbf{x}) \equiv \mathcal{A}\mathbf{x}^{\otimes m} = \sum_{i_1, \cdots, i_m = 1}^{n} v_{i_1 + \cdots + i_m - m} x_{i_1} \cdots x_{i_m} = v_0 f_0(\mathbf{x}) + v_1 f_1(\mathbf{x}), \tag{4.27}$$

where

$$f_j(\mathbf{x}) = \sum \left\{ x_{i_1} \cdots x_{i_m} : i_1 + \cdots + i_m = j, \ \mathrm{mod}(2), i_1, \cdots, i_m = 1, \cdots, n \right\}, \tag{4.28}$$

for $j = 0, 1$. We may see that

$$f_0(\mathbf{x}) + f_1(\mathbf{x}) = (x_1 + \cdots + x_n)^m. \tag{4.29}$$

Since we are concerned about PSD generalized anti-circulant tensors, we may assume that (4.3) holds, i.e., $v_0 \geq 0$.

Theorem 4.4. *Suppose that $\mathcal{A}$ is an mth order n dimensional generalized anti-circulant tensor with circulant index $r = 2$, where $m = 2k \geq 2$ and $n \geq 2$. Then $\mathcal{A}$*

is PSD if and only if $|v_1| \le v_0$. *In these cases, we may write* $v_1 = v_0(2t-1)$, *where* $t \in [0,1]$. *We have that*

$$f(\mathbf{x}) = tv_0(x_1 + \cdots x_n)^m + (1-t)v_0\left(x_1 - x_2 + x_3 - \cdots + (-1)^{n-1}x_n\right)^m,$$

and $\mathcal{A}$ *is a strong Hankel tensor.*

Proof. Suppose that $\mathcal{A}$ is PSD and $v_0 \ge 0$. Then we have $f(1,1,0,\cdots,0) \ge 0$. Note that

$$\begin{aligned}
& f_0(1,1,0,\cdots,0) \\
&= \sum \{x_{i_1}\cdots x_{i_m} : i_1 + \cdots + i_m = 0,\ \mathrm{mod}(2),\ i_1,\cdots,i_m = 1,2, x_1 = 1, x_2 = 1\} \\
&= \sum \{x_{i_1}\cdots x_{i_m} : \text{the number of } i_j = 1 \text{ is } m-p,\ \text{the number of } i_j = 2 \text{ is } p, \\
&\qquad p = 0,2,\cdots,m,\ x_1 = 1,\ x_2 = 1\} \\
&= \sum \left\{\binom{m}{p} : p = 0,2,\cdots,m\right\}.
\end{aligned}$$

Similarly, we can prove that

$$f_1(1,1,0,\cdots,0) = \sum \left\{\binom{m}{p} : p = 1,3,\cdots,m-1\right\},$$

$$f_0(1,-1,0,\cdots,0) = \sum \left\{\binom{m}{p} : p = 0,2,\cdots,m\right\}$$

and

$$f_1(1,-1,0,\cdots,0) = \sum \left\{(-1)^p\binom{m}{p} : p = 1,3,\cdots,m-1\right\}.$$

We may see that

$$\begin{aligned}
& f_0(1,-1,0,\cdots,0) + f_1(1,-1,0,\cdots,0) = 0, \\
& f_0(1,1,0,\cdots,0) = f_0(1,-1,0,\cdots,0) > 0, \\
& f_1(1,1,0,\cdots,0) = -f_1(1,-1,0,\cdots,0) > 0. \qquad (4.30)
\end{aligned}$$

By (4.27), we have $v_0 + v_1 \geq 0$. From $f(1, -1, 0, \cdots, 0) \geq 0$ and (4.27), we have $v_0 - v_1 \geq 0$. This implies that $v_0 \geq |v_1|$. On the other hand, suppose that $v_0 \geq |v_1|$. We may write $v_1 = v_0(2t - 1)$, where $t \in [0, 1]$. Write $f(\mathbf{x}) = v_0 f_0(\mathbf{x}) + v_1 f_1(\mathbf{x})$ such that $f_0(\mathbf{x}) + f_1(\mathbf{x}) = (x_1 + \cdots + x_n)^m$ and $f_0(\mathbf{x}) - f_1(\mathbf{x}) = (x_1 - x_2 + x_3 - \cdots + (-1)^{n-1}x_n)^m$ for all $\mathbf{x} = (x_1, \cdots, x_n)^\top \in \Re^n$. It then follows from (4.27) that

$$\begin{aligned} f(\mathbf{x}) &= v_0 f_0(\mathbf{x}) + v_1 f_1(\mathbf{x}) \\ &= v_0 f_0(\mathbf{x}) + (2t-1) v_0 f_1(\mathbf{x}) \\ &= t v_0 (f_0(\mathbf{x}) + f_1(\mathbf{x})) + (1-t) v_0 (f_0(\mathbf{x}) - f_1(\mathbf{x})) \\ &= t v_0 (x_1 + \cdots + x_n)^m + (1-t) v_0 (x_1 - x_2 + x_3 - \cdots + (-1)^{n-1} x_n)^m. \end{aligned}$$

Similarly, we have

$$\begin{aligned} g(\mathbf{y}) &= \mathbf{y}^\top A \mathbf{y} \qquad (4.31) \\ &= t v_0 (y_1 + \cdots + y_{nk-k+1})^2 + (1-t) v_0 (y_1 - y_2 + y_3 - \cdots + (-1)^{nk-k} y_{nk-k+1})^2, \end{aligned}$$

where A is the associated Hankel matrix of $\mathcal{A}$. The conclusions now follow from the definitions of PSD, SOS and strong Hankel tensors. □

4.4.2 The case that $GCD(m, r) = 2$

In this section, we allow $r \leq 2n - 4$ instead of $r \leq n$, and still call such a tensor a generalized anti-circulant tensor. We have the following theorem.

Theorem 4.5. *Suppose that $\mathcal{A}$ is an mth order n dimensional generalized anti-circulant tensor with $m = 2k, k \geq 1$, $4 \leq r = 2p \leq 2n - 4$. If $GCD(m, r) = 2$, then $\mathcal{A}$ is PSD if and only if $v_0 = v_2 = \cdots = v_{r-2}$, $v_1 = v_3 = \cdots = v_{r-1}$ and $v_0 \geq |v_1|$. In this case, we may write $v_1 = v_0(2t - 1)$, where $t \in [0, 1]$. Then we have*

$$f(\mathbf{x}) = t v_0 (x_1 + \cdots x_n)^m + (1-t) v_0 \left(x_1 - x_2 + x_3 - \cdots + (-1)^{n-1} x_n\right)^m.$$

This implies that $\mathcal{A}$ is PSD if only if it is SOS. Furthermore, in this case, $\mathcal{A}$ is a strong Hankel tensor.

Proof. Suppose that $\mathcal{A}$ is PSD. Let $\mathbf{x} = \mathbf{e}_q - \mathbf{e}_{q+2}$ for $q = 1, \cdots, n$, with $\mathbf{e}_{n+1} \equiv \mathbf{e}_1$ and $\mathbf{e}_{n+2} \equiv \mathbf{e}_2$. By $f(\mathbf{x}) \geq 0$, we have that

$$\sum_{j=0}^{m} \binom{m}{j} (-1)^j v_{(q-1)m+2j} \geq 0, \tag{4.32}$$

for $q = 1, \cdots, n$. Since $GCD(m, r) = 2$, for each $i = 0, \cdots, p-1$, there is an integer q, $1 \leq q \leq n$ such that $(q-1)m = 2i$, $\mathrm{mod}(r)$. Then $v_{2(i+j)} = v_{(q-1)m+2j}$ for such i, q and $j = 0, \cdots, m$. Thus, (4.32) implies that

$$\sum_{j=0}^{m} \binom{m}{j} (-1)^j v_{2(i+j)} \geq 0,$$

for $i = 0, \cdots, p-1$. Applying Theorem 4.1 with $M = m$ and $u_j = v_{2j}$, we have $v_0 = v_2 = \cdots = v_{r-2}$.

Let $\mathbf{x} = \alpha \mathbf{e}_{q-1} + \mathbf{e}_q - \alpha \mathbf{e}_{q+1}$ for $q = 1, \cdots, n$ with $\mathbf{e}_0 \equiv \mathbf{e}_n$. Since $v_0 = v_2 = \cdots = v_{r-2}$, in the expression of $f(\mathbf{x})$, the coefficient for power α^m is zero. Hence, the highest power of α in $f(\mathbf{x})$ is the term for power α^{m-1}, which is

$$m\alpha^{m-1} \left(\sum_{j=0}^{m-1} \binom{m-1}{j} (-1)^j v_{mq-2m+1+2j} \right).$$

From $f(\mathbf{x}) \geq 0$, letting $\alpha \to \infty$, we have

$$\sum_{j=0}^{m-1} \binom{m-1}{j} (-1)^j v_{mq-2m+1+2j} \geq 0, \tag{4.33}$$

for $q = 1, \cdots, n$. Since $GCD(m, r) = 2$, as in the first part of this proof, (4.33) implies that

$$\sum_{j=0}^{m-1} \binom{m-1}{j} (-1)^j v_{2i+1+2j} \geq 0,$$

for $i = 0, \cdots, p-1$. Applying Theorem 4.1 with $M = m-1$ and $u_j = v_{2j+1}$, we have $v_1 = v_3 = \cdots = v_{r-1}$. The remaining conclusions now follow from Theorem 4.4. $\square$

4.4.3 The case that $GCD(m,r) = 2l$ for $l \geq 2$

In this case, we have the following theorem for $m = 4$, $n \geq r = 4$.

Theorem 4.6. *Suppose that $\mathcal{A}$ is a fourth order n dimensional generalized anti-circulant tensor with circulant index $r = 4$, where $n \geq 4$. Then $\mathcal{A}$ is PSD if and only if $v_0 = v_2, v_1 = v_3$ and $|v_1| \leq v_0$. In these cases, we may write $v_1 = v_0(2t-1)$, where $t \in [0,1]$. We have that*

$$f(\mathbf{x}) = tv_0(x_1 + \cdots x_n)^4 + (1-t)v_0\left(x_1 - x_2 + x_3 - \cdots + (-1)^{n-1}x_n\right)^4,$$

and $\mathcal{A}$ is a strong Hankel tensor.

Proof. In this case, (4.1) and (4.2) have the following form:

$$v_i = v_{i+4} \tag{4.34}$$

for $i = 0, \cdots, 4n-8$. From (4.2), for $\mathbf{x} = (x_1, \cdots, x_n)^\top \in \Re^n$, we have

$$\begin{aligned} f(\mathbf{x}) &\equiv \mathcal{A}\mathbf{x}^{\otimes 4} && (4.35)\\ &= \sum_{i_1,\cdots,i_4=1}^{n} v_{i_1+\cdots+i_4}x_{i_1}\cdots x_{i_4} = v_0f_0(\mathbf{x}) + v_1f_1(\mathbf{x}) + v_2f_2(\mathbf{x}) + v_3f_3(\mathbf{x}), \end{aligned}$$

where

$$f_j(\mathbf{x}) = \sum\left\{x_{i_1}\cdots x_{i_4} : i_1 + \cdots + i_4 = j,\ \mathrm{mod}(4), i_1, \cdots, i_4 = 1, \cdots, n\right\}, \tag{4.36}$$

for $j = 0, 1, 2, 3$. Furthermore, we have

$$\begin{aligned} &f_0(x_1, x_2, x_3, x_4, 0, \cdots 0) && (4.37)\\ &= x_1^4 + x_2^4 + x_3^4 + x_4^4 + 6(x_1^2x_3^2 + x_2^2x_4^2) + 12(x_1^2x_2x_4 + x_1x_2^2x_3 + x_2x_3^2x_4 + x_1x_3x_4^2), \end{aligned}$$

$$\begin{aligned} &f_1(x_1, x_2, x_3, x_4, 0, \cdots 0) && (4.38)\\ &= 4(x_1^3x_2 + x_2^3x_3 + x_3^3x_4 + x_1x_4^3) + 12(x_1^2x_3x_4 + x_1x_2^2x_4 + x_1x_2x_3^2 + x_2x_3x_4^2), \end{aligned}$$

$$f_2(x_1, x_2, x_3, x_4, 0, \cdots 0) \tag{4.39}$$
$$= 4(x_1^3x_3 + x_1x_3^3 + x_2^3x_4 + x_2x_4^3) + 6(x_1^2x_2^2 + x_2^2x_3^2 + x_3^2x_4^2 + x_1^2x_4^2) + 24x_1x_2x_3x_4,$$

$$f_3(x_1, x_2, x_3, x_4, 0, \cdots 0) \tag{4.40}$$
$$= 4(x_1x_2^3 + x_2x_3^3 + x_3x_4^3 + x_1^3x_4) + 12(x_1^2x_2x_3 + x_2^2x_3x_4 + x_1x_3^2x_4 + x_1x_2x_4^2).$$

Suppose now that $\mathcal{A}$ is PSD. From (4.38) and (4.40), we see that

$$f_1(1, 0, -1, 0, \cdots, 0) = f_3(1, 0, -1, 0, \cdots, 0) = 0.$$

From (4.37) and (4.39), we have

$$f_0(1, 0, -1, 0, \cdots, 0) = -f_2(1, 0, -1, 0, \cdots, 0) > 0.$$

Then by $f(1, 0, -1, 0, \cdots, 0) \geq 0$, we have $v_0 \geq v_2$.

Similarly, from (4.37-4.40), we have

$$f_1(1, -1, -1, 1, 0, \cdots, 0) = f_3(1, -1, -1, 1, 0, 0, \cdots, 0) = 0$$

and

$$f_0(1, -1, -1, 1, 0, \cdots, 0) = -f_2(1, -1, -1, 1, 0, \cdots, 0) < 0.$$

Then by $f(1, -1, -1, 1, 0, \cdots, 0) \geq 0$, we have $v_0 \leq v_2$. Thus, we derive that $v_0 = v_2$.

From $f(\alpha, 1, -\alpha, 0, \cdots, 0) \geq 0$, $f(\alpha, -1, -\alpha, 0, \cdots, 0) \geq 0$ and (4.36), we derive that $v_0 \geq \phi(\alpha)|v_3 - v_1|$, where $\phi(\alpha) \to \infty$ if $\alpha \to \infty$. Letting α tend to ∞, we have $v_1 = v_3$. The remaining conclusions now follow from Theorem 4.4. $\square$

Chapter 5

Three Dimensional Strongly Symmetric Circulant Tensors

In this chapter, we give a necessary and sufficient condition for an even order three dimensional strongly symmetric circulant tensor to be positive semi-definite. We show that this condition can be a sufficient condition for such a tensor to be sum-of-squares in some cases. There are no PNS strongly symmetric circulant tensors to be found in numerical tests.

5.1 Introduction

We consider even order three dimensional strongly symmetric circulant tensors. A general three dimensional strongly symmetric circulant tensor has only three independent entries: the diagonal entry d, the off-diagonal entries of value u with two different indices in $a_{i_1 \cdots i_m}$, and the off-diagonal entries of value c with three different indices in $a_{i_1 \cdots i_m}$. Let $\mathcal{A} = (a_{i_1 \cdots i_m})$ be an mth order three dimensional strongly symmetric circulant tensor. Here m can be even or odd. We denote

$$a_S \equiv a_{i_1 \cdots i_m}$$

if $S = \{i_1, \cdots, i_m\}$. Then there are seven cases for S: $\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}$, and $\{1,2,3\}$. Since $\mathcal{A}$ is also circulant, we have

$$a_{\{1\}} = a_{\{2\}} = a_{\{3\}}, \qquad \text{and} \qquad a_{\{1,2\}} = a_{\{2,3\}} = a_{\{1,3\}}.$$

Let $d = a_{\{1\}} = a_{\{2\}} = a_{\{3\}}$, $u = a_{\{1,2\}} = a_{\{2,3\}} = a_{\{1,3\}}$ and $c = a_{\{1,2,3\}}$. Then we see that d is the diagonal entry of $\mathcal{A}$: $d = a_{1\cdots 1} = a_{2\cdots 2} = a_{3\cdots 3}$, and an mth order three dimensional strongly symmetric circulant tensor has only three independent entries d, u and c. Thus, we may denote a general three dimensional strongly symmetric circulant tensor $\mathcal{A} = \mathcal{A}(m, d, u, c)$, where m is its order. When the context is clear, we only use $\mathcal{A}$ to denote it.

In our discussion, we need the concept of H-eigenvalues of symmetric tensors, which was introduced in [73] and is closely related to positive semi-definiteness of even order symmetric tensors. When m is even, H-eigenvalues always exist. $\mathcal{T}$ is PSD if and only if its smallest H-eigenvalue is nonnegative by Theorem 2.1. From now on, we denote the smallest H-eigenvalue of $\mathcal{A}(m, d, u, c)$ as $\lambda_{\min}(m, d, u, c)$.

Now, let $m = 2k$ be even. We show that there are two one-variable functions $M_c(u)$ and $N_c(u)$, such that $M_c(u) \geq N_c(u) \geq 0$, $\mathcal{A}$ is SOS if and only if $d \geq M_c(u)$, and $\mathcal{A}$ is PSD if and only if $d \geq N_c(u)$. If $M_c(u) = N_c(u)$, then three dimensional strongly symmetric PNS circulant tensors do not exist for such u and c. We show that if $u, c \leq 0$ or $u = c > 0$, then $M_c(u) = N_c(u)$. Explicit formulae for $M_c(u) = N_c(u)$ are given there in these cases. Thus, it is PNS-free for such u and c.

Note that $\mathcal{A}$ is PSD or SOS if and only if $\alpha\mathcal{A}$ is PSD or SOS, respectively. Thus, we only need to consider three cases that $c = 0$, $c = 1$ and $c = -1$.

We discuss the case that $c = 0$. In this case, for $u > 0$, we have $M_0(u) = uM_0(1)$ and $N_0(u) = uN_0(1)$. We show that $-N_0(1)$ is the smallest H-eigenvalue of $\mathcal{A}(m, 0, 1, 0)$. Numerical tests show that $M_0(1) = N_0(1)$ for $m = 6, 8, 10, 12$ and 14.

Next, we study the case that $c = -1$. We show that there is a $u_0 > 0$ such that if $u \le u_0$, $N_{-1}(u)$ is linear and the explicit formula of $N_{-1}(u)$ can be given, and if $u > u_0$, $N_{-1}(u)$ is the smallest H-eigenvalue of a tensor with u as a parameter. Numerical tests show that for $u > 0$, we still have $M_{-1}(u) = N_{-1}(u)$ for $m = 6, 8, 10$ and 12.

Furthermore, we study the case that $c = 1$. We show that there is a $v_0 < 0$ such that if $u \le v_0$, $N_1(u)$ is linear and the explicit formula of $N_1(u)$ can be given, and if $u > v_0$, $N_1(u)$ is the smallest H-eigenvalue of a tensor with u as a parameter. Numerical tests show that for $u \neq 1$, we still have $M_1(u) = N_1(u)$ for $m = 6, 8, 10$ and 12.

5.2 Functions $M_c(u)$ and $N_c(u)$

In this section and the next three sections, we assume that $n = 3$ and $m = 2k$ is even. Let $\mathcal{A}$ be an mth order three dimensional strongly symmetric circulant tensor. Then we may write $f(\mathbf{x}) = \mathcal{A}\mathbf{x}^m$ as

$$\begin{aligned} f(\mathbf{x}) &= d(x_1^m + x_2^m + x_3^m) + u \sum_{p=1}^{m-1} \binom{m}{p} (x_1^{m-p} x_2^p + x_1^{m-p} x_3^p + x_2^{m-p} x_3^p) \\ &\quad + c \sum_{p=1}^{m-2} \sum_{q=1}^{m-p-1} \binom{m}{p} \binom{m-p}{q} x_1^{m-p-q} x_2^p x_3^q. \end{aligned} \tag{5.1}$$

We now establish two functions $M_c(u)$ and $N_c(u)$, in the following theorem. Recall that for an mth order n dimensional tensor $\mathcal{A} = (a_{i_1 \cdots i_m})$, the sum of the absolute values of its ith off-diagonal entries, i.e.,

$$r_i = \sum_{i_2, \cdots, i_m = 1}^{n} |a_{i i_2 \cdots i_m}| - |a_{i i \cdots i}|.$$

If $a_{i \cdots i} \ge r_i$ for $i = 1, \cdots, n$, then $\mathcal{A}$ is called diagonally dominated. It is shown in [73] that all the H-eigenvalues of a diagonally dominated tensor, if there is, are nonnegative. Furthermore, an even order symmetric diagonally dominated tensor is PSD [73] and SOS [13].

Theorem 5.1. *Let $\mathcal{A}$ be an mth order three dimensional strongly symmetric circulant tensor. For given off-diagonal entries u and c, we define*

$$\begin{aligned} M_c(u) &\equiv \inf\{d \in \Re \ : \ \mathcal{A}(m,d,u,c) \text{ is SOS}\}, \\ N_c(u) &\equiv \inf\{d \in \Re \ : \ \mathcal{A}(m,d,u,c) \text{ is PSD}\}. \end{aligned}$$

Then, functions $M_c(u)$ and $N_c(u)$ are well-defined and convex. Furthermore, we have

$$0 \le N_c(u) \le M_c(u) \le |u|(2^m - 2) + |c|(3^{m-1} - 2^m + 1). \tag{5.2}$$

Proof. Since $\mathcal{A}$ is a circulant tensor, then it has the same off-diagonal entry absolute value sum for different rows, i.e., $r_1 = r_2 = r_3$. By (5.1), this row sum is equal to the right hand side of (5.2). Thus, if d is greater than or equal to this value, $\mathcal{A}$ is diagonally dominated and thus PSD and SOS. This shows that functions $M_c(u)$ and $N_c(u)$ are well-defined and the inequalities in (5.2) hold. As the set of PSD tensors and the set of SOS tensors are convex [58], $M_c(u)$ and $N_c(u)$ are convex. Since a necessary condition for an even order circulant tensor to be PSD is that its diagonal entry to be nonnegative [19], we have $N_c(u) \ge 0$ for all u and c.

By definition, we have $N_c(u) \le M_c(u)$. Clearly, if $M_c(u) = N_c(u)$, then mth order three dimensional PNS strongly symmetric circulant tensors do not exist for such u and c. The theorem is proved. □

Theorem 5.1 means that $\mathcal{A}$ is SOS if and only if $d \ge M_c(u)$, and $\mathcal{A}$ is PSD if and only if $d \ge N_c(u)$. If $M_c(u) = N_c(u)$, then mth order three dimensional PNS strongly symmetric circulant tensors do not exist for such u and c. If $M_c(u) = N_c(u)$, u is called a **PNS-free point** for c.

For the convenience, we present formally three ingredients used in theoretical proofs of PNS-free points. If a point u enjoys these ingredients, it is PNS-free.

Definition 5.1. *Suppose that $n = 3$ and m is even. Suppose that there is a number M such that $f^*(\mathbf{x}) \equiv \mathcal{A}(m, M, u, c)\mathbf{x}^m$ is SOS for given u and c, and a nonzero vector $\bar{\mathbf{x}} \in \Re^3$ such that $f^*(\bar{\mathbf{x}}) = 0$. Then we call M the* **critical value** *of $\mathcal{A}$ at u and c, the SOS decomposition $f^*(\mathbf{x})$ the* **critical SOS decomposition** *of $\mathcal{A}$ at u and c, and $\bar{\mathbf{x}}$ the* **critical minimizer** *of $\mathcal{A}$ at u and c.*

Theorem 5.2. *Let $u \in \Re$. Then u is PNS-free point for c if $\mathcal{A}$ has a critical value M, a critical SOS decomposition $f^*(\mathbf{x})$ and a critical minimizer $\bar{\mathbf{x}}$ at u and c.*

Proof. Suppose that $\mathcal{A}$ has a critical value M, a critical SOS decomposition $f^*(\mathbf{x})$ and a critical minimizer $\bar{\mathbf{x}}$ at u. Then we have $M \geq M_c(u)$ by the definition of $M_c(u)$. If $d < M$, then

$$f(\bar{\mathbf{x}}) = (d - M)(\bar{x}_1^m + \bar{x}_2^m + \bar{x}_3^m) + f^*(\bar{\mathbf{x}}) < 0.$$

This implies that $N_c(u) \geq M$ by the definition of $N_c(u)$. But $N_c(u) \leq M_c(u)$. Thus, $M_c(u) = N_c(u) = M$, i.e., u is PNS-free point for c. $\square$

Corollary 5.1. *If $u, c \leq 0$, then*

$$M_c(u) = N_c(u) = -u(2^m - 2) - c(3^{m-1} - 2^m + 1). \tag{5.3}$$

Thus, it is PNS-free for such u and c.

Proof. Suppose that $u, c \leq 0$. Let M be the value of the right hand side of (5.2), and $\bar{\mathbf{x}} = (1, 1, 1)^\top$. If $d = M$, then $f(\mathbf{x}) = f^*(\mathbf{x})$ has an SOS decomposition as $\mathcal{A}$ is an even order diagonally dominated symmetric tensor [13]. We also see that $f^*(\bar{\mathbf{x}}) = 0$. The result follows. $\square$

Corollary 5.2. *If $u = c > 0$, then*

$$M_c(u) = N_c(u) = u = c.$$

Thus, it is PNS-free for such u and c.

Proof. Suppose that $u = c > 0$. Let $M = u = c$, and $\bar{\mathbf{x}} = (2, -1, -1)^\top$. If $d = M$, then $f(\mathbf{x}) = f^*(\mathbf{x}) = (x_1 + x_2 + x_2)^m$ has an SOS decomposition. We also see that $f^*(\bar{\mathbf{x}}) = 0$. The result follows. $\square$

Corollary 5.3. *If $u > 0$, then*

$$M_0(u) = uM_0(1)$$

and

$$N_0(u) = uN_0(1).$$

Hence, for $c = 0$, it is PNS-free if and only if $M_0(1) = N_0(1)$.

Proof. Suppose that $u > 0$ and $d \geq uM_0(1)$. By (5.1), we have

$$\begin{aligned} f(\mathbf{x}) &= (d - uM_0(1))(x_1^m + x_2^m + x_3^m) \\ &\quad + u\left(M_0(1)(x_1^m + x_2^m + x_3^m) + \sum_{p=1}^{m-1} \binom{m}{p} (x_1^{m-p}x_2^p + x_1^{m-p}x_3^p + x_2^{m-p}x_3^p) \right). \end{aligned}$$

We see that $f(\mathbf{x})$ is SOS. Hence, $M_0(u) = uM_0(1)$. Similarly, we may prove that $N_0(u) = uN_0(1)$. By these and Corollary 5.1, we have the last conclusion. $\square$

As discussed in the introduction, for the PNS-free problem, we only need to consider three cases: $c = 0, 1$ and -1.

5.3 $c = 0$

If $u \leq 0$, by Corollary 5.1, we have $M_0(u) = N_0(u) = -u(2^m - 2)$. If $u > 0$, by Corollary 5.3, we have $M_0(u) = uM_0(1)$ and $N_0(u) = uN_0(1)$. We only need to consider the case that $u = 1$.

Proposition 5.1. *We have that $N_0(1) = -\lambda_{\min}(m, 0, 1, 0)$.*

Proof. By [73], $\mathcal{A}(m, d, 1, 0)$ is PSD if and only if $\lambda_{\min}(m, d, 1, 0) \geq 0$. By the structure of circulant tensors, $\lambda_{\min}(m, d, 1, 0) = d + \lambda_{\min}(m, 0, 1, 0)$. Thus, $\mathcal{A}(m, d, 1, 0)$ is PSD if and only if $d \geq -\lambda_{\min}(m, 0, 1, 0)$. By the definition of $N_c(u)$, we have $N_0(1) = -\lambda_{\min}(m, 0, 1, 0)$. □

For $m = 6, 8, 10, 12$ and 14, we compute $M_0(1)$ and $N_0(1)$ by using softwares Matlab (YALMIP, GloptiPloy and SeDuMi) and Maple [25, 61, 60, 88], respectively. We find for such m, $M_0(1) = N_0(1)$. The results are displayed in Table 5.1.

m	$M_0(1)$	$N_0(1)$
6	1.737348471173345	1.737348471777547
8	1.882980354978972	1.882980356780414
10	1.947977161918168	1.947977172341075
12	1.976878006619490	1.976878047128592
14	1.989722829997529	1.989723542124766

Table 5.1: The values of $M_0(1)$ and $N_0(1)$.

5.4 $c = -1$

If $u \leq 0$, then Corollary 5.1 indicates that $M_{-1}(u) = N_{-1}(u) = -u(2^m - 2) + (3^{m-1} - 2^m + 1)$. We now discuss the case that $u > 0$.

In this section and the next section, we denote that $\mathcal{B} = \mathcal{A}(m, 3^{m-1} - 2^m + 1, 0, -1)$ and $\mathcal{T} = \mathcal{A}(m, 2^m - 2, -1, 0)$. Then, $\mathcal{B}$ and $\mathcal{T}$ are obviously diagonally dominated. Hence, they are PSD and SOS [13]. And all of their H-eigenvalues are nonnegative.

Theorem 5.3. *Let*

$$\varphi(u) \equiv \lambda_{\min}(\mathcal{B} - u\mathcal{T}),$$

where $\lambda_{\min}(\cdot)$ denotes the smallest H-eigenvalue. Then, $\varphi(u) \leq 0$. If $\varphi(u) = 0$, then we have

$$N_{-1}(u) = 3^{m-1} - 2^m + 1 - u(2^m - 2). \tag{5.4}$$

If $\varphi(u) < 0$, then we have

$$N_{-1}(u) = 3^{m-1} - 2^m + 1 - u(2^m - 2) - \lambda_{\min}(m, 3^{m-1} - 2^m + 1 - u(2^m - 2), u, -1). \quad (5.5)$$

Furthermore, the set $C = \{u : \varphi(u) = 0\}$ is a nonempty closed convex ray $(-\infty, u_0]$ for some $u_0 \geq 0$.

Proof. Let $\bar{x} = (1, 1, 1)^\top$. Then $\mathcal{B}\bar{\mathbf{x}}^m = 0$ and $\mathcal{T}\bar{\mathbf{x}}^m = 0$. Thus, $(\mathcal{B} - u\mathcal{T})\bar{\mathbf{x}}^m = 0$ for any u. By [73], we see that

$$\varphi(u) \equiv \lambda_{\min}(\mathcal{B} - u\mathcal{T}) \leq 0.$$

If $\varphi(u) = 0$, we let $d = 3^{m-1} - 2^m + 1 - u(2^m - 2)$. Then $\mathcal{A}(m, d, u, -1) = \mathcal{B} - u\mathcal{T}$. We have $\lambda_{\min}(m, d, u, -1) = 0$. This implies (5.4).

If $\varphi(u) < 0$, then, because

$$f(\mathbf{x}) = (d - (3^{m-1} - 2^m + 1) + u(2^m - 2))\mathcal{I}\mathbf{x}^m + \mathcal{B}\mathbf{x}^m - u\mathcal{T}\mathbf{x}^m \geq 0,$$

we have

$$\begin{aligned} N_{-1}(u) &= \inf\{d : \lambda_{\min}((d - (3^{m-1} - 2^m + 1) + u(2^m - 2))\mathcal{I} + \mathcal{B} - u\mathcal{T}) \geq 0\} \\ &= (3^{m-1} - 2^m + 1) - u(2^m - 2) - \lambda_{\min}(\mathcal{B} - u\mathcal{T}) \\ &= 3^{m-1} - 2^m + 1 - u(2^m - 2) - \lambda_{\min}(m, 3^{m-1} - 2^m + 1 - u(2^m - 2), u, -1). \end{aligned}$$

We have (5.5).

By Corollary 5.1, C is nonempty and $u \in C$ as long as $u \leq 0$. By Theorem 5.1, $N_{-1}(u)$ is a convex function. It follows, together with (5.4) and (5.5), that C is convex. Since $\lambda_{\min}$ is a continuous function [73], C is closed. Since $u \in C$ for any $u \leq 0$, C is a ray, with the form $(-\infty, u_0]$ for some $u_0 \geq 0$. □

Corollary 5.4. *Let $u_0 \equiv \max\{\hat{u} : \varphi(\hat{u}) = 0\}$. Then u_0 is well-defined and $u_0 \geq 0$. Furthermore, for $u \leq u_0$, we have (5.4), and for $u > u_0$, we have (5.5).*

Proposition 5.2. *If* $M_{-1}(u_0) = N_{-1}(u_0) = 3^{m-1} - 2^m + 1 - u_0(2^m - 2)$, *then for* $u \leq u_0$, *we have* $M_{-1}(u) = N_{-1}(u) = 3^{m-1} - 2^m + 1 - u(2^m - 2)$.

Proof. By Theorem 5.1, $M_{-1}(u)$ is convex. By Corollary 1, $M_{-1}(u) = 3^{m-1} - 2^m + 1 - u(2^m - 2)$ for $u \leq 0$. Since $u_0 \geq 0$, the conclusion follows. $\square$

Proposition 5.3. *Suppose* $u_0 = \max\{\hat{u} \ : \ \varphi(\hat{u}) = 0\}$. *Then, we have*

$$0 \leq u_0 \leq \bar{u}_0(m) \equiv \frac{3^{m-1}+1}{2^m} - 1. \tag{5.6}$$

Proof. Since $\mathcal{B}$ is PSD and has a H-eigenvalue 0, we have $\varphi(0) = 0$ and $u_0 \geq 0$.

On the other hand, we consider the case $u > \bar{u}_0$. Let $\mathbf{x}_0 = (1, 1, -3)^\top$. We have

$$(\mathcal{B} - \bar{u}_0\mathcal{T})\mathbf{x}_0^m = 0 \qquad \text{and} \qquad \mathcal{T}\mathbf{x}_0^m = 2^m(3^m - 1).$$

Then,

$$(\mathcal{B} - u\mathcal{T})\mathbf{x}_0^m = (\mathcal{B} - \bar{u}_0\mathcal{T})\mathbf{x}_0^m - (u - \bar{u}_0)\mathcal{T}\mathbf{x}_0^m = -(u - \bar{u}_0)2^m(3^m - 1) < 0.$$

Hence, we have $\varphi(u) = \lambda_{\min}(\mathcal{B} - u\mathcal{T}) < 0$ when $u > \bar{u}_0$. Therefore, $u_0 \leq \bar{u}_0$. $\square$

For $m = 6, 8, 10, 12$ and 14, we find that $\mathcal{B} - \bar{u}_0\mathcal{T}$ is PSD. This shows that for such m, $\varphi(\bar{u}_0) = 0$, i.e.,

$$u_0 = \bar{u}_0(m) \equiv \frac{3^{m-1}+1}{2^m} - 1. \tag{5.7}$$

It remains a further research topic to show that $\mathcal{B} - \bar{u}_0\mathcal{T}$ is PSD for all even m with $m \geq 16$. If this is true, then (5.7) is true for all even m with $m \geq 6$.

In Tables 5.2-5.5, the values of $M_{-1}(u)$ and $N_{-1}(u)$ for $m = 6, 8, 10, 12$ and $u = 0.1, 2, \frac{45}{16}, 5, 10, 40, 300$, $u = 1, 3, \frac{483}{64}, 10, 40, 140, 300$, $u = 1, 10, \frac{4665}{256}, 20, 40, 140, 300$ and $u = 1, 20, \frac{43263}{1024}, 60, 100, 140, 300$ are reported, respectively. We find for such m and u, $M_{-1}(u) = N_{-1}(u)$.

u	$M_{-1}(u)$	$N_{-1}(u)$
0.1	173.799999999899	173.8
2	55.9999999995172	56
$\frac{45}{16}$	5.62499991033116	5.625
5	9.42544641511067	9.4254465011842588
10	18.1121860822789	18.112186280892696
40	70.2326321651344	70.232638183914150
300	521.943237017699	521.94324013633004

Table 5.2: The values of $M_{-1}(u)$ and $N_{-1}(u)$ for $m = 6$.

u	$M_{-1}(u)$	$N_{-1}(u)$
1	1677.99999992219	1678
3	1169.99999999356	1170
$\frac{483}{64}$	15.0937478786308	15.09375
10	19.7129359300341	19.7129361640501
40	76.2023466001335	76.2023468071730
140	264.500365037152	264.500382469583
300	565.777184078832	565.777239551091

Table 5.3: The values of $M_{-1}(u)$ and $N_{-1}(u)$ for $m = 8$.

u	$M_{-1}(u)$	$N_{-1}(u)$
1	17637.9999999549	17638
10	8439.99999783081	8440
$\frac{4665}{256}$	36.4452603485520	36.4453125
20	39.9075358817909	39.9075375522954
40	78.8670625326286	78.8670809985488
140	273.664775923815	273.664798232238
300	585.341085323688	585.341145806726

Table 5.4: The values of $M_{-1}(u)$ and $N_{-1}(u)$ for $m = 10$.

5.5 $c = 1$

Corollary 5.2 indicates that $M_1(1) = N_1(1) = 1$. Hence, we only need to consider the case that $u \neq 1$. Let $\mathcal{B}$ and $\mathcal{T}$ be the same as in the last section. We have the following theorem.

u	$M_{-1}(u)$	$N_{-1}(u)$
1	168957.999979042	168958
20	91171.9999996683	91172
$\frac{43263}{1024}$	84.4971787852022	84.498046875
60	119.589505579120	119.589562756497
100	198.664532858285	198.664684641639
140	277.739708996851	277.739806526784
300	594.040191670531	594.040294067366

Table 5.5: The values of $M_{-1}(u)$ and $N_{-1}(u)$ for $m = 12$.

Theorem 5.4. *Let*

$$\psi(u) \equiv \lambda_{\min}(-u\mathcal{T} - \mathcal{B}).$$

Then, $\psi(u) \leq 0$. If $\psi(u) = 0$, then we have

$$N_1(u) = -(3^{m-1} - 2^m + 1) - u(2^m - 2). \tag{5.8}$$

If $\psi(u) < 0$, then we have

$$N_1(u) = -(3^{m-1} - 2^m + 1) - u(2^m - 2) - \lambda_{\min}(m, -(3^{m-1} - 2^m + 1) - u(2^m - 2), u, 1). \tag{5.9}$$

Furthermore, if the set $C = \{u : \psi(u) = 0\}$ is nonempty, then it is a closed convex ray $(-\infty, v_0]$ for some $v_0 < 0$.

Proof. The proof of this theorem is similar to the proof of Theorem 5.3. However, we cannot apply Corollary 5.1 here.

If C is nonempty, we may show that C is closed and convex as in the proof of Theorem 5.3.

If there is a $\hat{u} \leq 0$ such that $\lambda_{\min}(-\hat{u}\mathcal{T} - \mathcal{B}) = 0$, for $u \leq \hat{u} \leq 0$, we have

$$\begin{aligned} \psi(u) &= \lambda_{\min}(-u\mathcal{T} - \mathcal{B}) \\ &= \lambda_{\min}(-\hat{u}\mathcal{T} - \mathcal{B} + (-u + \hat{u})\mathcal{T}) \\ &\geq \lambda_{\min}(-\hat{u}\mathcal{T} - \mathcal{B}) \\ &= 0. \end{aligned}$$

Hence, $\psi(u) = 0$ for all $u \le \hat{u} \le 0$. So if C is not empty, it is a ray with the form $(-\infty, v_0]$ for some v_0. Clearly, $\psi(0) \equiv \lambda_{\min}(-\mathcal{B}) < 0$ as $\mathcal{B}$ is PSD and not a zero tensor. Hence, $v_0 < 0$.

The other parts of the proof are similar to the proof of Theorem 5.3. $\square$

Corollary 5.5. *If there is one point $\hat{u}$ such that $\psi(\hat{u}) = 0$, let $v_0 \equiv \max\{\hat{u} : \psi(\hat{u}) = 0\}$. Then for $u \le v_0$, we have (5.8), and for $u \ge v_0$, we have (5.9).*

We also have the following proposition.

Proposition 5.4. *If $M_1(v_0) = N_1(v_0) = -(3^{m-1} - 2^m + 1) - v_0(2^m - 2)$, then for $u \le v_0$, we have $M_1(u) = N_1(u) = -(3^{m-1} - 2^m + 1) - u(2^m - 2)$.*

Proof. Suppose that $M_1(v_0) = N_1(v_0) = -(3^{m-1} - 2^m + 1) - v_0(2^m - 2)$. By (5.3), if $u \le v_0$ and $d = -(3^{m-1} - 2^m + 1) - u(2^m - 2)$, we have

$$f^*(\mathbf{x}) = -\bar{u}g_1(\mathbf{x}) + g_2(\mathbf{x}),$$

where

$$g_1(\mathbf{x}) = \mathcal{A}(m, 2^m - 2, -1, 0),$$

$$g_2(\mathbf{x}) = \mathcal{A}(m, -(3^{m-1} - 2^m + 1) - v_0(2^m - 2), v_0, 1)$$

and

$$\bar{u} \equiv u - v_0 \le 0.$$

We see that $g_2(\mathbf{x})$ is equal to the critical SOS decomposition of $\mathcal{A}$ at $c = 1$ and $u = v_0$, and $g_1(\mathbf{x})$ is equal to the critical SOS decomposition of $\mathcal{A}$ at $c = 0$ and $u = -1$. Hence both $g_1(\mathbf{x})$ and $g_2(\mathbf{x})$ are SOS polynomials. This implies that $f^*(\mathbf{x})$ is an SOS polynomial. Let $\bar{\mathbf{x}} = (1, 1, 1)^\top$, we see that $f^*(\bar{\mathbf{x}}) = 0$. Then the conclusion follows from Theorem 5.2. $\square$

Proposition 5.5. *Suppose that C is not empty. Let $v_0 = \max\{\hat{u} \ : \ \psi(\hat{u}) = 0\}$. Then, we have*

$$v_0 \leq \bar{v}_0(m) \equiv 1 - \frac{3^{m-1}}{2^{m-1}+1}. \tag{5.10}$$

Proof. Let $\mathbf{x}_0 = (1, 1, -\frac{1}{2})^\top$. We have

$$(-\mathcal{B} - \bar{v}_0\mathcal{T})\mathbf{x}_0^m = 0 \qquad \text{and} \qquad \mathcal{T}\mathbf{x}_0^m = 2^m + 1 - 2^{1-m}.$$

Then,

$$(-\mathcal{B} - u\mathcal{T})\mathbf{x}_0^m = (-\mathcal{B} - \bar{v}_0\mathcal{T})\mathbf{x}_0^m - (u - \bar{v}_0)\mathcal{T}\mathbf{x}_0^m = -(u - \bar{v}_0)(2^m + 1 - 2^{1-m}) < 0.$$

Hence, we have $\psi(u) = \lambda_{\min}(-\mathcal{B} - u\mathcal{T}) < 0$ when $u > \bar{v}_0$. Therefore, $v_0 \leq \bar{v}_0$. $\square$

By a similar discussion on u_0, we find that $-\bar{v}_0\mathcal{T} - \mathcal{B}$ is PSD for $m = 6, 8, 10, 12$ and 14. This shows that for such m, $\psi(\bar{v}_0) = 0$, i.e.,

$$v_0 = \bar{v}_0(m) \equiv 1 - \frac{3^{m-1}}{2^{m-1}+1}. \tag{5.11}$$

This also shows that C is not empty for such m. It remains a further research topic to show that $-\bar{v}_0\mathcal{T} - \mathcal{B}$ is PSD for all even m with $m \geq 16$. If this is true, then (5.11) is true for all even m with $m \geq 6$.

In Tables 5.6-5.9, the values of $M_1(u)$ and $N_1(u)$ for $m = 6, 8, 10, 12$ and $u = -40, -10, -\frac{70}{11}, -5, -1, 10, 40$, $u = -40, -20, -\frac{686}{43}, -10, -1, 10, 40$, $u = -100, -40, -\frac{710}{19}, -20, -1, 10, 40$ and $u = -140, -100, -\frac{58366}{683}, -60, -1, 10, 40$ are reported, respectively. We find for such m and u, $M_1(u) = N_1(u)$.

u	$M_1(u)$	$N_1(u)$
-40	2299.99999993444	2300
-10	439.999999987168	440
$-\frac{70}{11}$	214.545454213196	214.54545454545454
-5	173.991050854352	173.99105151869704
-1	55.8846973214056	55.884697712412670
10	16.6347897201042	16.634789948247836
40	68.7552353830704	68.755241242186800

Table 5.6: The values of $M_1(u)$ and $N_1(u)$ for $m = 6$.

u	$M_1(u)$	$N_1(u)$
-40	8228.00000000754	8228
-20	3147.99999997053	3148
$-\frac{686}{43}$	2120.18604151092	2120.18604651163
-10	1371.80748977461	1371.80749709544
-1	243.740078469126	243.740080311110
10	17.9466697015668	17.9466711544215
40	74.4360734714431	74.4360817805826

Table 5.7: The values of $M_1(u)$ and $N_1(u)$ for $m = 8$.

u	$M_1(u)$	$N_1(u)$
-100	83539.9999994888	83540
-40	22219.9999995661	22220
$-\frac{710}{19}$	19530.5255392283	19530.5263157895
-20	10678.1156381743	10678.1156702343
-1	1004.40451207948	1004.40454284172
10	18.5317674915799	18.5317776218259
40	76.9710868541002	76.9710927899669

Table 5.8: The values of $M_1(u)$ and $N_1(u)$ for $m = 10$.

u	$M_1(u)$	$N_1(u)$
-140	400107.999992756	400108
-100	236347.999998551	236348
$-\frac{58366}{683}$	176802.173593347	176802.170881802
-60	124727.840916646	124727.840144917
-1	4063.38103939314	4063.38106552746
10	18.7918976770375	18.7919005425937
40	78.0982265029468	78.0982419563963

Table 5.9: The values of $M_1(u)$ and $N_1(u)$ for $m = 12$.

Chapter 6

Computing Extreme Eigenvalues of Large Scale Hankel Tensors

Large scale tensors, including large scale Hankel tensors, have many applications in science and engineering. In this chapter, we propose an inexact curvilinear search optimization method to compute Z- and H-eigenvalues of mth order n dimensional Hankel tensors, where n is large. Owing to the fast Fourier transform, the computational cost of each iteration of the new method is about $\mathcal{O}(mn\log(mn))$. Using the Cayley transform, we obtain an effective curvilinear search scheme. Then, we show that every limiting point of iterates generated by the new algorithm is an eigen-pair of Hankel tensors. Without the assumption of a second-order sufficient condition, we analyze the linear convergence rate of iterate sequence by the Kurdyka-Łojasiewicz property. Finally, numerical experiments for Hankel tensors, whose dimension may up to one million, are reported to show the efficiency of the proposed curvilinear search method.

6.1 Introduction

With the coming era of massive data, large scale tensors have important applications in science and engineering. How to store and analyze these tensors? This is a pressing and challenging problem. In the literature, there are two strategies for manipulating large scale tensors. The first one is to exploit their structures such as sparsity [5]. For example, we consider an online store (e.g. Amazon.com) where users may review various products [65]. Then, a third order tensor with modes: users, items, and words could be formed naturally and it is sparse. The other one is to use distributed and parallel computation [28, 23]. This technique could deal with large scale dense tensors, but it depends on a supercomputer. Recently, researchers applied these two strategies simultaneously for large scale tensors [46, 21].

In this chapter, we consider a class of large scale dense tensors with a special Hankel structure. Hankel tensors appear in many engineering problems such as signal processing [10, 30], automatic control [85], and geophysics [70, 89]. For instance, in nuclear magnetic resonance spectroscopy [45], a Hankel matrix was formed to analyze the time-domain signals, which is important for brain tumour detection. Papy et al. [72, 71] improved this method by using a high order Hankel tensor to replace the Hankel matrix. Ding et al. [30] proposed a fast computational framework for products of a Hankel tensor and vectors. On the mathematical properties, Luque and Thibon [64] explored the Hankel hyperdeterminants. Qi [74] and Xu [92] studied the spectra of Hankel tensors and gave some upper bounds and lower bounds for the smallest and the largest eigenvalues. In [74], Qi raised a question: Can we construct some efficient algorithms for the largest and the smallest H- and Z-eigenvalues of a Hankel tensor?

Numerous applications of the eigenvalues of higher order tensors have been found in science and engineering, such as automatic control [68], medical imaging [84, 80, 16], quantum information [67], and spectral graph theory [24]. For example, in magnetic resonance imaging [80], the principal Z-eigenvalues of an even order tensor associated to the fiber orientation distribution of a voxel in white matter of human brain denote volume factions of several nerve fibers in this voxel, and the corresponding Z-eigenvectors express the orientations of these nerve fibers. The smallest eigenvalue of tensors reflects the stability of a nonlinear multivariate autonomous system in automatic control [68]. For a given even order symmetric tensor, it is positive semidefinite if and only if its smallest H- or Z-eigenvalue is nonnegative [73].

The conception of eigenvalues of higher order tensors was defined independently by Qi [73] and Lim [59] in 2005. Unfortunately, it is an NP-hard problem to compute eigenvalues of a tensor even though the involved tensor is symmetric [41]. For two and three dimensional symmetric tensors, Qi et al. [78] proposed a direct method to compute all of its Z-eigenvalues. It was pointed out in [50, 51] that the polynomial system solver, `NSolve` in *Mathematica*, could be used to compute all of the eigenvalues of lower order and low dimensional tensors. We note that the mathematical software *Maple* has a similar command `solve` which is also applicable for the polynomial systems of eigenvalues of tensors.

For general symmetric tensors, Kolda and Mayo [50] proposed a shifted symmetric higher order power method to compute its Z-eigenpairs. Recently, they [51] extended the shifted power method to generalized eigenpairs of tensors and gave an adaptive shift. Based on the nonlinear optimization model with a compact unit spherical constraint, the power methods [52] project the gradient of the objective at the current iterate onto the unit sphere at each iteration. Its computation is very simple but may not converge [49]. Kolda and Mayo [50, 51] introduced a shift to force the objective to be (locally) concave/convex. Then the power method produces increasing/decreasing steps for computing maximal/minimal eigenvalues. The sequence of objectives converges to eigenvalues since the feasible region is compact. The convergence of the sequence of iterates to eigenvectors is established under the assumption that the tensor has finitely many real eigenvectors. The linear convergence rate is estimated by a fixed-point analysis.

Inspired by the power method, various optimization methods have been established. Han [37] proposed an unconstrained optimization model, which is indeed a quadratic penalty function of the constrained optimization for generalized eigenvalues of symmetric tensors. Hao et al. [39] employed a subspace projection method for Z-eigenvalues of symmetric tensors. Restricted by a unit spherical constraint, this method minimizes the objective in a big circle of n dimensional unit sphere at each iteration. Since the objective is a homogeneous polynomial, the minimization of the subproblem has a closed-form solution. Additionally, Hao et al. [38] gave a trust region method to calculate Z-eigenvalues of symmetric tensors. The sequence of iterates generated by this method converges to a second order critical point and enjoys a locally quadratic convergence rate.

Since nonlinear optimization methods may produce a local minimizer, some convex optimization models have been studied. Hu et al. [42] addressed a sequential semi-definite programming method to compute the extremal Z-eigenvalues of tensors. A sophisticated Jacobian semi-definite relaxation method was explored by Cui et al. [11]. A remarkable feature of this method is the ability to compute all of the real eigenvalues of symmetric tensors. Recently, Chen et al. [15] proposed homotopy continuation methods to compute all of the complex eigenvalues of tensors. When the order or the dimension of a tensor grows larger, the CPU times of these methods become longer and longer.

In some applications [45, 70], the scales of Hankel tensors can be quite huge. This highly restricted the applications of the above mentioned methods in this case. How to compute the smallest and the largest eigenvalues of a Hankel tensor? Can we propose a method to compute the smallest and the largest eigenvalues of a relatively large Hankel tensor, say $1,000,000$ dimension?

Owing to the multi-linearity of tensors, we model the problem of eigenvalues of Hankel tensors as a nonlinear optimization problem with a unit spherical constraint. Our algorithm is an inexact steepest descent method on the unit sphere. To preserve iterates on the unit sphere, we employ the Cayley transform to generate an orthogonal matrix such that the new iterate is this orthogonal matrix times the current iterate. By the Sherman-Morrison-Woodbury formula, the product of the orthogonal matrix and a vector has a closed-form solution. So the subproblem is straightforward. A curvilinear search is employed to guarantee the convergence. Then, we prove that every accumulation point of the sequence of iterates is an eigenvector of the involved Hankel tensor, and its objective is the corresponding eigenvalue. Furthermore, using the Kurdyka-Łojasiewicz property of the eigen-problem of tensors, we prove that the sequence of iterates converges without an assumption of second order sufficient condition. Under mild conditions, we show that the sequence of iterates has a linear or a sublinear convergence rate. Numerical experiments show that this strategy is successful.

6.2 Hankel tensors

Suppose $\mathcal{A}$ is an mth order n dimensional real symmetric tensor

$$\mathcal{A} = (a_{i_1,i_2,\ldots,i_m}), \qquad \text{for } i_j = 1,\ldots,n, j = 1,\ldots,m,$$

where all of the entries are real and invariant under any index permutation. Two products of the tensor $\mathcal{A}$ and a column vector $\mathbf{x} \in \Re^n$ are defined as follows.

- $\mathcal{A}\mathbf{x}^m$ is a scalar

$$\mathcal{A}\mathbf{x}^m = \sum_{i_1,\ldots,i_m=1}^{n} a_{i_1,\ldots,i_m} x_{i_1} \cdots x_{i_m}.$$

- $\mathcal{A}\mathbf{x}^{m-1}$ is a column vector

$$\left(\mathcal{A}\mathbf{x}^{m-1}\right)_i = \sum_{i_2,\ldots,i_m=1}^{n} a_{i,i_2,\ldots,i_m} x_{i_2} \cdots x_{i_m}, \qquad \text{for } i = 1,\ldots,n.$$

When the tensor $\mathcal{A}$ is dense, the computations of products $\mathcal{A}\mathbf{x}^m$ and $\mathcal{A}\mathbf{x}^{m-1}$ require $\mathcal{O}(n^m)$ operations, since the tensor $\mathcal{A}$ has n^m entries and we must visit all of them in the process of calculation. When the tensor is symmetric, the computational cost for these products is about $\mathcal{O}(n^m/m!)$ [83]. Obviously, they are expensive. In this section, we will study a special tensor, the Hankel tensor, whose elements are completely determined by a generating vector. So there exists a fast algorithm to compute products of a Hankel tensor and vectors. Let us give the definitions of two structured tensors.

Definition 6.1. *An mth order n dimensional tensor $\mathcal{H}$ is called a Hankel tensor if its entries satisfy*

$$h_{i_1,i_2,\ldots,i_m} = v_{i_1+i_2+\cdots+i_m-m}, \qquad \text{for } i_j = 1,\ldots,n, j = 1,\ldots,m.$$

The vector $\mathbf{v} = (v_0, v_1, \ldots, v_{m(n-1)})^\top$ with length $\ell \equiv m(n-1)+1$ is called the generating vector of the Hankel tensor $\mathcal{H}$.

An mth order ℓ dimensional tensor $\mathcal{C}$ is called an anti-circulant tensor if its entries satisfy

$$c_{i_1,i_2,\ldots,i_m} = v_{(i_1+i_2+\cdots+i_m-m \bmod \ell)}, \qquad \text{for } i_j = 1,\ldots,\ell, j = 1,\ldots,m.$$

It is easy to see that $\mathcal{H}$ is a sub-tensor of $\mathcal{C}$. Since for the same generating vector $\mathbf{v}$, we have

$$c_{i_1,i_2,\ldots,i_m} = h_{i_1,i_2,\ldots,i_m}, \qquad \text{for } i_j = 1,\ldots,n, j = 1,\ldots,m.$$

For example, a third order two dimensional Hankel tensor with a generating vector $\mathbf{v} = (v_0, v_1, v_2, v_3)^\top$ is

$$\mathcal{H} = \left[\begin{array}{cc|cc} v_0 & v_1 & v_1 & v_2 \\ v_1 & v_2 & v_2 & v_3 \end{array}\right].$$

It is a sub-tensor of an anti-circulant tensor with the same order and a larger dimension

$$\mathcal{C} = \left[\begin{array}{cccc|cccc|cccc|cccc} v_0 & v_1 & v_2 & v_3 & v_1 & v_2 & v_3 & v_0 & v_2 & v_3 & v_0 & v_1 & v_3 & v_0 & v_1 & v_2 \\ v_1 & v_2 & v_3 & v_0 & v_2 & v_3 & v_0 & v_1 & v_3 & v_0 & v_1 & v_2 & v_0 & v_1 & v_2 & v_3 \\ v_2 & v_3 & v_0 & v_1 & v_3 & v_0 & v_1 & v_2 & v_0 & v_1 & v_2 & v_3 & v_1 & v_2 & v_3 & v_0 \\ v_3 & v_0 & v_1 & v_2 & v_0 & v_1 & v_2 & v_3 & v_1 & v_2 & v_3 & v_0 & v_2 & v_3 & v_0 & v_1 \end{array}\right].$$

As discovered in [30, Theorem 3.1], the mth order ℓ dimensional anti-circulant tensor $\mathcal{C}$ could be diagonalized by the ℓ-by-ℓ Fourier matrix F_ℓ, i.e., $\mathcal{C} = \mathcal{D}F_\ell^m$, where $\mathcal{D}$ is a diagonal tensor whose diagonal entries are $\mathrm{diag}(\mathcal{D}) = F_\ell^{-1}\mathbf{v}$. It is well-known that the computations involving the Fourier matrix and its inverse times a vector are indeed the fast (inverse) Fourier transform `fft` and `ifft`, respectively. The computational cost is about $\mathcal{O}(\ell \log \ell)$ multiplications, which is significantly smaller than $\mathcal{O}(\ell^2)$ for a dense matrix times a vector when the dimension ℓ is large.

Now, we are ready to show how to compute the products introduced in the beginning of this section, when the involved tensor has a Hankel structure. For any $\mathbf{x} \in \Re^n$, we define another vector $\mathbf{y} \in \Re^\ell$ such that

$$\mathbf{y} \equiv \begin{bmatrix} \mathbf{x} \\ \mathbf{0}_{\ell-n} \end{bmatrix},$$

where $\ell = m(n-1)+1$ and $\mathbf{0}_{\ell-n}$ is a zero vector with length $\ell - n$. Then, we have

$$\mathcal{H}\mathbf{x}^m = \mathcal{C}\mathbf{y}^m = \mathcal{D}(F_\ell\mathbf{y})^m = \texttt{ifft}(\mathbf{v})^\top \left(\texttt{fft}(\mathbf{y})^{\circ m}\right).$$

To obtain $\mathcal{H}\mathbf{x}^{m-1}$, we first compute

$$\mathcal{C}\mathbf{y}^{m-1} = F_\ell\left(\mathcal{D}(F_\ell\mathbf{y})^{m-1}\right) = \texttt{fft}\left(\texttt{ifft}(\mathbf{v}) \circ \left(\texttt{fft}(\mathbf{y})^{\circ(m-1)}\right)\right).$$

Then, the entries of vector $\mathcal{H}\mathbf{x}^{m-1}$ is the leading n entries of $\mathcal{C}\mathbf{y}^{m-1}$. Here, $\circ$ denotes the Hadamard product such that $(A \circ B)_{i,j} = A_{i,j}B_{i,j}$. Three matrices A, B and $A \circ B$ have the same size. Furthermore, we define $A^{\circ k} = A \circ \cdots \circ A$ as the Hadamard product of k copies of A.

Since the computations of $\mathcal{H}\mathbf{x}^m$ and $\mathcal{H}\mathbf{x}^{m-1}$ require 2 and 3 `fft/iffts`, the computational cost is about $\mathcal{O}(mn\log(mn))$ and obviously cheap. Another advantage of this approach is that we do not need to store and deal with the tremendous Hankel tensor explicitly. It is sufficient to keep and work with the compact generating vector of that Hankel tensor.

6.3 A curvilinear search algorithm

We consider the generalized eigenvalue [12, 32] of an mth order n dimensional Hankel tensor $\mathcal{H}$

$$\mathcal{H}\mathbf{x}^{m-1} = \lambda\mathcal{B}\mathbf{x}^{m-1},$$

where m is even, $\mathcal{B}$ is an mth order n dimensional symmetric tensor and it is positive definite. If there is a scalar λ and a real vector $\mathbf{x}$ satisfying this system, we call λ a generalized eigenvalue and $\mathbf{x}$ its associated generalized eigenvector. Particularly, we find the following definitions from the literature, where the computation on the tensor $\mathcal{B}$ is straightforward.

- Qi [73] called a real scalar λ a Z-eigenvalue of a tensor $\mathcal{H}$ and a real vector $\mathbf{x}$ its associated Z-eigenvector if they satisfy

  $$\mathcal{H}\mathbf{x}^{m-1} = \lambda\mathbf{x} \quad \text{and} \quad \mathbf{x}^\top\mathbf{x} = 1.$$

  This definition means that the tensor $\mathcal{B}$ is an identity tensor $\mathcal{E}$ such that $\mathcal{E}\mathbf{x}^{m-1} = \|\mathbf{x}\|^{m-2}\mathbf{x}$.

- If $\mathcal{B} = \mathcal{I}$, where

  $$(\mathcal{I})_{i_1,\ldots,i_m} = \begin{cases} 1 & \text{if } i_1 = \cdots = i_m, \\ 0 & \text{otherwise }, \end{cases}$$

  the real scalar λ is called an H-eigenvalue and the real vector $\mathbf{x}$ is its associated H-eigenvector [73]. Obviously, we have $(\mathcal{I}\mathbf{x}^{m-1})_i = x_i^{m-1}$ for $i = 1, \ldots, n$.

To compute a generalized eigenvalue and its associated eigenvector, we consider the following optimization model with a spherical constraint

$$\min \quad f(\mathbf{x}) \equiv \frac{\mathcal{H}\mathbf{x}^m}{\mathcal{B}\mathbf{x}^m} \qquad \text{s.t.} \quad \|\mathbf{x}\| = 1, \tag{6.1}$$

where $\|\cdot\|$ denotes the Euclidean norm or its induced matrix norm. The denominator of the objective is positive since the tensor $\mathcal{B}$ is positive definite. By some calculations, we get its gradient and Hessian, which are formally presented in the following lemma.

Lemma 6.1. *Suppose that the objective is defined as in (6.1). Then, its gradient is*

$$\mathbf{g}(\mathbf{x}) = \frac{m}{\mathcal{B}\mathbf{x}^m}\left(\mathcal{H}\mathbf{x}^{m-1} - \frac{\mathcal{H}\mathbf{x}^m}{\mathcal{B}\mathbf{x}^m}\mathcal{B}\mathbf{x}^{m-1}\right). \tag{6.2}$$

And its Hessian is

$$\begin{aligned} H(\mathbf{x}) \quad = \quad & \frac{m(m-1)\mathcal{H}\mathbf{x}^{m-2}}{\mathcal{B}\mathbf{x}^m} - \frac{m(m-1)\mathcal{H}\mathbf{x}^m\mathcal{B}\mathbf{x}^{m-2} + m^2(\mathcal{H}\mathbf{x}^{m-1} \circledcirc \mathcal{B}\mathbf{x}^{m-1})}{(\mathcal{B}\mathbf{x}^m)^2} \\ & + \frac{m^2\mathcal{H}\mathbf{x}^m(\mathcal{B}\mathbf{x}^{m-1} \circledcirc \mathcal{B}\mathbf{x}^{m-1})}{(\mathcal{B}\mathbf{x}^m)^3}, \end{aligned} \tag{6.3}$$

where $\mathbf{x} \circledcirc \mathbf{y} \equiv \mathbf{x}\mathbf{y}^\top + \mathbf{y}\mathbf{x}^\top$.

Let $\mathbb{S}_{n-1} \equiv \{\mathbf{x} \in \Re^n \mid \mathbf{x}^\top\mathbf{x} = 1\}$ be the spherical feasible region. Suppose the current iterate is $\mathbf{x} \in \mathbb{S}_{n-1}$ and the gradient at $\mathbf{x}$ is $\mathbf{g}(\mathbf{x})$. Because

$$\mathbf{x}^\top\mathbf{g}(\mathbf{x}) = \frac{m}{\mathcal{B}\mathbf{x}^m}\left(\mathbf{x}^\top\mathcal{H}\mathbf{x}^{m-1} - \frac{\mathcal{H}\mathbf{x}^m}{\mathcal{B}\mathbf{x}^m}\mathbf{x}^\top\mathcal{B}\mathbf{x}^{m-1}\right) = 0, \tag{6.4}$$

the gradient $\mathbf{g}(\mathbf{x})$ of $\mathbf{x} \in \mathbb{S}_{n-1}$ is located in the tangent plane of $\mathbb{S}_{n-1}$ at $\mathbf{x}$.

Lemma 6.2. *Suppose* $\|\mathbf{g}(\mathbf{x})\| = \epsilon$, *where* $\mathbf{x} \in \mathbb{S}_{n-1}$ *and* ϵ *is a small number. Denote* $\lambda = \frac{\mathcal{H}\mathbf{x}^m}{\mathcal{B}\mathbf{x}^m}$. *Then, we have*

$$\|\mathcal{H}\mathbf{x}^{m-1} - \lambda\mathcal{B}\mathbf{x}^{m-1}\| = \mathcal{O}(\epsilon).$$

Moreover, if the gradient $\mathbf{g}(\mathbf{x})$ *at* $\mathbf{x}$ *vanishes, then* $\lambda = f(\mathbf{x})$ *is a generalized eigenvalue and* $\mathbf{x}$ *is its associated generalized eigenvector.*

Proof. Recalling the definition of gradient (6.2), we have

$$\|\mathcal{H}\mathbf{x}^{m-1} - \lambda\mathcal{B}\mathbf{x}^{m-1}\| = \frac{\mathcal{B}\mathbf{x}^m}{m}\epsilon.$$

Since the tensor $\mathcal{B}$ is positive definite and the vector $\mathbf{x}$ belongs to a compact set $\mathbb{S}_{n-1}$, $\mathcal{B}\mathbf{x}^m$ has a finite upper bound. Thus, the first assertion is valid.

If $\epsilon = 0$, we immediately know that $\lambda = f(\mathbf{x})$ is a generalized eigenvalue and $\mathbf{x}$ is its associated generalized eigenvector. $\square$

Next, we construct the curvilinear search path using the Cayley transform [36]. Cayley transform is an effective method which could preserve the orthogonal constraints. It has various applications in the inverse eigenvalue problem [34], p-harmonic flow [35], and matrix optimization [91].

Suppose the current iterate is $\mathbf{x}_k \in \mathbb{S}_{n-1}$ and the next iterate is $\mathbf{x}_{k+1}$. To preserve the spherical constraint $\mathbf{x}_{k+1}^\top\mathbf{x}_{k+1} = \mathbf{x}_k^\top\mathbf{x}_k = 1$, we choose the next iterate $\mathbf{x}_{k+1}$ such that

$$\mathbf{x}_{k+1} = Q\mathbf{x}_k, \tag{6.5}$$

where $Q \in \Re^{n\times n}$ is an orthogonal matrix, whose eigenvalues do not contain -1. Using the Cayley transform, the matrix

$$Q = (I + W)^{-1}(I - W) \tag{6.6}$$

is orthogonal if and only if the matrix $W \in \Re^{n\times n}$ is skew-symmetric.[1] Now, our task is to select a suitable skew-symmetric matrix W such that $\mathbf{g}(\mathbf{x}_k)^\top(\mathbf{x}_{k+1} - \mathbf{x}_k) < 0$. For simplicity, we take the matrix W as

$$W = \mathbf{a}\mathbf{b}^\top - \mathbf{b}\mathbf{a}^\top, \tag{6.7}$$

[1] See "http://en.wikipedia.org/wiki/Cayley_transform".

where $\mathbf{a}, \mathbf{b} \in \Re^n$ are two undetermined vectors. From (6.5) and (6.6), we have

$$\mathbf{x}_{k+1} - \mathbf{x}_k = -W(\mathbf{x}_k + \mathbf{x}_{k+1}).$$

Then, by (6.7), it yields that

$$\mathbf{g}(\mathbf{x}_k)^\top(\mathbf{x}_{k+1} - \mathbf{x}_k) = -[(\mathbf{g}(\mathbf{x}_k)^\top\mathbf{a})\mathbf{b}^\top - (\mathbf{g}(\mathbf{x}_k)^\top\mathbf{b})\mathbf{a}^\top](\mathbf{x}_k + \mathbf{x}_{k+1}).$$

For convenience, we choose

$$\mathbf{a} = \mathbf{x}_k \quad \text{and} \quad \mathbf{b} = -\alpha\mathbf{g}(\mathbf{x}_k). \tag{6.8}$$

Here, α is a positive parameter, which serves as a step size, so that we have some freedom to choose the next iterate. According to this selection and (6.4), we obtain

$$\begin{aligned}
\mathbf{g}(\mathbf{x}_k)^\top(\mathbf{x}_{k+1} - \mathbf{x}_k) &= -\alpha\|\mathbf{g}(\mathbf{x}_k)\|^2\mathbf{x}_k^\top(\mathbf{x}_k + \mathbf{x}_{k+1}) \\
&= -\alpha\|\mathbf{g}(\mathbf{x}_k)\|^2(1 + \mathbf{x}_k^\top Q\mathbf{x}_k).
\end{aligned}$$

Since -1 is not an eigenvalue of the orthogonal matrix Q, we have $1+\mathbf{x}_k^\top Q\mathbf{x}_k > 0$ for $\mathbf{x}_k^\top\mathbf{x}_k = 1$. Therefore, the conclusion $\mathbf{g}(\mathbf{x}_k)^\top(\mathbf{x}_{k+1} - \mathbf{x}_k) < 0$ holds for any positive step size α.

We summarize the iterative process in the following Theorem.

Theorem 6.1. *Suppose that the new iterate $\mathbf{x}_{k+1}$ is generated by (6.5), (6.6), (6.7), and (6.8). Then, the following assertions hold.*

- *The iterative scheme is*

$$\mathbf{x}_{k+1}(\alpha) = \frac{1 - \alpha^2\|\mathbf{g}(\mathbf{x}_k)\|^2}{1 + \alpha^2\|\mathbf{g}(\mathbf{x}_k)\|^2}\mathbf{x}_k - \frac{2\alpha}{1 + \alpha^2\|\mathbf{g}(\mathbf{x}_k)\|^2}\mathbf{g}(\mathbf{x}_k). \tag{6.9}$$

- *The progress made by $\mathbf{x}_{k+1}$ is*

$$\mathbf{g}(\mathbf{x}_k)^\top(\mathbf{x}_{k+1}(\alpha) - \mathbf{x}_k) = -\frac{2\alpha\|\mathbf{g}(\mathbf{x}_k)\|^2}{1 + \alpha^2\|\mathbf{g}(\mathbf{x}_k)\|^2}. \tag{6.10}$$

Proof. From the equality (6.4) and the Sherman-Morrison-Woodbury formula, we have

$$
\begin{aligned}
&\mathbf{x}_{k+1}(\alpha) \\
&= (I - \alpha \mathbf{x}_k \mathbf{g}(\mathbf{x}_k)^\top + \alpha \mathbf{g}(\mathbf{x}_k)\mathbf{x}_k^\top)^{-1}(I + \alpha \mathbf{x}_k \mathbf{g}(\mathbf{x}_k)^\top - \alpha \mathbf{g}(\mathbf{x}_k)\mathbf{x}_k^\top)\mathbf{x}_k \\
&= (I + \alpha \mathbf{g}(\mathbf{x}_k)\mathbf{x}_k^\top - \alpha \mathbf{x}_k \mathbf{g}(\mathbf{x}_k)^\top)^{-1}(\mathbf{x}_k - \alpha \mathbf{g}(\mathbf{x}_k)) \\
&= \left(I - \begin{bmatrix} \alpha \mathbf{g}(\mathbf{x}_k) & -\mathbf{x}_k \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \mathbf{x}_k^\top \\ \alpha \mathbf{g}(\mathbf{x}_k)^\top \end{bmatrix} I \begin{bmatrix} \alpha \mathbf{g}(\mathbf{x}_k) & -\mathbf{x}_k \end{bmatrix} \right)^{-1} \cdot \right. \\
&\qquad\qquad\qquad \left. \begin{bmatrix} \mathbf{x}_k^\top \\ \alpha \mathbf{g}(\mathbf{x}_k)^\top \end{bmatrix} \right)(\mathbf{x}_k - \alpha \mathbf{g}(\mathbf{x}_k)) \\
&= \mathbf{x}_k - \alpha \mathbf{g}(\mathbf{x}_k) - \begin{bmatrix} \alpha \mathbf{g}(\mathbf{x}_k) & -\mathbf{x}_k \end{bmatrix} \begin{bmatrix} 1 & -1 \\ \alpha^2 \|\mathbf{g}(\mathbf{x}_k)\|^2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ -\alpha^2 \|\mathbf{g}(\mathbf{x}_k)\|^2 \end{bmatrix} \\
&= \frac{1 - \alpha^2 \|\mathbf{g}(\mathbf{x}_k)\|^2}{1 + \alpha^2 \|\mathbf{g}(\mathbf{x}_k)\|^2} \mathbf{x}_k - \frac{2\alpha}{1 + \alpha^2 \|\mathbf{g}(\mathbf{x}_k)\|^2} \mathbf{g}(\mathbf{x}_k).
\end{aligned}
$$

The proof of (6.10) is straightforward. □

Whereafter, we devote to choose a suitable step size α by an inexact curvilinear search. At the beginning, we give a useful theorem.

Theorem 6.2. *Suppose that the new iterate $\mathbf{x}_{k+1}(\alpha)$ is generated by (6.9). Then, we have*

$$
\left. \frac{\mathrm{d} f(\mathbf{x}_{k+1}(\alpha))}{\mathrm{d}\alpha} \right|_{\alpha=0} = -2\|\mathbf{g}(\mathbf{x}_k)\|^2.
$$

Proof. By some calculations, we get

$$
\mathbf{x}'_{k+1}(\alpha) = \frac{-2}{1 + \alpha^2 \|\mathbf{g}(\mathbf{x}_k)\|^2} \mathbf{g}(\mathbf{x}_k) + \frac{-4\alpha \|\mathbf{g}(\mathbf{x}_k)\|^2}{(1 + \alpha^2 \|\mathbf{g}(\mathbf{x}_k)\|^2)^2} (\mathbf{x}_k - \alpha \mathbf{g}(\mathbf{x}_k)).
$$

Hence, $\mathbf{x}'_{k+1}(0) = -2\mathbf{g}(\mathbf{x}_k)$. Furthermore, $\mathbf{x}_{k+1}(0) = \mathbf{x}_k$. Therefore, we obtain

$$
\left. \frac{\mathrm{d} f(\mathbf{x}_{k+1}(\alpha))}{\mathrm{d}\alpha} \right|_{\alpha=0} = \mathbf{g}(\mathbf{x}_{k+1}(0))^\top \mathbf{x}'_{k+1}(0) = \mathbf{g}(\mathbf{x}_k)^\top(-2\mathbf{g}(\mathbf{x}_k)) = -2\|\mathbf{g}(\mathbf{x}_k)\|^2.
$$

Algorithm 1 A curvilinear search algorithm (ACSA).

1: Give the generating vector $\mathbf{v}$ of a Hankel tensor $\mathcal{H}$, the symmetric tensor $\mathcal{B}$, an initial unit iterate $\mathbf{x}_1$, parameters $\eta \in (0, \frac{1}{2}]$, $\beta \in (0,1)$, $\bar{\alpha}_1 = 1 \leq \alpha_{\max}$, and $k \leftarrow 1$.
2: **while** the sequence of iterates does not converge **do**
3: Compute $\mathcal{H}\mathbf{x}_k^m$ and $\mathcal{H}\mathbf{x}_k^{m-1}$ by the fast computational framework introduces in Section 2.
4: Calculate $\mathcal{B}\mathbf{x}_k^m$, $\mathcal{B}\mathbf{x}_k^{m-1}$, $\lambda_k = f(\mathbf{x}_k) = \frac{\mathcal{H}\mathbf{x}_k^m}{\mathcal{B}\mathbf{x}_k^m}$ and $\mathbf{g}(\mathbf{x}_k)$ by (6.2).
5: Choose the smallest nonnegative integer ℓ and determine $\alpha_k = \beta^\ell \bar{\alpha}_k$ such that

$$f(\mathbf{x}_{k+1}(\alpha_k)) \leq f(\mathbf{x}_k) - \eta\alpha_k \|\mathbf{g}(\mathbf{x}_k)\|^2, \tag{6.11}$$

where $\mathbf{x}_{k+1}(\alpha)$ is calculated by (6.9).
6: Update the iterate $\mathbf{x}_{k+1} = \mathbf{x}_{k+1}(\alpha_k)$.
7: Choose an initial step size $\bar{\alpha}_{k+1} \in (0, \alpha_{\max}]$ for the next iteration.
8: $k \leftarrow k + 1$.
9: **end while**

The proof is completed. □

According to Theorem 6.2, for any constant $\eta \in (0, 2)$, there exists a positive scalar $\tilde{\alpha}$ such that for all $\alpha \in (0, \tilde{\alpha}]$,

$$f(\mathbf{x}_{k+1}(\alpha)) - f(\mathbf{x}_k) \leq -\eta\alpha \|\mathbf{g}(\mathbf{x}_k)\|^2.$$

Hence, the curvilinear search process is well-defined.

Now, we present a curvilinear search algorithm (ACSA) formally in Algorithm 1 for the smallest generalized eigenvalue and its associated eigenvector of a Hankel tensor. If our aim is to compute the largest generalized eigenvalue and its associated eigenvector of a Hankel tensor, we only need to change respectively (6.9) and (6.11) used in Steps 5 and 6 of the ACSA algorithm to

$$\mathbf{x}_{k+1}(\alpha) = \frac{1 - \alpha^2 \|\mathbf{g}(\mathbf{x}_k)\|^2}{1 + \alpha^2 \|\mathbf{g}(\mathbf{x}_k)\|^2} \mathbf{x}_k + \frac{2\alpha}{1 + \alpha^2 \|\mathbf{g}(\mathbf{x}_k)\|^2} \mathbf{g}(\mathbf{x}_k),$$

and

$$f(\mathbf{x}_{k+1}(\alpha_k)) \geq f(\mathbf{x}_k) + \eta\alpha_k \|\mathbf{g}(\mathbf{x}_k)\|^2.$$

When the Z-eigenvalue of a Hankel tensor is considered, we have $\mathcal{E}\mathbf{x}^m = \|\mathbf{x}\|^m = 1$ and the objective $f(\mathbf{x})$ is a polynomial. Then, we could compute the global minimizer of the step size α_k (the exact line search) in each iteration as [39]. However, we use a cheaper inexact line search here. The initial step size of the next iteration follows Dai's strategy [26]

$$\bar{\alpha}_{k+1} = \frac{\|\Delta \mathbf{x}_k\|}{\|\Delta \mathbf{g}_k\|}, \tag{6.12}$$

which is the geometric mean of Barzilai-Borwein step sizes [6].

6.4 Convergence analysis

Since the optimization model (6.1) has a nice algebraic nature, we will use the Kurdyka-Łojasiewicz property [62, 9] to analyze the convergence of the proposed ACSA algorithm. Before we start, we give some basic convergence results.

6.4.1 Basic convergence results

If the ACSA algorithm terminates finitely, there exists a positive integer k such that $\mathbf{g}(\mathbf{x}_k) = 0$. According to Lemma 6.2, $f(\mathbf{x}_k)$ is a generalized eigenvalue and $\mathbf{x}_k$ is its associated generalized eigenvector.

Next, we assume that ACSA generates an infinite sequence of iterates.

Lemma 6.3. *Suppose that the even order symmetric tensor $\mathcal{B}$ is positive definite. Then, all the functions, gradients, and Hessians of the objective (6.1) at feasible points are bounded. That is to say, there is a positive constant M such that for all $\mathbf{x} \in \mathbb{S}_{n-1}$*

$$|f(\mathbf{x})| \le M, \quad \|\mathbf{g}(\mathbf{x})\| \le M, \quad \text{and} \quad \|H(\mathbf{x})\| \le M. \tag{6.13}$$

Proof. Since the spherical feasible region $\mathbb{S}_{n-1}$ is compact, the denominator $\mathcal{B}\mathbf{x}^m$ of the objective is positive and bounds away from zero. Recalling Lemma 6.1, we get this theorem immediately. $\square$

Theorem 6.3. *Suppose that the infinite sequence $\{\lambda_k\}$ is generated by ACSA. Then, the sequence $\{\lambda_k\}$ is monotonously decreasing. And there exists a λ_* such that*

$$\lim_{k\to\infty} \lambda_k = \lambda_*.$$

Proof. Since $\lambda_k = f(\mathbf{x}_k)$ which is bounded and monotonously decreasing, the infinite sequence $\{\lambda_k\}$ must converge to a unique λ_*. $\square$

This theorem means that the sequence of generalized eigenvalues converges. To show the convergence of iterates, we first prove that the step sizes bound away from zero.

Lemma 6.4. *Suppose that the step size α_k is generated by ACSA. Then, for all iterations k, we get*

$$\alpha_k \geq \frac{(2-\eta)\beta}{5M} \equiv \alpha_{\min} > 0. \tag{6.14}$$

Proof. Let $\underline{\alpha} \equiv \frac{2-\eta}{5M}$. According to the curvilinear search process of ACSA, it is sufficient to prove that the inequality (6.11) holds if $\alpha_k \in (0, \underline{\alpha}]$.

From the iterative formula (6.9) and the equality (6.4), we get

$$\begin{aligned}
\|\mathbf{x}_{k+1}(\alpha) - \mathbf{x}_k\|^2 &= \left\| \frac{-2\alpha^2\|\mathbf{g}(\mathbf{x}_k)\|^2}{1+\alpha^2\|\mathbf{g}(\mathbf{x}_k)\|^2}\mathbf{x}_k - \frac{2\alpha}{1+\alpha^2\|\mathbf{g}(\mathbf{x}_k)\|^2}\mathbf{g}(\mathbf{x}_k) \right\|^2 \\
&= \frac{4\alpha^4\|\mathbf{g}(\mathbf{x}_k)\|^4\|\mathbf{x}_k\|^2 + 4\alpha^2\|\mathbf{g}(\mathbf{x}_k)\|^2}{(1+\alpha^2\|\mathbf{g}(\mathbf{x}_k)\|^2)^2} \\
&= \frac{4\alpha^2\|\mathbf{g}(\mathbf{x}_k)\|^2}{1+\alpha^2\|\mathbf{g}(\mathbf{x}_k)\|^2}.
\end{aligned}$$

Hence,

$$\|\mathbf{x}_{k+1}(\alpha) - \mathbf{x}_k\| = \frac{2\alpha\|\mathbf{g}(\mathbf{x}_k)\|}{\sqrt{1+\alpha^2\|\mathbf{g}(\mathbf{x}_k)\|^2}}. \tag{6.15}$$

From the mean value theorem, (6.9), (6.4), and (6.15), we have

$$f(\mathbf{x}_{k+1}(\alpha)) - f(\mathbf{x}_k) \leq \mathbf{g}(\mathbf{x}_k)^\top(\mathbf{x}_{k+1}(\alpha) - \mathbf{x}_k) + \frac{1}{2}M\|\mathbf{x}_{k+1}(\alpha) - \mathbf{x}_k\|^2$$

$$= \frac{1}{1+\alpha^2\|\mathbf{g}(\mathbf{x}_k)\|^2}\left(-2\alpha^2\|\mathbf{g}(\mathbf{x}_k)\|^2\mathbf{g}(\mathbf{x}_k)^\top\mathbf{x}_k - 2\alpha\|\mathbf{g}(\mathbf{x}_k)\|^2 + \frac{M}{2}4\alpha^2\|\mathbf{g}(\mathbf{x}_k)\|^2\right)$$

$$\leq \frac{\alpha\|\mathbf{g}(\mathbf{x}_k)\|^2}{1+\alpha^2\|\mathbf{g}(\mathbf{x}_k)\|^2}\left(4\alpha M - 2\right).$$

It is easy to show that for all $\alpha \in (0, \underline{\alpha}]$

$$4\alpha M - 2 \leq -\eta(1+\alpha^2M^2).$$

Therefore, we have

$$f(\mathbf{x}_{k+1}(\alpha)) - f(\mathbf{x}_k) \leq \frac{-\eta(1+\alpha^2M^2)}{1+\alpha^2\|\mathbf{g}(\mathbf{x}_k)\|^2}\alpha\|\mathbf{g}(\mathbf{x}_k)\|^2 \leq -\eta\alpha\|\mathbf{g}(\mathbf{x}_k)\|^2.$$

The proof is completed. □

Theorem 6.4. *Suppose that the infinite sequence $\{\mathbf{x}_k\}$ is generated by ACSA. Then, the sequence $\{\mathbf{x}_k\}$ has an accumulation point at least. And we have*

$$\lim_{k\to\infty}\|\mathbf{g}(\mathbf{x}_k)\| = 0. \tag{6.16}$$

That is to say, every accumulation point of $\{\mathbf{x}_k\}$ is a generalized eigenvector whose associated generalized eigenvalue is λ_*.

Proof. Since the sequence of objectives $\{f(\mathbf{x}_k)\}$ is monotonously decreasing and bounded, by (6.11) and (6.14), we have

$$2M \geq f(\mathbf{x}_1) - \lambda_* = \sum_{k=1}^{\infty} f(\mathbf{x}_k) - f(\mathbf{x}_{k+1}) \geq \sum_{k=1}^{\infty}\eta\alpha_k\|\mathbf{g}(\mathbf{x}_k)\|^2 \geq \eta\alpha_{\min}\sum_{k=1}^{\infty}\|\mathbf{g}(\mathbf{x}_k)\|^2.$$

It yields that

$$\sum_k \|\mathbf{g}(\mathbf{x}_k)\|^2 \leq \frac{2M}{\eta\alpha_{\min}} < +\infty. \tag{6.17}$$

Thus, the limit (6.16) holds.

Let $\mathbf{x}_\infty$ be an accumulation point of $\{\mathbf{x}_k\}$. Then $\mathbf{x}_\infty$ belongs to the compact set $\mathbb{S}_{n-1}$ and $\|\mathbf{g}(\mathbf{x}_\infty)\| = 0$. According to Lemma 6.2, $\mathbf{x}_\infty$ is a generalized eigenvector whose associated eigenvalue is $f(\mathbf{x}_\infty) = \lambda_*$. $\square$

6.4.2 Further results based on the Kurdyka-Łojasiewicz property

In this subsection, we will prove that the iterates $\{\mathbf{x}_k\}$ generated by ACSA converge without an assumption of the second-order sufficient condition. The key tool of our analysis is the Kurdyka-Łojasiewicz property. This property was first discovered by S. Łojasiewicz [62] in 1963 for real-analytic functions. Bolte et al. [9] extended this property to nonsmooth subanalytic functions. Whereafter, the Kurdyka-Łojasiewicz property was widely applied to analyze regularized algorithms for nonconvex optimization [2, 3]. Significantly, it seems to be new to use the Kurdyka-Łojasiewicz property to analyze an inexact line search algorithm, e.g., ACSA proposed in Section 3.

We now write down the Kurdyka-Łojasiewicz property [9, Theorem 3.1] for completeness.

Theorem 6.5 (Kurdyka-Łojasiewicz (KL) property)**.** *Suppose that $\mathbf{x}_*$ is a critical point of $f(\mathbf{x})$. Then there is a neighborhood $\mathscr{U}$ of $\mathbf{x}_*$, an exponent $\theta \in [0,1)$, and a constant C_1 such that for all $\mathbf{x} \in \mathscr{U}$, the following inequality holds*

$$\frac{|f(\mathbf{x}) - f(\mathbf{x}_*)|^\theta}{\|\mathbf{g}(\mathbf{x})\|} \le C_1. \tag{6.18}$$

Here, we define $0^0 \equiv 1$.

Lemma 6.5. *Suppose that $\mathbf{x}_*$ is one of the accumulation points of $\{\mathbf{x}_k\}$. For the convenience of using the Kurdyka-Lojasiewicz property, we assume that the initial* *iterate* $\mathbf{x}_1$ *satisfies* $\mathbf{x}_1 \in \mathscr{B}(\mathbf{x}_*, \rho) \equiv \{\mathbf{x} \in \Re^n \mid \|\mathbf{x} - \mathbf{x}_*\| < \rho\} \subseteq \mathscr{U}$ *where*

$$\rho > \frac{2C_1}{\eta(1-\theta)}|f(\mathbf{x}_1) - f(\mathbf{x}_*)|^{1-\theta} + \|\mathbf{x}_1 - \mathbf{x}_*\|.$$

Then, we have the following two assertions:

$$\mathbf{x}_k \in \mathscr{B}(\mathbf{x}_*, \rho), \qquad \forall\ k = 1, 2, \ldots, \tag{6.19}$$

and

$$\sum_k \|\mathbf{x}_{k+1} - \mathbf{x}_k\| \leq \frac{2C_1}{\eta(1-\theta)}|f(\mathbf{x}_1) - f(\mathbf{x}_*)|^{1-\theta}. \tag{6.20}$$

Proof. We prove (6.19) by the induction. First, it is easy to see that $\mathbf{x}_1 \in \mathscr{B}(\mathbf{x}_*, \rho)$. Next, we assume that there is an integer K such that

$$\mathbf{x}_k \in \mathscr{B}(\mathbf{x}_*, \rho), \qquad \forall\ 1 \leq k \leq K.$$

Hence, the KL property (6.18) holds in these iterates. Finally, we prove that $\mathbf{x}_{K+1} \in \mathscr{B}(\mathbf{x}_*, \rho)$.

For the convenience of presentation, we define a scalar function

$$\varphi(s) \equiv \frac{C_1}{1-\theta}|s - f(\mathbf{x}_*)|^{1-\theta}.$$

Obviously, $\varphi(s)$ is a concave function and its derivative is $\varphi'(s) = \frac{C_1}{|s-f(\mathbf{x}_*)|^{\theta}}$ if $s >$

$f(\mathbf{x}_*)$. Then, for any $1 \leq k \leq K$, we have

$$
\begin{aligned}
\varphi(f(\mathbf{x}_k)) - \varphi(f(\mathbf{x}_{k+1})) &\geq \varphi'(f(\mathbf{x}_k))(f(\mathbf{x}_k) - f(\mathbf{x}_{k+1})) \\
&= \frac{C_1}{|f(\mathbf{x}_k) - f(\mathbf{x}_*)|^\theta}(f(\mathbf{x}_k) - f(\mathbf{x}_{k+1})) \\
\text{[by KL property]} \qquad &\geq \frac{1}{\|\mathbf{g}(\mathbf{x}_k)\|}(f(\mathbf{x}_k) - f(\mathbf{x}_{k+1})) \\
\text{[since (6.11)]} \qquad &\geq \frac{1}{\|\mathbf{g}(\mathbf{x}_k)\|}\eta\alpha_k\|\mathbf{g}(\mathbf{x}_k)\|^2 \\
&\geq \frac{\eta\alpha_k\|\mathbf{g}(\mathbf{x}_k)\|}{\sqrt{1+\alpha_k^2\|\mathbf{g}(\mathbf{x}_k)\|^2}} \\
\text{[because of (6.15)]} \qquad &\geq \frac{\eta}{2}\|\mathbf{x}_{k+1} - \mathbf{x}_k\|.
\end{aligned}
$$

It yields that

$$
\begin{aligned}
\sum_{k=1}^{K} \|\mathbf{x}_{k+1} - \mathbf{x}_k\| &\leq \frac{2}{\eta}\sum_{k=1}^{K} \varphi(f(\mathbf{x}_k)) - \varphi(f(\mathbf{x}_{k+1})) \\
&= \frac{2}{\eta}(\varphi(f(\mathbf{x}_1)) - \varphi(f(\mathbf{x}_{K+1}))) \\
&\leq \frac{2}{\eta}\varphi(f(\mathbf{x}_1)). \qquad (6.21)
\end{aligned}
$$

So, we get

$$
\begin{aligned}
\|\mathbf{x}_{K+1} - \mathbf{x}_*\| &\leq \sum_{k=1}^{K} \|\mathbf{x}_{k+1} - \mathbf{x}_k\| + \|\mathbf{x}_1 - \mathbf{x}_*\| \\
&\leq \frac{2}{\eta}\varphi(f(\mathbf{x}_1)) + \|\mathbf{x}_1 - \mathbf{x}_*\| \\
&< \rho.
\end{aligned}
$$

Thus, $\mathbf{x}_{K+1} \in \mathscr{B}(\mathbf{x}_*, \rho)$ and (6.19) holds.

Moreover, let $K \to \infty$ in (6.21). We obtain (6.20). $\square$

Theorem 6.6. *Suppose that the infinite sequence of iterates* $\{\mathbf{x}_k\}$ *is generated by ACSA. Then, the total sequence* $\{\mathbf{x}_k\}$ *has a finite length, i.e.,*

$$\sum_k \|\mathbf{x}_{k+1} - \mathbf{x}_k\| < +\infty,$$

and hence the total sequence $\{\mathbf{x}_k\}$ *converges to a unique critical point.*

Proof. Since the domain of $f(\mathbf{x})$ is compact, the infinite sequence $\{\mathbf{x}_k\}$ generated by ACSA must have an accumulation point $\mathbf{x}_*$. According to Theorem 6.4, $\mathbf{x}_*$ is a critical point. Hence, there exists an index k_0, which could be viewed as an initial iteration when we use Lemma 6.5, such that $\mathbf{x}_{k_0} \in \mathscr{B}(\mathbf{x}_*, \rho)$. From Lemma 6.5, we have $\sum_{k=k_0}^{\infty} \|\mathbf{x}_{k+1} - \mathbf{x}_k\| < +\infty$. Therefore, the total sequence $\{\mathbf{x}_k\}$ has a finite length and converges to a unique critical point. □

Finally, we give an estimation for the convergence rate of ACSA, which is a specialization of Theorem 2 in Attouch and Bolte [2]. The proof here is clearer since we have a new bound in Lemma 6.6.

Lemma 6.6. *There exists a positive constant* C_2 *such that*

$$\|\mathbf{x}_{k+1} - \mathbf{x}_k\| \geq C_2 \|\mathbf{g}(\mathbf{x}_k)\|. \tag{6.22}$$

Proof. Since $\alpha_{\max} \geq \alpha_k \geq \alpha_{\min} > 0$ and (6.15), we have

$$\|\mathbf{x}_{k+1} - \mathbf{x}_k\| = \frac{2\alpha_k \|\mathbf{g}(\mathbf{x}_k)\|}{\sqrt{1 + \alpha_k^2 \|\mathbf{g}(\mathbf{x}_k)\|^2}} \geq \frac{2\alpha_{\min}}{1 + \alpha_{\max} M} \|\mathbf{g}(\mathbf{x}_k)\|.$$

Let $C_2 \equiv \frac{2\alpha_{\min}}{1+\alpha_{\max} M}$. We get this lemma. □

Theorem 6.7. *Suppose that* $\mathbf{x}_*$ *is the critical point of the infinite sequence of iterates* $\{\mathbf{x}_k\}$ *generated by ACSA. Then, we have the following estimations.*

- *If* $\theta \in (0, \frac{1}{2}]$*, there exists a* $\gamma > 0$ *and* $\varrho \in (0,1)$ *such that*

$$\|\mathbf{x}_k - \mathbf{x}_*\| \leq \gamma \varrho^k.$$

- *If* $\theta \in (\frac{1}{2}, 1)$*, there exists a* $\gamma > 0$ *such that*

$$\|\mathbf{x}_k - \mathbf{x}_*\| \leq \gamma k^{-\frac{1-\theta}{2\theta-1}}.$$

Proof. Without loss of generality, we assume that $\mathbf{x}_1 \in \mathscr{B}(\mathbf{x}_*, \rho)$. For convenience of following analysis, we define

$$\Delta_k \equiv \sum_{i=k}^{\infty} \|\mathbf{x}_i - \mathbf{x}_{i+1}\| \geq \|\mathbf{x}_k - \mathbf{x}_*\|.$$

Then, we have

$$\begin{aligned}
\Delta_k &= \sum_{i=k}^{\infty} \|\mathbf{x}_i - \mathbf{x}_{i+1}\| \\
\text{[since (6.20)]} \quad &\leq \frac{2C_1}{\eta(1-\theta)} |f(\mathbf{x}_k) - f(\mathbf{x}_*)|^{1-\theta} \\
&= \frac{2C_1}{\eta(1-\theta)} \left(|f(\mathbf{x}_k) - f(\mathbf{x}_*)|^{\theta}\right)^{\frac{1-\theta}{\theta}} \\
\text{[KL property]} \quad &\leq \frac{2C_1}{\eta(1-\theta)} \left(C_1 \|\mathbf{g}(\mathbf{x}_k)\|\right)^{\frac{1-\theta}{\theta}} \\
\text{[for (6.22)]} \quad &\leq \frac{2C_1}{\eta(1-\theta)} \left(C_1 C_2^{-1} \|\mathbf{x}_k - \mathbf{x}_{k+1}\|\right)^{\frac{1-\theta}{\theta}} \\
&= \frac{2C_1^{\frac{1}{\theta}} C_2^{-\frac{1-\theta}{\theta}}}{\eta(1-\theta)} \left(\Delta_k - \Delta_{k+1}\right)^{\frac{1-\theta}{\theta}} \\
&\equiv C_3 \left(\Delta_k - \Delta_{k+1}\right)^{\frac{1-\theta}{\theta}},
\end{aligned} \tag{6.23}$$

where C_3 is a positive constant.

If $\theta \in (0, \frac{1}{2})$, we have $\frac{1-\theta}{\theta} \geq 1$. When the iteration k is large enough, the inequality (6.23) implies that

$$\Delta_k \leq C_3(\Delta_k - \Delta_{k+1}).$$

That is

$$\Delta_{k+1} \leq \frac{C_3 - 1}{C_3} \Delta_k.$$

Hence, recalling $\|\mathbf{x}_k - \mathbf{x}_*\| \leq \Delta_k$, we obtain the estimation if we take $\varrho \equiv \frac{C_3-1}{C_3}$.

Otherwise, we consider the case $\theta \in (\frac{1}{2}, 1)$. Let $h(s) = s^{-\frac{\theta}{1-\theta}}$. Obviously, $h(s)$ is monotonously decreasing. Then, the inequality (6.23) could be rewritten as

$$\begin{aligned} C_3^{-\frac{\theta}{1-\theta}} &\leq h(\Delta_k)(\Delta_k - \Delta_{k+1}) \\ &= \int_{\Delta_{k+1}}^{\Delta_k} h(\Delta_k) \, \mathrm{d}s \\ &\leq \int_{\Delta_{k+1}}^{\Delta_k} h(s) \, \mathrm{d}s \\ &= -\frac{1-\theta}{2\theta - 1}(\Delta_k^{-\frac{2\theta-1}{1-\theta}} - \Delta_{k+1}^{-\frac{2\theta-1}{1-\theta}}). \end{aligned}$$

Denote $\nu \equiv -\frac{2\theta-1}{1-\theta} < 0$ since $\theta \in (\frac{1}{2}, 1)$. Then, we get

$$\Delta_{k+1}^{\nu} - \Delta_k^{\nu} \geq -\nu C_3^{-\frac{\theta}{1-\theta}} \equiv C_4 > 0.$$

It yields that for all $k > K$,

$$\Delta_k \leq [\Delta_K^{\nu} + C_4(k - K)]^{\frac{1}{\nu}} \leq \gamma k^{\frac{1}{\nu}},$$

where the last inequality holds when the iteration k is sufficiently large. $\square$

We remark that if the Hessian $H(\mathbf{x}_*)$ at the critical point $\mathbf{x}_*$ is positive definite, the key parameter θ in the Kurdyka-Łojasiewicz property is $\theta = \frac{1}{2}$. Under Theorem 6.7, the sequence of iterates generated by ACSA has a linear convergence rate. In this viewpoint, the Kurdyka-Łojasiewicz property is weaker than the second order sufficient condition of $\mathbf{x}_*$ being a minimizer.

6.5 Numerical experiments

To show the efficiency of the proposed ACSA algorithm, we perform some numerical experiments. The parameters used in ACSA are

$$\eta = .001, \qquad \beta = .5, \qquad \alpha_{\max} = 10000.$$

We terminate the algorithm if the objectives satisfy

$$\frac{|\lambda_{k+1} - \lambda_k|}{\max(1, |\lambda_k|)} < 10^{-12}\sqrt{n}$$

or the number of iterations exceeds 1000. The codes are written in MATLAB R2012a and run in a desktop computer with Intel Core E8500 CPU at 3.17GHz and 4GB memory running Windows 7.

We will compare the following four algorithms in this section.

- An adaptive shifted power method [50, 51] (Power M.) is implemented as `eig_sshopm` and `eig_geap` in Tensor Toolbox 2.6 for Z- and H-eigenvalues of even order symmetric tensors.

- An unconstrained optimization approach [37] (Han's UOA) is solved by `fminunc` in MATLAB with settings: `GradObj:on, LargeScale:off, TolX:1.e-10, TolFun:1.e-8, MaxIter:10000, Display:off.`

- For general symmetric tensors without considering a Hankel structure, we implement ACSA as ACSA-general.

- The ACSA algorithm (ACSA-Hankel) is proposed in Section 3 for Hankel tensors.

6.5.1 Small Hankel tensors

First, we examine some small tensors, whose Z- and H-eigenvalues could be computed exactly.

Example 6.1 ([69]). *A Hankel tensor $\mathcal{A}$ whose entries are defined as*

$$a_{i_1 i_2 \cdots i_m} = \sin(i_1 + i_2 + \cdots + i_m), \qquad i_j = 1, 2, \ldots, n, \; j = 1, 2, \ldots, m.$$

Its generating vector is $\mathbf{v} = (\sin(m), \sin(m+1), \ldots, \sin(mn))^\top$.

If $m = 4$ and $n = 5$, there are five Z-eigenvalues which are listed as follows [11, 15]

$$\lambda_1 = 7.2595, \quad \lambda_2 = 4.6408, \quad \lambda_3 = 0.0000, \quad \lambda_4 = -3.9204, \quad \lambda_5 = -8.8463.$$

Table 6.1: Computed Z-eigenvalues of the Hankel tensor in Example 6.1.

Algorithms	Power M.	Han's UOA	ACSA-general	ACSA-Hankel
-8.846335	54%	58%	72%	72%
-3.920428	46%	42%	28%	28%
CPU t. (sec)	23.09	9.34	8.39	0.67

We test four kinds of algorithms: power method, Han's UOA, ACSA-general and ACSA-Hankel. For the purpose of obtaining the smallest Z-eigenvalue of the Hankel tensor, we select 100 random initial points on the unit sphere. The entries of each initial point is first chosen to have a Gaussian distribution, then we normalize it to a unit vector. The resulting Z-eigenvalues and CPU times are reported in Table 6.1. All of the four methods find the smallest Z-eigenvalue -8.846335. But the occurrences for each method finding the smallest Z-eigenvalue are different. We say that the ACSA algorithm proposed in Section 3 could find the extremal eigenvalues with a higher probability.

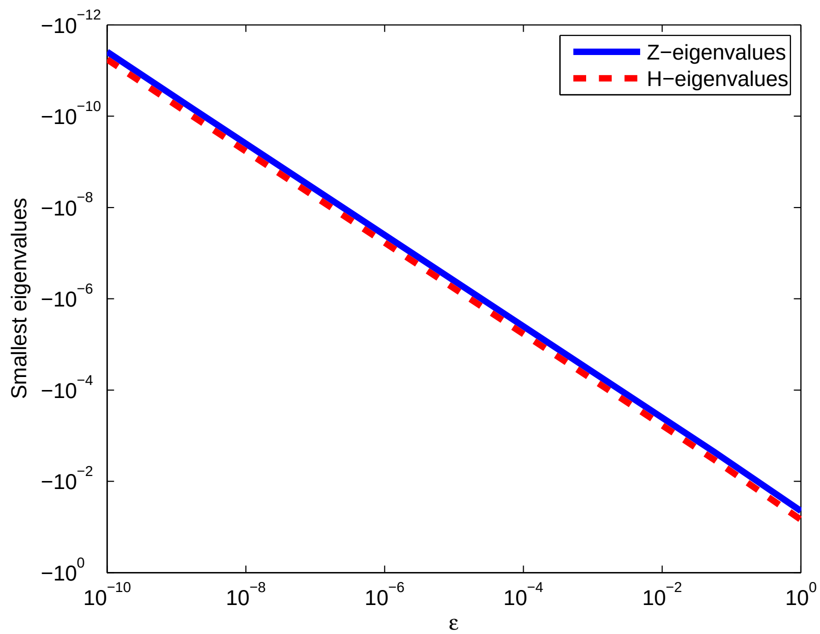

Figure 6.1: The smallest Z- and H-eigenvalues of the parameterized fourth order four dimensional Hankel tensors.

Form the viewpoint of totally computational times, ACSA-general, and ACSA-Hankel are faster than the power method and Han's UOA. When the Hankel structure of a fourth order five dimensional symmetric tensor $\mathcal{A}$ is exploited, it is unexpected that the new method is about 30 times faster than the power method.

Example 6.2. *We study a parameterized fourth order four dimensional Hankel tensor $\mathcal{H}_\epsilon$ whose generating vector has the following form*

$$\mathbf{v}_\epsilon = (8-\epsilon, 0, 2, 0, 1, 0, 1, 0, 1, 0, 2, 0, 8-\epsilon)^\top.$$

If $\epsilon = 0$, $\mathcal{H}_0$ is positive semidefinite but not positive definite [18]. When the parameter ϵ is positive and trends to zero, the smallest Z- and H-eigenvalues are negative and trends to zero. In this example, we will illustrate this phenomenon by a numerical approach.

Table 6.2: CPU times (second) for computing Z- and H-eigenvalues of the parameterized Hankel tensors shown in Example 6.2.

Algorithms	Power M.	Han's UOA	ACSA-general	ACSA-Hankel
Z-eigenvalues	41.980	46.629	17.878	1.498
H-eigenvalues	29.562	45.833	16.973	1.544
Total CPU times	71.542	92.462	34.851	3.042

Again, we compare the power method, Han's UOA, ACSA-general, and ACSA-Hankel for computing the smallest Z- and H-eigenvalues of the parameterized Hankel tensors in Example 6.2. For the purpose of accuracy, we slightly modify the setting `TolX:1.e-12, TolFun:1.e-12` for Han's UOA. In each case, thirty random initial points on a unit sphere are selected to obtain the smallest Z- or H-eigenvalues. When the parameter ϵ decreases from 1 to 10^{-10}, the smallest Z- and H-eigenvalues returned by these four algorithm are congruent. We show this results in Figure 6.1. When ϵ trends to zero, the smallest Z- and H-eigenvalues are negative and going to zero too.

The detailed CPU times for these four algorithms computing the smallest Z- and H-eigenvalues of the parameterized fourth order four dimensional Hankel tensors are drawn in Table 6.2. Obviously, even without exploiting the Hankel structure, ACSA-general is two times faster than the power method and Han's UOA. Furthermore, when the fast computational framework for the products of a Hankel tensor time vectors is explored, ACSA-Hankel saves about 90% CPU times.

6.5.2 Large scale problems

When the Hankel structure of higher order tensors is explored, we could compute eigenvalues and associated eigenvectors of large scale Hankel tensors.

Example 6.3. *A Vandermonde tensor [74, 92] is a special Hankel tensor. Let*

$$\alpha = \frac{n}{n-1} \qquad \text{and} \qquad \beta = \frac{1-n}{n}.$$

Then, $\mathbf{u}_1 = (1, \alpha, \alpha^2, \ldots, \alpha^{n-1})^\top$ *and* $\mathbf{u}_2 = (1, \beta, \beta^2, \ldots, \beta^{n-1})^\top$ *are two Vandermonde vectors. The following* m*th order* n *dimensional symmetric tensor*

$$\mathcal{H}_V = \underbrace{\mathbf{u}_1 \otimes \mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_1}_{m \text{ times}} + \underbrace{\mathbf{u}_2 \otimes \mathbf{u}_2 \otimes \cdots \otimes \mathbf{u}_2}_{m \text{ times}}$$

is a Vandermonde tensor which satisfies the Hankel structure. Here $\otimes$ *is the outer product. Obviously, the generating vector of* $\mathcal{H}_V$ *is* $\mathbf{v} = (2, \alpha + \beta, \ldots, \alpha^{m(n-1)} + \beta^{m(n-1)})^\top$.

Proposition 6.1. *Suppose the* m*th order* n *dimensional Hankel tensor* $\mathcal{H}_V$ *is defined as in Example 6.3. Then, when* n *is even, the largest Z-eigenvalue of* $\mathcal{H}_V$ *is* $\|\mathbf{u}_1\|^m$ *and its associated eigenvector is* $\frac{\mathbf{u}_1}{\|\mathbf{u}_1\|}$.

Proof. Since $\alpha\beta = -1$ and n is even, $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthogonal. We consider the optimization problem

$$\begin{aligned} \max \quad & \mathcal{H}_V \mathbf{x}^m = (\mathbf{u}_1^\top \mathbf{x})^m + (\mathbf{u}_2^\top \mathbf{x})^m, \\ \text{s.t.} \quad & \mathbf{x}^\top \mathbf{x} = 1. \end{aligned}$$

Since $\|\mathbf{u}_1\| > \|\mathbf{u}_2\|$, when $\mathbf{x} = \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|}$, the above optimization problem obtains its maximal value $\|\mathbf{u}_1\|^m$. We write down its KKT condition, and it is easy to see that $(\|\mathbf{u}_1\|^m, \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|})$ is a Z-eigenpair of $\mathcal{H}_V$. □

Now, we employ the proposed ACSA algorithm which works with the generating vector of a Hankel tensor to compute the largest Z-eigenvalue of the Vandermonde tensor defined in Example 6.3. We consider different orders $m = 4, 6, 8$ and various dimension $n = 10, \ldots, 10^6$. For each case, we choose ten random initial points, which has a Gaussian distribution on a unit sphere. Table 6.3 shows the computed largest Z-eigenvalues and the associated CPU times. For all case, the resulting largest Z-eigenvalue is agree with Proposition 6.1. When the dimension of the tensor is

Table 6.3: The largest Z-eigenvalues of Vandermonde tensor in Example 6.3.

m	n	largest Z-eigenvalues	Occurrences	CPU times (sec.)
4	10	9.487902e02	8	0.062
4	100	1.013475e05	8	0.140
4	1,000	1.019800e07	7	0.889
4	10,000	1.020431e09	8	9.048
4	100,000	1.020494e11	10	150.245
4	1,000,000	1.020500e13	5	2066.592
6	10	2.922505e04	5	0.140
6	100	3.226409e07	5	0.234
6	1,000	3.256659e10	7	1.919
6	10,000	3.259683e13	7	17.753
6	100,000	3.259985e16	9	211.537
6	1,000,000	3.260016e19	4	3190.439
8	10	9.002029e05	5	0.359
8	100	1.027131e10	5	0.437
8	1,000	1.039992e14	7	2.917
8	10,000	1.041279e18	7	30.561
8	100,000	1.041408e22	8	1058.248

one million, the computational times for fourth order and sixth order Vandermonde tensors are about 35 and 55 minutes respectively.

Example 6.4. *An mth order n dimensional Hilbert tensor [86] is defined as*

$$\mathcal{H}_H = \frac{1}{i_1 + i_2 + \cdots + i_m - m + 1} \qquad i_j = 1, 2, \ldots, n, j = 1, 2, \ldots, m.$$

Its generating vector is $\mathbf{v} = (1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{m(n-1)+1})^\top$. *When the order m is even, the Hilbert tensors are positive definite. Its largest Z-eigenvalue and largest H-eigenvalues are bounded by* $n^{\frac{m}{2}} \sin \frac{\pi}{n}$ *and* $n^{m-1} \sin \frac{\pi}{n}$ *respectively.*

We illustrate by numerical experiments to show whether these bounds are tight? First, for the dimension varying from ten to one million, we calculate the theoretical upper bounds of the largest Z-eigenvalues of corresponding fourth order and sixth order Hilbert tensors. Then, for each Hilbert tensor, we choose ten initial points

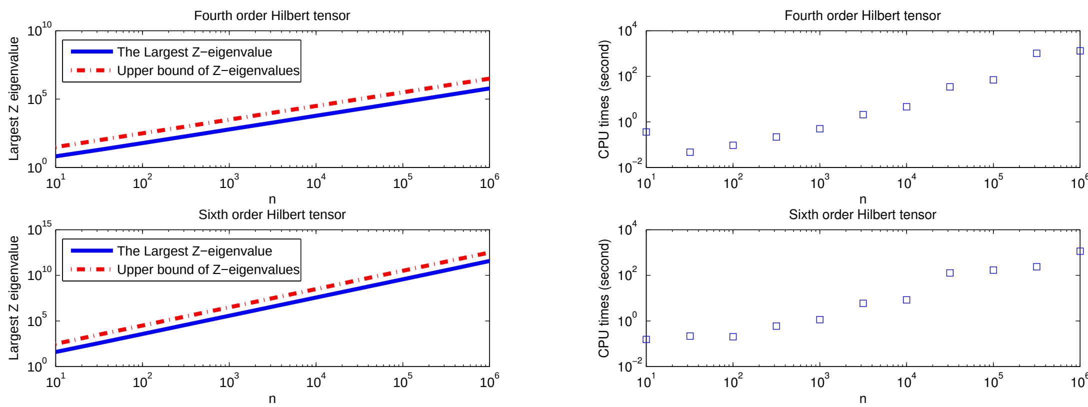

Figure 6.2: The largest Z-eigenvalue and its upper bound for Hilbert tensors.

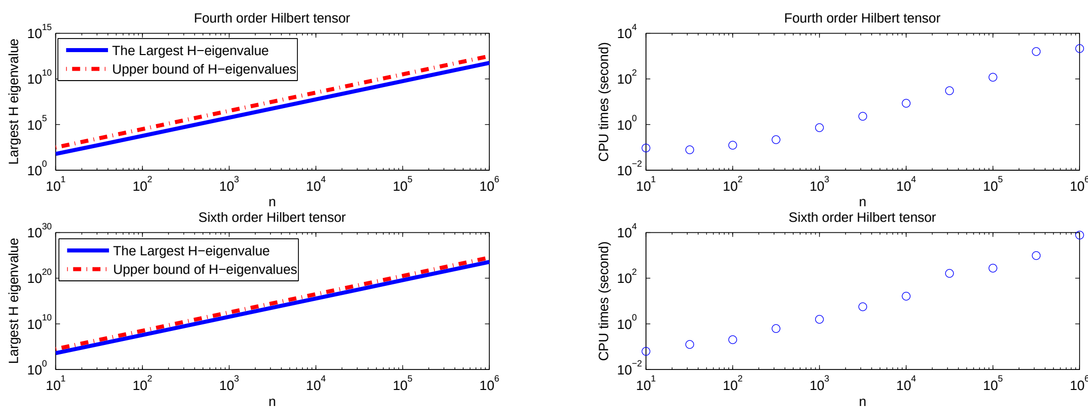

Figure 6.3: The computed largest H-eigenvalue and its upper bound for Hilbert tensors.

and employ the ACSA algorithm equipped with a fast computational framework for products of a Hankel tensor and vectors to compute the largest Z-eigenvalues. These results are shown in the left sub-figure of Figure 6.2. The right sub-figure of Figure 6.2 shows the corresponding CPU times for ACSA-Hankel. We can see that the theoretical upper bounds for the largest Z-eigenvalues of the Hilbert tensors are almost tight up to a constant multiple.

Similar results for the largest H-eigenvalues and their theoretical upper bounds of Hilbert tensors are illustrated in Figure 6.3.

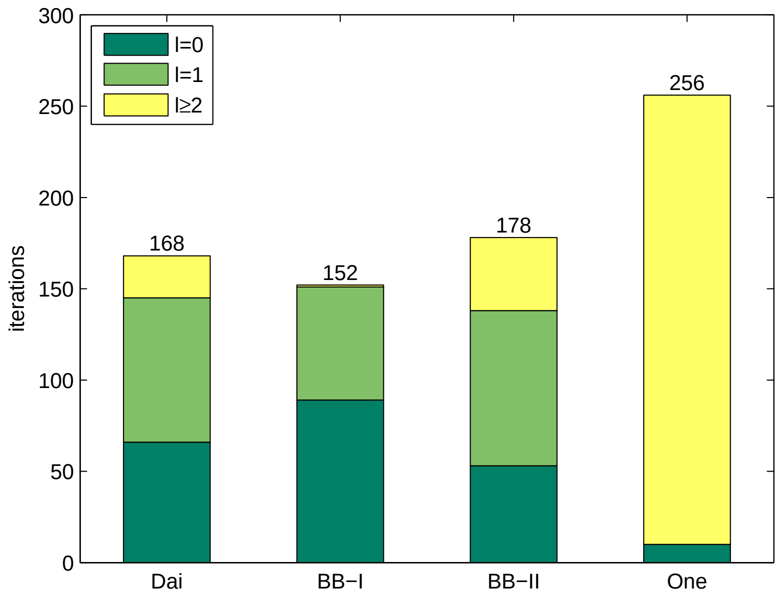

Figure 6.4: Comparisons of four sorts of step size strategies.

6.5.3 Initial step sizes

In the process of the curvilinear search, how to determine a suitable step size is a critical problem. Barzilai and Borwein [6] provided two candidates

$$\bar{\alpha}_{k+1}^{\mathrm{BB-I}} := \frac{\Delta\mathbf{x}_k^\top \Delta\mathbf{g}_k}{\|\Delta\mathbf{g}_k\|^2} \qquad \text{and} \qquad \bar{\alpha}_{k+1}^{\mathrm{BB-II}} := \frac{\|\Delta\mathbf{x}_k\|^2}{\Delta\mathbf{x}_k^\top \Delta\mathbf{g}_k},$$

which satisfy the quasi-Newton condition approximately. However, when the optimization problem is nonconvex, the inner product $\Delta\mathbf{x}_k^\top \Delta\mathbf{g}_k$ maybe zero or negative, which could destroy the curvilinear search. Dai [26] proposed to use their geometric mean.

Next, we compare four sorts of strategies for the initial step size of curvilinear search: (i) Dai's step size (6.12), (ii)-(iii) absolute values of $\bar{\alpha}_{k+1}^{\mathrm{BB-I}}$ and $\bar{\alpha}_{k+1}^{\mathrm{BB-II}}$, (iv) a fixed step size $\bar{\alpha}_{k+1}^{\mathrm{One}} = 1$. Using these strategies, we compute the largest Z-eigenvalue of a fourth order $10,000$ dimensional Hilbert tensor. All of the four approaches start from the same ten initial points and reach the same Z-eigenvector. Figure

6.4 illustrates counting results of the curvilinear search parameter ℓ. Obviously, the fixed step size one performs poorly since ℓ is always great than or equal to 2. By exploiting the quasi-Newton condition approximately, BB-I and BB-II perform satisfactory, where BB-I seems better. The performance of Dai's step size is in the medium place of BB-I and BB-II. It only requires $\ell = 0.78$ times curvilinear search per iteration on average. We employ Dai's step size since it is positive and hence safe in theory.

Chapter 7

Conclusions and Future Work

This chapter draws conclusions on the thesis, and points out some possible research directions related to the work done in this thesis.

We investigate the problem whether there exist PNS Hankel tensors, including sixth order three dimensional Hankel tensors, fourth order four dimensional Hankel tensors and anti-circulant tensors. In Chapter 3, we examine four classes of sixth order three dimensional Hankel tensors and give the sufficient conditions and necessary conditions for these cases. We also randomly generate several thousands sixth order three dimensional PSD Hankel tensors and identify that they are SOS or not. There are no PNS Hankel tensors to be found by this way. For fourth order four dimensional PNS Hankel tensors with symmetric generating vectors, we prove that PNS Hankel tensors do not exist on a segment, a cone, a ray and a point. Numerical tests also indicate that PNS Hankel tensors do not exist. However, a complete proof that sixth order three dimensional and fourth order four dimensional PNS Hankel tensors do not exist may not be easy.

In Chapter 4, we extend the definition of anti-circulant tensors to generalized anti-circulant tensors by introducing a circulant index. For some cases, including the matrix case, we give necessary and sufficient conditions for even order PSD generalized anti-circulant tensors, and show that in these cases, they are SOS tensors.

This shows that, there are no PNS Hankel tensors in these cases. We see that Theorem 4.2 may still hold as long as r is odd, even if $GCD(m,r) > 1$; and that Theorem 4.5 may still hold as long as r is even, even if $GCD(m,r) > 2$. Are these true in general? How can we prove these? We may see that the proofs of Theorems 4.2 and 4.5 rely on Theorem 4.1, but the proofs of Theorems 4.3 and 4.6 do not use a unified technique like Theorem 4.1. Can we have a unified technique to study the case that r is odd, $GCD(m,r) > 1$, and the case that r is even, $GCD(m,r) > 2$?

In Chapter 5, we give a sufficient and necessary condition for an even order three dimensional strongly symmetric circulant tensor to be positive semi-definite, and circulant tensor is a special class of Toeplitz tensor. For $u, c \leq 0$ and $u = c > 0$, we show that this condition is also sufficient for this tensor to be sum-of-squares. Numerical tests indicate that this is also true in the other cases. How can $\mathcal{B} - \bar{u}_0\mathcal{T}$ and $-\bar{v}_0\mathcal{T} - \mathcal{B}$ be shown to be PSD for all even $m \geq 6$? If these are true, then (5.7) and (5.11) are true for all even $m \geq 6$. More efforts are needed to prove that this problem is PNS-free eventually.

In Chapter 6, we propose an inexact steepest descent method processing on a unit sphere for generalized eigenvalues and associated eigenvectors of Hankel tensors. Owing to the fast computation framework for the products of a Hankel tensor and vectors, the new algorithm is fast and efficient as shown by some preliminary numerical experiments. Since the Hankel structure is well-exploited, the new method could deal with some large scale Hankel tensors, whose dimension is up to one million in a desktop computer.

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