DYNAMIC PRICING FOR STOCHASTIC CONTAINER LEASING SYSTEM

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CERTIFICATE OF ORIGINALITY

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ABSTRACT

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by

Wen JIAO

With the substantial upsurge of container traffic, the container leasing company thrives on the financial benefits and operations flexibility of leasing containers requested by shippers. In practice, container lease pricing problem is different from consumer product pricing in consideration of the fair value of container, limited customer types and monopolistic supply market. In view of the durability of container and the diversified lease time and quantity, the pricing is a challenging task for the leasing company.

In the first part, the monopolist's nonlinear pricing problems in static and dynamic environments are examined. In particular, the leasing company designs and commits a menu of price and hire quantity/time pairs to maximize the expected profit and in turn customers choose hire quantities/time to maximize their surpluses according to their hire preferences. In a static environment, closed-form solutions are obtained for different groups of customers with multiple types subject to capacity constraint. In a dynamic environment with contemporaneous arrivals, we address two customer types and derive closed-form solutions for the problem of customers with hire time preference. We show that the effect of the capacity constraint increases with time

of the planning horizon when customers have the same hire time preference; while in the case with different hire time preferences, the capacity constraint has opposite effects on the low and high type customers. Next the case of customers with hire quantity preference is discussed. We focus on the lease with alternative given sets of hire time and use dynamic programming to derive the numerical optimal hire time sequence. Further we investigate the nonlinear pricing problem with dynamic arrivals and hire time preference. We derive the closed-form solutions and discuss the effects of capacity constraints and dynamic arrivals on the optimal solution. Compared with the solution with contemporaneous arrivals, the dynamic arrivals only aggravate the effect of capacity constraint for the consistent low type customers.

The leasing company provides customer-oriented services to increase fleet efficiency and maximize profit. Advance reservation could be a segment fence for container leasing firm to vary the base price according to the supply and demand conditions. In the second part, we consider a dynamic pricing problem of a container leasing firm with unit capacity request and reservations. A reserved customer books containers some time before the pickup date and settles the rent at booking time. A walk-in customer arrives at the firm and requests the immediate lease service. The problem is modelled as a continuous-time Markov decision process. Using value iteration, the properties of the optimal allocation and pricing policy are derived. We show that there exists a state-dependent rationing policy with bounded sensitivity. The optimal posted price is nondecreasing with the leased amount and the number of advance demands. Numerical experiments are conducted to study the effect of reservation on the optimal policy.

In the third part, we examine a dynamic pricing problem of a container leasing firm facing reserved customers and walk-in customers with multiple units of capacity request and fixed lease durations. We first discuss the case with same lease duration and the optimal prices for two customer types are nonincreasing in the system state.

Next, we propose myopic pricing policy to the dynamic pricing problem. Finally, we partially characterize the optimal policies for different lease durations. The optimal policies have bounded and monotone sensitivity.

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TABLE OF CONTENTS

LIST OF FIGURES

Figure

CHAPTER 1

Introduction

1.1 Background and Motivation

In the past two decades, global container trade has witnessed a substantial upsurge growing from 28.7 million TEU (Twenty-foot Equivalent Unit)s in 1990 to 161 million TEUs in 2013 (*UNCTAD* 2014). In contrast to the thriving container trade, the leasing companies' share of world container fleet does not change much, from 43.2% of 6.4 million in 1990 to 46.2% of 34.4 million in 2013 (*Drewry* 2014). The relatively stable share of leasing company in ownership partially reveals the strong demand of shipping companies whose needs are satisfied by lessors flexible services. From the lessor's perspective, the leasing company[1] could enjoy the economies of scale by the procurement of large numbers of containers, efficient utilization and access to raise capital at a competitive rate in a volatile economy. From the lessee's perspective, renting containers could serve as a financial tool with the following advantages.

- **Conserving capital.** Instead of purchasing containers, the shipping companies are relieved from the burden of the huge expenditure on containers. It is reasonable especially when the new container price is too high or it is difficult to raise finance for container investment such as in the recession year

[1]For variety, we use lessor, lease firm, leasing company, monopolist interchangeably without confusion.

of 2009. This reserves lessee's limited borrowing capacity for more profitable investments, such as infrastructure depots and IT facilities.

- **Providing a better fiscal picture.** The lease is usually qualified as pre-tax expense and considered as 'off-balance-sheet financing'. The monthly payment appears on the balance sheet as expense rather than long term debt.

- **Avoiding risk.** With fixed and predictable payment on container lease, the shipper is protected from inflation.

Besides the financial benefits, the operational advantages of renting containers to supplement their own fleet are as follows.

- **Quick response to demand changes.** Some shipping lines have extremely high imbalance container flows owing to imbalance of trade volume between continents. For instance, in 2013, the container moving from Asia to North America (13.8 Million TEUs) is about twice of that from North America to Asia (7.4 million TEUs). The imbalance between Asia and Europe is even bigger with ratio 14.1:6.4 (*UNCTAD* 2014). The consequence of such a imbalance cargo flow is the higher cost per TEU for these routes, which is a difficult task in capacity management to the shipping companies. In addition, the trade volume and the demand of containers are high during peak seasons such as Christmas. After holidays, the demand falls back to a low level. Therefore, it is reasonable to rent containers for fluctuating seasonal demand or imbalance cargo routes.

- **High flexibility.** It is convenient for a shipper to pick up/drop off containers at the nearest depot and select the most suitable lease contracts to satisfy his needs. When the lease period expires, the lessee could return, purchase, re-lease or replace the leased containers.

- **Cost Saving.** The shippers receive carefully designed services with high quality control, unique depot selection, professional repair and the disposal of used containers. The leasing service reduces costs such as overhead cost, maintenance cost, finance cost and administrative cost.

The container lease contracts can be divided into two categories: master lease and term lease. Master lease is also referred to a full service lease. Both parties agree on a master contract: the shipper has the right to pick up/drop off container at his convenience and changes the number of leased containers under the basic terms. The lessor is responsible for repositioning the empty containers and the maintenance and repair. The term lease has fixed lease duration including short and long terms, ranging from a single-trip lease up to eight years rent. Unlike a master lease, the lessee is responsible for the maintenance and repair of containers.

In the container leasing industry, pricing is a very challenging factor for a leasing company. The main characteristics of container lease are the fair value of container, stable and limited customer types and monopolistic supply market. **(1) Fair value of container.** A container is labeled as an industrial product and durable good. Its value is much higher than those of daily commodities but lower than those of precision equipment. The average ex-factory price for newbuild TEU and resale price for used TEU in 2013 are US$2150 and US$1260 respectively (*Drewry* 2014). **(2) Limited customer types.** The target customers of container lease are big shipping companies with long-term contractual relationship. There are limited discrete customer types. In the changing lease market, each customer (shipper) requests large numbers of container with diverse hire time (from one month to five years) depending on his own demand. The varied lease time and quantities of different customer types in container lease meet the requirement for second-degree price discrimination mechanism (different prices for distinct quantities). In other words, the leasing company should pay attention to the characteristics of each customer in the price determination process which is the essence of the nonlinear pricing problem. By contrast, in service pricing, the target customers are individual customers who usually demand for one unit without any contractual relationship and have wide variety of customer types. **(3) Monopolistic supply market.** The container leasing industry has been dominated by a small group of influential companies about two decades. The top ten companies control 87.5% of the entire lease fleet. The top tier is headed by long-standing number one, Taxtainer Group. Its fleet is about 40% bigger than its nearest competitor (*Drewry*, 2014). This is the reason why the monopolist supply market is studied in our paper. In practice, the lease rate determined by a leasing company is usually based on the past leasing experience. Thus it is necessary to have a scientific method assisting the leasing company on the pricing determination process.

Based on the main features of container lease, the price and discount affect customers' intention to deal, hire time and quantity. The more favorable price offered to longer hire time and larger lease quantity incurs an opportunity cost, resulting in inadequate capacity of containers that affects the lessor from gaining future profit from other customers. On the other hand, higher price deters consumers' interest to rent and cause more idle capacity. There is clearly a need for identifying the hire discount and time discount in the lease system given a capacity constraint.

Although a container is a durable product as the subject of lease service, it should be seen as a perishable product on a daily basis. Two important factors for a container leasing firm to stay profitable and competitive in the leasing industry are high level of utilization and lease rate. The utilization rate of the container leasing industry leaders, Taxtainer Group and CAI are 96.1% and 91.5%, respectively (*Textainer* 2014, *CAI* 2014). Lease rate hinges on a number of elements including the supply and demand for leased containers, the price of new containers, interest rates for leasing company, shipping lines and the quantity of containers available by competitors. Except for the first element, other elements are beyond the control of the leasing firm. The container leasing firm exploits the different features to decide the base price based on supply and demand conditions. One feature to differentiate the customers is the lead time. A reserved lessee usually books the containers in advance, settles the payment and is protected by the availability of containers reserved, while a walk-in lessee rents the containers in the last minute before the pickup date and the availability of containers may not guaranteed.

Advance reservations could serve as one of segment fences to differentiate demand between walk-in and reserved customers. However, the advance reservation could be a double-edged sword. On one hand, a reserved customer books the containers and settles the payments before the actual pickup date. Such advance reservations give the leasing firm more information about future demand, more flexibility to manage its fleet, more operational cash flow (prepay) and more effective price to control the customer flow. On the other hand, if the firm allocates all the available containers for reserved demand, it definitely hurts the profit of the firm since more revenue can be generated by saving a portion of containers from reserved customers and leasing the container to the forthcoming walk-in customers. Thus, a challenge for the operation manager of the container leasing company is given the information on the total available fleet how to deploy the fleet across two customer types and how to use the pricing tool to achieve the maximum net profit.

1.2 Organization of the thesis

After introducing the background and motivation in Chapter 1, Chapter 2 investigates the revenue management problem considering both short-term lease and long-term lease in practice. First, the firm allocates the capacity once in a static environment. Closed form solutions are obtained for different groups of customers subject to capacity constraint. Next, the firm allocates the finite capacity repeatedly in a dynamic environment. For the case of customers with simultaneous arrivals and hire time preference, closed-form solutions are derived and the effect of capacity constraint is discussed. For the case of customers with hire quantity preference, dynamic programming is adopted to acquire the numerical optimal solution and discuss the effect of parameters on the optimal solution. Further the problem is investigated with dynamic arrivals and hire time preference. We derive the closed-form solutions and discuss the effect of capacity constraints and dynamic arrivals on the solution.

Chapter 3 examines a dynamic pricing and capacity management problem for stochastic lease system facing two types–reserved and walk-in customers with unit capacity request. In this chapter, customers have stochastic lease duration and s-tochastic lead time for reserved customers. After reviewing the related literature, the pricing and capacity rationing problem is modeled as a queuing model and the structural properties of the optimal policy are shown. Numerical results and insights about the model are stated.

Chapter 4 addresses a container leasing firm facing two customer types with multiple units of capacity request. After the literature review, some preliminary results are present. The case with same lease duration is discussed and the optimal prices for two customers are nonincreasing in the system state. The optimal walk-in demand is most sensitive to reserved demand of current and next periods. The optimal reserved demand is most sensitive to the latest booking. Further, we relax the same lease duration assumption and study the optimal policies under different lease durations. Myopic pricing policy is proposed for the dynamic pricing problem.

Last, Chapter 5 concludes the thesis and points out future research directions.

CHAPTER 2

Dynamic Nonlinear Pricing for Stochastic Container Leasing System[1]

This part investigates the static and dynamic rental revenue management problem considering several situations (short-term lease and long-term lease) in practice. The firm commits a price menu with hire quantity (time) to maximize the expected profit and in turn customers choose their hire quantities/time to maximize their surpluses based on their hire preferences. In a static environment, the firm allocates the capacity once. Closed form solutions are obtained for different groups of customers subject to capacity constraint. The leasing company provides various lease contracts based on the type of customers. In a dynamic environment with contemporaneous arrivals, the firm allocates the finite capacity repeatedly in the planning horizon. For the case of customers with hire time preference, closed-form solutions are derived and the effect of capacity constraint are discussed. We show that the effect of the capacity constraint increases with time of the planning horizon when customers have the same hire time preference; while in the case with different hire time preferences, the effect of capacity constraint becomes smaller whenever the last customer type is the high type and this effect becomes larger whenever the last customer type is the low type.

[1] The static and dynamic nonlinear pricing problems in Sections 2.2 and 2.3 are published as "Jiao, W., H. Yan, and K.-W. Pang (2016). Nonlinear pricing for stochastic container leasing system. Transportation Research Part B: Methodological 89, 1-18".

In other words, the influence of capacity constraint depends only on the realization of the customer type in the previous period. For the case of customers with hire quantity preference, we adopt dynamic programming to acquire the numerical optimal solution and discuss the effect of parameters on the optimal solution.

Further we investigate the nonlinear pricing problem with dynamic arrivals and hire time preference in a dynamic environment. We derive the closed-form solutions and discuss the effect of capacity constraints on the solution. In the setting with same hire time preference, the effect of capacity constraint at each time still increases over time and but the effect reduces when customers arrive at the system dynamically. In the setting with different hire time preferences, the dependent effect of capacity constraint on the last customer type still holds. For the consistent high type customers and inconsistent customer, the effect of capacity constraint is the same as in the contemporaneous arrivals. For the consistent low type customers, the dynamic arrivals accentuate the capacity effect.

2.1 Literature Review

The first part of this dissertation is built on three streams of literature: study about rental service system, static and dynamic mechanism design.

There is a burgeoning scholarly literature on the rental/leasing systems. Recent related study about rental systems mostly concentrate on the following major problems, empty container reposition problem in some regions with imbalance inbound and outbound traffic (*Song and Dong* 2012, *Bell et al.* 2013), capacity rationing problem for different customer types (*Savin et al.* 2005, *Papier and Thonemann* 2008, 2010). Besides, *Gans and Savin* (2007) and *Cachon and Feldman* (2011) study the rental/service revenue management and capacity allocation problem as queuing model. In queuing model, once the price is accepted by the customer, the rental duration is a given parameter following exponential distribution rather than a specific rental time selected by each consumer. *Dobbs* (1995) examines a monopolist's inter-temporal nonlinear pricing problem with unit/excess capacity where customers arrive randomly and choose their hire time. Polynomial function is utilized to represent the price schedule. The optimal pricing policy is sensitive to customer arrival frequency but insensitive to changes in time discount rate. *Dobbs* captures the main feature of rental system—customers have the option of selecting the hire time. In his model, same type customers select the same hire time which is history independent in the infinite horizon. To the best of our knowledge, there is no paper analyzing the pricing and capacity rationing problem in the context of container leasing industry.

Various study about static mechanism design problem addresses the dynamics between customer information and the firm's pricing schedule. This line of research starts with the seminal work of *Mussa and Rosen* (1978). They explore a monopolist's price-quality schedule allocating quality-differentiated goods to customers under self-selection constraint. In *Myerson* (1981), his contribution to the literature on unidimensional continuous customer types is the identification of the optimal auction structure to attain the criterion (e.g. social welfare maximization, customer purchasing cost minimization). *Maskin and Riley* (1984) demonstrate that under a separability assumption, the seller's optimal price-quantity schedule has the quantity-discount structure. Based on these pioneer works, the mechanism design theory of pricing has been applied to information goods (*Sundararajan* 2004), multiproduct (*Armstrong* 1996), parking slot assignment (*Zou et al.* 2015) and transportation service procurement (*Huang and Xu* 2013).

The dynamic mechanism design literature can be classified by two strands of literature according to the nature of dynamics. One strand of literature primarily focuses on the setting that a fixed population whose preference evolves over time and allocation for each customer is determined repeatedly. *Battaglini* (2005) investigates the optimal contract between a monopolist and a customer whose valuation follows a two-stage Markov process in an infinite horizon. The optimal contract is non-stationary and converges to efficient contract over time. The differences between this part and *Battaglini*'s work are that we consider multiple customers, one more dimension of product characteristics (hire time) and capacity constraint. *Athey and Segal* (2013), *Kakade et al.* (2013), *Pavan et al.* (2014) study the social welfare maximization and incentive compatible mechanisms in dynamic environments. *Athey and Segal* (2013) construct a budge-balanced mechanism in general dynamic environments. *Kakade et al.* (2013) explore the optimal mechanism design in separable environments with dynamic private information. They employ a relaxation method to first find an allocation rule in the relaxed environment and then determine the allocation rule is ex post incentive compatible under the restriction that each agent needs to report his entire type history in each period. *Pavan et al.* (2014) adopt the first-order approach to study mechanism design in dynamic quasilinear environment which each agent has a dynamic unidimensional private information. *Battaglini and Lamba* (2014) examine a dynamic principal-agent model in which the agent's types are serially correlated and follow a Markov process. They show a dynamic envelope formula considering only local incentive compatibility constraints but the formula fails to characterize the optimal dynamic contract in general dynamic environments. So they present the suboptimal monotonic contracts which works well in complex environments. *Garrett* (2014) considers dynamic mechanism design in a setting that buyer arrive over time and their values changes over time. The author fully characterizes the optimal mechanism under the two customer types setting and a continuum of values setting. The late arrivals are punished with less surplus to earn.

The other strand considers that a dynamic population arrives over time with fixed preference and the allocation for each customer is determined only once. *Gershkov and Moldovanu* (2009) investigate a dynamic allocation of finite heterogeneous objects to privately informed agents who arrive randomly before a deadline. They characterize implementable dynamic allocation rules through a method combining the payoff equivalence principle and a variational argument. *Pai and Vohra* (2013) solve a monopolist's multi-period dynamic auction design problem for multiple identical items. Customers with unit demand arrive over time with private arrival time and deadline. The optimal allocation rule is a simple index rule which is calculated by a dynamic knapsack algorithm. The customer's index depends on the reported value and its distribution with entry and exit time. The allocation rule is monotone in valuation. *Board and Skrzypacz* (2013) study a seller's revenue maximization problem in the finite horizon. Buyers enter the market over time and forward looking, thus strategically time their purchases. The optimal mechanism includes a sequence of deterministic cutoffs in continuous time and can be implemented by posed prices. When the number of entrants decreases over time, the cutoffs also reduce and satisfy the one-period-look-ahead property. *Bergemann and Said* (2011) provide a detailed review of dynamic mechanism design.

However, the above studies on dynamic mechanism design assume that the characteristic of product only has the quantity dimension or quality dimension under unit demand assumption, in this part, we extend the problem with consideration of two dimensions, hire quantity and hire time. We believe that this is the first work to consider two dimensional features of product in a dynamic environment.

2.2 Static Nonlinear Pricing Problem

In this section, we discuss an atemporal monopolist nonlinear pricing problem where the capacity is rationed once. In the next section, we study the intertemporal nonlinear pricing problem where the capacity is allocated repeatedly in the finite horizon.

In this paper, the cases we focus on that the lessee has hire quantity/time preference are also motivated by industry operations. In practice, the leasing company enters into long-term leases for a fixed term normally ranging from three to eight years, with five-year term leases being most common (*Textainer*, 2014). The lessee open calls for tenders to choose the alternative lessor with specific hire time preference. The final winner may be a sole bidder or two/three bidders who share the contract. Under the agreement of preferred hire time, the hire quantity is decided based on the price offered by leasing companies. In the trip lease, the sailing time between origin and destination is also known in advance.

2.2.1 One group of customers

The monopolist (leasing company) has C units of container available to rent by M customers at one time. In this subsection, we assume that all M customers have the same hire time d. This assumption is reasonable as five year is the most common hire time in the long-term lease contract according to the 2014 annual report of *Textainer*. A customer usually has a preferred hire time and then choose the hire quantity depending on the posted price. Further, in view of the sufficient long hire time, the market would be quite different from now when the containers are returned. Hence, it is more sensible to model this pricing problem in the long-term lease contract as an atemporal nonlinear pricing problem. The firm determines a vector of quantity and price pairs associated with each customer type to maximize the expected profit and each customer selects the hire quantity and price pair designed for his type.

The M customers are classified into N types, where $M < N$. A customer's valuation refers to the benefit that a customer obtains from the leasing service. A customer's valuation, $\bar{\theta}_i \in \bar{\Theta}$, is private information and drawn from a known iid probability mass function $f(\bar{\theta}_i)$ and cumulative distribution function $F(\bar{\theta}_i)$, shortened as f_i and F_i respectively. $\bar{\Theta} = \{\bar{\theta}_1, \bar{\theta}_2, \ldots, \bar{\theta}_N\}$ is a finite set with $\bar{\theta}_i = i\psi$ for some $\psi > 0$. Assume that f_i and F_i satisfy the monotone hazard rate condition, if $j > i$ then $\frac{1-F_j}{f_j} > \frac{1-F_i}{f_i}$. This assumption is quite usual in literature (*Armstrong* 1996, *Myerson*

1981, *Anderson and Dana Jr* 2009). A customer rents q_i units of container for hire time d_i enjoys utility $U(\bar{\theta}_i, d_i, q_i) = \bar{\theta}_i d_i q_i - \frac{1}{2}(d_i^2 + q_i^2)$. The quadratic utility function follows the tradition of the literature (*Wilson* 1993, *Rochet and Stole* 2003). Let Y be the fixed lease contract setup cost. The operating cost per time per unit for the leasing company is a and $a < \psi$. The unit benefit ψ obtained by the customer could not lower than the operating cost per unit a of the leasing company. The direct operating expense includes storage, handling, maintenance, and reposition (*Textainer*, 2014). Such operating cost is a component of the objective function of these empty container reposition articles (*Cheung and Chen* 1998, *Song and Dong* 2012, *Bell et al.* 2013), which is linear to the number of containers and the duration of the lease contract. The adoption of linear cost function is just to simplify exposition, and it is easy to extend to some other forms of cost function. In this section with same hire time d, let $\theta_i = \bar{\theta}_i d$ and $U(\bar{\theta}_i, d_i, q_i)$ becomes to $U(\theta_i, q_i) = \theta_i q_i - \frac{1}{2}(d^2 + q_i^2)$.

The Firm's Problem

The objective of the leasing company is to maximize the expected profit with finite capacity. In the direct revelation mechanism, the leasing company first announces and commits a menu of quantity q_i and total price P_i pairs, shortened as $\{Q, P\}$. When a customer arrives at the leasing company, the customer reports a type i based on the announced menu $\{Q, P\}$ to maximize his surplus $U(\theta_i, q_i) - P_i$. The customer receives q_i units of container and issues the payment P_i.

There are some constraints in which the direct revelation mechanism must satisfy.

Incentive Compatibility (IC_{ij}) Constraint. Each customer reports his type i truthfully and selects the quantity and price pair $\{q_i, P_i\}$ offered to his type. Customer has no incentive to deviate from his type i as the consumer surplus of reporting type i is greater than that reporting other types j ($j \neq i$). Thus, truthfulness is the best strategy to maximize his own consumer surplus.

$$U(\theta_i, q_i) - P_i \geq U(\theta_i, q_j) - P_j \quad \forall i \text{ and } j \text{ where } j \neq i \tag{2.1}$$

Refer $IC_{i,i+1}$ to the upward IC constraint and $IC_{i,i-1}$ to the downward IC constraint.

Individual Rationality (IR_i) Constraint. Each customer only rents the containers if he has nonnegative consumer surplus.

$$U(\theta_i, q_i) - P_i \geq 0 \quad \forall i \tag{2.2}$$

Due to the IR_i constraint, there exists some customers obtaining positive consumer surpluses, meanwhile some types of customers are priced out of the market. There exists a type j such that $q_j = 0$ which divides all customers into two categories, for customers with type $k \leq j$, $q_k = 0$; otherwise, for $j < k \leq N$, $q_k > 0$. In other words, the types of customers who belong to $[1, j]$ fail to accept the price and leave the firm; and the types of customers who belong to $(j, N]$ hire a positive quantity with nonnegative consumer surplus.

Capacity Constraint (CC). The total units of container allocated cannot exceed the available capacity.

$$M \sum_i f_i q_i \leq C \tag{2.3}$$

The expected profit of the leasing company is the expected revenue minus the operating cost. The atemporal nonlinear pricing problem is written as follows.

$$\Pi(Q, P) = \max_{\{Q,P\}} M \sum_i f_i (P_i - Y - adq_i) \tag{2.4}$$

$$s.t. \; IC_{ij}, IR_i, CC \quad \forall i, j \in [1, \cdots, N]$$

Lemma 2.1. *If a mechanism $\{Q, P\}$ is implementable, then $q_i \geq q_j$ for any $1 \leq j <$*

$i \leq N$.

Proof. Suppose that $q_i < q_j$ for $i > j$. From $IC_{i,j}$ and $IC_{j,i}$ constraints, we have

$$\begin{aligned} IC_{i,j} \qquad & U(\theta_i, q_i) - P_i \geq U(\theta_i, q_j) - P_j, \\ IC_{j,i} \qquad & U(\theta_j, q_j) - P_j \geq U(\theta_j, q_i) - P_i. \end{aligned}$$

The above two inequalities imply that

$$\theta_i(q_j - q_i) \leq P_j - P_i + \frac{1}{2}(q_j^2 - q_i^2) \leq \theta_j(q_j - q_i).$$

According to assumption $\theta_i > \theta_j$ and $q_j > q_i$, we obtain that $\theta_i(q_j - q_i) > \theta_j(q_j - q_i)$, yielding a contradiction. □

Denote the consumer surplus of type i as $S(\theta_i) = U(\theta_i, q_i) - P_i$.

Lemma 2.2. *If a mechanism $\{Q, P\}$ is implementable, then for all customer types i,*

$$S(\theta_i) \geq S(\theta_1) + \psi d \sum_{k=1}^{i-1} q_k \tag{2.5}$$

$$S(\theta_i) \leq S(\theta_1) + \psi d \sum_{k=2}^{i} q_k \tag{2.6}$$

Proof. From $IC_{i,i-1}$, we have

$$S(\theta_i) = U(\theta_i, q_i) - P_i \geq U(\theta_i, q_{i-1}) - P_{i-1}.$$

And the right side of the above equation can be rewritten as

$$
\begin{aligned}
& U(\theta_i, q_{i-1}) - P_{i-1} \\
=& U(\theta_{i-1}, q_{i-1}) - P_{i-1} + U(\theta_i, q_{i-1}) - U(\theta_{i-1}, q_{i-1}) \\
=& S(\theta_{i-1}) + \psi d q_{i-1}.
\end{aligned}
$$

Accordingly, we have $S(\theta_i) \geq S(\theta_{i-1}) + \psi d q_{i-1}$. Summing up the constraints from $IC_{i,i-1}$ to $IC_{2,1}$, we get

$$
S(\theta_i) \geq S(\theta_1) + \psi d \sum_{k=1}^{i-1} q_k.
$$

Similarly, summing up the constraints from $IC_{i-1,i}$ to $IC_{1,2}$ and we obtain (4.2). □

Lemma 2.3. *If the adjacent downward (upward) IC constraint binds, then the corresponding upward (downward) IC constraint is satisfied.*

Proof. If $IC_{i,i-1}$ binds, $U(\theta_i, q_i) - U(\theta_i, q_{i-1}) = P_i - P_{i-1}$. Recall that $U(\theta_i, q_i) = \theta_i q_i - \frac{1}{2}(d^2 + q_i^2)$, then

$$
\begin{aligned}
\theta_{i-1}(q_i - q_{i-1}) + \frac{1}{2}(q_{i-1}^2 - q_i^2) &= U(\theta_{i-1}, q_i) - U(\theta_{i-1}, q_{i-1}) \\
< U(\theta_i, q_i) - U(\theta_i, q_{i-1}) &= \theta_i(q_i - q_{i-1}) + \frac{1}{2}(q_{i-1}^2 - q_i^2).
\end{aligned}
$$

It follows that $U(\theta_{i-1}, q_i) - U(\theta_{i-1}, q_{i-1}) < P_i - P_{i-1}$. The upward $IC_{i-1,i}$ constraint is satisfied. □

The atemporal nonlinear pricing problem with capacity constraint can be transformed to a standard static nonlinear pricing problem using the Lagrange multiplier approach. Based on the above lemmas, we could consider the relaxed problem which the adjacent downward $IC_{i,i-1}$ and IR_1 constraints bind. The binding constraint IR_1 means that $S(\theta_1) = 0$ and the binding constraints $IC_{i,i-1}$ indicates that

$S(\theta_i) = \psi d \sum_{k=1}^{i-1} q_k$. As a result, the optimal menu of prices can be derived from the binding downward IC constraints. Replace $P_i = U(\theta_i, q_i) - S(\theta_i)$ and reformulate the problem (2.4) as

$$\Pi(Q,\lambda) = \max_{\{Q,\lambda\}} M \sum_i f_i[U(\theta_i, q_i) - S(\theta_i) - Y - adq_i] + \lambda(C - M \sum_i f_i q_i) \quad (2.7)$$

$$\begin{aligned} s.t. \quad & S(\theta_1) = 0 \qquad\qquad IR_1 \\ & S(\theta_i) = \psi d \sum_{k=1}^{i-1} q_k \quad IC_{i,i-1} \text{ for } i \in [2, N] \end{aligned}$$

Theorem 2.1. *The optimal allocation policy of atemporal nonlinear pricing problem is characterized as follows. Let* $w_i = i - \frac{1-F_i}{f_i}$ *and* $i^* = \arg\min\{i | w_i > \frac{a}{\psi}\}$.

- *If* $(i^* - \frac{a}{\psi})(1 - F_{i^*-1}) \leq \frac{C}{Md\psi}$, *then the capacity constraint is not binding. For* $i < i^*$, *the optimal quantity* q_i *is* 0; *for* $i^* \leq i \leq N$, $q_i = d\psi w_i - ad$.

- *If* $(i^* - \frac{a}{\psi})(1 - F_{i^*-1}) > \frac{C}{Md\psi}$, *then the capacity constraint is binding. For* $i < i^*$, *the optimal quantity* q_i *is* 0; *for* $i^* \leq i \leq N$, $q_i = d\psi w_i - ad - \lambda$ *where* $\lambda = d\psi i^* - ad - \frac{C}{M(1-F_{i^*-1})}$.

Proof. *(i)* When the capacity constraint is not binding, then $\lambda = 0$ and the problem becomes a standard nonlinear pricing problem. Take the derivative of (4.2) w.r.t θ_i, we can get

$$f_i(id\psi - ad - q_i - d\psi \frac{1 - F_i}{f_i}) = 0,$$
$$q_i = d\psi w_i - ad.$$

For $i < i^*$, $q_i = 0$; otherwise, $q_i = d\psi w_i - ad$. Since capacity constraint is not binding, from $M \sum_{i=i^*}^{N} f_i q_i \leq C$, we have $(i^* - \frac{a}{\psi})(1 - F_{i^*-1}) \leq \frac{C}{Md\psi}$.

(ii) When the capacity constraint is binding, using the Lagrange multiplier ap-

proach, (4.2) can be rewritten as

$$\Pi(Q,\lambda) = \max_{\{Q,\lambda\}} M \sum_i f_i[\theta_i q_i - \frac{1}{2}(d^2 + q_i^2) - S(\theta_i) - Y - adq_i - \lambda q_i] + \lambda C$$
$$s.t.\ IR_1, IC_{i,i-1} \qquad \text{for } i \in [2, N]$$

The pointwise maximization solution is derived from the derivative about θ_i combining with the binding capacity constraint.

$$\begin{cases} f_i(id\psi - ad - q_i - d\psi\frac{1-F_i}{f_i} - \lambda) = 0, \\ M\sum_{i=i^*}^{N} f_i q_i = C \end{cases} \Rightarrow \begin{cases} q_i = d\psi w_i - ad - \lambda \\ \lambda = d\psi i^* - ad - \frac{C}{M(1-F_{i^*-1})} \end{cases}$$

The optimal allocation policy corresponding to the binding status of capacity constraint is derived. □

Remark. The leasing company provides different rental contracts based on the type of customers. The total number of customers who received the lease contract is $M(1 - F_{i^*-1})$.

2.2.2 Two different groups of customers

In this subsection, we consider the case that the M customers are classified into two groups: one group of customers has hire time preference, denoted by Group I; another group of customers has hire quantity preference, denoted by Group J. Let $\bar{d}_i$ be the preferred hire time in Group I, $\bar{d}_i \leq \bar{d}_{i+1}$ ($i \in I$) and $\bar{q}_j$ is the preferred hire quantity in Group J, $\bar{q}_j \leq \bar{q}_{j+1}$ ($j \in J$). Assume that each group still has N types. A customer's valuation in Group I is $\theta_i^I = i\bar{d}_i\psi$ and in Group J is $\theta_j^J = j\bar{q}_j\xi$ for $i, j \in \{1, 2, \cdots, N\}$, $\psi > 0$ and $\xi > 0$. The other assumptions are the same as those stated in Section 3.1. The atemporal nonlinear pricing problem with two different groups of customers can be formulated as

$$\max_{\{Q_I,P_I,D_J,P_J\}} M[\sum_i f_i(P_i - Y - a\bar{d}_i q_i) + \sum_j f_j(P_j - Y - a\bar{q}_j d_j)]$$

$$\begin{aligned}
s.t. \qquad & U(\theta_i^I, q_i) - P_i \geq U(\theta_i^I, q_{i'}) - P_{i'} && IC_{i,i'}^I \\
& U(\theta_j^J, d_j) - P_j \geq U(\theta_j^J, d_{j'}) - P_{j'} && IC_{j,j'}^J \\
& U(\theta_i^I, q_i) - P_i \geq 0, U(\theta_j^J, d_j) - P_j \geq 0 && IR_i^I, IR_j^J \\
& M[\textstyle\sum_i f_i q_i + \sum_j f_j \bar{q}_j \mathbf{1}(d_j > 0)] \leq C && CC \\
& \forall i, i' \in I, \forall j, j' \in J
\end{aligned}$$

The first two constraints are incentive compatible constraints for both groups. The next two constraints are individual rationality constraints. The last is the capacity constraint, and $\mathbf{1}(\cdot)$ is an indicator function. The inequality (4.1) in Lemma 3.2 becomes to

$$S(\theta_i^I) \geq S(\theta_1^I) + \frac{1}{2}(\bar{d}_1^2 - \bar{d}_i^2) + \psi \sum_{k=1}^{i-1} q_k[(k+1)\bar{d}_{k+1} - k\bar{d}_k];$$
$$S(\theta_j^J) \geq S(\theta_1^J) + \frac{1}{2}(\bar{q}_1^2 - \bar{q}_i^2) + \xi \sum_{k=1}^{j-1} d_k[(k+1)\bar{q}_{k+1} - k\bar{q}_k].$$

To solve this problem, we can still use Lagrange multiplier approach to obtain the optimal solution.

Theorem 2.2. *The optimal allocation policy of atemporal nonlinear pricing problem with two different groups of customers is characterized as follows. The superscripts s and b signify the slack and binding status of the capacity constraint.*

- *When the capacity constraint is not binding, the optimal allocation policy is that for Group I,* $q_i^s = \psi \left\{ i\bar{d}_i - \frac{a\bar{d}_i}{\psi} - [(i+1)\bar{d}_{i+1} - i\bar{d}_i]\frac{1-F_i}{f_i} \right\}^+$*; for Group J,*

$d_j^s = \xi \left\{ j\bar{q}_j - \frac{a}{\psi} - [(j+1)\bar{q}_{j+1} - j\bar{q}_j]\frac{1-F_j}{f_i} \right\}^+.$

- *When the capacity constraint is binding, the optimal allocation policy is that for Group I,* $q_i^b = \psi \left\{ i\bar{d}_i - \frac{a\bar{d}_i}{\psi} - [(i+1)\bar{d}_{i+1} - i\bar{d}_i]\frac{1-F_i}{f_i} \right\}^+ - \lambda$ *where* $\lambda = \frac{1}{1-F_{i^*-1}} \left(\sum_{i \geq i^*} f_i q_i^s + \sum_{j \geq j^*} f_j \bar{q}_j - \frac{C}{M} \right)$ *and* $i^* = \arg\min\{i | q_i^s \geq 0\}$, $j^* = \arg\min\{i | d_j^s \geq 0\}$*; for Group J,* $d_j^b = d_j^s$.

Proof. The proof is similar to the proof of Theorem 2.1, thus the proof is omitted. □

For Group I, when customers have the same preferred hire time, the optimal allocation policy reduces to the optimal policy in Theorem 3.1 except with different λ values. The capacity constraint only binds the optimal hire quantity q_i^b of Group I, but not the optimal hire time d_j of Group J owing to the fact that customers in this group have preferred hire quantities.

2.3 Dynamic Nonlinear Pricing Problem with Contemporaneous Arrivals

In this section, we discuss the monopolist's nonlinear pricing problem in a dynamic environment with contemporaneous arrivals. In particular, the monopolist determines a menu of quantity (time) and price pairs to maximize the expected profit in discrete-time and finite horizon setting. All customers first show up in the leasing company at the beginning of the horizon. To better capture the characteristics of the two different groups of customers, we explore the nonlinear pricing problem for customers with preferred hire time and preferred hire quantity separately. Besides, suppose that there are two customer types in each group, the low type L and the high type H.

2.3.1 Hire time preference

In this subsection, we describe the situation of a trip lease contract in the short term lease category. The liner ship carrier announces the ports of call of a specific route with specific estimated time of arrival and departure. This information is usually known in advance and considered constant. The shippers will then make reservation to occupy certain capacity (in TEUs) of a voyage with specific origin-destination pair. They will consider rent the containers from the leasing company for this specific shipment if they don't have their own containers. If the route of the liner shipping service has only one origin-destination port, the hire time of all customers are considered the same; if there are multiple ports of call, this leads to the situation that customers may have different hire times. Denote the preferred hire time of two customer types by $\bar{d}_H$ and $\bar{d}_L$. Corresponding to the different situations in practice, we first study the case that $\bar{d}_H = \bar{d}_L$ and then discuss the case when $\bar{d}_H > \bar{d}_L$. Let $\theta_i = \bar{\theta}_i \bar{d}_i$ and $\bar{\theta}_i > a$.

2.3.1.1 Same hire time preference ($\bar{d}_H = \bar{d}_L = d$)

As the M customers of the leasing company have the same hire time preference d, there are equidistant time points in finite horizon, $\{0 = t_0, t_1, \cdots, t_K, t_{K+1} = T\}$, where $t_k = kd$ and t_K is the last pricing decision point. In a finite period setting, customer type is affected by the shipper's demand and fleet capacity, world trade and economic conditions, the price of new and used containers, shifting trend of cargo traffic and fluctuation in interest rates and currency exchange rates. The above factors directly or indirectly affect the customer type at the leasing time. Therefore, we model the customer valuation as stochastic variable. At time t_0, the firm has a prior information about the proportion of customers being classified into the low and high type, f_L and f_H. The customer type evolves over time according to a two-stage Markov process. Let f_{ij} be the probability that a type i customer at time point t_k

becomes a type j customer at time point t_{k+1} for any t_k, where $i, j \in \{L, H\}$. The probability f_{ij} is independent of the time point. Moreover, assume that customer types are positively correlated, $f_{HH} - f_{HL} > 0$ and $f_{LL} - f_{LH} > 0$. $\boldsymbol{H}_k$ is the set of all possible history up to time t_k. h_k^z is the public history up to time point t_k and the customer type at t_{k-1} is z ($z \in \{L, H\}$), $h_k^z := \{h_{k-1}, z\}$ and $h_0 := \emptyset$. h_k stands for the general history up to time point t_k. Let $f(h_k)$ be the expected probability of history h_k.

In each time point t_k, the sequence of events is listed as follows: (1) if $k \geq 1$, contracted customers return the rented containers; (2) based on the public history h_k, the leasing company designs and commits a menu of quantity and price pairs $\{Q, P\} = \{q_i(h_k), P_i(h_k)\}$ to maximize the expected profit with time discount factor δ; (3) each customer reports his type to maximize his expected consumer surplus, receives the corresponding quantity of his type and settles the payment.

Let $S(\theta_i|h_k)$ be a customer's expected surplus with type i up to history h_k, $S(\theta_i|h_k) = U(\theta_i, q_i(h_k)) - P_i(h_k) + \delta^d \sum_{j \in \{L,H\}} f_{ij} S(\theta_j|h_{k+1}^i)$. For notational brevity, we denote $q_i(h_0) := q_i, S(\theta_i|h_0) := S(\theta_i)$.

Incentive Compatibility ($IC_{ij}(h_k)$) Constraint. In a dynamic environment, the incentive compatibility constraint for the high type and history h_k ($k = 0, \cdots, K$) can be written as

$$U(\theta_H, q_H(h_k)) - P_H(h_k) + \delta^d \sum_j f_{Hj}[U(\theta_j|h_{k+1}^H) - P_j(h_{k+1}^H)] \geq$$
$$U(\theta_H, q_L(h_k)) - P_L(h_k) + \delta^d \sum_j f_{Hj}[U(\theta_j|h_{k+1}^L) - P_j(h_{k+1}^L)]$$

Simplify the above inequality, it becomes

$$S(\theta_H|h_k) \geq S(\theta_L|h_k) + \Delta\theta q_L(h_k) + \delta^d \sum_j (f_{Hj} - f_{Lj}) S(\theta_j|h_{k+1}^L)$$

where $\Delta\theta = \theta_H - \theta_L = (\bar{\theta}_H - \bar{\theta}_L)d$.

Individual Rationality ($IR_i(h_k)$) Constraint. The individual rationality constraint in a dynamic environment is $S(\theta_i|h_k) \geq 0$ for $h_k \in \boldsymbol{H}_k$.

Capacity Constraint ($CC(h_k)$). In view of the same hire time for all customers, the capacity constraint at each time point t_k is that the total number of allocated units cannot exceed C. At time t_0, the capacity constraint is $M(f_H q_H + f_L q_L) \leq C$. When $k > 0$, $\boldsymbol{H}_k$ can be divided into three subsets, $\boldsymbol{H}_k = \{h_k^L, h_k^{\hat{H}}, h_k^{\hat{L}}\}$, where $h_k^L = (L, \cdots, L)$ is the history from time t_0 to time t_{k-1} where the customer types are all L in the first k time points; $h_k^{\hat{H}} = \{h_{k-1}, H\}$ refers to the history where the customer type at t_{k-1} is H and $h_k^{\hat{L}} = \{h_{k-1}, L\}$ is the history where the customer type at t_{k-1} is L and $h_{k-1} \neq h_{k-1}^L$. Note that $f(h_1^{\hat{L}}) = 0$. The containers rented out at time point t_{k-1} will return to the firm at the time point t_k. The capacity constraint at each time point t_k can be expressed as

$$M\left[f(h_k^L)\sum_i f_{Li} q_i(h_k^L) + f(h_k^{\hat{H}})\sum_i f_{Hi} q_i(h_k^{\hat{H}}) + f(h_k^{\hat{L}})\sum_i f_{Li} q_i(h_k^{\hat{L}})\right] \leq C$$

The monopolist's problem boils down to as follows.

$$\begin{aligned}
\Pi(Q,P) &= \max_{\{Q,P\}} M\{\sum_i f_i(P_i - Y - adq_i) + \sum_{k=1}^{K} \delta^{kd}\mathbb{E}_{h_k^z}\sum_i f_{zi}[P_i(h_k^z) - Y - adq_i(h_k^z)]\} \\
&= \max_{\{Q,S(\theta)\}} M\{\sum_i f_i[U(\theta_i, q_i) - S(\theta_i) - Y - adq_i] \\
&\qquad + \sum_{k=1}^{K} \delta^{kd}\mathbb{E}_{h_k^z}\sum_i f_{zi}[U(\theta_i, q_i(h_k^z)) - Y - adq_i]\} \qquad (2.8) \\
&s.t. \quad IC_{ij}(h_k), IR_i(h_k), CC(h_k) \\
&\qquad \forall i,j \in \{L,H\}, h_k \in \boldsymbol{H}_k \ (k = 0, \cdots, K)
\end{aligned}$$

Battaglini and Lamba (2014) proved that the first-order approach is valid for the monotonic allocations in a dynamic environment, $q(h_k) \geq q(h_{k'})$ if $h_k > h_{k'}$, e.g.

$h_k = \{H, L\}$ and $h_{k'} = \{L, L\}$ in our problem. Since we only have two customer types and unidimensional allocation under the monotone hazard rate assumption, the monotonic allocation requirement is satisfied in our problem. We adopt the relaxed method as presented in the standard static nonlinear pricing problem where the incentive compatibility constraints for the high type $IC_{HL}(h_k)$ and the individual rationality constraints for the low type $IR_L(h_k)$ are binding for any $h_k \in \boldsymbol{H}_k$. Define the relaxed problem as $IR_L(h_k)$ and $IC_{HL}(h_k)$ are binding constraints for any $h_k \in \boldsymbol{H}_k$.

Lemma 2.4. *In a dynamic environment with same hire time preference, the optimal solution of the relaxed problem is also an optimal solution of the original problem.*

Proof. Suppose that $\{Q, P\}$ is an optimal solution of the original problem which $IR_L(h_k)$ and $IC_{HL}(h_k)$ are not binding constraints for some $h_k \in \boldsymbol{H}_k$.
If $IR_L(h_k)$ are not binding constraints for some $h_k \in \boldsymbol{H}_k$, that is, $S(\theta_L|h_k) = \omega$, where ω is a positive number.

- When $k = 0$, consider an alternative solution $\{Q', S'(\theta)\}$ such that $S'(\theta_L) = S(\theta_L) - \omega$. Then the expected optimal profit increases by $f_L\omega$, $\Pi(Q', S'(\theta)) = \Pi(Q, P) + Mf_L\omega$.

- When $k > 0$, let $S'(\theta_L|h_k) = S(\theta_L|h_k) - \omega$, the expected profit remains the same satisfying all constraints, $\Pi(Q', S'(\theta)) = \Pi(Q, P)$.

If $IC_{HL}(h_k)$ are not binding constraints for some $h_k \in \boldsymbol{H}_k$, $S(\theta_H|h_k) = \Delta\theta q_L(h_k) + \delta^d \Delta f S(\theta_H|h^L_{k+1}) + \omega$, where $\Delta f = f_{HH} - f_{LH}$. Consider an alternative solution $\{Q', S'(\theta)\}$ such that $S'(\theta_H|h_k) = S(\theta_H|h_k) - \omega$.

- When $k = 0$, the net change is $f_H\omega$, $\Pi(Q', S'(\theta)) = \Pi(Q, P) + Mf_H\omega$.

- When $k > 0$ and $k' > k$, $S'(\theta_H|h_{k'})$ remains the same as in the original solution, $S'(\theta_H|h_{k'}) = S(\theta_H|h_{k'})$. When $k' = k - 1$, we have $S'(\theta_H|h_{k-1}) =$

$S(\theta_H|h_{k-1}) - \delta^d\omega$. By repeatedly applying the above modifications until $k' = 0$, $S'(\theta_H) = S(\theta_H) - \delta^{kd}\omega$. The expected optimal profit increases by $Mf_H\delta^{kd}\omega$, $\Pi(Q', S'(\theta)) = \Pi(Q, P) + Mf_H\delta^{kd}\omega$.

Based on the above, the alternative solution satisfying the binding constraints $IC_{HL}(h_k)$ and $IR_L(h_k)$ in the relaxed problem yields an equal or higher profit, which contradicts the optimality of the assumption. □

From the binding constraint $IC_{HL}(h_k)$ in the relaxed problem,

$$S(\theta_H|h_k) = \Delta\theta q_L + \delta^d \Delta f S(\theta_H|h^L_{k+1}), \text{ where } \Delta f = f_{HH} - f_{LH},$$

the expected consumer surplus $S(\theta_H)$ is

$$S(\theta_H) = S(\theta_H|h_0) = \Delta\theta \sum_{k=0}^{K} (\delta^d \Delta f)^k q_L(h^L_k).$$

As $S(\theta_L|h_k) = 0$ and $S(\theta_H)$ depends on $q_L(h^L_k)$ for $k = \{0, 1, \cdots, K\}$, reformulate (2.8) as the Lagrangean objective function. The relaxed problem is written as follows.

$$\begin{aligned}
\Pi(Q, S(\theta_H), \Lambda) = &\max_{\{Q, S(\theta_H), \Lambda\}} M[\sum_i f_i(U(\theta_i, q_i) - Y - adq_i) - f_H S(\theta_H)] \\
&+ M \sum_{k=1}^{K} \delta^{kd} \mathbb{E}_{h^z_k} \sum_i f_{zi}[U(\theta_i, q_i(h^z_k)) - Y - adq_i(h^z_k)] + \lambda_0(\frac{C}{M} - f_H q_H - f_L q_L) \\
&+ \sum_{k=1}^{K} \lambda_k[\frac{C}{M} - f(h^L_k) \sum_i f_{Li} q_i(h^L_k) - f(h^{\hat{H}}_k) \sum_i f_{Hi} q_i(h^{\hat{H}}_k) - f(h^{\hat{L}}_k) \sum_i f_{Li} q_i(h^{\hat{L}}_k)]
\end{aligned}$$

$$\begin{array}{lll}
s.t. & S(\theta_L|h_k) = 0 & IR_L(h_k) \\
& S(\theta_H) = \Delta\theta \sum_{k=0}^{K} (\delta^d \Delta f)^k q_L(h^L_k) & IC_{HL}(h_k) \\
& \forall i \in \{L, H\}, h_k \in \boldsymbol{H}_k \ (k = 0, \cdots, K) &
\end{array}$$

Theorem 2.3. *For any $h_k \in \boldsymbol{H}_k$, the optimal intertemporal allocation policy for customers with same hire time preference is characterized as follows.*
At time 0,

$$\begin{cases} q_H = \theta_H - ad - \Lambda_0; \\ q_L = \theta_L - ad - \frac{f_H}{f_L}\Delta\theta - \Lambda_0; \end{cases} \qquad \textit{where } \Lambda_0 = \frac{\lambda_0}{M} = (\theta_L - ad - \frac{C}{M})^+;$$

when $h_k \in \{h_k^{\hat{H}}, h_k^{\hat{L}}\}$, *when* $h_k = h_k^L$,

$$\begin{cases} q_H(h_k) = \theta_H - ad - \Lambda_k; \\ q_L(h_k) = \theta_L - ad - \Lambda_k; \end{cases} \quad \begin{cases} q_H(h_k^L) = \theta_H - ad - \Lambda_k; \\ q_L(h_k^L) = \theta_L - ad - \frac{f_H\Delta\theta}{f_L}(\frac{\Delta f}{f_{LL}})^k - \Lambda_k; \end{cases}$$

where $\Lambda_k = \frac{\lambda_k}{M\delta^{kd}} = \left\{\theta_L - ad - \frac{C}{M} + \Delta\theta\left[f_{LH}(f(h_k^L) + f(h_k^{\hat{L}})) + f(h_k^{\hat{H}})f_{HH} - f_H\Delta f^k\right]\right\}^+$.

Proof. When $h_k \in \{h_k^{\hat{H}}, h_k^{\hat{L}}\}$, the first-order conditions w.r.t $q_H(h_k)$ and $q_L(h_k)$ are given by the following equations.

$$q_H(h_k): \qquad M\delta^{kd}[\theta_H - ad - q_H(h_k)] - \lambda_k = 0$$

$$q_L(h_k): \qquad M\delta^{kd}[\theta_L - ad - q_L(h_k)] - \lambda_k = 0$$

When $h_k = h_k^L$, the first-order conditions about $q_H(h_k^L)$ and $q_L(h_k^L)$ are given by the following equations.

$$q_H(h_k^L): \qquad M\delta^{kd}[\theta_H - ad - q_H(h_k)] - \lambda_k = 0$$

$$q_L(h_k^L): \qquad M\delta^{kd}[\theta_L - ad - q_L(h_k) - \tfrac{f_H\Delta\theta}{f_L}(\tfrac{\Delta f}{f_{LL}})^k] - \lambda_k = 0$$

And Λ_k is obtained from the following equation
$\left[f(h_k^L)\sum_i f_{Li}q_i(h_k^L) + f(h_k^{\hat{H}})\sum_i f_{Hi}q_i(h_k^{\hat{H}}) + f(h_k^{\hat{L}})\sum_i f_{Li}q_i(h_k^{\hat{L}}) - \frac{C}{M}\right]^+ = 0.$ □

Corollary 2.1. *For $0 \le k \le K$, Λ_k is increasing*[2] *in k.*

Proof. We use induction to prove this lemma. For $k = 1$, $\Lambda_1 - \Lambda_0 = f_{LH}\Delta\theta > 0$. Next, we show that if $\Lambda_k - \Lambda_{k-1} > 0$, then also $\Lambda_{k+1} - \Lambda_k > 0$ holds. Note that

$$\begin{cases} f(h_k^L) = f(h_{k-1}^L) f_{LL}; \\ f(h_k^{\hat{L}}) = f(h_{k-1}^{\hat{L}}) f_{LL} + f(h_{k-1}^{\hat{H}}) f_{HL}; \\ f(h_k^{\hat{H}}) = f(h_{k-1}^{\hat{H}}) f_{HH} + f(h_{k-1}^{\hat{L}}) f_{LH} + f(h_{k-1}^{L}) f_{LH}. \end{cases}$$

$$\begin{aligned} \frac{\Lambda_k - \Lambda_{k-1}}{\Delta\theta} =& \Delta f \left[f_{LH} \left(f(h_{k-1}^{\hat{L}}) + f(h_{k-1}^{L}) \right) - f_{HL} f(h_{k-1}^{\hat{H}}) \right] \\ &+ f_H \Delta f^{k-1} (1 - \Delta f) > 0 \end{aligned}$$

Suppose that if the above equation is positive,

$$\begin{aligned} \frac{\Lambda_{k+1} - \Lambda_k}{\Delta\theta} =& f_{LH} \left[f(h_{k+1}^L) - f(h_k^L) + f(h_{k+1}^{\hat{L}}) - f(h_k^{\hat{L}}) \right] \\ &+ f_{HH} \left[f(h_{k+1}^{\hat{H}}) - f(h_k^{\hat{H}}) \right] + f_H \Delta f^k (1 - \Delta f) \\ =& \Delta f \Big\{ \Delta f \left[f_{LH} \left(f(h_{k-1}^{\hat{L}}) + f(h_{k-1}^{L}) \right) - f_{HL} f(h_{k-1}^{\hat{H}}) \right] \\ &+ f_H \Delta f^{k-1} (1 - \Delta f) \Big\} > 0 \end{aligned}$$

□

As Λ_k increases with k, it indicates that the effect of capacity constraint at each time point increases over time. The optimal quantity $q_i(h_k)$ for $h_k \in \{h_k^{\hat{H}}, h_k^{\hat{L}}\}$ and $q_H(h_k^L)$ diminishes as Λ_k grows; while the trend of $q_L(h_k^L)$ as k increases depends on the mixed effects which are the decrement of $\frac{f_H \Delta\theta}{f_L}(\frac{\Delta f}{f_{LL}})^k(1 - \frac{\Delta f}{f_{LL}})$ and the increment of $\Lambda_k - \Lambda_{k-1}$. Additionally, the capacity constraint at each time point binds the optimal

[2]The increase and decrease in our paper refer to the weak sense.

quantity after any history h_k. In *Battaglini* (2005), the uncapacitated and infinite version of our problem, the optimal quantity converges to the efficient quantity θ_i ($\theta_i - ad$ in our context) due to $\frac{\Delta f}{f_{LL}} < 1$.

2.3.1.2 Different hire time preferences ($\bar{d}_H > \bar{d}_L$)

In this subsection, we discuss the problem with two types of customers having different preferred hire time $\bar{d}_H > \bar{d}_L$. Unlike the previous section, there are fixed but not equidistant time points in the finite horizon, $\{t_0 = 0, t_1 = \bar{d}_L, t_2 = \bar{d}_H, \cdots, t_K, t_{K+1} = T\}$. Assume that each customer has only one rental contract at any time point. The corresponding monopolist's optimization problem can be formulated as follows.

$$\max_{\{Q,S(\theta)\}} M\{\sum_i f_i(P_i - Y - a\bar{d}_i q_i) + \sum_{k=1}^{K} \delta^{t_k} \mathbb{E}_{h_k^z} \sum_i f_{zi}[P_i - Y - a\bar{d}_i q_i(h_k^z)]\}$$

$$\begin{aligned}
s.t. \qquad & S(\theta_i|h_k) \geq S(\theta_j|h_k) + \Delta\theta q_j(h_k) + \delta^{\bar{d}_j} \textstyle\sum_l (f_{il} - f_{jl}) S(\theta_l|h_{k+1}^j) && IC_{i,j}(h_k) \\
& S(\theta_i|h_k) \geq 0 && IR_i(h_k) \\
& M(f_H q_H + f_L q_L) \leq C && CC(h_0) \\
& f_{zL} q_L(h_k^z) + f_{zH} q_H(h_k^z) \leq q_z(h_{k-1}) \quad h_k^z = \{h_{k-1}, z\} && CC(h_k) \\
& \forall i, j, l \in \{L, H\},\ h_k \in \boldsymbol{H}_k\ (k = 0, \cdots, K)
\end{aligned}$$

Note that $\triangle\theta = \theta_H - \theta_L = \bar{\theta}_H \bar{d}_H - \bar{\theta}_L \bar{d}_L$. The capacity constraint $CC(h_k)(k > 0)$ in this model is different from that in the model of customers with the same hire time. In the previous section, with equidistant time points, the available capacity at each time point is C; whereas in this section, the time points in the horizon are not equidistant anymore, the available capacity at time point t_k is the number of rented containers at t_{k-1}, $q_z(h_{k-1})$. We still apply the relaxed method to solve this problem.

Lemma 2.5. *In a dynamic environment with different hire time preferences, the* *optimal solution of the relaxed problem is also an optimal solution of the original problem.*

Proof. The proof is similar to the proof of Lemma 2.4, except the part $IC_{HL}(h_k)$ when $k > 0$.

If $IC_{HL}(h_k)$ are not binding constraints in an optimal solution $\{Q, P\}$ of the original problem for some $h_k \in \boldsymbol{H}_k$, then $S(\theta_H|h_k) = \Delta\theta q_L(h_k) + \delta^{\bar{d}_L}\Delta f S(\theta_H|h^L_{k+1}) + \omega$. Consider an alternative solution $\{Q', S'(\theta)\}$ such that $S'(\theta_H|h_k) = S(\theta_H|h_k) - \omega$.

When $k > 0$ and $k' > k$, $S'(\theta_H|h_{k'})$ remains the same as in the original solution, $S'(\theta_H|h_{k'}) = S(\theta_H|h_{k'})$. When $k' = k-1$, we have $S'(\theta_H|h_{k-1}) = S(\theta_H|h_{k-1}) - \delta^{\bar{d}_L}\omega$. By repeatedly applying the above modifications until $k' = 0$, $S'(\theta_H) = S(\theta_H) - \delta^{k\bar{d}_L}\omega$. The expected optimal profit increases by $Mf_H\delta^{k\bar{d}_L}\omega$, $\Pi(Q', S'(\theta)) = \Pi(Q, P) + Mf_H\delta^{k\bar{d}_L}\omega$. It contradicts the optimality of the assumption. $\square$

Use the Lagrange multiplier approach to reformulate the relaxed problem as

$$\begin{aligned}\Pi(Q, S(\theta_H), \Lambda) = \max_{\{Q,S(\theta_H),\Lambda\}} & M\{\sum_i f_i[\theta_i q_i - \frac{1}{2}(q_i^2 + \bar{d}_i^2) - Y - a\bar{d}_i q_i] - f_H S(\theta_H) \\ & + \sum_{k=1}^{K} \delta^{t_k}\mathbb{E}_{h_k^z}\sum_i f_{zi}[\theta_i q_i(h_k^z) - \frac{1}{2}(q_i^2(h_k^z) + \bar{d}_i^2) - Y - a\bar{d}_i q_i(h_k^z)]\} \\ & + \lambda_0(\frac{C}{M} - f_H q_H - f_L q_L) + \sum_{k=1}^{K}\lambda_k[q_z(h_{k-1}) - f_{zL}q_L(h_k) - f_{zH}q_H(h_k)]\end{aligned}$$

$$\begin{aligned} s.t. \qquad & S(\theta_L|h_k) = 0 & IR_L(h_k) \\ & S(\theta_H) = \Delta\theta \textstyle\sum_{k=0}^{K}(\delta^{\bar{d}_L}\Delta f)^k q_L(h_k^L) & IC_{HL}(h_k) \\ & \forall i \in \{L, H\}, h_k \in \boldsymbol{H}_k \ (k = 0, \ldots, K) & \end{aligned}$$

Theorem 2.4. *For any $h_k \in \boldsymbol{H}_k$, the optimal intertemporal allocation policy for customers with different hire time preferences is characterized as follows.*

At time 0,

$$\begin{cases} q_H = \theta_H - a\bar{d}_H - \Lambda_0; \\ q_L = \theta_L - a\bar{d}_L - \frac{f_H}{f_L}\Delta\theta - \Lambda_0; \end{cases} \quad \textit{where } \Lambda_0 = \frac{\lambda_0}{M} = [\theta_L - a(f_H\bar{d}_H + f_L\bar{d}_L) - \frac{C}{M}]^+;$$

when $h_k \in \{h_k^{\hat{H}}, h_k^{\hat{L}}\}$,

$$\begin{cases} q_H(h_k) = \theta_H - a\bar{d}_H - \Lambda_k; \\ q_L(h_k) = \theta_L - a\bar{d}_L - \Lambda_k; \end{cases}$$

where $\Lambda_k = \frac{\lambda_k}{M\delta^{t_k}f(h_k)} = [f_{zH}\theta_H + f_{zL}\theta_L - \theta_z - a(f_{zH}\bar{d}_H + f_{zL}\bar{d}_L - \bar{d}_z) + \Lambda_{k-1}]^+$;

when $h_k = h_k^L$,

$$\begin{cases} q_H(h_k^L) = \theta_H - a\bar{d}_H - \Lambda_k; \\ q_L(h_k^L) = \theta_L - a\bar{d}_L - \frac{f_H\Delta\theta}{f_L}(\frac{\Delta f}{f_{LL}})^k - \Lambda_k; \end{cases}$$

where $\Lambda_k = \frac{\lambda_k}{M\delta^{k\bar{d}_L}f(h_k^L)} = \left[f_{LH}(\Delta\theta - a\bar{d}_H + a\bar{d}_L) + \frac{f_H\Delta\theta}{f_L}(\frac{\Delta f}{f_{LL}})^{k-1}(1-\Delta f) + \Lambda_{k-1}\right]^+$.

Proof. When $h_k \in \{h_k^{\hat{H}}, h_k^{\hat{L}}\}$, the first-order conditions w.r.t $q_H(h_k)$ and $q_L(h_k)$ are given by the following equations.

$$\begin{aligned} q_H(h_k): &\quad M\delta^{t_k}f(h_k)[\theta_H - a\bar{d}_H - q_H(h_k)] - \lambda_k = 0 \\ q_L(h_k): &\quad M\delta^{t_k}f(h_k)[\theta_L - a\bar{d}_L - q_L(h_k)] - \lambda_k = 0 \end{aligned}$$

And $\Lambda_k = [f_{zH}(\theta_H - a\bar{d}_H) + f_{zL}(\theta_L - a\bar{d}_L) - q_z(h_{k-1})]^+ = [f_{zH}\theta_H + f_{zL}\theta_L - \theta_z - a(f_{zH}\bar{d}_H + f_{zL}\bar{d}_L - \bar{d}_z) + \Lambda_{k-1}]^+$.

When $h_k = h_k^L$, the first-order conditions w.r.t $q_H(h_k^L)$ and $q_L(h_k^L)$ are given by the following equations.

$$
\begin{aligned}
q_H(h_k^L): &\qquad M\delta^{t_k} f(h_k^L)[\theta_H - a\bar{d}_H - q_H(h_k)] - \lambda_k = 0 \\
q_L(h_k^L): &\qquad M\delta^{t_k} f(h_k^L) f_{LL}[\theta_L - a\bar{d}_L - q_L(h_k)] - \lambda_k f_{LL} - M\delta^{t_k} f_H \Delta\theta \Delta f^k = 0
\end{aligned}
$$

$$
\begin{aligned}
\Lambda_k = &\ \Big[f_{LH}(\theta_H - a\bar{d}_H) + f_{LL}(\theta_L - a\bar{d}_L - \tfrac{f_H\Delta\theta}{f_L}(\tfrac{\Delta f}{f_{LL}})^k) - q_L(h_{k-1}^L)\Big]^+ \\
= &\ \Big[f_{LH}(\Delta\theta - a\bar{d}_H + a\bar{d}_L) + \tfrac{f_H\Delta\theta}{f_L}(\tfrac{\Delta f}{f_{LL}})^{k-1}(1-\Delta f) + \Lambda_{k-1}\Big]^+.
\end{aligned}
$$

□

Corollary 2.2. *For $0 \leq k \leq K$ and any $h_k^z \in \boldsymbol{H}_k$, if $z = L$, Λ_k increases with k; if $z = H$, Λ_k decreases with k.*

Proof. We prove this corollary by induction. To begin with, consider the case $z = L$.

Step 1. For $k = 1$, $h_1 = \{\text{L}\} = h_1^L$.

$$
\begin{aligned}
\Lambda_1 - \Lambda_0 &= f_{LH}(\Delta\theta - a\bar{d}_H + a\bar{d}_L) + \frac{f_H\Delta\theta}{f_L}(1-\Delta f) \\
&= f_{LH}[\bar{d}_H(\bar{\theta}_H - a) - \bar{d}_L(\theta_L - a)] + \frac{f_H\Delta\theta}{f_L}(1-\Delta f) \\
&> f_{LH}(\theta_L - a)(\bar{d}_H - \bar{d}_L) + \frac{f_H\Delta\theta}{f_L}(1-\Delta f) > 0
\end{aligned}
$$

The positivity is derived from $\bar{\theta}_L > a$, $\bar{d}_H > \bar{d}_L$ and $\Delta f = f_{HH} - f_{LH} < 1$.

Step 2. If $h_k = h_k^{\hat{L}}$, $\Lambda_k = [f_{LH}(\Delta\theta - a\bar{d}_H + a\bar{d}_L) + \Lambda_{k-1}]^+$; if $h_k = h_k^L$, $\Lambda_k = [f_{LH}(\Delta\theta - a\bar{d}_H + a\bar{d}_L) + \frac{f_H\Delta\theta}{f_L}(\frac{\Delta f}{f_{LL}})^{k-1}(1-\Delta f) + \Lambda_{k-1}]^+$. As $\Delta\theta - a\bar{d}_H + a\bar{d}_L > 0$ and $1 - \Delta f > 0$, we have $\Lambda_k \geq \Lambda_{k-1}$.

Consider the case when $z = H$. For $k \in \{1, \cdots, K\}$, $\Lambda_k = [-f_{HL}(\Delta\theta - a\bar{d}_H + a\bar{d}_L) + \Lambda_{k-1}]^+$, hence $\Lambda_k \leq \Lambda_{k-1}$. □

In this subsection, it is interesting that the effect of capacity constraint becomes smaller whenever the last customer type is the high type and this effect becomes larger whenever the last customer type is the low type no matter the history is $h_k^{\hat{L}}$ or h_k^L. In other words, the influence of capacity constraint depends only on the realization of the customer type in the previous period. Based on the monotonicity of Λ_k, the optimal quantity of the consistent high type converges to the efficient quantity level $(\theta_H - a\bar{d}_H)$ over time. At the same time, the distortion of the (highly consistent or inconsistent) low type customer becomes greater over time. The situation that the effect of capacity constraints depends on customer type contrasts with the effect of capacity constraint which is independent of customer type in Section 4.1.1. For the case with same hire time preference, the capacity constraint binds all the allocated quantities at each time point, while in this case with different hire time preferences, the capacity constraint only takes effect on the next allocation.

2.3.2 Hire quantity preference

Denote the preferred hire quantities of two customer types by $\bar{q}_H$ and $\bar{q}_L$. We first study the case that $\bar{q}_H = \bar{q}_L$ and then discuss the case when $\bar{q}_H > \bar{q}_L$. Let $\theta_i = \bar{\theta}_i \bar{q}_i$.

2.3.2.1 Same hire quantity preference ($\bar{q}_H = \bar{q}_L = q$)

In this section, we analyze the case that both customer types have the same hire quantity preference. The monopolist's problem is to determine a optimal hire time sequence $D = \{d_i(h_0), d_i(h_1), \cdots, d_i(h_k)\}$ for any $h_k \in \boldsymbol{H}_k$ $(k = 0, 1, \cdots, N)$ to maximize the expected profit, where $d_i(h_k)$ is the hire time of customer type i after history h_k and N is the total number of rental contracts for history h_k^L in the horizon. $t_i(h_k)$ is the beginning time of rental contract for type i after history h_k, $t_i(h_k) = \sum_{l=0}^{k-1} d(h_l)$ $(h_l \in h_k)$. The optimization problem in this setting is written as shown below.

$$\Pi(D, S(\theta)) = \max_{\{D,S(\theta)\}} M(\sum_i f_i[U(\theta_i, d_i(h_0)) - S(\theta_i) - Y - aqd_i(h_0)]$$
$$+ \sum_{k=1}^{N} \mathbb{E}_{h_k^z} \sum_j \delta^{t_j(h_k^z)} f_{zj}[U(\theta_j, d_j(h_k^z)) - Y - aqd_i(h_k^z)])$$

$$\begin{aligned} s.t. \quad & S(\theta_i|h_k) \geq S(\theta_j|h_k) + \Delta\theta d_j(h_k) + \delta^{d_j(h_k)} \textstyle\sum_l (f_{il} - f_{jl}) S(\theta_l|h_{k+1}^j) \quad IC_{i,j}(h_k) \\ & S(\theta_i|h_k) \geq 0 \qquad IR_i(h_k) \\ & t_i(h_k) + d_i(h_k) \leq T \\ & \forall i, j, l \in \{L, H\}, h_k \in \boldsymbol{H}_k, k = \{0, 1, \cdots, N\} \end{aligned}$$

The last constraint is the finite time constraint for hire time sequence D with respect to h_k. That is, for any $h_k \in \boldsymbol{H}_k$, the summation of the hire time of all rental contracts cannot exceed the planning horizon T. As customers have hire quantity preference, there is no capacity constraint for the problem. Unfortunately, as the following example shows, even $N = 2$, a closed form solution is unlikely derivable.

Example (N=2) Suppose that at time $t_0 = 0$ the types of all arriving customers are H, then at time point $t_1 = d_H(h_0)$, the hire time for the low type and the high type are $\theta_L - a\bar{q}_L$ and $\theta_H - a\bar{q}_H$, respectively. The objective function of the monopolist is

$$\Pi(D, S(\theta)) = \max_{d_H(h_0)} Mf_H\{\theta_H d_H(h_0) - \frac{1}{2}(d_H^2(h_0) + \bar{q}_H^2) - Y - a\bar{q}_H d_H(h_0)$$
$$+ \frac{1}{2}\delta^{d_H(h_0)} \pi(\theta_H, \theta_L)\}$$

where $\pi(\theta_H, \theta_L) = f_{HH}[(\theta_H - a\bar{q}_H)^2 - \bar{q}_H^2] + f_{HL}[(\theta_L - a\bar{q}_L)^2 - \bar{q}_L^2]$. The first order condition w.r.t $d_H(h_0)$ is

$$\theta_H - a\bar{q}_H - d_H(h_0) + \frac{1}{2}\delta^{d_H(h_0)} \ln d_H(h_0)\pi(\theta_H, \theta_L) = 0$$

As $d_H(h_0)$ lies in the exponent, the above equation is a transcendental equation which does not exist a closed form solution. The first decision variable $d_i(h_0)$ influences all the following decision variables $d_i(h_k)$ $(k = 1, \cdots, N)$. This contrasts with the problem in a dynamic environment when customers have hire time preference. Because in the problem discussed in Section 4.1 the decision variable such as $q_i(h_k)$ only affects the next decision variables $q_i(h_{k+1})$ through available capacity constraint rather than all subsequent decision variables.

In view of this, we limit the choices of possible hire time. Suppose that the alternative hire time sets for both types are $D_L = \{d_L^1, d_L^2, \cdots, d_L^K\}$ and $D_H = \{d_H^1, d_H^2, \cdots, d_H^K\}$, where $d_i^{k+1} > d_i^k$ $(i \in \{L, H\})$, $d_H^1 > d_L^K$ and $|D_L| = |D_H| = K$. Due to the given hire time set, there may exist some idle time after the last rental contract as the remaining time in the planning horizon is less than the minimum given hire time d_L^1. Let κ be the set of the last rental contract for each h_k in the planning horizon and the unit inventory cost per time period be $\bar{c}$. For short, let $R(d_i(h_k)) = \theta_i d_i(h_k) - \frac{1}{2}(d_i^2(h_k) + q^2) - Y - aqd_i(h_k)$. $d_i^* = \arg\max_{d_i \in D_i}\{R(d_i)\}$ refers to the efficient hire time of type i. Thus, the problem degenerates into selecting an optimal hire time from the given hire time set.

We still consider the relaxed problem where $IC_{HL}(h_k)$ and $IR_L(h_k)$ are binding constraints.

$$\Pi(D, S(\theta_H)) =$$

$$\max_{\{D,S(\theta_H)\}} M\{\sum_{i\in\{L,H\}} f_i[R(d_i(h_0)) - S(\theta_H)] + \sum_{k=1}^{N} \mathbb{E}_{h_k^z} \sum_{j\in\{L,H\}} \delta^{t_j(h_k^z)} f_{zj} R(d_j(h_k^z))\}$$
$$- \bar{c} \sum_{j\in\kappa} \mathbb{E}_{h_k}(T - t_j(h_k) + d_j(h_k))$$

$$\begin{aligned} s.t. \qquad & S(\theta_L|h_k) = 0 && IR_L(h_k) \\ & S(\theta_H) = \Delta\theta \textstyle\sum_{k=0}^{N} \delta^{t_L(h_k^L)} \Delta f^k d_L(h_k^L) && IC_{H,L}(h_k^L) \\ & t_j(h_k) + d_j(h_k) \le T, \quad j \in \kappa \\ & \forall h_k \in \boldsymbol{H}_k, k = \{0, 1, \cdots, N\} \end{aligned}$$

We use a binary tree (see Figure 2.1) to illustrate the computation process. Node $t = 0$ is the beginning of the binary tree. Nodes with odd numbers represent the low type customers and nodes with even numbers denote the high type customers. Let l be a general node in the tree and $\rho(l)$ be the type of that node. If $l \bmod 2 = 1$, $\rho(l) = L$; if $l \bmod 2 = 0$, $\rho(l) = H$. For short, let h_l be the history ended at node l and t_l be the beginning time of rental contract at node l. The objective function of this problem implies that given the alternative hire time sets, there is a trade-off between efficient hire discount of current state and future profit maximization. For a node in tree, if the efficient hire time $d_{\rho(l)}(h_k) = d^*_{\rho(l)}$ is selected by the leasing company, then the next rental contracts for both types are available from time $t_l + d^*_{\rho(l)}$; if the hire time of node l is less than the efficient hire time $d^*_{\rho(l)}$, the next rental contracts start earlier than the former case at the cost of sacrificing the profit at the current state. The optimal hire time sequence strikes a balance between current state maximization and expected future profit maximization.

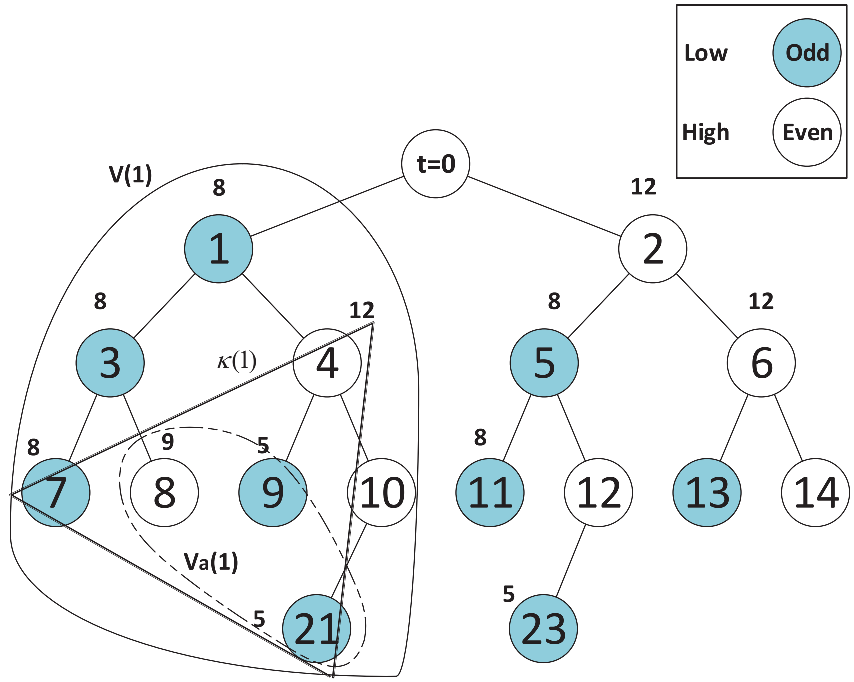


Figure 2.1. A binary tree, $d_L^* = 8$, $d_H^* = 12$ and $T = 25$

The dynamic programming algorithm is designed in the following way. First assign the efficient hire time of both types to the corresponding nodes until the second last contracts of different histories and calculate the hire time of last rental contract for each history subject to the time constraint. For example, in Figure 2.1, assign $d_L^* = 8$ for odd nodes and $d_H^* = 12$ for even nodes except the last rental contract. For this feasible solution, let $\kappa(l)$ be the set of last rental contract for each history rooted at node l and $V(l)$ be a subtree rooted at node l. $V(l)$ can be divided into two sets, $V(l) = \{V_a(l), V_n(l)\}$, where $V_a(l) = \{k | k \in \kappa(l) \text{ and } d_{\rho(k)}(h_{t_k}) \neq d^*_{\rho(k)}\}$ is the set of nodes in $\kappa(l)$ whose hire time do not equal to the efficient hire time of the corresponding types and $V_n(l) = V(l) \setminus V_a(l)$ is the set of rest nodes in $V(l)$ whose hire time equal to the efficient hire time of the corresponding types. Next, for each node with efficient hire time, adjust this feasible solution by decreasing node l by η and increasing nodes j in $V_a(l)$ by η. Let $\Delta_l(\eta)$ be the net change of the expected profit which includes the changes of expected profit of hire time at nodes l and $j \in V_a(l)$, the changes of time discount for nodes $j \in V_n(l)$ whose hire time remain the same, the changes of inventory cost of nodes in $\kappa(l)$ and the possible change of $S(\theta_H)$.

$$
\begin{aligned}
\Delta_l(\eta) =& f(h_l)\delta^{t_l}[R(d_{\rho(l)}(h_l)-\eta) - R(d_{\rho(l)}(h_l))] + \sum_{j\in V_n(l)} f(h_j)\delta^{t_j} R(d_{\rho(j)}(h_j))(\delta^{-\eta}-1) \\
&+ \sum_{j\in V_a(l)} f(h_j)[R(d_{\rho(j)}(h_j)+\eta)\delta^{t_j-\eta} - R(d_{\rho(j)}(h_j))\delta^{t_j}] \\
&+ \bar{c} \sum_{k\in\kappa(l)\setminus V_a(l)} f(h_k)\delta^{t_k}(T - t_k - d_{\rho(k)}(h_k)+\eta) - f_H \triangle\theta \triangle_l(S(\theta_H))
\end{aligned}
$$

where $\triangle_l(S(\theta_H)) = \mathbf{1}(l \in h_k^L)[-\eta\triangle f^k \delta^{t_l} + \sum_{j=k+1}^{N} \triangle f^j \delta^{t_L(h_j^L)} d_L(h_j^L)(\delta^{-\eta}-1) + \mathbf{1}(h_N^L \cap V_a(l))\eta\triangle f^N \delta^{t_L(h_N^L)-\eta}]$.

Compute $\Delta_l(\eta_{\min})$ where $\eta_{l\min} = \min_{k\in\{2,\cdots,K\}}\{d_{\rho(l)}^k - d_{\rho(l)}^{k-1}\}$ and $\Delta(\eta_{l\max})$ where $\eta_{l\max} = \min\{d_{\rho(l)}^* - d_{\rho(l)}^1, \min_{j\in V_a(l)}\{d_{\rho(j)}^* - d_{\rho(j)}(h_{t_j})\}\}$. $\eta_l^* = \arg\max\{\Delta_l(\eta)|\Delta_l(\eta) > 0\}$. If η_l^* is not empty set, then adjust the feasible solution, otherwise continue to next node. Subtrees $V(l)$ rooted at the same level can be calculated separately since the above adjustment does not affect the solution of other subtrees with the same level, e.g., $V(1)$ and $V(2)$. Apply the adjustment repeatedly until η_l^* is an empty set for each node in $V(l) \setminus \kappa(l)$.

Numerical Example 1. Consider an intertemporal nonlinear pricing problem in the planning horizon $T = 25$ with different time discount factors $\delta = 0.95, 0.85, 0.65$. There are $M = 10$ customers with same hire quantity preference $q = 5$. The high type customer with valuation $\bar{\theta}_H = 3.6$ arrives at the system with probability $f_H = 0.6$ at time 0 and with consistent probability $f_{HH} = 0.6$ during the rest of planning horizon. The low type customer with valuation $\bar{\theta}_L = 2.6$ enters the system with inconsistent probability $f_{LH} = 0.45$. The alternative hire time sets for both types are $D_L = [5, 6, 7, 8]$ and $D_H = [9, 10, 11, 12]$. Contract setup cost Y is 4, the unit inventory cost $\bar{c}$ is 0.5 and the unit operating cost a is 0.6. The optimal hire time sequence is shown in Figure 4.1.

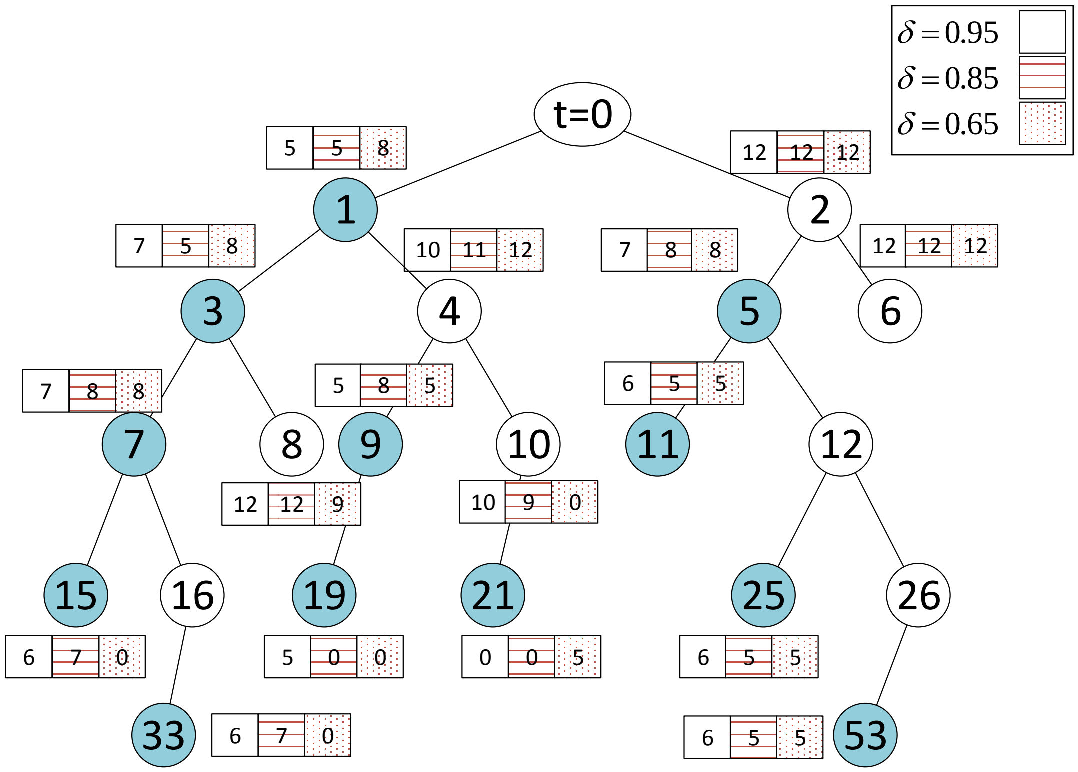


Figure 2.2. The Optimal Hire Time Sequence of Example 1, $\bar{\theta}_H = 3.6$ and $\bar{\theta}_L = 2.6$

Numerical Example 2. In this example, the valuations of both two types become $\bar{\theta}_H = 2.8$ and $\bar{\theta}_L = 2$. The other parameters remain the same. The optimal hire time sequence is illustrated in Figure 2.3.

Two interesting points can be drawn from these two examples. The first one is that when δ decreases, the optimal hire time sequence converges to the efficient hire time. The lessening time discount factor reduces the proportion of future profit in the total expected profit which results in the dominant effect of current profit maximization in the tradeoff. As the time discount factor diminishes, it is optimal for the leasing company to lease the containers at the efficient hire time in order to maximize the expected profit. In the two examples, when δ falls from from 0.95 to 0.65, the optimal hire time sequence except the last rental contract has the trend of converging to the efficient hire time.

Another finding is that the decrement of customer valuations lessens the effect of

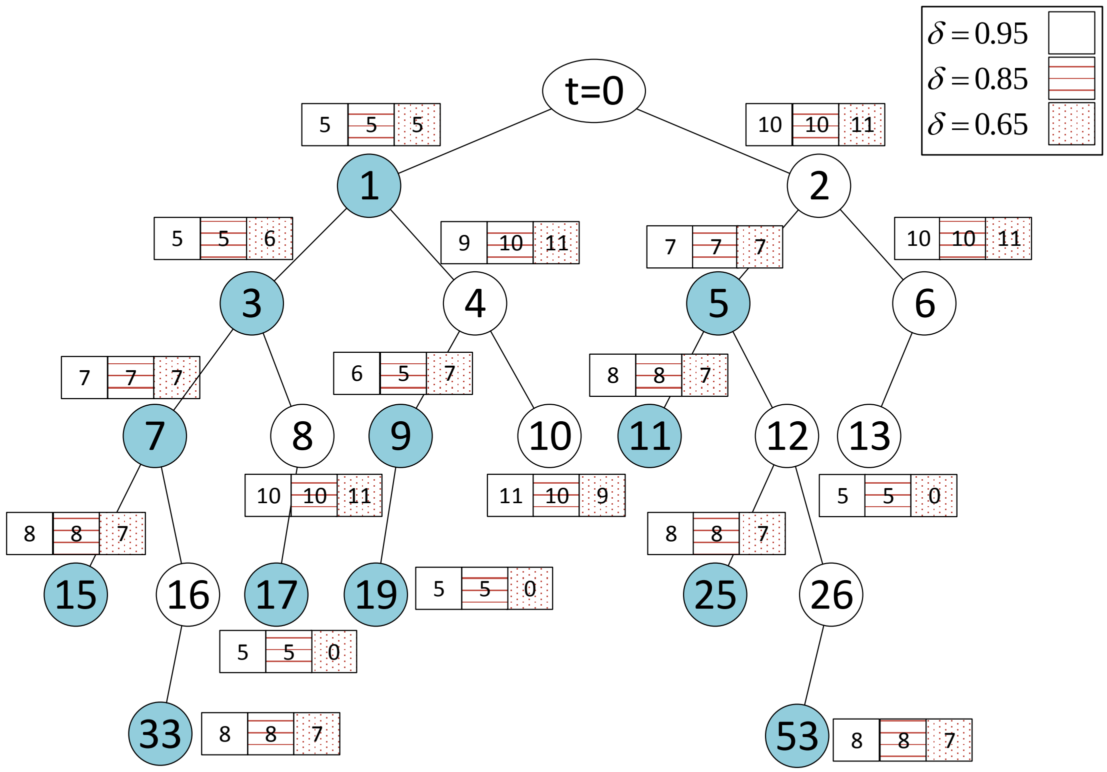


Figure 2.3. The Optimal Hire Time Sequence of Example 2, $\bar{\theta}_H = 2.8$ and $\bar{\theta}_L = 2$

current profit maximization. In Example 1, $(\bar{\theta}_i - a)q$ $(\theta_H = 15, \theta_L = 10)$ is greater than the maximal alternative hire time d_i^K $(d_H^K = 12, d_L^K = 8)$, the efficient hire time for both types are $d_H^* = 12$ and $d_L^* = 8$. In Example 2, $d_i^1 < \bar{\theta}_i - a)q < d_i^K$, the efficient hire time for both types becomes $d_H^* = 11$ and $d_L^* = 7$. The decreasing customer valuation leads to the reducing efficient hire time. Thus, the opportunity cost of leasing the containers at the efficient hire time reduces, which weakens the effect of efficient hire discount. The effect of future profit maximization becomes dominant in the trade-off. This is the reason that even $\delta = 0.65$, the optimal hire time of nodes 1 and 3 in Example 2 are 5 and 6 which are less than the efficient hire time of low type, $d_L^* = 7$. In a word, $(\bar{\theta}_i - a)q$ is the main factor of deciding the effect of current state maximization and δ is the major determinant of the effect of future profit maximization.

2.3.2.2 Different hire quantity preferences ($\bar{q}_H > \bar{q}_L$)

We further investigate the case that customers have different hire quantity preferences, $\bar{q}_H > \bar{q}_L$. In the previous section, because of the same hire quantity preference, there is no container left except the last rental contract in the planning horizon, whereas the varied hire quantity preferences in this section give rise to the phenomenon that after each rental contract, there may be several excess/inadequate units of container. Suppose that the consistent type customers have the priority of satisfying the demand and the inconsistent low type customers (the low type customer of last time changes to the high type customer) are patient and can wait until when idle containers are available. If $h_k^z = h_k^{\hat{H}}$, $Mf(h_k^z)\bar{q}_H$ units of container are returned at $t_i(h_k^z)(i \in \{L, H\})$. The total number of requested containers by both types is $Mf(h_k^z)(f_{HH}\bar{q}_H + f_{HL}\bar{q}_L)$ which is strictly less than the units of available container $Mf(h_k^z)\bar{q}_H$. It implies that $Mf(h_k^z)f_{HL}(\bar{q}_H - \bar{q}_L)$ units of container are left in the firm; if $h_k^z \in \{h_k^L, h_k^{\hat{L}}\}$, $Mf(h_k^z)\bar{q}_L$ units of container are returned at time $t_i(h_k^z)(i \in \{L, H\})$. The consistent low type customer requires $Mf(h_k^z)f_{LL}\bar{q}_L$, which is less than $Mf(h_k^z)\bar{q}_L$. But the inconsistent low type customer requests $Mf(h_k^z)f_{LH}\bar{q}_H$ which is greater than the rest of container, $Mf(h_k^z)f_{LH}\bar{q}_L$. Assume that the waiting cost per time is w. $x_i(h_k)$ represents the inventory level of containers for customer type i after history h_k. Let $R(d_i(h_k)) = \theta_i d_i(h_k) - \frac{1}{2}((d_i^2(h_k) + \bar{q}_i^2) - Y - a\bar{q}_i d_i(h_k)$. The corresponding relaxed optimization problem can be formulated as

$$\Pi(D, S(\theta_H)) = \max_{\{D,S(\theta_H)\}} M \sum_i f_i[R(d_i(h_0)) - S(\theta_i)] + M \sum_{k=1}^{N} \mathbb{E}_{h_k^z} \sum_j \delta^{t_j(h_k^z)} f_{zj} R(d_j(h_k))$$
$$- \bar{c}\mathbb{E}_{h_k}(x_i(h_k))^+ - w\mathbb{E}_{h_k}\mathbf{1}(t_j(h_k) - t_i(h_k))(x_i(h_k))^-$$

$$
\begin{aligned}
s.t. \qquad & S(\theta_L|h_k) = 0 && IR_L(h_k) \\
& S(\theta_H) = \Delta\theta \textstyle\sum_{k=0}^{N} \delta^{t_L(h_k^L)} \Delta f^k d_L(h_k^L) && IC_{H,L}(h_k^L) \\
& t_i(h_k) + d_i(h_k) \leq T \\
& \forall i,j \in \{L,H\}, \forall h_k \in \boldsymbol{H}_k, k = \{0,1,\cdots,N\}
\end{aligned}
$$

The objective function in this section is the rental revenue minus inventory cost, operating cost and waiting cost. The other components are the same as the model in the previous section.

The differences between this section and previous section are that (1) subtrees with the same level jointly determine the optimal hire time sequence. For example, the hire time of node 2 affects the available time of idle containers for inconsistent low type customer at node 4; (2) Besides the finite time constraint of the planning horizon, there exist hire quantity constraints for inconsistent low type customer due to different hire quantity preferences, which incurs the inventory cost and waiting cost. While in Section 4.2.1, there is only inventory cost after the last rental contract. The changes of the algorithm are that $\Delta_l(\eta)$ for nodes l in $V(2)$ include the waiting cost and the inventory cost of each node. The other parts remain the same.

Numerical Example 3. The preferred hire time for both types are $\bar{q}_H = 10$ and $\bar{q}_L = 5$. The waiting cost per time w is 1. The other parameters are the same as in Example 1. The optimal hire time sequence is shown in Figure 2.4. The number in triangle denotes the preferred hire quantity requested by customers, $Mf(h_k^z)f_{zi}\bar{q}_i$ ($z, i \in \{L,H\}$).

Compare the result in Example 3 with the result in Example 1, one implication for high type customers is that the optimal hire time are the efficient hire time, $d_H^* = 12$ and are insensitive to the variations of time discount factors. This is because the preferred hire quantity of high type in this example is $\bar{q}_H = 10$, the profit from high type customer becomes larger than that in Example 1. The effect of efficient

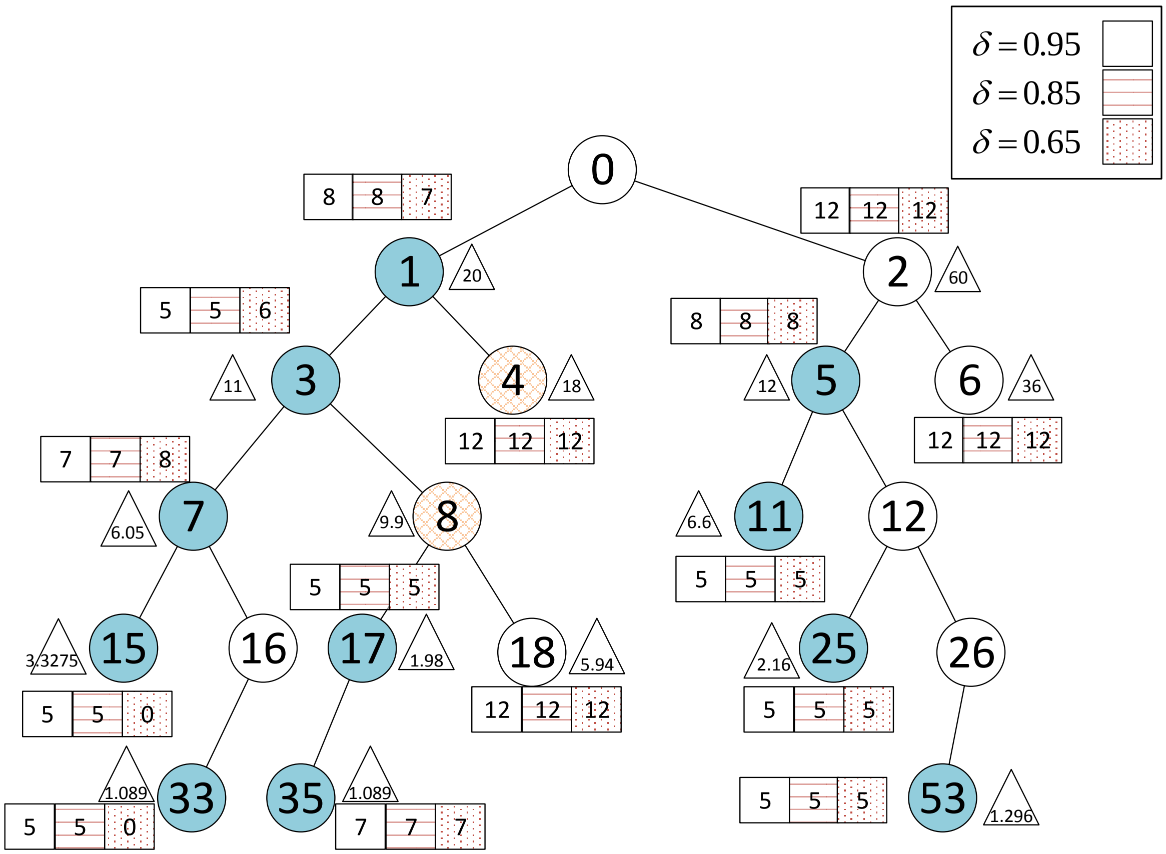


Figure 2.4. The Optimal Hire Time Sequence of Example 3, $\bar{q}_H = 10$ and $\bar{q}_L = 5$

hire discount (current state profit maximization) dominates the effect of future profit maximization. In addition, considering different hire quantity preferences, there is inventory left for high type customers. Thus, it is optimal for the leasing company to lease containers for high type customers at the efficient hire time. The second implication is that the optimal hire time of low type customers at node 1 is longer than that in Example 1, while the result at node 3 has the opposite trend. The most likely explanation is as follows. Nodes 4 and 8 are inconsistent low type customers. In the optimal solution, customers at node 4 wait until customers at node 2 return containers. The waiting cost of node 4 depends on the difference between the hire time of node 2 and 1. Then it is reasonable to increase the hire time of node 1 in order to reduce the waiting cost. But customers at node 8 are refused by the leasing company due to long waiting time and the finite horizon time constraint. Among the

12 units excess inventory of node 2, 9 units is used to satisfy the demand of node 4 and 2.97 units is used to meet the demand of node 18. The hire time of node 3 affects the inventory cost of 2.97 units, thus the optimal hire time of node 3 is less than the efficient hire time so as to minimize the time lag between node 2 and node 3 and reduce the inventory cost.

2.4 Dynamic Nonlinear Pricing Problem with Dynamic Arrivals

In this section, we discuss the monopolist's nonlinear pricing problem in a dynamic environment with dynamic arrivals. Considering the fluctuation in the shipping market, a lessee's eagerness to lease often depends on his own situation. Thus, the case that the lessees' first arrival dates are uncertain is examined.

2.4.1 Same hire time preference

Customers first arrive at the leasing company at some date τ before $T-d$ with probability ρ_τ. The possibility that a customer arrives before τ is $\beta_\tau = \sum_{k=1}^{\tau-1} \rho_k$. Upon the first arrival, the leasing company has a prior information about the probability being categorized by the low and high type, f_L and f_H. When the first lease contract of each lessee is over, the lessee keeps leasing from the monopolist until the end of planning horizon. The continuing leases reflect the long-term contractual relationship between the shippers and the leasing company. A customer informs the leasing company his initial arrival date τ and his customer type and reports the subsequent types truthfully in the following periods. The customer type evolves over time according to a Markov process. Let f_{ij} be the probability that a type i customer at the current rental contract becomes a type j customer at the next rental contract, where $i, j \in \{L, H\}$. The probability f_{ij} is independent of the time point. A customer remains his type of previous lease contract with a higher probability compared with the possibility of being the other type, that is, customer types are positively correlated, $\Delta f = f_{HH} - f_{HL} > 0$ and $f_{LL} - f_{LH} > 0$.

Let $h_{\tau,k}$ be the lease history with initial arrival date τ and k finished lease contracts from time point τ to $\tau + kd$ with hire duration d. $h^z_{\tau,k}$ specifies the fact that customer type at $\tau + (k-1)d$ is $z \in \{L, H\}$. $\boldsymbol{\mathcal{H}}_{\tau,k}$ is the set of all possible histories from time τ to time $\tau + kd$. When $k > 0$, $\boldsymbol{\mathcal{H}}_{\tau,k}$ can be divided into three subsets, $\boldsymbol{\mathcal{H}}_{\tau,k} = \{h^{\hat{L}}_{\tau,k}, \mathcal{H}^L_{\tau,k}, \mathcal{H}^H_{\tau,k}\}$. $h^{\hat{L}}_{\tau,k} = (L, \cdots, L)$ is the consistent low type history from time τ to time $\tau+(k-1)d$. $\mathcal{H}^L_{\tau,k} = \{\mathcal{H}_{\tau,k-1}, L\}$ is the set of inconsistent low type histories where the customer type at $\tau + (k-1)d$ is L and $h^{\hat{L}}_{\tau,k-1} \notin \mathcal{H}_{\tau,k-1}$. $\mathcal{H}^H_{\tau,k} = \{\mathcal{H}_{\tau,k-1}, H\}$ refers to the set of the histories where the customer type at $\tau + (k-1)d$ is H. Let $f(h_{\tau,k})$ be the probability for history $h_{\tau,k}$, e.g. $f(h^{\hat{L}}_{\tau,k}) = \rho_\tau f_L (f_{LL})^{k-1}$. $K_\tau = \lfloor (T-\tau)/d \rfloor - 1$ is defined as the final pricing point for customers with entry date τ.

A customer of type i is allocated $q_i(h_{\tau,k})$ units of container for preferred hire duration d_i[3] and enjoys utility $U(\theta_i, q_i(h_{\tau,k})) = \theta_i d q_i(h_{\tau,k}) - \frac{1}{2}(\alpha d_i^2 + q_i^2(h_{\tau,k}))$, where α is a scale factor to ensure that hire duration and hire quantity is comparable in the utility function. The quadratic utility function follows the tradition of the literature (*Sundararajan*, 2004; *Wilson*, 1993). The operations cost per time per unit for the leasing company is a and $a < \theta_L$. The direct operations expense includes storage, handling, maintenance, and reposition. The operations cost in our context is a component of the objective function of the empty container reposition articles (*Bell et al.*, 2013; *Cheung and Chen*, 1998), which is linear in the number of containers and the duration of the lease contract. The adoption of linear cost function is just to simplify exposition, and it is easy to extend to some other forms of cost function. Besides, we focus on the case that the finite capacity is insufficient to meet the demands of all first arrival customers, in other words, some first arrival customers are

[3]In this section, d_i denotes hire time preference and θ_i represents per-time customer valuation of type i.

denied the service by the leasing company.

The aim of the leasing company is to maximize the expected profit with finite capacity C by designing a menu of quantity $q_i(h_{\tau,k})$ and price $P_i(h_{\tau,k})$ given lease history $h_{\tau,k}$. Upon the first arrival, a customer reports a type i corresponding to $\{q_i(h_{\tau,k}), P_i(h_{\tau,k})\}$ and reports his type truthfully in the subsequent periods to maximize his expected customer surplus. $S(\theta_i|h_{\tau,k})$ is a type i customer's expected surplus up to history $h_{\tau,k}$,

$$S(\theta_i|h_{\tau,k}) = U(\theta_i, q_i(h_{\tau,k})) - P_i(h_{\tau,k}) + \delta^d \sum_{j\in\{L,H\}} f_{ij} S(\theta_j|h^i_{\tau,k+1}),$$

where δ is the time discount factor.

The sequence of events at time point $\tau + kd$ for history $h_{\tau,k}$ is listed as follows: (1) if $k > 0$, customers return leased containers on the due date; (2) the leasing company designs and commits a menu of quantity and price pairs $\{Q, P\} = \{q_i(h_{\tau,k}), P_i(h_{\tau,k})\}$ to maximize the expected profit; (3) each customer reports his type at time $\tau + kd$ truthfully to maximize his expected consumer surplus, receives the allocated quantity of his type and pays the lease revenue.

The direct revelation mechanism must satisfy the following constraints.

Initial Presence ($IP_i(h_{\tau,0})$) Constraint. A type i customer with claimed entry date τ must prefer to arrive at date τ instead of delaying presence at date $\tau + 1$.

$$S(\theta_i|h_{\tau,0}) \geq \delta[f_{ii} S(\theta_i|h_{\tau+1,0}) + f_{ij} S(\theta_j|h_{\tau+1,0})]$$

Incentive Compatibility ($IC_{ij}(h_{\tau,k})$) Constraint. The leasing company designs the quantity and price menu in the way that misreporting customer type is hurtful to customer surplus. Each customer reports his type i truthfully based on his report history $h_{\tau,k}$ and chooses the quantity and price pair $\{q_i(h_{\tau,k}), P_i(h_{\tau,k})\}$ designed for his type. Since the consumer surplus of reporting type i is greater than that reporting other types j ($j \neq i$), there is no incentive to deviate from his truthful type based on report history. The incentive compatibility constraint in a dynamic environment for the high type and history $h_{\tau,k}$ can be written as

$$S(\theta_H|h_{\tau,k}) \geq S(\theta_L|h_{\tau,k}) + \Delta\theta d q_L(h_{\tau,k}) + \delta^d \sum_j (f_{Hj} - f_{Lj}) S(\theta_j|h^L_{\tau,k+1}).$$

Individual Rationality ($IR_i(h_{\tau,k})$) Constraint. Each customer leases containers with nonnegative consumer surplus, $S(\theta_i|h_{\tau,k}) \geq 0$ for $h_{\tau,k} \in \boldsymbol{\mathcal{H}}_{\tau,k}$.

Capacity Constraint ($CC(h_{\tau,k})$). When $k = 0$, the capacity constraint for first arrival customers is

$$M \sum_{\tau=0}^{T-d} \rho_\tau (f_H q_H(h_{\tau,0}) + f_L q_L(h_{\tau,0})) \leq C.$$

In view of the same hire duration preference d, the capacity constraint for customers with the same entry date is that the total number of allocated units cannot exceed the number allocated at the initial date τ. The containers leased out at time $\tau + (k-1)d$ are returned to the company at the time point $\tau + kd$ for $1 \leq k \leq K_\tau$. The capacity constraint for customers with the same entry date can be expressed as

$$\begin{aligned} f(h^{\hat{L}}_{\tau,k}) \sum_i f_{Li} q_i(h^{\hat{L}}_{\tau,k}) + \sum_{h^H_{\tau,k} \in \mathcal{H}^H_{\tau,k}} f(h^H_{\tau,k}) \sum_i f_{Hi} q_i(h^H_{\tau,k}) + \sum_{h^L_{\tau,k} \in \mathcal{H}^L_{\tau,k}} f(h^L_{\tau,k}) \sum_i f_{Li} q_i(h^L_{\tau,k}) \\ \leq \rho_\tau [f_H q_H(h_{\tau,0}) + f_L q_L(h_{\tau,0})], i \in \{L, H\}. \end{aligned}$$

The monopolist's stochastic dynamic optimization problem is formulated as follows.

$$
\begin{aligned}
\Pi(Q,P) &= \max_{\{Q,P\}} \sum_{\tau=0}^{T-d} \sum_{k=0}^{K_\tau} \delta^{\tau+kd} \mathbb{E}_{h^z_{\tau,k}} M \sum_i f_{zi} [P_i(h^z_{\tau,k}) - adq_i(h^z_{\tau,k})] \\
&= \max_{\{Q,S(\theta)\}} M \Big\{ \sum_{\tau=0}^{T-d} \sum_{k=0}^{K_\tau} \delta^{\tau+kd} \mathbb{E}_{h^z_{\tau,k}} \sum_i f_{zi} [U(\theta_i, q_i(h^z_{\tau,k})) - adq_i(h^z_{\tau,k})] \\
&\qquad - \sum_{\tau=0}^{T-d} \rho_\tau \delta^\tau \sum_i f_i S(\theta_i | h_{\tau,0}) \Big\} \qquad (2.9) \\
&s.t. \quad IP_i(h_{\tau,0}), IC_{ij}(h_{\tau,k}), IR_i(h_{\tau,k}), CC(h_{\tau,k}) \\
&\qquad \forall i, z \in \{L,H\}, h_{\tau,k} \in \mathcal{H}_{\tau,k}\ 0 \le k \le K_\tau.
\end{aligned}
$$

When $k = 0$, it indicates that customers first arrive at the leasing company without lease history, we have $\mathbb{E}_{h^z_{\tau,0}} = \rho_\tau$, $z = \emptyset$ and $f_{zi} = f_i$.

The setting with two customer types and hire time preference allows us to model the problem as a one-dimensional dynamic screening problem. To motivate the high type to report his true type, the monopolist utilizes the future payoff rather than the present payoff to screen the customer types. Thus the high type customers enjoy more surplus than the low type customers in the optimal solution. The following lemma shows that there is no loss of generality in assuming constraints in the relaxed problem are binding constraints. The relaxed problem of the dynamic nonlinear pricing problem is defined as maximizing the monopolist's intertemporal expected profit where individual rationality constraint for the low type customers shown at the last moment $IR_L(h_{T-d,0})$, the incentive compatibility constraints for the high type $IC_{HL}(h_{\tau,k})$ and initial presence constraints for the low type $IP_L(h_{\tau,0})$ for $0 \le \tau \le T-d$ are satisfied as equalities.

Lemma 2.6. *In a dynamic environment with same hire duration preference, the optimal solution of the relaxed problem is also an optimal solution of the original problem.*

Proof. Suppose that $\{Q, S(\theta)\}$ is an optimal solution of the original problem which $IR_L(h_{T-d,0})$, $IC_{HL}(h_{\tau,k})$ and $IP_L(h_{\tau,0})$ are not binding constraints for $0 \le \tau \le T-d$.

If $IR_L(h_{T-d,0})$ is not a binding constraint, that is, $S(\theta_L|h_{T-d,0}) = \omega$, where ω is a positive number. Consider an alternative solution $\{Q', S'(\theta)\}$ such that $S'(\theta_L|h_{T-d,0}) = S(\theta_L|h_{T-d,0}) - \omega$. The expected profit increases by $M\rho_{T-d}\delta^{T-d} f_L \omega$, $\Pi(Q', S'(\theta)) = \Pi(Q, P) + M\rho_{T-d}\delta^{T-d} f_L \omega$.

If $IC_{HL}(h_{\tau,k})$ are not binding constraints for $0 \le k \le K_\tau$, $S(\theta_H|h_{\tau,k}) - S(\theta_L|h_{\tau,k}) = \Delta\theta d q_L(h_{\tau,k}) + \delta^d \triangle f [S(\theta_H|h^L_{\tau,k+1}) - S(\theta_L|h^L_{\tau,k+1})] + \omega$. Consider an alternative solution $\{Q', S'(\theta)\}$ such that $S'(\theta_H|h_{\tau,k}) - S'(\theta_L|h_{\tau,k}) = S(\theta_H|h_{\tau,k}) - S(\theta_L|h_{\tau,k}) - \omega$.

- When $k = 0$, the net increase is $M\rho_\tau \delta^\tau f_H \omega$.
- When $k > 0$ and $k' > k$, $S'(\theta_H|h_{\tau,k'}) - S'(\theta_L|h_{\tau,k'})$ remains the same as in the original solution. When $k' = k - 1$, we have $S'(\theta_H|h_{\tau,k-1}) - S'(\theta_L|h_{\tau,k-1}) = S(\theta_H|h_{\tau,k-1}) - S(\theta_L|h_{\tau,k-1}) - \delta^d \triangle f \omega$. By repeatedly applying the above modifications until $k' = 0$, $S'(\theta_H|h_{\tau,0}) - S'(\theta_L|h_{\tau,0}) = S(\theta_H|h_{\tau,0}) - S(\theta_L|h_{\tau,0}) - (\delta^d \triangle f)^k \omega$. The expected profit increases by $M\rho_\tau \delta^\tau f_H (\delta^d \triangle f)^k \omega$.

If $IP_L(h_{\tau,0})$ are not binding constraints for $0 \le \tau \le T-d$, then $S(\theta_L|h_{\tau,0}) = \delta[f_{LL} S(\theta_L|h_{\tau+1,0}) + f_{LH} S(\theta_H|h_{\tau+1,0})] + \omega$. Let $S'(\theta_L|h_{\tau,0}) = S(\theta_L|h_{\tau,0}) - \omega$. The expected profit grows by $M\rho_\tau \delta^\tau f_L \omega$.

Based on the above, the alternative solution satisfying the binding constraints $IR_L(h_{T-d,0})$, $IC_{HL}(h_{\tau,k})$ and $IP_L(h_{\tau,0})$ in the relaxed problem yields an equal or higher profit, which contradicts the optimality of the assumption. $\square$

Lemma 2.7. *(Garrett, 2014) For a mechanism in the relaxed problem with the same hire duration preference, the binding $IC_{HL}(h_{\tau,k})$ constraints for $0 \le \tau \le T-d$ and $0 \le k \le K_\tau$ imply that*

$$S(\theta_H|h_{\tau,k}) - S(\theta_L|h_{\tau,k}) = \Delta\theta d \sum_{l=0}^{K_\tau - k} (\delta^d \triangle f)^l q_L(h^L_{\tau,k+l}). \tag{2.10}$$

The binding $IP_L(h_{\tau,0})$ and $IR_L(h_{T-d,0})$ constraints for $0 \le \tau \le T-d$ imply that

$$S(\theta_L|h_{\tau,0}) = f_{LH}\Delta\theta d \sum_{l=1}^{T-d-\tau} \delta^l \sum_{k=0}^{K_{\tau+l}} (\delta^d \triangle f)^k q_L(h^{\hat{L}}_{\tau+l,k}).$$

Customers' expected surpluses are

$$\sum_{\tau=0}^{T-d} \rho_\tau \delta^\tau \sum_i f_i S(\theta_i|h_{\tau,0}) = \Delta\theta d \sum_{\tau=0}^{T-d} (f_{LH}\beta_\tau + f_H\rho_\tau)\delta^\tau \sum_{k=0}^{K_\tau} (\delta^d \triangle f)^k q_L(h^{\hat{L}}_{\tau,k}).$$

Proof. The binding $IC_{HL}(h_{\tau,k})$ constraints imply that the inequalities become to equalities.

$$S(\theta_H|h_{\tau,k}) - S(\theta_L|h_{\tau,k}) = \Delta\theta d q_L(h_{\tau,k}) + \delta^d \triangle f [S(\theta_H|h^L_{\tau,k+1}) - S(\theta_L|h^L_{\tau,k+1})]$$

where

$$S(\theta_H|h_{\tau,k+1}) - S(\theta_L|h_{\tau,k+1}) = \Delta\theta d q_L(h_{\tau,k+1}) + \delta^d \triangle f [S(\theta_H|h^L_{\tau,k+2}) - S(\theta_L|h^L_{\tau,k+2})].$$

Based on induction by τ, it is easy to derive (2.10).

For the binding $IP_L(h_{\tau,0})$ constraints, we have

$$\begin{aligned}
S(\theta_L|h_{\tau,0}) &= \delta[f_{LL}S(\theta_L|h_{\tau+1,0}) + f_{LH}S(\theta_H|h_{\tau+1,0})] \\
&= \delta[S(\theta_L|h_{\tau+1,0}) + f_{LH}(S(\theta_H|h_{\tau+1,0}) - S(\theta_L|h_{\tau+1,0}))] \\
&= \delta[S(\theta_L|h_{\tau+1,0}) + f_{LH}\Delta\theta d \sum_{l=0}^{K_{\tau+1}} (\delta^d \triangle f)^l q_L(h^{\hat{L}}_{\tau+1,l})] \\
&= \delta^{T-d-\tau} S(\theta_L|h_{T-d,0}) + f_{LH}\Delta\theta d \sum_{l=1}^{T-d-\tau} \delta^l \sum_{k=0}^{K_{\tau+l}} (\delta^d \triangle f)^k q_L(h^{\hat{L}}_{\tau+l,k}) \\
&= f_{LH}\Delta\theta d \sum_{l=1}^{T-d-\tau} \delta^l \sum_{k=0}^{K_{\tau+l}} (\delta^d \triangle f)^k q_L(h^{\hat{L}}_{\tau+l,k}).
\end{aligned}$$

The third equality follows (2.10), the fourth equality drives from the induction until $T-d$ and the last equality obtains from the binding $IR_L(h_{T-d,0})$ constraint.

Customers' expected surplus is

$$\begin{aligned}
&\sum_{\tau=0}^{T-d}\rho_\tau\delta^\tau\sum_i f_i S(\theta_i|h_{\tau,0})\\
=&\sum_{\tau=0}^{T-d}\rho_\tau\delta^\tau[S(\theta_L|h_{\tau,0})+f_H(S(\theta_H|h_{\tau,0})-S(\theta_L|h_{\tau,0}))]\\
=&\Delta\theta d f_{LH}\sum_{\tau=0}^{T-d}\delta^\tau\beta_\tau\sum_{k=0}^{K_\tau}(\delta^d\triangle f)^k q_L(h_{\tau,k}^{\hat{L}})+\Delta\theta d f_H\sum_{\tau=0}^{T-d}\delta^\tau\rho_\tau\sum_{k=0}^{K_\tau}(\delta^d\triangle f)^k q_L(h_{\tau,k}^{\hat{L}})\\
=&\Delta\theta d\sum_{\tau=0}^{T-d}(f_{LH}\beta_\tau+f_H\rho_\tau)\delta^\tau\sum_{k=0}^{K_\tau}(\delta^d\triangle f)^k q_L(h_{\tau,k}^{\hat{L}}).
\end{aligned}$$

$\square$

Use the Lagrange multiplier approach to reformulate (2.9) as follows.

$$\begin{aligned}
&\Pi(Q,S(\theta),\Lambda)\\
=&\max_{\{Q,S(\theta),\Lambda\}}\sum_{\tau=0}^{T-d}\sum_{k=0}^{K_\tau}\mathbb{E}_{h_{\tau,k}^z}\delta^{\tau+kd}M\sum_i f_{zi}[U(\theta_i,q_i(h_{\tau,k}^z))-adq_i(h_{\tau,k}^z)]\\
&+\lambda_{\tau,0}[C-M\sum_{\tau=0}^{T-d}\rho_\tau(f_Hq_H(h_{\tau,0})+f_Lq_L(h_{\tau,0}))]+\lambda_{\tau,k}M[\rho_\tau f_Hq_H(h_{\tau,0})+\rho_\tau f_Lq_L(h_{\tau,0})\\
&-f(h_{\tau,k}^{\hat{L}})\sum_i f_{Li}q_i(h_{\tau,k}^{\hat{L}})-\sum_{h_{\tau,k}^H\in\mathcal{H}_{\tau,k}^H}f(h_{\tau,k}^H)\sum_i f_{Hi}q_i(h_{\tau,k}^H)-\sum_{h_{\tau,k}^L\in\mathcal{H}_{\tau,k}^L}f(h_{\tau,k}^L)\sum_i f_{Li}q_i(h_{\tau,k}^L)]\\
&-M\Delta\theta d\sum_{\tau=0}^{T-d}(f_{LH}\beta_\tau+f_H\rho_\tau)\delta^\tau\sum_{k=0}^{K_\tau}(\delta^d\triangle f)^k q_L(h_{\tau,k}^{\hat{L}})
\end{aligned}$$

$$s.t. \qquad S(\theta_L|h_{T-d,0}) = 0 \qquad IR_L(h_{T-d,0})$$

$$S(\theta_H|h_{\tau,k}) - S(\theta_L|h_{\tau,k}) = \Delta\theta d \textstyle\sum_{l=0}^{K_\tau - k} (\delta^d \triangle f)^l q_L(h^L_{\tau,k+l}) \quad IC_{HL}(h_{\tau,k})$$

$$S(\theta_L|h_{\tau,0}) = f_{LH}\Delta\theta d \textstyle\sum_{l=1}^{T-d-\tau} \delta^l \sum_{k=0}^{K_{\tau+l}} (\delta^d \triangle f)^k q_L(h^{\hat{L}}_{\tau+l,k}) \quad IP_L(h_{\tau,0})$$

$$\forall i, z \in \{L, H\}, h_{\tau,k} \in \boldsymbol{\mathcal{H}}_{\tau,k} \ (0 \le k \le K_\tau)$$

In the following theorem, we describe the optimal allocation policy. Let $\Lambda_{\tau,k}$ be to the constrained effect of capacity constraint $CC(h_{\tau,k})$ caused by the finite number of containers, otherwise with infinite capacity, the leasing company could allocate the number requested by both types of customers.

Theorem 2.5. *For any $h_{\tau,k} \in \boldsymbol{\mathcal{H}}_{\tau,k}$, the optimal intertemporal allocation policy for customers with same hire duration preference is characterized as follows.*
For $h_{\tau,0}$,

$$\begin{cases} q_H(h_{\tau,0}) = (\theta_H - a)d - \Lambda_{\tau,0}; \\ q_L(h_{\tau,0}) = (\theta_L - a)d - \Delta\theta d \frac{f_{LH}\beta_\tau + f_H\rho_\tau}{f_L\rho_\tau} - \Lambda_{\tau,0}; \end{cases}$$

where

$$\Lambda_{\tau,0} = \frac{\lambda_{\tau,0} - \sum_{k=1}^{K_\tau} \lambda_{\tau,k}}{\delta^\tau} = \frac{1}{\rho_\tau}\big[\beta_{\tau+1}(\theta_L - a)d - \Delta\theta d f_{LH} \sum_{l=0}^{\tau} \beta_l - \frac{C}{M} - \sum_{l=0}^{\tau-1} \rho_l \Lambda_{l,0}\big]^+;$$

when $h_{\tau,k} \in \{\mathcal{H}^H_{\tau,k}, \mathcal{H}^L_{\tau,k}\}$, *when $h_{\tau,k} = h^{\hat{L}}_{\tau,k}$,*

$$\begin{cases} q_H(h_{\tau,k}) = (\theta_H - a)d - \Lambda_{\tau,k}; \\ q_L(h_{\tau,k}) = (\theta_L - a)d - \Lambda_{\tau,k}; \end{cases} \quad \begin{cases} q_H(h^{\hat{L}}_{\tau,k}) = (\theta_H - a)d - \Lambda_{\tau,k}; \\ q_L(h^{\hat{L}}_{\tau,k}) = (\theta_L - a)d - \Delta\theta d \frac{f_{LH}\beta_\tau + f_H\rho_\tau}{f_L\rho_\tau} (\frac{\Delta f}{f_{LL}})^k - \Lambda_{\tau,k}; \end{cases}$$

where $\Lambda_{\tau,k} = \frac{\lambda_{\tau,k}}{\delta^{\tau+kd}} = \Big\{ f_{LH}\Delta\theta d(1 - \triangle f^k)(\frac{1}{1-\triangle f} + \frac{\beta_\tau}{\rho_\tau}) + \Lambda_{\tau,0} \Big\}^+.$

Proof. When $k = 0$, the first-order conditions w.r.t $q_H(h_{\tau,0})$ and $q_L(h_{\tau,0})$ are given by the following equations.

$$q_H(h_{\tau,0}): \qquad \delta^\tau[(\theta_H - a)d - q_H(h_{\tau,0})] - \lambda_{\tau,0} + \textstyle\sum_{k=1}^{K_\tau} \lambda_{\tau,k} = 0$$
$$q_L(h_{\tau,0}): \qquad \delta^\tau[(\theta_L - a)d - q_L(h_{\tau,0}) - \tfrac{f_{LH}\beta_\tau + f_H\rho_\tau}{f_L\rho_\tau}\Delta\theta d] - \lambda_{\tau,0} + \textstyle\sum_{k=1}^{K_\tau} \lambda_{\tau,k} = 0$$

For $0 \leq \tau \leq T-d$, the initial presence capacity constraint refers to

$$\sum_{\tau=0}^{T-d} \rho_\tau \Lambda_{\tau,0} = \Big[\sum_{\tau=0}^{T-d} \rho_\tau(\theta_L - a)d - \Delta\theta d f_{LH}\beta_\tau - \frac{C}{M}\Big]^+.$$

$\Lambda_{\tau,0}$ can be obtained by the induction on τ from 0 to $T-d$.

When $h_{\tau,k} \in \{\mathcal{H}^H_{\tau,k}, \mathcal{H}^L_{\tau,k}\}$, the first-order conditions w.r.t $q_H(h_{\tau,k})$ and $q_L(h_{\tau,k})$ are given by the following equations.

$$q_H(h_{\tau,k}): \qquad \delta^{\tau+kd}[(\theta_H - a)d - q_H(h_{\tau,k})] - \lambda_{\tau,k} = 0$$
$$q_L(h_{\tau,k}): \qquad \delta^{\tau+kd}[(\theta_L - a)d - q_L(h_{\tau,k})] - \lambda_{\tau,k} = 0$$

When $h_{\tau,k} = h^{\hat{L}}_{\tau,k}$, the first-order conditions about $q_H(h^{\hat{L}}_{\tau,k})$ and $q_L(h^{\hat{L}}_{\tau,k})$ are given by the following equations.

$$q_H(h^{\hat{L}}_{\tau,k}): \qquad \delta^{\tau+kd}[(\theta_H - a)d - q_H(h^{\hat{L}}_{\tau,k})] - \lambda_{\tau,k} = 0$$
$$q_L(h^{\hat{L}}_{\tau,k}): \qquad \delta^{\tau+kd}[(\theta_L - a)d - q_L(h^{\hat{L}}_{\tau,k}) - \Delta\theta d \tfrac{f_{LH}\beta_\tau + f_H\rho_\tau}{f_L\rho_\tau}(\tfrac{\Delta f}{f_{LL}})^k] - \lambda_{\tau,k} = 0$$

For $k > 0$, $\Lambda_{\tau,k}$ is obtained from the following equation.

$$f(h^{\hat{L}}_{\tau,k}) \sum_i f_{Li} q_i(h^{\hat{L}}_{\tau,k}) + \sum_{h^H_{\tau,k} \in \mathcal{H}^H_{\tau,k}} f(h^H_{\tau,k}) \sum_i f_{Hi} q_i(h^H_{\tau,k}) + \sum_{h^L_{\tau,k} \in \mathcal{H}^L_{\tau,k}} f(h^L_{\tau,k}) \sum_i f_{Li} q_i(h^L_{\tau,k})$$
$$= \rho_\tau[f_H q_H(h_{\tau,0}) + f_L q_L(h_{\tau,0})].$$

$\square$

Theorem 2.5 states the optimal allocation policy in the case with same hire duration preference. The finite capacity is first allocated for first arrival customers. Due to the same hire duration preference, the capacity is allocated repeatedly in the following periods for customers with same entry date constrained by the quantity allocated on that entry date. Besides the capacity distortion, there exists a distortion for consistent low type customers away from the efficient quantity level $(\theta_i - a)d$.

From Lemma (2.6), the customer surpluses of both customer types can be derived from the optimal allocation policy. Based on the definition of the customer surplus, the optimal pricing policy is obtained accordingly.

The next corollary discusses the effect of capacity constraints and the trend of allocated quantity over time. When $\rho_0 = 1$, it indicates that all customers arrive at the company at the beginning of the horizon, we use superscript S to denote the simultaneous arrival case. And let $\triangle\Lambda_{\tau,k} = \Lambda_{\tau,k+1} - \Lambda_{\tau,k}$.

Corollary 2.3. *(i) For fixed τ, $\Lambda_{\tau,k}$ increases with k.*

(ii) $q_i(h_{\tau,k})$ decreases with k for $h_{\tau,k} \in \{\mathcal{H}^H_{\tau,k}, \mathcal{H}^L_{\tau,k}\}$.

(iii) $\triangle\Lambda_{\tau,k} > \triangle^S\Lambda_{\tau,k}$.

(iv) Let $\tau^* = \min\{\tau|\Lambda_{\tau,0} > 0\}$. The company denies the acceptance of first arrivals customers from period $\tau + 1$ until $T - d$.

Proof. (i) For customers with same entry date τ,

$$\triangle\Lambda_{\tau,k} = f_{LH}\Delta\theta d(\Delta f)^k[1 + \frac{\beta_\tau}{\rho_\tau}(1 - \Delta f)] > 0.$$

(ii) For $h_{\tau,k} \in \{\mathcal{H}^H_{\tau,k}, \mathcal{H}^L_{\tau,k}\}$, $q_i(h_{\tau,k+1}) - q_i(h_{\tau,k+1}) = -\triangle\Lambda_{\tau,k} < 0$.

(iii)

$$\triangle^S \Lambda_{\tau,k} = f_{LH}\Delta\theta d(\Delta f)^k$$
$$\triangle\Lambda_{\tau,k} = \triangle^S \Lambda_{\tau,k} + f_{LH}\Delta\theta d(\Delta f)^k \frac{\beta_\tau}{\rho_\tau}(1-\Delta f).$$

(iv) According to the definition of τ^*, $\Lambda_{\tau,0} = 0$ for $\tau < \tau^*$,

$$\Lambda_{\tau^*,0} = \frac{1}{\rho_\tau^*}\left[\beta_{\tau^*+1}(\theta_L - a)d - \Delta\theta d f_{LH}\sum_{l=0}^{\tau^*}\beta_l - \frac{C}{M}\right]^+.$$

Then the total number of containers rent out until τ^* is

$$\begin{aligned}
&M\sum_{\tau=0}^{\tau^*}\rho_\tau(f_H q_H(h_{\tau,0}) + f_L q_L(h_{\tau,0})) \\
=&M\sum_{\tau=0}^{\tau^*-1}(f_H q_H(h_{\tau,0}) + f_L q_L(h_{\tau,0})) + \rho_{\tau^*}(f_H q_H(h_{\tau^*,0}) + f_L q_L(h_{\tau^*,0})) \\
=&M\sum_{\tau=0}^{\tau^*-1}\rho_\tau[(\theta_L - a)d - \Delta\theta d\frac{f_{LH}\beta_\tau}{\rho_\tau}] + \rho_{\tau^*}[(\theta_L - a)d - \Delta\theta d\frac{f_{LH}\beta_{\tau^*}}{\rho_{\tau^*}} - \Lambda_{\tau^*,0}] \\
=&C.
\end{aligned}$$

Thus the leasing company denies the acceptance of first arrival customers from time $\tau^* + 1$ to $T - d$.

$\square$

In the setting with same hire duration preference, Corollary 2.3(i) demonstrates that the effect of capacity constraint on the optimal allocation policy increases over time for customers with same initial arrival date. In Corollary 2.3(ii), the optimal quantity for customers who are not consistent low type customers decreases with the number of finished contracts k. For $q_L(h^{\hat{L}}_{\tau,k})$, the tendency towards k is determined by the combined effect including the decrement of $\Delta\theta d\frac{f_{LH}\beta_\tau + f_H\rho_\tau}{f_L\rho_\tau}(\frac{\Delta f}{f_{LL}})^k$ and the in-

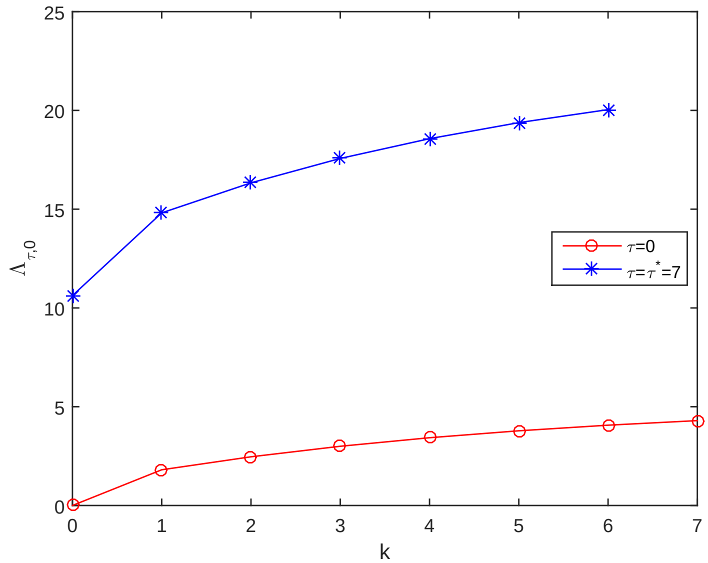


Figure 2.5. The Capacity Effect $\Lambda_{\tau,k}$ of the Optimal Solution

crement of $\triangle\Lambda_{\tau,k}$. Corollary 2.3(iv) shows that the leasing company has a deadline for the first arrival customers. If $\Lambda_{0,0} > 0$, then the system is the same as the system with simultaneous arrival case.

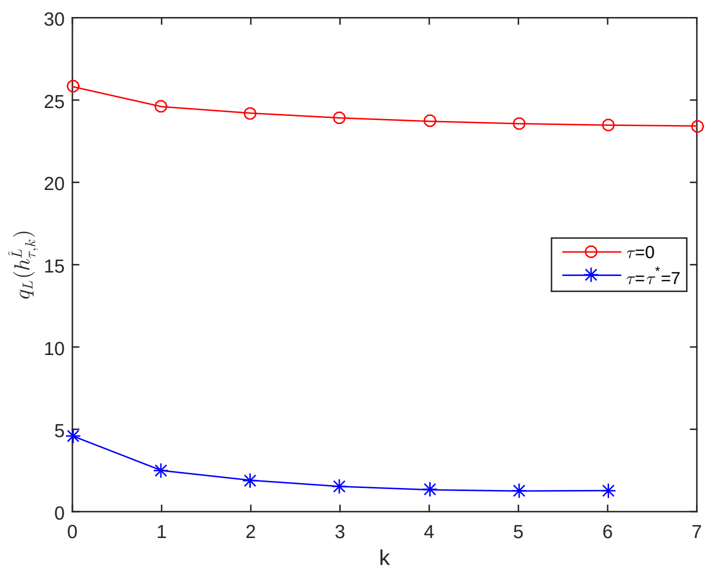


Figure 2.6. The Optimal Allocated Quantity $q_L(h^{\hat{L}}_{\tau,k})$ for Consistent Low-Type Customers, $f_H = 0.34$, $f_{HH} = 0.93$, $f_{LH} = 0.13$

We use a numerical example to illustrate the properties of the optimal allocation policy. The leasing company has $M = 30$ customers to allocate the finite capacity

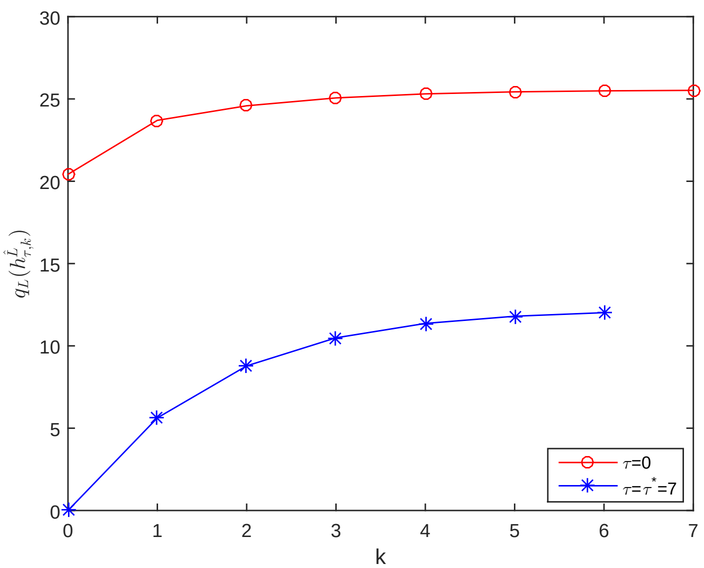


Figure 2.7. The Optimal Allocated Quantity $q_L(h^{\hat{L}}_{\tau,k})$ for Consistent Low-Type Customers, $f_H = 0.54, f_{HH} = 0.69, f_{LH} = 0.39$

$C = 200$ with time discount factor $\delta = 0.95$. Customers have the same hire duration preference $d = 4$. Customers enter the leasing company with probability ρ_τ which is generated randomly and the sum of ρ_τ equals to 1. High type customers have per time valuation $\theta_H = 10$ arriving at the company with probability $f_H = 0.34$ upon the first arrival customers and consistent high type probability $f_{HH} = 0.93$. Low type customers enjoy per time valuation $\theta_L = 8$ and remain the low type customer with probability $f_{LL} = 0.87$. The operations cost is 0.5 per time per unit. In Fig.(2.5), $\Lambda_{0,k}$ and $\Lambda_{\tau^*,k}$ share the increasing over time property. It is clear that $\Lambda_{\tau^*,k} > \Lambda_{0,k}$ and $\Lambda_{\tau^*,k}$ has a steeper slope with k compared with the slope of $\Lambda_{0,k}$. The possible explanation for this phenomenon is as follows. Considering the two customer types in the horizon with same hire duration, current profit maximization outweighs future profit maximization. Given the continuous lease behavior after initial arrival, thus the leasing company has a inclination to lease out all units of container as early as possible. Fig.(2.6) and Fig.(2.7) display the declining and climbing trends of the allocated quantities for consistent low-type customer under different parameter settings. This corroborates the trend of $q_L(h_k^{\hat{L}})$ is determined by the decrement of $\Delta\theta d\frac{f_{LH}\beta_\tau+f_H\rho_\tau}{f_L\rho_\tau}(\frac{\Delta f}{f_{LL}})^k$ and the increment of $\triangle\Lambda_{\tau,k}$.

2.4.2 Different hire duration preferences

The different hire time preferences $d_H > d_L$ is investigated in this section. The $d_H < d_L$ case can be solved similarly. The basic thought is solving the dynamic nonlinear pricing problem by the binding initial presence constraints of low type customers, binding incentive compatibility constraints for high type customers and binding individual rationality constraint of low type customers shown at last allowable entry date.

The allowable entry dates start from 0 to $T - d_H$. $h_{\tau,l,k}$ refers to the history with initial arrival date τ and l lease contracts with hire duration d_L and k lease contracts with hire duration d_H. $\boldsymbol{\mathcal{H}}_{\tau,l,k}$ is the set of all possible histories from time τ to time $\tau + ld_L + kd_H$. $K^L_{\tau,k} = \lfloor (T-\tau-kd_H)/d_L \rfloor - 1$ is the last pricing point for low-type customers with entry date τ and k complete lease contracts being high type customers and $K^H_{\tau,l} = \lfloor (T-\tau-ld_L)/d_H \rfloor - 1$ is the last pricing point for high-type customers with entry date τ and l lease contracts being low type customers.

The direct revelation mechanism must satisfy the following constraints.

$IP_i(h_{\tau,0,0})$ Constraint. $S(\theta_i|h_{\tau,0,0}) \geq \delta[f_{ii}S(\theta_i|h_{\tau+1,0,0}) + f_{ij}S(\theta_j|h_{\tau+1,0,0})]$.

$IC_{ij}(h_{\tau,l,k})$ Constraint.

$$S(\theta_H|h_{\tau,l,k}) \geq S(\theta_L|h_{\tau,l,k}) + \Delta\theta d_L q_L(h_{\tau,l,k}) + \delta^{d_L}\sum_j (f_{Hj} - f_{Lj})S(\theta_j|h^L_{\tau,l+1,k}).$$

$$S(\theta_L|h_{\tau,l,k}) \geq S(\theta_H|h_{\tau,l,k}) - \Delta\theta d_H q_H(h_{\tau,l,k}) + \delta^{d_H}\sum_j (f_{Hj} - f_{Lj})S(\theta_j|h^H_{\tau,l,k+1}).$$

$IR_i(h_{\tau,l,k})$ Constraint. $S(\theta_i|h_{\tau,l,k}) \geq 0$ for $h_{\tau,l,k} \in \boldsymbol{\mathcal{H}}_{\tau,l,k}$.

Capacity Constraint ($CC(h_{\tau,l,k})$). The capacity constraint for first arrival customers with entry date τ for $0 \leq \tau \leq T - d_H$ is

$$M \sum_{\tau=0}^{T-d_H} \rho_\tau (f_H q_H(h_{\tau,0,0}) + f_L q_L(h_{\tau,0,0})) \leq C.$$

Owing to different hire duration preferences and the binding capacity constraint for first arrival customers, the available capacity for history $h^z_{\tau,l,k}$ at time $t + ld_L + kd_H$ is the number of units rent out $q_z(h_{\tau,l-\mathbf{1}_L(z),k-\mathbf{1}_H(z)})$, where $\mathbf{1}_i(z)$ for $i \in \{L, H\}$ is an indicator function, e.g. $\mathbf{1}_L(L) = 1$ and $\mathbf{1}_L(H) = 0$.

The capacity constraint for history $h^z_{\tau,l,k}$ can be expressed as

$$f_{zL} q_L(h^z_{\tau,l,k}) + f_{zH} q_H(h^z_{\tau,l,k}) \leq q_z(h_{\tau,l-\mathbf{1}_L(z),k-\mathbf{1}_H(z)}).$$

The corresponding monopolist's optimization problem can be formulated as follows.

$$\begin{aligned}
\Pi(Q,S) = \max_{\{Q,S\}} M \Big\{ & \sum_{\tau=0}^{T-d} \sum_{l=0}^{K^L_{\tau,0}} \sum_{k=0}^{K^H_{\tau,l}} \delta^{\tau+ld_L+kd_H} \mathbb{E}_{h^z_{\tau,l,k}} \sum_i f_{zi} [U(\theta_i, q_i(h^z_{\tau,l,k})) \\
& - ad_i q_i(h^z_{\tau,l,k})] - \sum_{\tau=0}^{T-d} \rho_\tau \delta^\tau \sum_i f_i S(\theta_i | h_{\tau,0,0}) \Big\} \\
s.t. \quad & IP_i(h_{\tau,0,0}), IC_{ij}(h_{\tau,l,k}), IR_i(h_{\tau,l,k}), CC(h_{\tau,l,k}) \\
& \forall i, z \in \{L, H\}, h_{\tau,l,k} \in \boldsymbol{\mathcal{H}}_{\tau,l,k}
\end{aligned}$$

The relaxed problem in the dynamic nonlinear pricing problem with dynamic arrivals are defined by the binding constraints $IR_L(h_{T-d_H,0,0})$, $IC_{HL}(h_{\tau,l,k})$ and $IP_L(h_{\tau,0,0})$ for $0 \leq \tau \leq T - d_H$, $0 \leq L \leq K^L_{\tau,0}$ and $0 \leq k \leq K^H_{\tau,l}$.

Lemma 2.8. *In a dynamic environment with different hire duration preferences, the optimal solution of the relaxed problem is also an optimal solution of the original problem.*

Proof. The proof is similar to the proof of Lemma 2.6, except the part for $IC_{HL}(h_{\tau,l,k})$.

If $IC_{HL}(h_{\tau,l,k})$ are not binding constraints for $0 \le l \le K^L_{\tau,0}$, $S(\theta_H|h_{\tau,l,k}) - S(\theta_L|h_{\tau,l,k}) = \Delta\theta d_L q_L(h_{\tau,l,k}) + \delta^{d_L} \triangle f[S(\theta_H|h_{\tau,l+1,k}) - S(\theta_L|h_{\tau,l+1,k})] + \omega$. Consider an alternative solution $\{Q', S'(\theta)\}$ such that $S'(\theta_H|h_{\tau,l,k}) - S'(\theta_L|h_{\tau,l,k}) = S(\theta_H|h_{\tau,l,k}) - S(\theta_L|h_{\tau,l,k}) - \omega$.

- When $l = 0$ and $k = 0$, the net increase is $M\rho_\tau \delta^\tau f_H \omega$.

- When $l > 0$ and $l' = l-1$, we have $S'(\theta_H|h_{\tau,l-1,k}) - S'(\theta_L|h_{\tau,l-1,k}) = S(\theta_H|h_{\tau,l-1,k}) - S(\theta_L|h_{\tau,l-1,k}) - \delta^{d_L}\triangle f\omega$. By repeatedly applying the above modifications until $l' = 0$, $S'(\theta_H|h_{\tau,0,k}) - S'(\theta_L|h_{\tau,0,k}) = S(\theta_H|h_{\tau,0,k}) - S(\theta_L|h_{\tau,0,k}) - (\delta^{d_L}\triangle f)^l\omega$. Apply the same modification for $k > 0$. The expected profit increases by $M\rho_\tau f_H \delta^{\tau+ld_L}\triangle f^l\omega$. It contradicts the optimality of the assumption.

□

Lemma 2.9. *For a mechanism in the relaxed problem with different hire duration preferences, the binding $IC_{HL}(h_{\tau,l,k})$ constraints for $0 \le \tau \le T - d_H$, $0 \le l \le K^L_{\tau,0}$ and $0 \le k \le K^H_{\tau,l}$ imply that*

$$S(\theta_H|h_{\tau,l,k}) - S(\theta_L|h_{\tau,l,k}) = \Delta\theta d_L \sum_{u=0}^{K^L_{\tau,k}-l} (\delta^{d_L}\triangle f)^u q_L(h_{\tau,l+u,k}). \tag{2.11}$$

The binding $IP_L(h_{\tau,0,0})$ and $IR_L(h_{T-d_H,0,0})$ constraints for $0 \le \tau \le T - d_H$ imply that

$$S(\theta_L|h_{\tau,0,0}) = f_{LH}\Delta\theta d_L \sum_{u=1}^{T-d_H-\tau} \delta^u \sum_{v=0}^{K^L_{\tau+u,0}} (\delta^{d_L}\triangle f)^v q_L(h_{\tau+u,v,0}).$$

Customers' expected surplus is

$$\sum_{\tau=0}^{T-d_H} \rho_\tau\delta^\tau \sum_i f_i S(\theta_i|h_{\tau,0,0}) = \Delta\theta d_L \sum_{\tau=0}^{T-d_H} (f_{LH}\beta_\tau + f_H\rho_\tau)\delta^\tau \sum_{v=0}^{K^L_{\tau,0}} (\delta^{d_L}\triangle f)^v q_L(h_{\tau,v,0})$$

Use the Lagrange multiplier approach to reformulate the relaxed problem as

$$\Pi(Q,S,\Lambda)=\sum_{\tau=0}^{T-d_H}\sum_{l=0}^{K^L_{\tau,0}}\sum_{k=0}^{K^H_{\tau,l}}\delta^{\tau+ld_L+kd_H}\mathbb{E}_{h^z_{\tau,l,k}}M\sum_i f_{zi}[U(\theta_i,q_i(h^z_{\tau,l,k}))$$

$$-ad_iq_i(h^z_{\tau,l,k})]+\lambda_{\tau,0,0}[C-M\sum_{\tau=0}^{T-d_H}\rho_\tau(f_Hq_H(h_{\tau,0,0})+f_Lq_L(h_{\tau,0,0}))]$$

$$+\lambda_{\tau,l,k}Mf(h^z_{\tau,l,k})[q_z(h_{\tau,l-\mathbf{1}_L(z),k-\mathbf{1}_H(z)})-f_{zL}q_L(h^z_{\tau,l,k})-f_{zH}q_H(h^z_{\tau,l,k})]$$

$$-M\Delta\theta d_L\sum_{\tau=0}^{T-d_H}(f_{LH}\beta_\tau+f_H\rho_\tau)\delta^\tau\sum_{v=0}^{K^L_{\tau,0}}(\delta^{d_L}\triangle f)^v q_L(h_{\tau,v,0})$$

$$s.t. \qquad S(\theta_L|h_{T-d_H,0,0})=0 \qquad IR_L(h_{T-d_H,0,0})$$

$$S(\theta_H|h_{\tau,l,k})-S(\theta_L|h_{\tau,l,k})=\Delta\theta d_L\textstyle\sum_{u=0}^{K^L_{\tau,k}-l}(\delta^{d_L}\triangle f)^u q_L(h_{\tau,l+u,k}) \qquad IC_{HL}(h_{\tau,l,k})$$

$$S(\theta_L|h_{\tau,0,0})=f_{LH}\Delta\theta d_L\textstyle\sum_{u=1}^{T-d_H-\tau}\delta^u\sum_{v=0}^{K^L_{\tau+u,0}}(\delta^{d_L}\triangle f)^v q_L(h_{\tau+u,v,0}) \qquad IP_L(h_{\tau,0,0})$$

$$\forall i,z\in\{L,H\},h_{\tau,l,k}\in\boldsymbol{\mathcal{H}}_{\tau,l,k}(0\le l\le K^L_{\tau,0},0\le k\le K^H_{\tau,l})$$

Theorem 2.6 presents the optimal allocation policy in the case with different hire duration preferences.

Theorem 2.6. *For any $h_{\tau,l,k}\in\boldsymbol{\mathcal{H}}_{\tau,l,k}$, the optimal intertemporal allocation policy for customers with different hire duration preferences is characterized as follows.*
For $h_{\tau,0,0}$,

$$\begin{cases} q_H(h_{\tau,0,0})=(\theta_H-a)d_H-\Lambda_{\tau,0,0};\\ q_L(h_{\tau,0,0})=(\theta_L-a)d_L-\Delta\theta d_L\frac{f_{LH}\beta_\tau+f_H\rho_\tau}{f_L\rho_\tau}-\Lambda_{\tau,0,0};\end{cases}$$

where

$$\Lambda_{\tau,0,0} = \frac{\lambda_{\tau,0,0}}{\delta^{\tau}} = \frac{1}{\rho_\tau}\Big\{\beta_{\tau+1}[f_H(\theta_H - a)(d_H - d_L) + (\theta_L - a)d_L] - \Delta\theta d_L f_{LH}\sum_{u=0}^{\tau}\beta_u - \frac{C}{M} - \sum_{u=0}^{\tau-1}\rho_u\Lambda_{u,0,0}\Big\}^+;$$

when $k > 0$,

$$\begin{cases} q_H(h^z_{\tau,l,k}) = (\theta_H - a)d_H - \Lambda_{\tau,l,k}; \\ q_L(h^z_{\tau,l,k}) = (\theta_L - a)d_L - \Lambda_{\tau,l,k}; \end{cases}$$

where

$$\Lambda_{\tau,l,k} = \frac{\lambda_{\tau,l,k}}{\delta^{\tau+ld_L+kd_H}} = \Big\{[(\theta_H - a)d_H - (\theta_L - a)d_L][f_{LH}\mathbf{1}_L(z) - f_{HL}\mathbf{1}_H(z)] + \Lambda_{\tau,l-\mathbf{1}_L(z),k-\mathbf{1}_H(z)}\Big\}^+;$$

when $k = 0$,

$$\begin{cases} q_H(h_{\tau,l,0}) = (\theta_H - a)d_H - \Lambda_{\tau,l,0}; \\ q_L(h_{\tau,l,0}) = (\theta_L - a)d_L - \Delta\theta d_L \frac{f_{LH}\beta_\tau + f_H\rho_\tau}{f_L\rho_\tau}(\frac{\Delta f}{f_{LL}})^l - \Lambda_{\tau,l,0}; \end{cases}$$

where

$$\Lambda_{\tau,l,0} = \frac{\lambda_{\tau,l,0}}{\delta^{\tau+ld_L}} = \Big\{\Lambda_{\tau,l-1,0} + \Delta\theta d_L\frac{f_{LH}\beta_\tau + f_H\rho_\tau}{\rho_\tau f_L}(\frac{\triangle f}{f_{LL}})^{l-1}(1 - \triangle f) + f_{LH}[(\theta_H - a)d_H - (\theta_L - a)d_L]\Big\}^+.$$

Proof. $q_i(h_{\tau,l,k})$ exists in the capacity constraints $CC(h_{\tau,l,k})$ and $CC(h_{\tau,l+\mathbf{1}_L(i),k+\mathbf{1}_H(i)})$. $CC(h_{\tau,l,k})$ binds the optimal allocated quantity $q_i(h_{\tau,l,k})$, while $CC(h_{\tau,l+\mathbf{1}_L(i),k+\mathbf{1}_H(i)})$ reduces the binding effect due to the returning process. Thus, we consider the optimal allocation policy under the tightened binding capacity effect $CC(h_{\tau,l,k})$.

When $k > 0$, the first-order conditions w.r.t $q_H(h^z_{\tau,l,k})$ and $q_L(h^z_{\tau,l,k})$ are given by the following equations.

$$\begin{aligned}
q_H(h^z_{\tau,l,k}) : \quad & \delta^{\tau+ld_L+kd_H} f(h^z_{\tau,l,k})[(\theta_H - a)d_H - q_H(h^z_{\tau,l,k})] - \lambda_{\tau,l,k} = 0 \\
q_L(h^z_{\tau,l,k}) : \quad & \delta^{\tau+ld_L+kd_H} f(h^z_{\tau,l,k})[(\theta_L - a)d_L - q_L(h^z_{\tau,l,k})] - \lambda_{\tau,l,k} = 0
\end{aligned}$$

$$\begin{aligned}
\Lambda_{\tau,l,k} = \frac{\lambda_{\tau,l,k}}{\delta^{\tau+ld_L+kd_H}} &= [f_{zH}(\theta_H - a)d_H + f_{zL}(\theta_L - a)d_L - q_z(h_{\tau,l-\mathbf{1}_L(z),k-\mathbf{1}_H(z)})]^+ \\
&= \Big\{[(\theta_H - a)d_H - (\theta_L - a)d_L][f_{LH}\mathbf{1}_L(z) - f_{HL}\mathbf{1}_H(z)] + \Lambda_{\tau,l-\mathbf{1}_L(z),k-\mathbf{1}_H(z)}\Big\}^+.
\end{aligned}$$

When $k = 0$, the first-order conditions w.r.t $q_H(h_{\tau,l,0})$ and $q_L(h_{\tau,l,0})$ are given by the following equations.

$$\begin{aligned}
q_H(h_{\tau,l,0}) : \quad & \delta^{\tau+ld_L} f(h_{\tau,l,0})[(\theta_H - a)d_H - q_H(h_{\tau,l,0})] - \lambda_{\tau,l,0} = 0 \\
q_L(h_{\tau,l,0}) : \quad & \delta^{\tau+ld_L}[(\theta_L - a)d_L - q_L(h_{\tau,l,0})] - \lambda_{\tau,l,0} - \delta^{\tau+ld_L}\Delta\theta d_L \tfrac{f_{LH}\beta_\tau + f_H\rho_\tau}{f_L\rho_\tau}(\tfrac{\Delta f}{f_{LL}})^l = 0
\end{aligned}$$

$$\begin{aligned}
\Lambda_{\tau,l,0} = \frac{\lambda_{\tau,l,0}}{\delta^{\tau+ld_L}} =& \Big[f_{LL}[(\theta_L - a)d_L - \Delta\theta d_L \frac{f_{LH}\beta_\tau + f_H\rho_\tau}{f_L\rho_\tau}(\frac{\Delta f}{f_{LL}})^l] \\
& + f_{LH}(\theta_H - a)d_H - q_L(h_{\tau,l-1,0})\Big]^+ \\
=& \Big\{\Lambda_{\tau,l-1,0} + \Delta\theta d_L \frac{f_{LH}\beta_\tau + f_H\rho_\tau}{f_L\rho_\tau}(\frac{\Delta f}{f_{LL}})^{l-1}(1 - \Delta f) \\
& + f_{LH}[(\theta_H - a)d_H - (\theta_L - a)d_L]\Big\}^+.
\end{aligned}$$

□

The next corollary discusses the effect of capacity constraints and the trend of allocated quantity over time. Let $\triangle_i\Lambda_{\tau,l,k} = \Lambda_{\tau,l+\mathbf{1}_L(i),k+\mathbf{1}_H(i)} - \Lambda_{\tau,l,k}$. Recall that the superscript S represents the simultaneous arrival case.

Corollary 2.4. *For fixed τ, $0 \le l \le K^L_{\tau,0}$ and $0 \le k \le K^H_{\tau,l}$,*

(i) $\Lambda_{\tau,l,k}$ *increases with* l *but decreases with* k*;*

(ii) $q_i(h_{\tau,l,k})$ *decreases with* l *and increases with* k *for* $i \in \{L, H\}$*;*

(iii) $\triangle_L \Lambda_{\tau,l,0} \ge \triangle^S_L \Lambda_{0,l,0}$*;*

$\triangle_i \Lambda_{\tau,l,k} = \triangle^S_i \Lambda_{0,l,k}$ *for* $k, \tau > 0$ *and* $i \in \{L, H\}$*.*

Proof. (i) If $k > 0$, $\triangle_L \Lambda_{\tau,l,k} = f_{LH}[(\theta_H - a)d_H - (\theta_L - a)d_L] > 0$; if $k = 0$,

$$\triangle_L \Lambda_{\tau,l,k} = f_{LH}[(\theta_H - a)d_H - (\theta_L - a)d_L] + \Delta\theta d_L \frac{f_{LH}\beta_\tau + f_H\rho_\tau}{f_L\rho_\tau}(\frac{\Delta f}{f_{LL}})^l(1 - \Delta f) > 0.$$

The positivity is derived from $\theta_H > \theta_L, d_H > d_L$ and $\Delta f < 1$.

$$\triangle_H \Lambda_{\tau,l,k} = -f_{HL}[(\theta_H - a)d_H - (\theta_L - a)d_L] < 0.$$

Thus, $\Lambda_{\tau,l,k+1} \le \Lambda_{\tau,l,k}$.

(ii) If $i = L, k > 0$ or $i = H, k \ge 0$

$$q_i(h_{\tau,l,k+1}) - q_i(h_{\tau,l,k}) = f_{HL}[(\theta_H - a)d_H - (\theta_L - a)d_L] > 0;$$
$$q_i(h_{\tau,l+1,k}) - q_i(h_{\tau,l,k}) = -f_{LH}[(\theta_H - a)d_H - (\theta_L - a)d_L] < 0;$$

if $k = 0$,

$$\begin{aligned} q_L(h_{\tau,l,1}) - q_L(h_{\tau,l,0}) =& f_{HL}[(\theta_H - a)d_H - (\theta_L - a)d_L] + \Delta\theta d_L \frac{f_{LH}\beta_\tau + f_H\rho_\tau}{f_L\rho_\tau}(\frac{\Delta f}{f_{LL}})^l > 0; \\ q_L(h_{\tau,l+1,0}) - q_L(h_{\tau,l,0}) =& - f_{LH}[(\theta_H - a)d_H - (\theta_L - a)d_L] \\ &+ \Delta\theta d_L \frac{f_{LH}\beta_\tau + f_H\rho_\tau}{f_L\rho_\tau}(\frac{\Delta f}{f_{LL}})^l \Delta f(1 - \frac{1}{f_{LL}}) < 0. \end{aligned}$$

Thus $q_i(h_{\tau,l,k})$ increases with k and decreases with l.

(iii)

$$\begin{aligned}
\triangle_L^S\Lambda_{0,l,0} =&\Big\{f_{LH}[(\theta_H-a)d_H-(\theta_L-a)d_L]+\frac{f_H\Delta\theta d_L}{f_L}(\frac{\Delta f}{f_{LL}})^l(1-\Delta f)\Big\}^+\\
\triangle_L\Lambda_{\tau,l,0} =&\Big[\triangle_L^S\Lambda_{0,l,0}+\Delta\theta d_L\frac{f_{LH}\beta_\tau}{f_L\rho_\tau}(\frac{\Delta f}{f_{LL}})^l(1-\Delta f)\Big]^+\\
\triangle_i\Lambda_{\tau,l,k} =&\triangle_i^S\Lambda_{0,l,k}=[(\theta_H-a)d_H-(\theta_L-a)d_L](f_{LH}\mathbf{1}_L(i)-f_{HL}\mathbf{1}_H(i)).
\end{aligned}$$

□

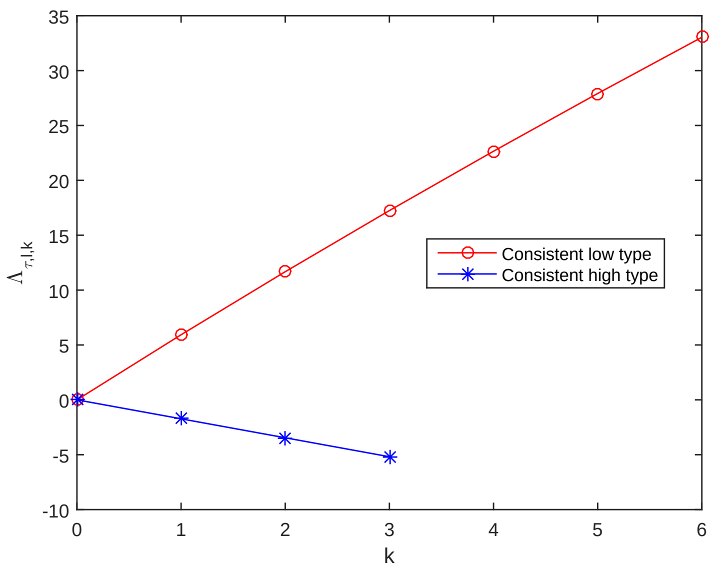


Figure 2.8. The Capacity Effect $\Lambda_{\tau,k}$ of the Optimal Solution

The parameters of numerical example in this subsection are the same as in the above subsection except $d_H = 6$ and $d_L = 4$. Corollary 2.4(i) suggests that the effect of capacity constraint hinges on the customer type of the previous period. Fig.(2.8) depicts the effect of capacity constraint for consistent low and high type customers. The impact of capacity constraint becomes smaller when the last customer type is low type and greater when last customer type is high type. While for customers with same hire time preference in Section 4.3, capacity constraints have the same increasing effect for both customer types with same entry date. Corollary 2.4(ii) means that given the same initial presence date, the leasing company allocates less quantity for low type

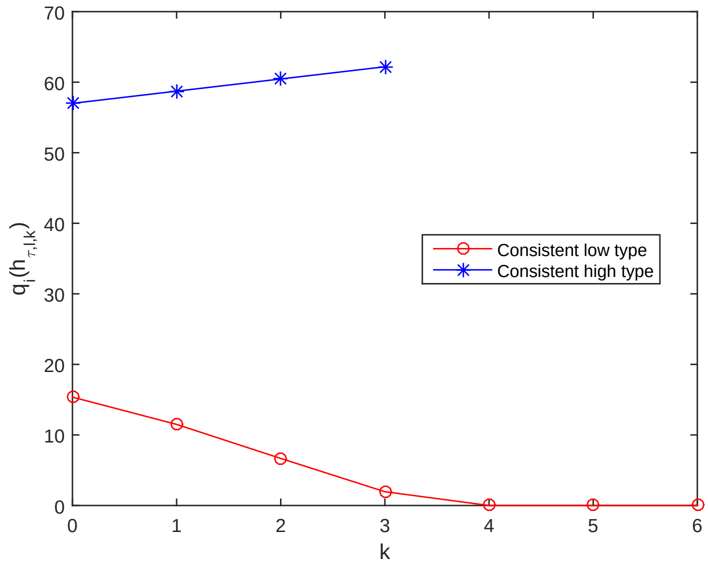


Figure 2.9. The Optimal Allocated Quantity $q_i(h_{\tau,l,k})$ for Consistent Low-Type/ High-type Customers

customers and more quantity for high type customers (See Fig.(2.9)). That is, the growing distortion of the low type customers over time contrasts with the declining distortion of the high type customers. In Corollary 2.4(iii) we compare the effect of capacity constraint in dynamic arrivals case with the effect of capacity constraint in the simultaneous arrival case. The results show that the dynamic arrivals accentuate the capacity effect for the consistent low-type customers, but have the same effect for other types customers. The reason behind this is that capacity constraints in the different hire duration preferences are contingent on the allocated amount of last customer type. The dynamic arrivals only increase the effect of capacity constraints for consistent low type customers by $\frac{\Delta\theta d_L f_{LH} \beta_\tau}{\rho_\tau f_L}(\frac{\triangle f}{f_{LL}})^l(1-\triangle f)$ but have the same effect for the other types of customers.

2.5 Summary

In this part, we explore the monopolist's dynamic nonlinear pricing problems in static and dynamic environments. In a static environment, we obtain the closed-form solutions for the capacity-constrained nonlinear pricing problems under different customer groups with multiple types. In a dynamic environment with contemporaneous arrivals, we concentrate on two customer types in each group. The optimal closed-form solutions are derived for customers with hire time preference. The capacity constraint has distinct effects when customers have same or different hire time preference(s). For customers with preferred hire quantity, on the given alternative hire time sets, we use dynamic programming approach to obtain the numerical optimal solution. Further we investigate the dynamic arrivals in a dynamic environment with hire time preference. We derive the closed-form solution and discuss the effect of capacity constraints on the solution. Compared with the solution with contemporaneous arrivals, the dynamic arrivals aggravate the effect of capacity constraint for the consistent low type customers but have the same effect of capacity constraint compared with the simultaneous arrival case.

CHAPTER 3

Dynamic Pricing with Reservation and Unit Demand

This chapter examines the pricing and capacity management problem for stochastic rental systems with advance demand information. In particular, we consider a container leasing company that manages fixed container units to maximize the expected profit with discount factor. The firm confronts two types of customers: advance-demand-information (ADI) customers and walk-in customers. ADI customers inform the leasing company some time before the actual demand happens and the lead time follows exponential distribution. We make this assumption due to the fact that reservations may not be very accurate. The demand realization will be earlier or later than the expected date. Walk-in customers arrive the system and require the immediate service. Customers with unit capacity request arrive at the system according to a Poisson process and the rental duration of each customer class follows exponential distribution. Next, the firm employs a pricing mechanism to control the number of customers in service. By modeling the pricing and capacity rationing problem as a continuous time Markov decision process, we show that the objective function is an anti-multimodular function and there exists a state-dependent rationing policy with bounded sensitivity such that walk-in customers' rationing threshold nonincreases in the number of ADI demand with at most unit decrease and ADI customers' rationing threshold nondecreases in the leased amount with at most two-unit decrement. The optimal posted price is nondecreasing with the leased amount and the number of advance demands. Numerical experiments are conducted to study the effect of reservations on the optimal policy.

3.1 Literature Review

Three areas of research are connected with the second part of the dissertation: customer-based pricing, pricing and admission control in rental systems and inventory control with ADI.

The first one comprises studies about the customer-based pricing, which is to decide the prices based on customers type. There are two criteria to categorize the customers. On the one hand, according to the customers' behavior, customers are divided to two types: strategic and myopic. Strategic customers wait for a sale price or a promotion sacrificing the immediate gratification. They are forward-looking to maximize their utility across time periods. And myopic customers are impulsive and willing to buy goods without hesitation. *Stokey* (1979) is the earliest pricing work considering strategic customers. Stokey analyzes a monopolist model of selling a new product in continuous time and derive the optimal pricing strategies under zero and positive production cost cases. Since then, the customer behavior-based pricing has been extensively studied in the literature. One set of papers (*Besanko and Winston* 1990, *Su* 2007 , *Aviv and Pazgal* 2008, *Liu and van Ryzin* 2008) apply game theory to study the interaction between seller and customers. Other papers solve the pricing problem from the seller's point. For example, *Zhang and Cooper* (2008) discuss a seller's pricing and rationing decisions with mixed type of customers for a single product over two periods under four different scenarios–finite(infinite) capacity and fixed pricing (flexible pricing). The results indicate that rationing in the second period has no additional benefit to the revenue under the infinite capacity and pricing flexibility. But for fixed price, rationing does contribute to the revenue. In general, rationing has a mild effect on the optimal revenue compared with pricing without rationing. *Mersereau and Zhang* (2012) extend the model of Zhang and Cooper and discuss a two-period selling season in which the seller knows the aggregate demand curve but without the information of the proportion of strategic customer. Using a robust approach, the authors derive the preannounced pricing strategy which continues to work well in stochastic demand situation.

On the other hand, according to the customers serving cost, two customer segments are a high-cost segment and a low-cost segment. *Shin et al.* (2012) study a two-period monopoly model where a firm set price in the second period based on the customer information at the first period. The results show that when heterogeneity is small, there is no need to impose price discrimination. When heterogeneity is large, the customer cost-based pricing is effective. By firing some high-cost customer, the firm will have a new mixture of good and bad customer who is more profitable than the original high-cost customer. Even the customer type is endogenous, the profit of customer cost-based pricing is higher and the price gap between two types of customer is also larger than that under exogenous case.

Another stream is the literature on admission control and pricing in rental and queuing systems. *Miller* (1969) formulates a n-sever and m-customer class queuing system as continuous time Markov decision process and present a specialized algorithm to solve it. *Lippman* (1975) proposes a new definition of the time of transition, to uniformize Markovian queuing systems. The uniformization is that the times between transition is exponentially distributed with a constant number. *Altman et al.* (2001) use event-based dynamic programming to examine a call admission control system with multiple classes without waiting room and derive structural properties of optimal policy. Then they use a fluid model to approximate the large-capacity case. *Savin et al.* (2005) investigate a capacity allocation problem in rental system with two customer classes. They characterize the switching-curve policy and conditions for preferred classes. An aggregate threshold policy is developed for the fluid approximation model and the effect of capacity rationing on the optimal fleet size is analyzed. *Gans and Savin* (2007) extend their model by incorporating the price interaction between walk-in customer and rental company. They model the pricing and capacity rationing problem for a rental system as a Markov decision process and demonstrate that the optimal policy parameters are monotone with the system parameters. They further obtain the condition of preferred customer class. This area of research neglects the effect of advance demand information, except *Papier and Thonemann* (2010). They study a stochastic rental system with two types of customers–advance demand information(ADI) customers and walk-in customers and show that the optimal admission policy is a threshold type policy. Due to the computational intractability, a close-to-optimal ADI policy is proposed and performs better than the policy that ignores ADI.

For the literature on with pricing in the queuing system, *Leeman* (1964) discusses qualitatively the use of pricing to reduce queues in real life applications where additional charge may be helpful for a peak-load queue. *Naor* (1969) is the first analytical work of pricing in queuing models. He considers an M/M/1 queuing model with equal and constant reward for each customer. The strategies under self-optimization and overall optimization objectives are analyzed. He shows that the strategy under self-interest does not result in the overall optimality. A toll is charged to new coming customer on some critical points so that the overall optimality is achieved. *Knudsen* (1972), *Lippman and Stidham Jr* (1977), *Stidham* (1978) extend Noar's work to more general settings, such as multi-server with a general cost-benefit function, birth-death process with holding cost and general interarrival time distribution. *Johansen* (1994) studies the optimal dynamic pricing of an M/G/1 jobshop under profit maximization and welfare maximization objectives. The price is charged at the completion of each job.

The third area related to our research is the literature on production-inventory systems with advance demand information. There is a growing interest in the production-inventory systems with ADI, see flexible delivery (*Wang and Toktay* 2008), imperfect ADI (*Gayon et al.* 2009), sharing ADI (*Zhu and Thonemann* 2004). Here we only review the literature with ADI which is an endogenous outcome of pricing. *Weng and Parlar* (1999) propose a model to evaluate the multiple effects of joint-stocking and prior discount on the expected profit. The retailer provides prior-sale discount to encourage customer's early purchase before the selling season. The authors obtain the optimal stocking quantity and discount rate when the two decisions are determined jointly. *Tang et al.* (2004) study a similar problem but focus more on the benefits of advance book discount. *McCardle et al.* (2004) extend the model of Tang to capture the market competition by considering a duopoly model. *Boyacı and Özer* (2010) discuss the strategy of collecting revenue and information through advance sales to a capacity decision under different cases–price are determined exogenously or optimally. The control band policy is obtained and shows that advance selling can improve the profit significantly. *Li and Zhang* (2012) analyze a seller's preorder strategy who sells a perishable product in two periods to heterogeneous customers. They obtain the seller's optimal price, quantity decisions and the timing of offering a price guarantee. They find that the seller's profit decrease with the accuracy degree of advance demand information.

3.2 Model Description

Consider a container leasing company that manages C units of containers to maximize the expected profit with discount factor α. The firm confronts with two classes of customers: advance-demand-information (ADI) customers and walk-in customers. We denote the set of ADI customers by a and walk-in customers by w, respectively. Customers arrive at the system according to a Poisson process with rate $\lambda_i (i \in \{a, w\})$. The lease duration of both customer classes follows exponential distribution with mean $1/\mu$ and each customer requests one unit of containers.

ADI customers inform the leasing company L time before the actual demand happens. The lead time L follows exponential distribution with mean $1/\nu$. We make this assumption due to the fact that the advance demand information may not be very accurate. The demand realization will be earlier or later than the expected date.

Each customer class has a maximum price that they are willing to pay for one unit container. The price is called reservation price. Customers lease one unit of containers only if the posted price is lower than their reservation price. Assume that the cumulative probability distribution function of the reservation price for class-i customers, $F_i(p)$ where $p \in P_i = [\underline{P}^i, \bar{P}^i]$, $\bar{P}^w > \bar{P}^a$ and the price range set P_i is a finite set. $F_i(p)$ is known to the leasing company and $\bar{F}_i(\bar{P}^i) = 1 - F_i(\bar{P}^i) = 0$ refers to the case that no customers arrive when the posted price equals to $\bar{P}^i$. $\bar{P}^{i-1}$ is the highest effective price of class i when it is optimal to accept this class. Considering the current system state and advance demand information, the firm employs a pricing mechanism to determine the admission control and pricing policy. The pricing mechanism is as follows: (1) the leasing company posts unit price for both customer classes depending on the inventory level and accepted ADI demands; (2) customers who accept the posted price arrive, request one unit of containers and accepted customers settle the lease revenue; (3) walk-in customers are served immediately, while ADI customers are served when they realize demands. If the remaining inventory level is not available, the high emergency cost transshipped from other place or leased from other leasing companies η is occurred and $\eta > \bar{P}^w$.

The assumptions of Poisson arrival, exponential distributed lead time and holding time allow us to model the pricing and capacity rationing problem as an infinite-horizon, discounted reward, continuous time Markov decision process (MDP). To simplify the analysis, we uniformize the continuous time system to discrete time system at rate $\Omega = \lambda_w + \lambda_a + C(\nu + \mu) + \alpha$ and rescale time by letting $\Omega = 1$. (*Lippman*, 1975). Let λ_i, μ and ν divide by Ω so that the transformed problem is equivalent to the original problem. Therefore, the MDP in our problem can be characterized by three objects: (1) system state (x, y): x represents the number of containers leased out by both classes, y denotes the number of container booked by ADI customers. The system space is $\boldsymbol{C}^2$, where $\boldsymbol{C} = \{0, 1, \cdots, C\}$; (2) action space $p^i_{x,y} \in P^i$: for each customer class i, the lessor first determines the optimal price $p^i_{x,y}$ from P^i. If $p^i_{x,y} \leq \bar{P}^{i-1}$ provides one unit of containers for walk-in customers and reserves one unit of containers for reserved customers, otherwise customer of class i is rejected; (3) transition probability after events given state (x, y) are given as follows:

$$(x', y') = \begin{cases} (x, y) & \text{if } p^i_{x,y} = \bar{P}^i \text{ with prob. } \lambda_i \\ (x+1, y) & \text{if } p^w_{x,y} \leq \bar{P}^{w-1} \text{ with prob. } \lambda_w \\ (x, y+1) & \text{if } p^a_{x,y} \leq \bar{P}^{a-1} \text{ with prob. } \lambda_a \\ (x+1, y-1) & \text{with prob. } y\nu \\ (x-1, y) & \text{with prob. } \mu x \\ (x, y) & \text{with prob. } \nu(C-y) + \mu(C-x) \end{cases}$$

where 'with prob.' stands for 'with probability'. Based on the above, the finiteness of system states and action spaces implies that there exists an optimal switching stationary pricing and rationing policy (*Puterman*, 1994). Let $g(x, y)$ be the expected discounted profit given the system is now in state (x, y). Let π^* be the optimal pricing policy for both customer classes. The optimal value function $g^*(x, y) \equiv Tg^{\pi^*}(x, y)$

holds for all (x, y) under the value-iteration operator T:

$$Tg = \lambda_w W + \lambda_a A^1 + \nu A^2 + \mu B$$

where

$$Wg(x,y) = \begin{cases} \max_{p^w_{x,y} \in P^w} \{\bar{F}_w(p^w_{x,y})[p^w_{x,y} + g(x+1, y)] + F_w(p)g(x,y)\} & \text{if } x < C \\ g(x,y) & \text{if } x = C \end{cases}$$

$$A^1 g(x,y) = \begin{cases} \max_{p^a_{x,y} \in P^a} \{\bar{F}_a(p^a_{x,y})[p^a_{x,y} + g(x, y+1)] + F_a(p)g(x,y)\} & \text{if } y < C \\ g(x,y) & \text{if } y = C \end{cases}$$

$$A^2 g(x,y) = \begin{cases} yg(x+1, y-1) + (C-y)g(x,y) & \text{if } x < C \\ y[g(x, y-1) - \eta] + (C-y)g(x,y) & \text{if } x = C \end{cases}$$

$$Bg(x,y) = xg(x-1,y) + (C-x)g(x,y)$$

W denotes the lessor's pricing operator for walk-in customers. To illustrate, if there are no available containers, that is $x = C$, walk-in customer is denied at system state (x, y); otherwise if $x < C$, $p^{w*}_{x,y}$ is the optimal posted price for walk-in customer, if $p^{w*}_{x,y} < \bar{P}^w$, the system state moves from (x, y) to $(x + 1, y)$. The operator A^1 is the lessor's pricing operator for ADI customers. A^2 is the advance demand realization operator. When an ADI customer realizes his demand, the lessor provides one unit of containers. B represents the rental duration operator.

3.3 Structure of the Optimal Policy

In this section, the structure of value function and the properties of the optimal policy are characterized. First, the properties of revenue function are discussed. Let b, c, d, e are decreasing numbers and $R(b) = \bar{F}(p)[p + c] + F(p)b$ be general revenue function, $p_b = \arg\max_{p \in P}\{R(b)\}$ and $R^*(b) = R(b|p_b)$.

Lemma 3.1. *If $b + d - 2c < 0$, then*

(i) $R^*(b)$ is increasing in b;

(ii) p_b is nondecreasing in b;

(iii) $R^*(b) + R^*(d) - 2R^*(c) < 0$.

Proof. (i).

$$R^*(c) - R^*(b) \leq R(c|p_c) - R(b|p_c) = c - b + \bar{F}(p_c)(b + d - 2c) < 0$$

(ii). Suppose that $p_c < p_b$, we have $\bar{F}(p_c) > \bar{F}(p_b)$. Since p_c is the maximizer of $R(c)$,

$$\begin{aligned} R^*(c) - R(c|p_b) &= R(c|p_c) - R(c|p_b) \\ =&\bar{F}(p_c)p_c - \bar{F}(p_b)p_b + (d - c)[\bar{F}(p_c) - \bar{F}(p_b)] \geq 0. \end{aligned}$$

And

$$\begin{aligned} 0 \geq R(b|p_c) - R^*(b) &= R(b|p_c) - R(b|p_b) \\ =&\bar{F}(p_c)p_c - \bar{F}(p_b)p_b + (c - b)[\bar{F}(p_c) - \bar{F}(p_b)] \\ >&\bar{F}(p_c)p_c - \bar{F}(p_b)p_b + (d - c)[\bar{F}(p_c) - \bar{F}(p_b)] \end{aligned}$$

A contradiction is obtained. Thus p_b is nondecreasing as b increases.

(iii).

$$
\begin{aligned}
& R^*(b)+R^*(d)-2R^*(c) \\
\leq & R^*(b)-R(c|p_b)+R^*(d)-R(c|p_d) \\
= & \bar{F}(p_b)(c-d)+F(p_b)(b-c)+\bar{F}(p_d^*)(e-d)+F(p_d))(d-c) \\
\leq & \bar{F}(p_b)(c-d)+\bar{F}(p_d)(e-d)+F(p_d)(b+d-2c)<0
\end{aligned}
$$

□

Define $\aleph$ as a set of anti-multimodular functions with respect to M in $\boldsymbol{C}^2$, satisfying the following properties.

$\mathcal{P}.1$	$\triangle_{1,2}g(x,y) \leq 0$	if $x<C$ and $y<C$
$\mathcal{P}.2$	$\triangle_1 g(x+1,y) \leq \triangle_1 g(x,y+1)$	if $x<C-1$ and $y<C$
	$\triangle_2 g(x,y+2) \leq \triangle_2 g(x+1,y)$	if $x<C$ and $y<C-2$
$\mathcal{P}.3$	$\triangle_{1,1}g(x,y) \leq 0$	if $x<C-1$
	$\triangle_{2,2}g(x,y) \leq 0$	if $y<C-1$

Given $g(x,y) \in \aleph$, define a series of switching curves.

$$S_1(y)=\max\{x|\triangle_1 g(x,y)>-\bar{P}^{w-1}\} \tag{3.1}$$

$$S_2(x)=\max\{y|\triangle_2 g(x,y)>-\bar{P}^{r-1}\} \tag{3.2}$$

The definition of switching curves indicates that when $x \leq S_1(y)$, $\triangle_1 g(x,y)+\bar{P}^{w-1}>0$, then it is optimal to accept the walk-in customer; when $y \leq S_2(x)$, $\triangle_2 g(x,y)+\bar{P}^{r-1}>0$, it is optimal to accept the ADI customer.

Lemma 3.2. *Given* $g(x,y) \in \aleph$,

$$S_1(y) - 1 \le S_1(y+1) \le S_1(y), \ \textit{if } 0 \le y < C;$$
$$S_2(x) - 2 \le S_2(x+1) \le S_2(x), \ \textit{if } 0 \le x < C.$$

Proof. When $0 \le y < C$, from the definition of $S_1(y)$ and $\mathcal{P}.2$,

$$-\bar{P}^{w-1} < \triangle_1 g(S_1(y), y) \le \triangle_1 g(S_1(y) - 1, y+1)$$

We have $S_1(y) - 1 \le S_1(y+1)$.

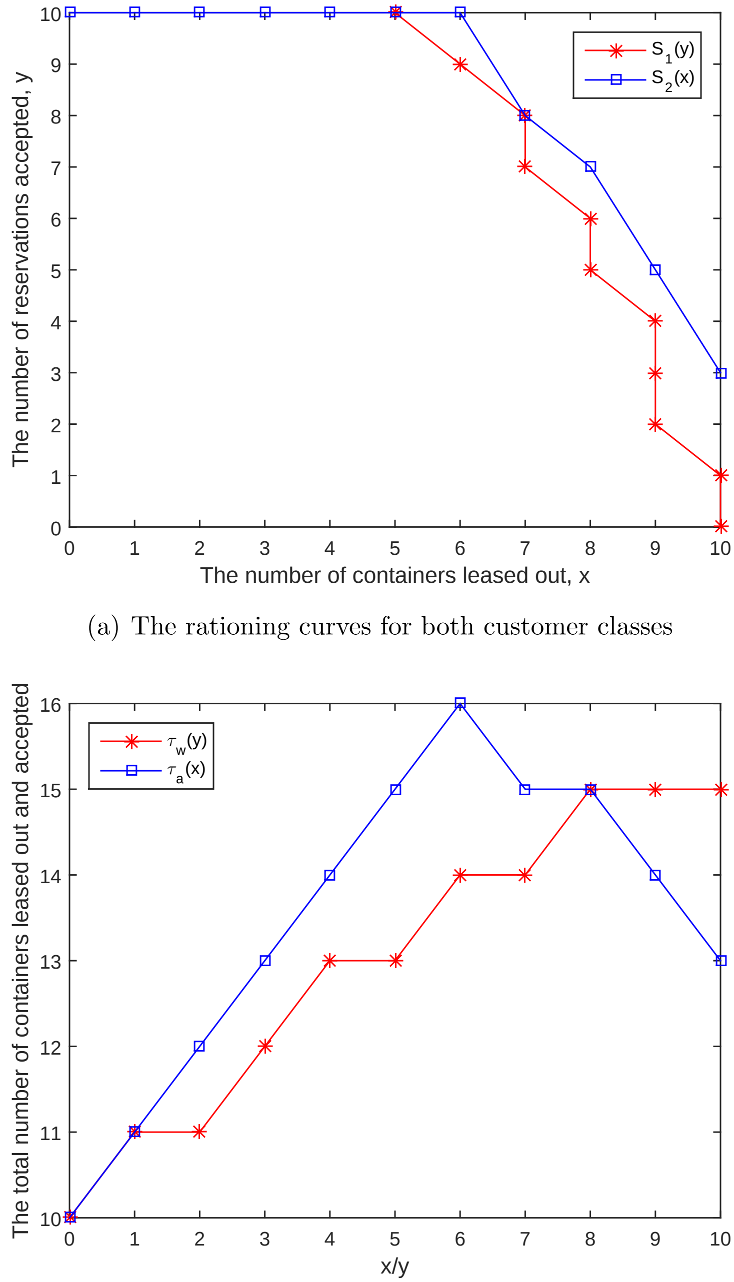
$$-\bar{P}^{w-1} < \triangle_1 g(S_1(y+1), y+1) \le \triangle_1 g(S_1(y+1), y)$$

Then $S_1(y+1) \le S_1(y)$. Similarly, When $0 \le x < C$,

$$-\bar{P}^{r-1} < \triangle_2 g(x, S_2(x)) \le \triangle_2 g(x+1, S_2(x) - 2)$$
$$-\bar{P}^{r-1} < \triangle_2 g(x+1, S_2(x+1)) \le \triangle_2 g(x, S_2(x+1))$$

The inequality $S_2(x) - 2 \le S_2(x+1) \le S_2(x)$ is obtained. $\square$

The Lemma 3.2 proves the existence of monotone rationing curves. $S_1(y)$ nonincreases in y with bounded sensitivity which means that $S_1(y+1)$ has at most unit decrease in $S_1(y)$. The similar property holds for $S_2(x)$. This is also reflect the diagonal dominance in the anti-modularity. As x, y are integer-valued, the bounded sensitivity results in at most unit/two-unit decrease. In Example. 3.1, the parameter setting is $C = 10, \lambda_a = \lambda_w = 2, \mu = 1, \nu = 2, \underline{P}^w = 8, \bar{P}^w = 18, \underline{P}^a = 6.8, \bar{P}^a = 16.8, \alpha = 0.95, \eta = 67.2$. Fig.3.1(a) illustrate the rationing switching curves for both customer classes. The rationing curve of walk-in customers $S_1(y)$ nonincreases in y with at most unit decrease. The rationing curve of ADI customers $S_2(x)$ equals to



(a) The rationing curves for both customer classes

(b) The total number of containers leased out and accepted

Figure 3.1. The structure of the optimal rationing policy

the total capacity when x is relatively small, and then decreases with at most two units when x achieves some critical point (6 in this case). Fig.3.1(b) shows the total number of containers leased out and accepted given x/y value. It is easy to find that when $x = y \leq 7$, $\tau_a(x) \geq \tau_w(y)$ which means that the total number of containers leased out and accepted $\tau_a(x) = x + S_2(x)$ when the coming customer is an ADI customer is greater than $\tau_w(y) = y + S_1(y)$ when the coming customer is a walk-in customer; when $7 < x = y \leq 10$, $\tau_a(x) < \tau_w(y)$.

Intuitively, unit increase in the accepted ADI demand y has limited effect on $S_1(y)$ due to the fact that the leasing company just reduce the number of walk-in customers by one unit and reserved for the unit increment of reserved demand. By contrast, when x is larger than some critical point, unit increase in x results in more than one unit decrease in $S_2(x)$. The reason behind this is that when the number of containers leased out x is relatively small, the benefit of accepting ADI customers is the increase of the lease revenue, when x achieves some point, the high emergency cost to satisfy accepted ADI demands counteracts the lease revenue benefit. So the leasing company cut the rationing threshold by more than unit decrement. Fig. 3.1(b) also reflects the above phenomenon.

Based on Lemma 3.2, the operators W and A^1 can be formulated as follows.

$$Wg(x,y) = \begin{cases} \max_{p^w_{x,y} \in P^w} \{\bar{F}_w(p^w_{x,y})[p^w_{x,y} + g(x+1,y)] + F_w(p)g(x,y)\} & \text{if } x \leq S_1(y) \\ g(x,y) & \text{if } x > S_1(y) \end{cases}$$

$$A^1 g(x,y) = \begin{cases} \max_{p^a_{x,y} \in P^a} \{\bar{F}_a(p^a_{x,y})[p^a_{x,y} + g(x,y+1)] + F_a(p)g(x,y)\} & \text{if } y \leq S_2(x) \\ g(x,y) & \text{if } y > S_2(x) \end{cases}$$

After identifying the structural properties of the optimal pricing and rationing policy, the following proposition shows that the operator T preserves the properties of a function g through dynamic programming recursion if g belongs to $\aleph$.

Proposition 3.3.1. If $g \in \aleph$, then $Tg \in \aleph$.

Proof. $\mathcal{P}$**.1** $\triangle_{1,2}g(x,y) \le 0$

For short, let $r(x,y|p^i_{x,y}) = \bar{F}(p^i_{x,y})(p^i_{x,y} + \triangle_i g(x,y))$.

If $x < S_1(y) - 1$ or $x = S_1(y) - 1 = S_1(y+1) - 1$,

$$\begin{aligned}
\triangle_{1,2}Wg(x,y) &= \triangle_{1,2}g(x,y) + r(x+1,y+1|p^w_{x+1,y+1}) \\
&\quad - r(x+1,y|p^w_{x+1,y}) - r(x,y+1|p^w_{x,y+1}) + r(x,y|p^w_{x,y}) \\
&\le \triangle_{1,2}g(x,y) + r(x+1,y+1|p^w_{x+1,y+1}) - r(x+1,y|p^w_{x+1,y+1}) \\
&\quad - r(x,y+1|p^w_{x,y}) + r(x,y|p^w_{x,y}) \\
&= \bar{F}(p^w_{x+1,y+1})\triangle_{1,2}g(x+1,y) + F(p^w_{x,y})\triangle_{1,2}g(x,y) < 0.
\end{aligned}$$

The first inequality follows that as $p^i_{x,y}$ is the maximizer of $R(x,y) = r(x,y|p^i_{x,y}) + g(x,y)$, $r(x+1,y|p^w_{x+1,y+1}) \le r(x+1,y|p^w_{x+1,y})$ and $r(x,y+1|p^w_{x,y}) \le r(x,y+1|p^w_{x,y+1})$.

If $x = S_1(y) - 1 = S_1(y+1)$, then $x+1 > S_1(y+1)$, we have

$$\begin{aligned}
\triangle_{1,2}Wg(x,y) &= \triangle_{1,2}g(x,y) - r(x+1,y|p^w_{x+1,y}) - r(x,y+1|p^w_{x,y+1}) + r(x,y|p^w_{x,y}) \\
&\le \triangle_{1,2}g(x,y) - r(x+1,y|p^w_{x+1,y}) - r(x,y+1|p^w_{x,y}) + r(x,y|p^w_{x,y}) \\
&= F(p^w_{x,y})\triangle_{1,2}g(x,y) - r(x+1,y|p^w_{x+1,y}) < 0.
\end{aligned}$$

The last inequality derives that $x+1 = S_1(y)$ and $\triangle_1 g(x+1,y) + \bar{P}^{w-1} > 0$, thus $r(x+1,y|p^w_{x+1,y}) > 0$ by setting $p^w_{x+1,y} = \bar{P}^{w-1}$.

If $x = S_1(y) = S_1(y+1)$,

$$\begin{aligned}
\triangle_{1,2}Wg(x,y) &= \triangle_{1,2}g(x,y) - r(x,y+1|p^w_{x,y+1}) + r(x,y|p^w_{x,y}) \\
&\le \triangle_{1,2}g(x,y) - \bar{F}(p^w_{x,y})\triangle_{1,2}g(x,y) \\
&= F(p^w_{x,y})\triangle_{1,2}g(x,y) < 0.
\end{aligned}$$

If $x = S_1(y) = S_1(y+1) + 1$,

$$\begin{aligned}
\triangle_{1,2} Wg(x,y) &= \triangle_{1,2} g(x,y) + r(x,y|p^w_{x,y}) \\
&= \triangle_{1,2} g(x,y) + \triangle_1 g(x,y) + p^w_{x,y} - F(p^w_{x,y})(p^w_{x,y} + \triangle_1 g(x,y)) \\
&= \triangle_1 g(x,y+1) + p^w_{x,y} - F(p^w_{x,y})(p^w_{x,y} + \triangle_1 g(x,y)) < 0.
\end{aligned}$$

Owing to $x = S_1(y)$, $p^w_{x,y} + \triangle_1 g(x,y) > 0$ by setting $p^w_{x,y} = \bar{P}^{w-1}$. Since $x > S_1(y+1)$ and $p^w_{x,y} \le \bar{P}^{w-1}$, we have $\triangle_1 g(x,y+1) + p^w_{x,y} < 0$. , The negativity of $\triangle_{1,2} Wg(x,y)$ is obtained.

If $x > S_1(y)$, $\triangle_{1,2} Wg(x,y) = \triangle_{1,2} R(x,y) < 0$ from Lemma 3.1.

The other operators are proved similarly.

$$\triangle_{1,2} A^1 g(x,y) \le \begin{cases}
F(p^a_{x,y})\triangle_{1,2} g(x,y) + \bar{F}(p^a_{x+1,y+1})\triangle_{1,2} g(x,y+1) < 0 & \\
\qquad\qquad \text{if } y < S_2(x) = S_2(x+1) \text{ or } y < S_2(x+1) < S_2(x) & \\
F(p^a_{x,y})\triangle_{1,2} g(x,y) - r(x,y+1|p^a_{x,y+1}) < 0 & \\
\qquad\qquad\qquad\qquad \text{if } y = S_2(x+1) < S_2(x) & \\
F(p^a_{x,y})\triangle_{1,2} g(x,y) < 0 \qquad \text{if } y = S_2(x) = S_2(x+1) & \\
\triangle_2 g(x+1,y) + p^a_{x,y} - F(p^a_{x,y})(p^w_{x,y} + \triangle_2 g(x,y)) & \\
\qquad\qquad\qquad\qquad \text{if } y = S_2(x) > S_2(x+1) & \\
\triangle_{1,2} g(x,y) < 0 \qquad\qquad \text{if } y > S_2(x) &
\end{cases}$$

$$\triangle_{1,2} A^2 g(x,y) = \begin{cases}
(C-y)\triangle_{1,2} g(x,y) + y\triangle_{1,2} g(x+1,y-1) & \\
\quad +\triangle_1 g(x+1,y) - \triangle_1 g(x,y+1) < 0 & \text{if } x < C-1 \\
(C-y-1)\triangle_{1,2} g(x,y) - \triangle_1 g(x,y) - \eta < 0 & \text{if } x = C-1
\end{cases}$$

$$\triangle_{1,2}Bg(x,y) = x\triangle_{1,2}g(x-1,y) + (C-x-1)\triangle_{1,2}g(x,y) < 0$$

$\mathcal{P}.2$ $\triangle_{1,-2}g(x,y) = \triangle_1 g(x+1,y) - \triangle_1 g(x,y+1) \le 0$

$$\triangle_{1,-2}Wg(x,y) \le \begin{cases} F(p^w_{x,y+1})\triangle_{1,-2}g(x,y) + \bar{F}(p^w_{x+2,y})\triangle_{1,-2}g(x+1,y) < 0 & \\ \qquad\qquad\qquad\qquad \text{if } x < S_1(y) - 1 & \\ F(p^w_{x,y+1})\triangle_{1,-2}g(x,y) - r(x+1,y+1|p^w_{x+1,y+1}) < 0 & \\ \qquad\qquad\qquad\qquad \text{if } x = S_1(y) - 1 = S_1(y+1) - 1 & \\ F(p^w_{x,y})\triangle_{1,-2}g(x,y) < 0 \quad \text{if } x = S_1(y) - 1 = S_1(y+1) & \\ \triangle_{1,-2}g(x,y) < 0 \qquad\quad \text{if } x \ge S_1(y) & \end{cases}$$

$$\triangle_{1,-2}A^1g(x,y) \le \begin{cases} F(p^a_{x,y+1})\triangle_{1,-2}g(x,y) + \bar{F}(p^a_{x+2,y})\triangle_{1,-2}g(x+1,y) < 0 \\ \qquad \text{if } y < S_2(x+2) = S_2(x+1) \le S_2(x) \\ \qquad \text{or } y \le S_2(x+2) < S_2(x+1) \le S_2(x) \\ F(p^a_{x,y+1})\triangle_{1,-2}g(x,y) + \bar{F}(p^a_{x+2,y})\triangle_{1,2}g(x+1,y) < 0 \\ \qquad \text{if } y = S_2(x+2) = S_2(x+1) < S_2(x) \\ \triangle_{1,-2}g(x,y) + F(p^a_{x+2,y})\triangle_{1,2}g(x+1,y) < 0 \\ \qquad \text{if } y = S_2(x) = S_2(x+1) = S_2(x+2) \\ \triangle_{1,-2}g(x,y) - r(x+1,y|p^a_{x+1,y}) \\ \qquad \text{if } y = S_2(x) = S_2(x+1) > S_2(x+2) \\ \triangle_{1,-2}g(x,y) < 0 \\ \qquad \text{if } y > S_2(x) = S_2(x+1) \ge S_2(x+2) \\ \qquad \text{or } y \ge S_2(x) > S_2(x+1) \ge S_2(x+2) \end{cases}$$

$$\triangle_{1,-2}A^2g(x,y) = \begin{cases} (C-y)\triangle_{1,-2}g(x,y) + y\triangle_{1,-2}g(x+1,y-1) \\ \quad +\triangle_{-1,2}g(x,y) < 0 \qquad \text{if } x < C-2 \\ (C-y-1)\triangle_{1,-2}g(x,y) - y(\triangle_1 g(x+1,y)+\eta) < 0 \\ \qquad\qquad \text{if } x = C-2 \end{cases}$$

$$\triangle_{1,-2}Bg(x,y) \leq x\triangle_{1,-2}g(x-1,y) + (C-x-2)\triangle_{1,-2}g(x,y) < 0$$

$\triangle_2 g(x,y+2) \leq \triangle_2 g(x+1,y)$ can be proved similarly.

$\mathcal{P}.3$ $\triangle_{1,1}g(x,y) \leq 0$

$$\triangle_{1,1}Wg(x,y) \leq \begin{cases} \bar{F}(p^w_{x+2,y})\triangle_{1,1}g(x+1,y) + F(p^w_{x,y})\triangle_{1,1}g(x,y) < 0 & \text{if } x < S_1(y)-1 \\ F(p^w_{x,y})\triangle_{1,1}g(x,y) - r(x+1,y|p^w_{x+1,y}) < 0 & \text{if } x = S_1(y)-1 \\ F(p^w_{x,y})\triangle_{1,1}g(x,y) < 0 & \text{if } x = S_1(y) \\ \triangle_{1,1}g(x,y) < 0 & \text{if } x > S_1(y) \end{cases}$$

$$\triangle_{1,1}A^2g(x,y) = \begin{cases} y\triangle_{1,1}g(x+1,y-1) + (c-y)\triangle_{1,1}g(x,y) < 0 & \text{if } x < c-2 \\ (c-y)\triangle_{1,1}g(x,y) - y(\triangle_1 g(x+1,y-1)+\eta) < 0 & \text{if } x = c-2 \end{cases}$$

$$\triangle_{1,1}Bg(x,y) = x\triangle_{1,1}g(x-1,y) + (c-x-2)\triangle_{1,1}g(x,y) < 0$$

$\mathcal{P}.3\ \triangle_{2,2}g(x,y) \le 0$

$$\triangle_{2,2}A^1 g(x,y) \le \begin{cases} F(p^a_{x,y})\triangle_{2,2}g(x,y) + \bar{F}(p^a_{x,y+2})\triangle_{2,2}g(x,y+1) < 0 & \text{if } y < S_2(x)-1 \\ F(p^a_{x,y})\triangle_{2,2}g(x,y) + r(x,y+1|p^a_{x,y+1}) < 0 & \text{if } y = S_2(x)-1 \\ F(p^a_{x,y})\triangle_{2,2}g(x,y) < 0 & \text{if } y = S_2(x) \\ \triangle_{2,2}g(x,y) < 0 & \text{if } y > S_2(x) \end{cases}$$

$$\triangle_{2,2}A^2 g(x,y) = \begin{cases} y\triangle_{2,2}g(x+1,y-1) + (c-y-2)\triangle_{2,2}g(x,y) & \\ \qquad +2\triangle_{1,2}g(x,y) < 0 & \text{if } x < c \\ y(\triangle_{2,2}g(x,y) - \eta) + (c-y-2)\triangle_{2,2}g(x,y) < 0 & \text{if } x = c \end{cases}$$

$$\triangle_{2,2}Bg(x,y) = x\triangle_{2,2}g(x-1,y) + (c-x)\triangle_{2,2}g(x,y) < 0$$

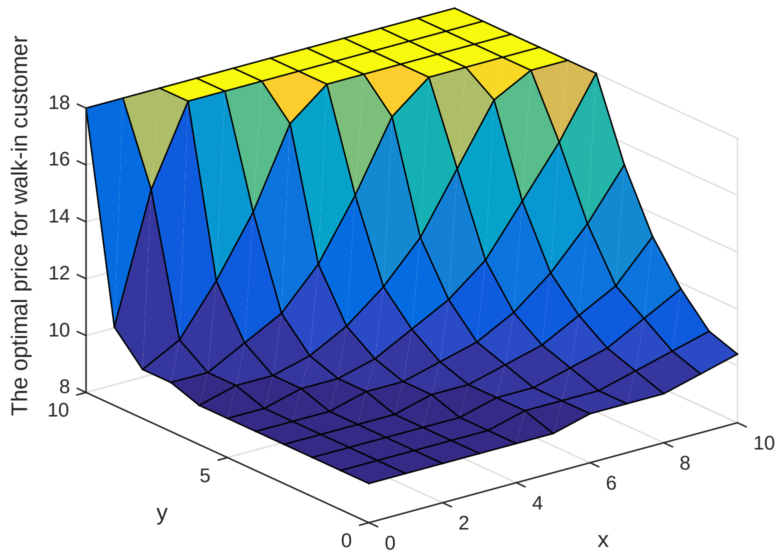
$\square$

Theorem 3.1. *The optimal value function $g^*(x,y)$ belongs to $\aleph$. There exists an optimal switching stationary policy satisfying the following properties.*

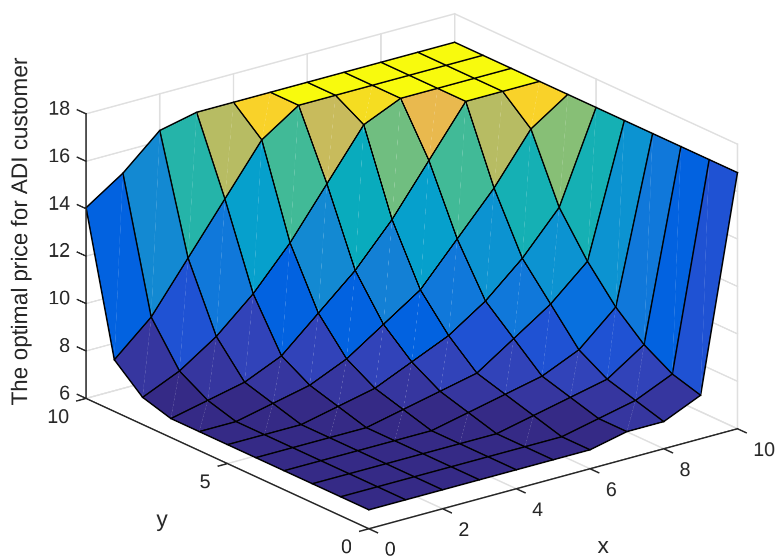
1. *For walk-in customers, if $x \le S_1(y)$, it is optimal to accept the walk-in customer; otherwise, deny the walk-in customer; For ADI customers, the leasing company accepts the reserved demand as long as $y \le S_2(x)$.*

2. *The walk-in customers' rationing threshold $S_1(y)$ nonincreases in y with bounded sensitivity $S_1(y) - 1 \le S_1(y+1) \le S_1(y)$. The ADI customers' rationing threshold $S_2(x)$ nonincreases in x with bounded sensitivity $S_2(x) - 2 \le S_2(x+1) \le S_2(x)$.*

3. *When it is optimal to accept the class i customer, the optimal price $p^{i*}_{x,y}$ is nondecreasing with x and y.*



(a) The optimal price for walk-in customer



(b) The optimal price for ADI customer

Figure 3.2. The optimal pricing policy

To illustrate Theorem 3.1, a simple example is used to illustrate the properties of the optimal policy. Under the same parameter setting as in Example. 3.1, Figure 3.2(a) and 3.2(b) display the optimal price for walk-in customer and ADI customer, respectively. It can be seen that when y is less than 5 and x approaches the capacity, the slope of optimal ADI price is greater than the slope of the optimal walk-in price. That could be explained by the fact that when the total number of container is about to leased out, the leasing company can still accept the walk-in customers with relatively fair price, but the firm posts high price for ADI customers to avoid the high emergency cost. When y is near capacity, the leasing company denies more walk-in customers to fulfill the reserved demands.

From Theorem 3.1, it is easy to extend the optimal policy for stochastic rental system without advance demand information.

Corollary 3.1. *The optimal value function g^* belongs to $\aleph$. There exits a optimal switching stationary policy satisfying the following properties.*

1. *The rationing threshold is $\bar{S}_1(x) = \max\{x | \triangle_1 g(x, y) > \bar{P}^{w-1}\}$. The leasing company provides the container as long as $x \leq \bar{S}_1(x)$.*

2. *The rationing threshold $\bar{S}_1(x)$ nonincreases with x with $\bar{S}_1(x) - 1 \leq \bar{S}_1(x+1) \leq \bar{S}_1(x)$.*

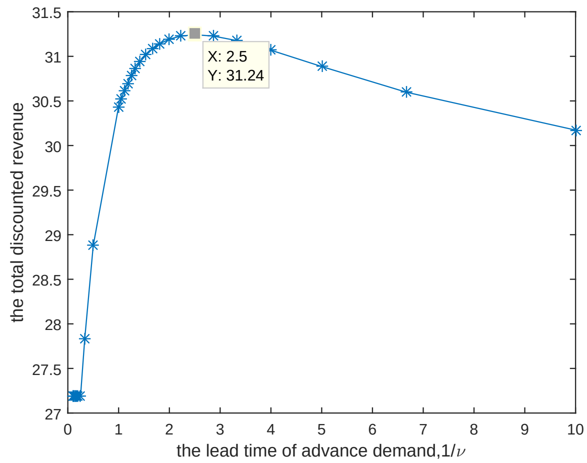
3. *When it is optimal to accept the customer, the optimal price p_x^{i*} is nondecreasing with x.*

3.4 Numerical Study

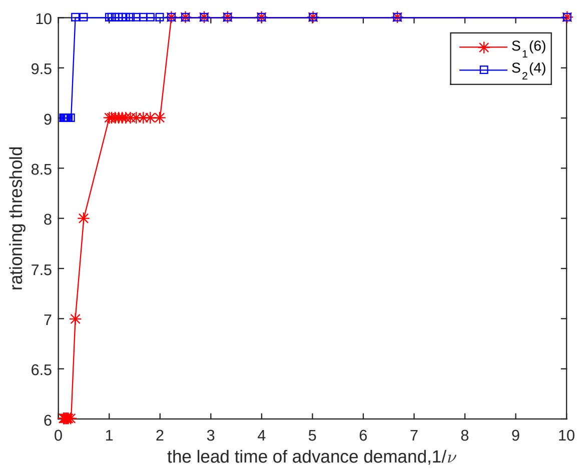
In this section, numerical experiments are conducted to explore the influence of ADI on the optimal rationing and pricing policy, which provides insights for the leasing company to maximize its revenue. The numerical results are obtained from value-iteration dynamic programming algorithm. The algorithm is terminated when the difference of total discounted revenue between iterations is less than 10^{-7}.

Effect of lead time on the optimal policy. In this case, the mean of demand lead time $1/\nu_i$ ranges from 0.1 to 10 and other parameters setting remains the same as in Example. 3.1. We take the system state (4,6) as example to discuss the effect of lead time on the optimal rationing and pricing policy. Fig. 3.3(a) displays the total discounted revenue at system state (4,6). It can be seen that the total discounted revenue increases as the mean of lead time grows from 0.1 to 2.5, when the mean of lead time is greater than 2.5, the total revenue begins to decrease. As the mean of lead time increases, the lease revenue benefit is offset by the emergency lease cost. This indicates that the leasing company should set a maximum allowable lead time. Fig. 3.3(b) shows that the rationing curves $S_1(6)$ for walk-in customers and $S_2(4)$ for ADI customers increases with the mean of lead time. Fig. 3.3(c) exhibits that the optimal prices have a declining trend as the lead time becomes longer. The optimal walk-in price has a greater drop compared with the change of optimal ADI price. The ascending of lead time gives the leasing company more time and flexibility to manage its capacity. Especially when $1/v$ is greater than the duration $\mu = 1$, the firm could give priority to walk-in customers and then fulfill the reserved demand later. Thus, $S_1(y)$ is more sensitive to the change of $1/\nu$ compare to $S_2(x)$, the reduction of $p^{w*}_{4,6}$ is larger than the contraction of $p^{a*}_{4,6}$.

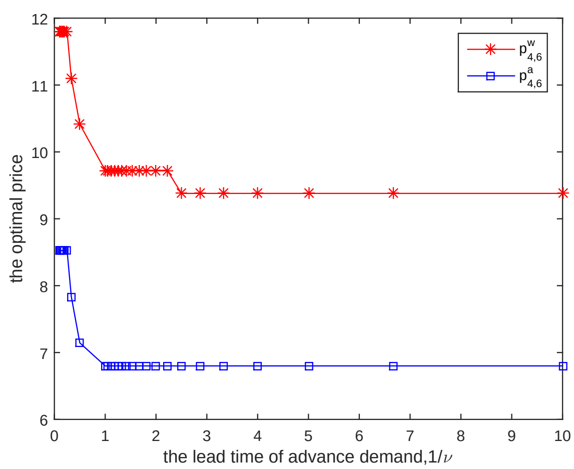
Effect of Emergency lease cost on the optimal policy. In this case, the emergency lease cost η changes from $\bar{P}^a$ to $7\bar{P}^a$, the other parameter setting remains the same as in Example. 3.1. In Fig. 3.4(a), when $s = 1$, $S_1(y)$ equals to the capacity as the emergency lease cost is small. When η grows, $S_1(y)$ becomes to more sensitive to y and unit decrease sensitivity is still kept. In Fig. 3.4(b), as η increases, $S_1(y)$ is more sensitive to x and the curve is more steep with at most two-unit decrease sensitivity. Fig. 3.4(c) shows that the optimal walk-in price and ADI price shares the



(a) The total discounted revenue at (4,6)

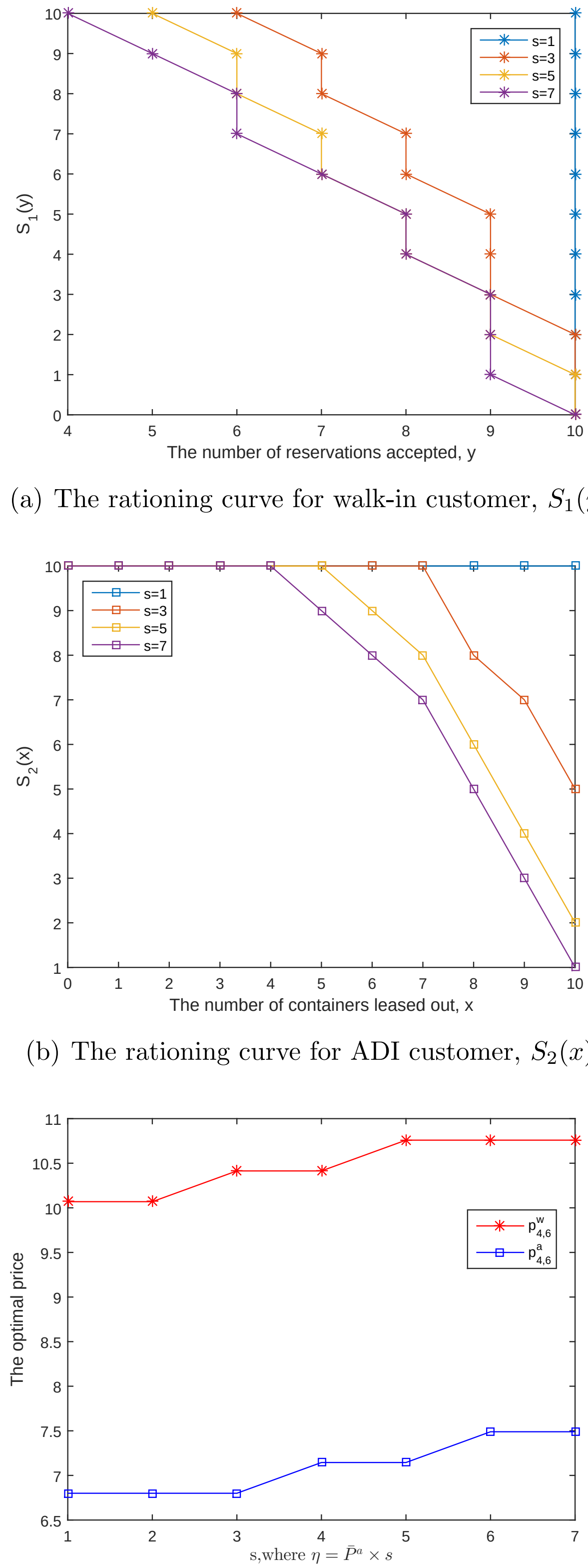


(b) The rationing curves



(c) The optimal price for both customer classes

Figure 3.3. The effect of lead time on the total revenue, optimal rationing and pricing policy



(a) The rationing curve for walk-in customer, $S_1(y)$

(b) The rationing curve for ADI customer, $S_2(x)$

(c) The optimal price for both classes at state (4,6)

Figure 3.4. The effect of emergency lease cost on the optimal rationing and pricing policy

similar increasing pattern when η becomes greater. In a word, the high emergency cost results in the decreasing rationing threshold and increasing optimal prices.

3.5 Summary

In this part, we consider a dynamic pricing problem of a container leasing company with reservations and unit capacity request. The problem is modelled as a continuous-time Markov decision process. Using value iteration, the properties of the optimal allocation and pricing policy are derived. We show that the objective function is an anti-multimodular function and there exists a state-dependent rationing policy with bounded sensitivity such that the walk-in customers' rationing threshold nonincreases in the number of ADI reservations and the ADI customers' rationing threshold nonincreases in the leased amount. When it is optimal to accept this customer class, the optimal posted price is nondecreasing with the inventory level and the number of advance demands. Numerical experiments are conducted to study the effect of reservations on the optimal policy.

CHAPTER 4

Dynamic Pricing with Reservation and Multiple-Unit Demand

This chapter focuses on a container leasing company dealing with two customer types, reserved customers and walk-in customers. A reserved customer books the containers before the pickup date and pays the rent at the booking. A walk-in customer requests an immediate rental service. We first discuss the case with same lease duration and the optimal prices for two customers are nonincreasing in the system state. The optimal walk-in demand is most sensitive to reserved demand of current period and the following period compared with reserved demands and rental amount in other periods. The optimal reserved demand is most sensitive to the latest booking. Next we propose an effective heuristic to derive the myopic pricing policy to the dynamic pricing problem. Finally, the optimal policies are partially characterized for the case with different lease durations. The distinct lease durations cause the movement of the vector of reserved in the system state. The optimal pricing policy still nonincreases in the system state with monotone bounded sensitivity.

4.1 Literature Review

There are three streams of literature related to this work: capacity control and pricing, advance selling in restaurant industry and study about multimodularity.

The literature about capacity control and pricing for stochastic rental system has not received so much academic attention. *Savin et al.* (2005) investigate a capacity allocation problem in rental system with two customer classes. They characterize the switching-curve raioning policy and conditions for preferred classes. An aggregate threshold policy is developed for the fluid approximation model and the effect of capacity rationing on the optimal fleet size is analyzed. *Gans and Savin* (2007) extend their model by incorporating the price interaction between walk-in customer and rental company. They model the pricing and capacity rationing problem for a rental system as a Markov decision process and demonstrate that the optimal policy parameters are monotone with the system parameters. They further obtain the condition of preferred customer class. *Papier and Thonemann* (2010) study a stochastic rental system with two types of customers–advance demand information(ADI) customers and walk-in customers and show that the optimal admission policy is a threshold type policy. Due to the computational intractability, a close-to-optimal ADI policy is proposed and performs better than the policy that ignores ADI. (*Jain et al.*, 2015) discuss a rental model with limited inventory, decreasing demand and two customer classes differentiated by return behavior and penalty cost. They adopt the optimal control theory to analyze the optimal allocation policy. The allocation policy is a priority policy but the priority class changes at different points in the horizon. The models in the above study fail to capture the time sensitivity of demand which is a main feature for the container leasing system.

The second stream of work explores how to balance between reserved sales and walk-in sales. This topic is well discussed in restaurant reservation. *Alexandrov and*

Lariviere (2012) investigate a restaurant selling its service to strategic customer (a single customer type) by advance reservation or walk-ins. The authors define the conditions of offering reservations and various policies to reduce the chance of no-show. Competition makes reservation more attractive to customers. *Cil and Lariviere* (2013) consider a service provider with a fixed capacity facing two customer types (reserved customers and walk-in customers) and clarify how much of the limited capacity available to reserved customers in a single sale period. When reserved demand is more profitable but uncertain while the number of walk-in customers is fixed, it may be optimal to set aside a capacity for walk-in customers. When walk-in demand is more profitable but uncertain, it is possible that the firm allocates all the capacity for reserved customers. The studies in this stream of literature are usually based on a single-sale period and a multiple-period reservation pricing problem is discussed in this part.

The last stream of the related literature is about the multimodularity. *Hajek* (1985) first proposes the definition of multimodularity on integer space and selects the deterministic integer sequence to minimize the long-term average queue size through proving that the mean queue size is a limit of multimodular functions of the sequence. *Altman et al.* (2000) study the properties of multimodular functions and the relation between the multimodularity and convexity of its linear interpolation. Based on these properties, the authors establish the lower bounds which are achieved by regular sequences for the expected average cost problem. Finally, they apply this theory in admission control to a G/G/1 queue and D/D/1 queue with fixed batch arrivals. *Murota* (2005) relates multimodular functions and $L^\natural$-convex function through a unimodular coordinate transformation and show that a discrete separation theorem holds for multimodular functions. *Zhuang and Li* (2012) present a unified framework to analyze monotone optimal control through multimodularity for a class of Markov decision processes. They demonstrate that each system in this class could be a substitution or complement and present a generic proof of the structural properties for both type systems. *Li and Yu* (2014) define multimodularity in real space and develop basic properties of multimodularity. They apply multimodularity into three stochastic dynamic inventory problems where the system state and decision variables are economic substitutes and derive monotone optimal policies with bounded sensitivity. They also prove that multimodular function and L♮-convex function can be related through unimodular coordinate transformations. With the theoretical development of multimodularity, to our knowledge, the computational attempt for multimodularity is scarce. *Chen et al.* (2014) propose an effective heuristic policy for the L♮-concave objective function in the joint pricing and inventory control model. They develop analytical bounds on the optimal order-up-to levels and propose an approximate dynamic programming scheme through extending L♮-concave function to a multidimensional domain from a finite number of points. The multimodularity will be employed in the stochastic rental system and further analyze the sensitivity of system state in the optimal pricing practice.

4.2 Preliminary

This section introduces the definition and basic properties of anti-multimodularity in real space. Let $V \in \Re^n$ be a polyhedron that forms a general lattice.

Definition 4.1. A function $f : V \to \Re$ is supermodular on V if

$$f(u) + f(v) \leq f(u \vee v) + f(u \wedge v)$$

for all $u, v \in V$, where $\vee$ and $\wedge$ refer to the componenwise maximum and minimum.

The concept of multimodularity is first proposed by *Hajek* (1985), being an efficient tool for queuing network problems (*Altman et al.* 2000, *Hordijk and Van der*

Laan 2005). The multimodularity is traditionally defined on integer variables (*Hajek* 1985, *Murota* 2005). In this part we follow *Li and Yu* (2014) to define the multimodularity in the real space. Let W be a polyhedron, $W = \{w \in \Re^n | a_i v \geq b_i, i = 1, 2 \cdots, m\}$, satisfying the property

Property (P1). The nonzero components of each n-dimensional vector a_i are consecutive 1s or -1s.

Let $W' \in \Re$ a polyhedron satisfying (P1).

Definition 4.2. (*Li and Yu* 2014) A function $f : W \to \Re$ is anti-multimodular (multimodular) if the function $\psi(\mathbf{x}, y) = f(x_1 - y, x_2 - x_1, \cdots, x_n - x_{n-1})$ is supermodular (submodular) on $S = \{(\mathbf{x}, y) \in \Re^n \times \Re | y \in W', (x_1 - y, x_2 - x_1, \cdots, x_n - x_{n-1}) \in W\}$.

Given that W and W' satisfy (P1) and the constraints defined on S involves only two variables with opposite signs, the domain of the function $\psi(x, y)$, forms a lattice according to *Topkis* (1998) Example 2.2.7(b). Anti-multimodularity implies decreasing difference, component concavity, joint concavity and diagonal dominance. An anti-multimodular function with decreasing difference further indicates the variables are economic substitutes, which means having one more unit of one variable results in the decreasing of another variable.

The next lemma describes the properties of anti-multimodularity presented in *Li and Yu* (2014).

Lemma 4.1. *(a) If $f(v)$ is anti-multimodular and α is a positive real number, then $\alpha g(v)$ is also anti-multimodular.*

(b) If $g(v)$ and $f(v)$ is anti-multimodular, then $f(v) + g(v)$ is anti-multimodular.

(c) If $g(v, d)$ is anti-multimodular in v for any given d and D is a random variable, then $\mathbb{E}g(v, D)$ is anti-multimodular in v.

(d) If $g(v)$ is anti-multimodular, then $\tilde{g}(\boldsymbol{v}_{1:i-1}, \boldsymbol{w}, \boldsymbol{v}_{i+1:n}) = g(v_1, \cdots, v_{i-1}, w_1 + w_2 + \cdots + w_m, v_{i+1}, \cdots, v_n)$ is anti-multimodular in $(v_1, \cdots, v_{i-1}, w_1, w_2, \cdots, w_m, v_{i+1}, \cdots, v_n)$.

(e) If $g(v)$ is anti-multimodular, $g(v_n, v_{n-1}, \ldots, v_1)$ is also anti-multimodular in v.

(f) If $U = \{(v, w) | v \in S \subseteq \Re^n, w \in A(v) \subseteq \Re^m\}$ is a polyhedron satisfying (P1), $g(v, w)$ is anti-multimodular in (v, w) on U, then $f(v) = \max_{v \in A(v)}\{g(v, w)\}$ is anti-multimodular in v on S.

Proof. Part (a), (b), (c) are directly follow from lemma 2.6.1 and Corollary 2.6.2 in *Topkis* (1998).

Part (d) It suffices to show that the supermodularity of

$$
\begin{aligned}
&\psi(v_1, \cdots, v_{i-1}, w_1, w_2, \cdots, w_m, v_{i+1}, \cdots, v_n, y) \\
=&\tilde{g}(v_1 - y, v_2 - v_1, \cdots, w_1 - v_{i-1}, \cdots, w_m - w_{m-1}, v_{i+1} - w_m, \cdots, v_n - v_{n-1}) \\
=&g(v_1 - y, v_2 - v_1, \cdots, v_{i-1} - v_{i-2}, w_m - v_{i-1}, v_{i+1} - w_m, \cdots, v_n - v_{n-1})
\end{aligned}
$$

Since $g(v)$ is anti-multimodular, $g(v_1 - y, v_2 - v_1, \cdots, v_{i-1} - v_{i-2}, w_m - v_{i-1}, v_{i+1} - w_m, \cdots, v_n - v_{n-1})$ is supermodular in $(v_1, v_2, \cdots, v_{i-1}, w_m, v_{i+1}, \cdots, v_n, y)$, further supermodular in $(v_1, \cdots, v_{i-1}, w_1, w_2, \cdots, w_m, v_{i+1}, \cdots, v_n, y)$. The result holds.

Part (e) and (f) are directly follows Theorem 1 in *Li and Yu* (2014). □

Lemma 4.2. *Assume that $g(\boldsymbol{v}, \xi, \boldsymbol{w})$ is anti-multimodular on $U \in \Re^n \times \Re \times \Re^m$, where U is a polyhedron satisfying (P1) and $\boldsymbol{v} = (v_1, v_2, \cdots, v_n)$, $\boldsymbol{w} = (w_1, w_2, \cdots, w_m)$. $\xi^*(\boldsymbol{v}, \boldsymbol{w})$ is the smallest value of ξ that maximizes $g(\boldsymbol{v}, \xi, \boldsymbol{w})$. Then $\xi^*(\boldsymbol{v}, \boldsymbol{w})$ is non-increasing in $(\boldsymbol{v}, \boldsymbol{w})$, and satisfies*

$$
-1 \leq \triangle_{v_n}\xi^* \leq \triangle_{v_{n-1}}\xi^* \leq \cdots \leq \triangle_{v_1}\xi^* \leq 0
$$

$$
-1 \leq \triangle_{w_1}\xi^* \leq \triangle_{w_2}\xi^* \leq \cdots \leq \triangle_{w_m}\xi^* \leq 0
$$

Proof. According to the definition of anti-multimodularity, $g(\boldsymbol{v}, \xi, \boldsymbol{w})$ is anti-multimodular, we have

$$\psi(\boldsymbol{v}, \xi, \boldsymbol{w}, y) = g(v_1 - y, v_2 - v_1, \cdots, \xi - v_n, w_1 - \xi, \cdots, w_m - w_{m-1})$$

is a supermodular function on U. Let $z = (v_1 + y, v_1 + v_2 + y, \cdots, \sum_{k=1}^{n} v_k + y, \sum_{k=1}^{n} v_k + y, \sum_{k=1}^{n} v_k + y + w_1, \cdots, \sum_{k=1}^{n} v_k + y + \sum_{k=1}^{m} w_k, y)$. Then $g(\boldsymbol{v}, \xi, \boldsymbol{w}) = \psi(z + \xi \sum_{k=n+1}^{n+m+1} e_k)$.

For $0 < i \leq n$, $\xi^*_{v_i} = \xi^*(\boldsymbol{v} + \boldsymbol{\delta e_i}, \boldsymbol{w})$, we first prove $\xi^*_{v_i} \leq \xi^*_{v_{i-1}}$. For any $\delta > 0$ and $\xi < \xi^*_{v_i}$,

$$\begin{aligned}
&g(\boldsymbol{v} + \boldsymbol{\delta e_{i-1}}, \xi, \boldsymbol{w}) - g(\boldsymbol{v} + \boldsymbol{\delta e_{i-1}}, \xi^*_{v_i}, \boldsymbol{w}) \\
=&\psi(z + \delta \sum_{k=i-1}^{n+m+1} e_k + \xi \sum_{k=n+1}^{n+m+1} e_k) - \psi(z + \delta \sum_{k=i-1}^{n+m+1} e_k + \xi^*_{v_i} \sum_{k=n+1}^{n+m+1} e_k) \\
\leq&\psi(z + \delta \sum_{k=i}^{n+m+1} e_k + \xi \sum_{k=n+1}^{n+m+1} e_k) - \psi(z + \delta \sum_{k=i}^{n+m+1} e_k + \xi^*_{v_i} \sum_{k=n+1}^{n+m+1} e_k) \\
=&g(\boldsymbol{v} + \boldsymbol{\delta e_i}, \xi, \boldsymbol{w}) - g(\boldsymbol{v} + \boldsymbol{\delta e_i}, \xi^*_{v_i}, \boldsymbol{w}) \\
<&0
\end{aligned}$$

The first inequality follows the supermodularity of $\psi(\boldsymbol{v}, \xi, \boldsymbol{w}, y)$ and the last inequality derives from the fact that $\xi^*_{v_i}$ is the smallest value to maximize $g(\boldsymbol{v} + \boldsymbol{\delta e_i}, \xi, \boldsymbol{w})$ and the assumption $\xi < \xi^*_{v_i}$. Thus, $g(\boldsymbol{v} + \boldsymbol{\delta e_{i-1}}, \xi, \boldsymbol{w}) < g(\boldsymbol{v} + \boldsymbol{\delta e_{i-1}}, \xi^*_{v_i}, \boldsymbol{w})$, ξ could not optimal for $g(\boldsymbol{v} + \boldsymbol{\delta e_{i-1}}, \xi, \boldsymbol{w})$. Therefore, $\xi^*_{v_{i-1}} \geq \xi^*_{v_i}$.

Next, we prove $\xi^*(\boldsymbol{v},\boldsymbol{w}) - \delta \le \xi^*_{v_n}$. For any $\delta > 0$ and $\xi < \xi^*(\boldsymbol{v},\boldsymbol{w}) - \delta$,

$$
\begin{aligned}
&g(\boldsymbol{v}+\boldsymbol{\delta e_n},\xi,\boldsymbol{w}) - g(\boldsymbol{v}+\boldsymbol{\delta e_n},\xi^*(\boldsymbol{v},\boldsymbol{w})-\delta,\boldsymbol{w})\\
=&\psi(z+\delta\sum_{k=n}^{n+m+1} e_k + \xi\sum_{k=n+1}^{n+m+1} e_k) - \psi(z+\delta\sum_{k=n}^{n+m+1} e_k + (\xi^*(\boldsymbol{v},\boldsymbol{w})-\delta)\sum_{k=n+1}^{n+m+1} e_k)\\
\le&\psi(z+(\delta+\xi)\sum_{k=n+1}^{n+m+1} e_k) - \psi(z+\xi^*(\boldsymbol{v},\boldsymbol{w})\sum_{k=n+1}^{n+m+1} e_k)\\
=&g(\boldsymbol{v},\xi+\delta,\boldsymbol{w}) - g(\boldsymbol{v},\xi^*(\boldsymbol{v},\boldsymbol{w}),\boldsymbol{w})\\
<&0
\end{aligned}
$$

The first inequality follows the supermodularity of $\psi(\boldsymbol{v},\xi,\boldsymbol{w},y)$ and the last inequality derives from the definition of $\xi^*(\boldsymbol{v},\boldsymbol{w})$ and the assumption $\xi < \xi^*(\boldsymbol{v},\boldsymbol{w}) - \delta$. Thus, ξ could not optimal for $g(\boldsymbol{v}+\boldsymbol{\delta e_n},\xi,\boldsymbol{w})$. Therefore, $\xi^*(\boldsymbol{v},\boldsymbol{w}) - \delta \le \xi^*_{v_n}$.

For $1 \le j \le m$, $\xi^*_{w_j} = \xi^*(\boldsymbol{v},\boldsymbol{w}-\boldsymbol{\delta e_{n+j+1}})$, we first prove $\xi^*_{w_j} \le \xi^*_{w_{j+1}}$. For any $\delta > 0$ and $\xi < \xi^*_{w_j}$,

$$
\begin{aligned}
&g(\boldsymbol{v},\xi,\boldsymbol{w}-\boldsymbol{\delta e_{n+j+2}}) - g(\boldsymbol{v},\xi^*_{w_j},\boldsymbol{w}-\boldsymbol{\delta e_{n+j+2}})\\
=&\psi(z-\delta\sum_{k=n+j+2}^{n+m+1} e_k + \xi\sum_{k=n+1}^{n+m+1} e_k) - \psi(z-\delta\sum_{k=n+j+2}^{n+m+1} e_k + \xi^*_{w_j}\sum_{k=n+1}^{n+m+1} e_k)\\
\le&\psi(z-\delta\sum_{k=n+j+1}^{n+m+1} e_k + \xi\sum_{k=n+1}^{n+m+1} e_k) - \psi(z-\delta\sum_{k=n+j+1}^{n+m+1} e_k + \xi^*_{w_j}\sum_{k=n+1}^{n+m+1} e_k)\\
=&g(\boldsymbol{v},\xi,\boldsymbol{w}-\boldsymbol{\delta e_{n+j+1}}) - g(\boldsymbol{v},\xi^*_{w_j},\boldsymbol{w}-\boldsymbol{\delta e_{n+j+1}})\\
<&0
\end{aligned}
$$

The first inequality follows the supermodularity of $\psi(\boldsymbol{v},\xi,\boldsymbol{w},y)$ and the last inequality derives from the fact that $\xi^*_{w_j}$ is the smallest value to maximize $g(\boldsymbol{v},\xi,\boldsymbol{w}-\boldsymbol{\delta e_{n+j+1}})$ and the assumption $\xi < \xi^*_{w_j}$. Thus, ξ could not optimal for $g(\boldsymbol{v},\xi,\boldsymbol{w}-\boldsymbol{\delta e_{n+j+2}})$. Therefore, $\xi^*_{w_j} \le \xi^*_{w_{j+1}}$.

Last, we prove $\xi^*(\boldsymbol{v}, \boldsymbol{w}) + \delta \leq \xi^*_{w_1}$. For any $\delta > 0$ and $\xi < \xi^*(\boldsymbol{v}, \boldsymbol{w}) + \delta$,

$$
\begin{aligned}
& g(\boldsymbol{v}, \xi, \boldsymbol{w} - \boldsymbol{\delta e_{n+2}}) - g(\boldsymbol{v}, \xi^*(\boldsymbol{v}, \boldsymbol{w}) + \delta, \boldsymbol{w} - \boldsymbol{\delta e_{n+2}}) \\
=& \psi(z - \delta \sum_{k=n+2}^{n+m+1} e_k + \xi \sum_{k=n+1}^{n+m+1} e_k) - \psi(z - \delta \sum_{k=n+2}^{n+m+1} e_k + (\xi^*(\boldsymbol{v}, \boldsymbol{w}) + \delta) \sum_{k=n+1}^{n+m+1} e_k) \\
\leq& \psi(z + (\xi - \delta) \sum_{k=n+1}^{n+m+1} e_k) - \psi(z + \xi^*(\boldsymbol{v}, \boldsymbol{w}) \sum_{k=n+1}^{n+m+1} e_k) \\
=& g(\boldsymbol{v}, \xi - \delta, \boldsymbol{w}) - g(\boldsymbol{v}, \xi^*(\boldsymbol{v}, \boldsymbol{w}), \boldsymbol{w}) \\
<& 0
\end{aligned}
$$

The first inequality follows the supermodularity of $\psi(\boldsymbol{v}, \xi, \boldsymbol{w}, y)$ and the last inequality derives from the definition of $\xi^*(\boldsymbol{v}, \boldsymbol{w})$ and the assumption $\xi - \delta < \xi^*(\boldsymbol{v}, \boldsymbol{w})$. Thus, ξ could not optimal for $g(\boldsymbol{v}, \xi, \boldsymbol{w} - \boldsymbol{\delta e_{n+2}})$. Therefore, $\xi^*(\boldsymbol{v}, \boldsymbol{w}) + \delta \leq \xi^*_{w_1}$. □

Unlike the supermodular functions, the anti-multimodular function is sensitive to the order of variables.

4.3 Model Formulation and the Optimal Policy

We consider a pricing problem of container leasing system with reservations in the finite planning horizon of length T. There are C units of containers of the same type available to two customer classes, reserved customers and walk-in customers. Reserved customers book containers at a reserved unit price, settle payments ahead of service and pick up containers at L periods later. The reserved quantities are guaranteed to be available at the pickup date. Walk-in customers arrive at the front desk and decide to take the service based on the unit price of the lease service on spot. The lease duration of all customers are μ periods. h is the unit holding cost, b is the lost sales penalty cost in walk-in customer context and emergency lease cost per unit of container in reserved customer context and $\alpha \in (0, 1]$ is the time discount factor. The objective of the leasing company is to maximize the expected profit over the finite horizon by setting prices P_t^w for walk-in customers and P_{t+L}^r for reserved customers.

The sequence of events in each period t is specified as follows: (1) containers leased at period $t-\mu$ are returned at the beginning of period and the firm reviews the amounts of leased containers for the last $\mu - 1$ period and reserved containers for the following $L-1$ periods; (2) the firm decides the current period walk-in price and reserved price of each unit; (3) reserved customers realize the demand and walk-in customers rent containers depend on the current walk-in price; (4) both holding cost or penalty cost for this period are calculated.

Following the literature on revenue management (*Chen and Simchi-Levi* 2004, *Huh and Janakiraman* 2008, *Pang et al.* 2012), stochastic aggregate demand has the additive form

$$d_t^i = D_t^i(P_t^i) + \epsilon_t, \text{ where } i \in \{r, w\}$$

where $D_t^i(P_t^i)$ is the mean demand with respect to P_t^i in period t and a strictly decreasing in the price P_t^i and the inverse demand function exists. ϵ_t is a zero-mean random variable with bounded support $[A, B]$, $(A < 0 < B \leq +\infty)$. Let $F(\cdot)$ be the probability density function of ϵ_t and $\bar{F}(\cdot) = 1 - F(\cdot)$ be the lost-sales rate. In addition, $P^i \in [\underline{P}^i, \bar{P}^i]$ for $i \in \{r, w\}$ and $\underline{P}^r < \underline{P}^w, \bar{P}^r < \bar{P}^w$. The one-to-one correspondence between the aggregated demand and price within the same customer type implies that $d_t^i \in D_t^i = [\underline{d}_t^i, \bar{d}_t^i]$, where $\underline{d}_t^i = D_t(\bar{P}^i)$ and $\bar{d}_t^i = D_t(\underline{P}^i)$. Thus, we choose d_t^i instead of P_t^i as decision variables. Now the decision variables of the firm are the walk-in demand d_t^w and the accepted booking for period $t+L$, d_{t+L}^r. x_k is the number of containers rent out at period k. $z_{t,l}$ refers to the total number of leased containers at period $t+l$ given the number of leased and reserved containers at period

t. z_t is used to represent $z_{t,0}$ for short.

$$z_{t,l} = \begin{cases} \sum_{j=l+1}^{\mu-1} x_{t-\mu+j} + \sum_{k=0}^{l} d_{t+k}^r, & 0 \le l < L \wedge \mu \\ \sum_{j=l+1}^{\mu-1} x_{t-\mu+j} + \sum_{k=0}^{L-1} d_{t+k}^r, & L \le l \le \mu - 1 \text{ and } \mu \ge L \\ \sum_{k=l-\mu+1}^{l} d_{t+k}^r, & \mu \le l \le L \text{ and } \mu < L \end{cases}$$

The expected revenue function for walk-in customers and reserved customers are

$$R_t^w(z_t, d_t^w) = P^w(d_t^w)\mathbb{E}[\min(d_t^w + \epsilon_t, C - z_t)]$$
$$R_t^r(z_{t,L}, d_{t+L}^r) = P(d_{t+L}^r)\mathbb{E}[\min(d_{t+L}^r + \epsilon_t, C - z_{t,L})].$$

Two assumptions are imposed on the demand function and the probability density function $F(\epsilon_t)$. (A1) $\rho(d^i, x) = \frac{P^i(d^i)\bar{F}'(C-d^i-x)}{P^{i'}(d^i)\bar{F}(C-d^i-x)} \ge 1$ for $d^i \in [\underline{d}^i, \bar{d}^i]$ and $0 \le x \le C - d$. $\rho(d^i, x)$ is first proposed by *Kocabiyikoglu and Popescu* (2011) and defined as lost-sales rate elasticity. (A2) $P''(d^i)d^i + P'(d^i) \le 0$ for $d^i \in [\underline{d}^i, \bar{d}^i]$. *Chen et al.* (2014) develop this assumption and show that this assumption works well for general demand functions, such as linear, log, logit and exponential demand functions, except for iso-elasticity demand function.

Lemma 4.3. *If assumptions (A1) and (A2) hold, the expected revenue functions $R_t^w(z_t, d_t^w)$ and $R_t^r(z_{t,L}, d_{t+L}^r)$ are anti-multimodular in (z_t, d_t^w) and $(z_{t,L}, d_{t+L}^r)$, respectively.*

Proof. We first prove that $R_t^w(z_t, d_t^w)$ is anti-multimodular in (z_t, d_t^w). According to the definition of anti-multimodularity, it is need to show the supermodularity of function $\psi(z_t, d_t^w, y) = R_t(z_t - y, d_t^w - z_t)$ for $z_t - C \le y \le z_t$. Rewrite $R_t^w(z_t, d_t^w) = P^w(d_t^w)(d_t^w + \int_0^{C-z_t-d_t^w} \bar{F}(v)dv)$, thus $R_t(z_t - y, d_t^w - z_t) = P^w(d_t^w - z_t)(d_t^w - z_t +$

$\int_0^{C-d_t^w+y} \bar{F}(v)dv)$. It suffices to show that $\psi(z_t, d_t^w, y)$ is supermodular in (z_t, d_t^w, y).

$$\frac{\partial^2 \psi}{\partial z_t \partial y} = -P^{w'}(d_t^w - z_t)\bar{F}(C + y - d_t^w) \geq 0$$

The nonnegativity of $\frac{\partial^2 R}{\partial z_t \partial y}$ derives from the fact that $P'(d) < 0$.

$$\begin{aligned}
\frac{\partial^2 \psi}{\partial d_t^w \partial y} &= P^{w'}(d_t^w - z_t)\bar{F}(C + y - d_t^w) - P^w(d_t^w - z_t)\bar{F}'(C + y - d_t^w) \\
&= -P^{w'}(d_t^w - z_t)\bar{F}(C + y - d_t^w)[\frac{P^w(d_t^w - z_t)\bar{F}'(C + y - d_t^w)}{P^{w'}(d_t^w - z_t)\bar{F}(C + y - d_t^w)} - 1] \\
&= -P^{w'}(d_t^w - z_t)\bar{F}(C + y - d_t^w)[\rho(d_t^w - z_t, z_t - y) - 1]
\end{aligned}$$

$\frac{\partial^2 \psi}{\partial d_t^w \partial y} > 0$ follows the assumption that $\rho(d, z_t) \geq 1$ and $P'(d) < 0$.

$$\frac{\partial^2 \psi}{\partial d_t^w \partial z_t} = -P^{w''}(d_t^w - z_t)(d_t^w - z_t + \int_0^{C+y-d_t^w} \bar{F}(v)dv) - P^{w'}(d_t^w - z_t)[2 + \bar{F}(C + y - d_t^w)]$$

If $P''(d_t^w - z_t) \leq 0$, then all terms in the equation are positive, $\frac{\partial^2 R}{\partial d_t^w \partial z_t} > 0$.

If $P''(d_t^w - z_t) > 0$, we have

$$\begin{aligned}
\frac{\partial^2 \psi}{\partial d_t^w \partial z_t} &\geq -P^{w''}(d_t^w - z_t)d_t^w - P^{w'}(d_t^w - z_t)[2 - \bar{F}(C + y - d_t^w)] \\
&= -P^{w''}(d_t^w - z_t)d_t^w - P'(d_t^w - z_t) - P^{w'}(d_t^w - z_t)F(C + y - d_t^w) \geq 0
\end{aligned}$$

The first inequality is derived from the fact that $-P^{w''}(d_t^w - z_t) \int_0^{C+y-d_t^w} \bar{F}(v)dv \geq 0$. Since ϵ_t is a zero-mean random variable with bounded support $[A, B]$, $(A < 0 < B \leq +\infty)$, $\int_0^{C+y-d_t^w} \bar{F}(v)dv \leq \int_A^B \bar{F}(v)dv = 0$. The last inequality is obtained from the assumption (A2).

In summary, the second-order partial derivative with respect to any two variables are nonnegative. Therefore, $R_t^w(z_t, d_t^w)$ is anti-multimodular in (z_t, d_t^w). The proof for $R_t^r(z_{t,L}, d_{t+L}^r)$ is similar to the abovep. $\square$

Let $\boldsymbol{x} = (x_{t-\mu+1}, \cdots, x_{t-1}, d_t^r)$ and $\boldsymbol{d^r} = (d_{t+1}^r, \cdots, d_{t+L-1}^r)$ (For conciseness, the subscript t is omitted when there is no confusion). The system state of period t can be represented by $(\boldsymbol{x}, \boldsymbol{d^r})$ and the system dynamics are

$$(\boldsymbol{x_+}, \boldsymbol{d_+^r}) = (x_{t-\mu+2}, \cdots, x_{t-1}, d_t^r + d_t^w, d_{t+1}^r, \cdots, d_{t+L-1}^r, d_{t+L}^r).$$

The inventory related holding-penalty cost at the end of period t is $H(z_t, d_t^w) = \mathbb{E}[h(C - z_t - d_t^w)^+ + b(z_t + d_t^w - C)^+]$. $f_t(\boldsymbol{x}, \boldsymbol{d^r})$ denotes the maximal expected profit of operating the rental system at state $(\boldsymbol{x}, \boldsymbol{d^r})$ from period t to the end of the planning horizon. The problem can be expressed as the stochastic dynamic programming formulation.

$$g_t(\boldsymbol{x}, d_t^w, \boldsymbol{d^r}, d_{t+L}^r) = R_t^w(z_t, d_t^w) + R_t^r(z_{t,L}, d_{t+L}^r) - H(z_t, d_t^w) + \alpha \mathbb{E} f_{t+1}(\boldsymbol{x_+}, \boldsymbol{d_+^r}) \tag{4.1}$$

$$f_t(\boldsymbol{x}, \boldsymbol{d^r}) = \max_{d_t^w, d_{t+L}^r \in A(d)} g_t(\boldsymbol{x}, d_t^w, \boldsymbol{d^r}, d_{t+L}^r) \tag{4.2}$$

where when $\mu \geq L$,

$$\begin{aligned} A(d) = \{(\boldsymbol{x}, d_t^w, \boldsymbol{d^r}, d_{t+L}^r) | d_t^w \in D^w, d_t^w + z_{t,l} \leq C, \text{ for } 1 \leq l \leq L-1, \\ d_{t+L}^r \in D^r, d_{t+L}^r + d_t^w + z_{t,L} \leq C\}; \end{aligned}$$

when $\mu < L$,

$$\begin{aligned} A(d) = \{(\boldsymbol{x}, d_t^w, \boldsymbol{d^r}, d_{t+L}^r) | d_t^w \in D^w, d_t^w + z_{t,l} \leq C, \text{ for } 1 \leq l \leq \mu-1, \\ d_{t+L}^r \in D^r, d_{t+L}^r + z_{t,L} \leq C\}. \end{aligned}$$

The boundary condition is $f_{T+1}(\boldsymbol{x}, \boldsymbol{d^r}) = 0$ for any $(\boldsymbol{x}, \boldsymbol{d^r})$. In Eq. (4.1), the first two terms are expected revenues for walk-in customers and reserved customers.

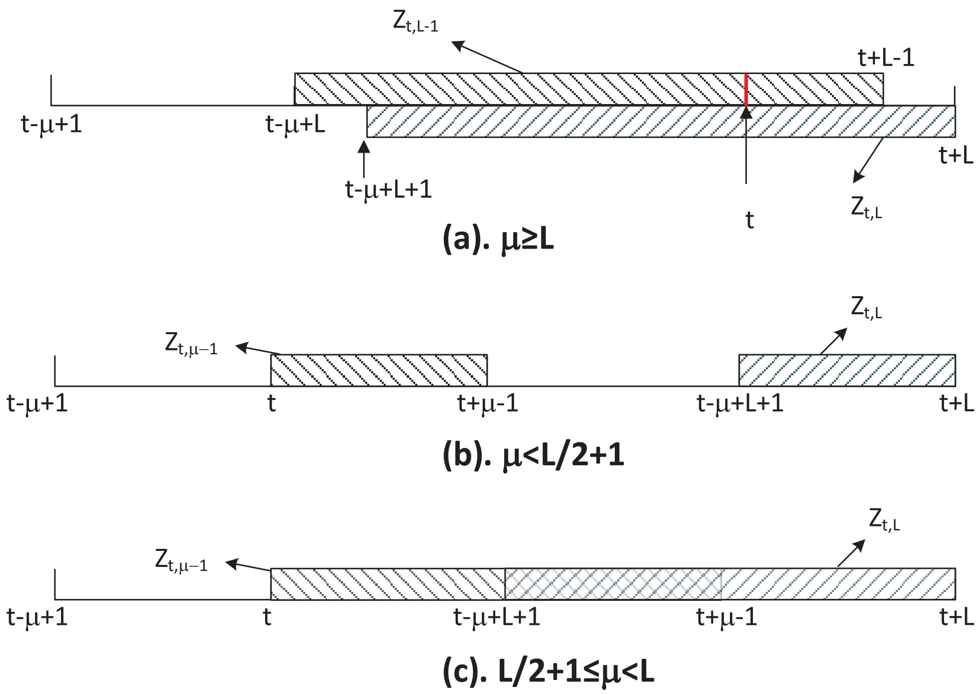


Figure 4.1. Capacity constraints

The third term is the possible holding or lost sales penalty costs depend on the value of $z_t - d_t^w$. The last term is the maximal expected profit of period $t+1$ at state $(\boldsymbol{x}_+, \boldsymbol{d}_+^r)$. Next, we explain the capacity constraints $A(d)$ for d_t^w and d_{t+L}^r. The amount of walk-in demand d_t^w is leased out for μ periods once the lease period is over, the due containers will be returned back. That is, d_t^w affects the capacity constraints from period t to period $t+\mu-1$. When $\mu \geq L$, the last realized reserved demand occurs at period $t+L-1$ and $z_{t,l}$ decreases in l for $L \leq l \leq \mu-1$. Thus the effective capacity constraints for d_t^w are the guaranteed reserved demands, $d_t^w + z_{t,l} \leq C$, for $1 \leq l \leq L-1$. When $\mu < L$, the effective capacity constraints for d_t^w are $d_t^w + z_{t,l} \leq C$, for $1 \leq l \leq \mu-1$. The other decision variable d_{t+L}^r is the latest reserved demand. When $\mu \geq L$, the walk-in demand and $z_{t,L}$ binds d_{t+L}^r, $d_{t+L}^r + d_t^w + z_{t,L} \leq C$; when $\mu < L$, although its effect on the capacity constraint still holds for μ periods, there is no further information after this period, then the capacity constraint is $d_{t+L}^r + z_{t,L} \leq C$

when $\mu < L$. We can check that the constraint set $A(d)$ is a polyhedron satisfying (P1).

It is easy to transform our model to a traditional joint inventory-pricing control model. The system state can be represented by $\boldsymbol{z} = (z_t, w_{1,t}, \cdots, w_{L\vee\mu-1,t})$ where $w_{l,t}$ denotes the net quantity to be leased in l periods. When $\mu \geq L$, $w_{l,t} = d^r_{t+l} - x_{t-\mu+l}$ for $1 \leq l < L$ and $w_{l,t} = -x_{t-\mu+l}$ for $L \leq l \leq L+\mu-1$; when $\mu < L$, $w_{l,t} = d^r_{t+l} - x_{t-\mu+l}$ for $1 \leq l \leq \mu - 1$ and $w_{l,t} = d^r_{t+l} - d^r_{t+l-\mu}$ for $\mu \leq l \leq L-1$. In the joint inventory-pricing model, $w_{l,t}$ is nonnegative owing to the nonnegative order quantity and thus $z_t \leq z_{t,1} \leq z_{t,2} \leq \cdots z_{t,L\vee\mu-1}$. But in our model, $w_{l,t}$ could be either positive or negative. For example, when $1 \leq l \leq \mu$, $w_{l,t}$ depends on the realized reserved demand d^r_{t+l} and leased quantity $x_{t-\mu+l}$ or reserved demand $d^r_{t+l-\mu}$. Hence the nondecreasing inventory level $z_{t,l}$ property does not hold for our model.

In addition, the decision variables of the joint inventory-pricing model with lead time are the expected walk-in demand (in our context) and the order quantity of current period which arrives at L periods later. The objective is to find a joint pricing and ordering policy which strikes a balance between the expected revenue and the inventory-related cost to maximize the expected discounted profit. The decision variables in our model are the expected walk-in demand and expected reserved demand. The optimization is a tradeoff between current state profit maximization and expected future profit maximization.

Theorem 4.1 partially characterizes the optimal policy with bounded sensitivity.

Theorem 4.1. *(i) For each $t \in \{1, \cdots, T\}$, the functions $g_t(\boldsymbol{x}, d^w_t, \boldsymbol{d^r}, d^r_{t+L})$ and $f_t(\boldsymbol{x}, \boldsymbol{d^r})$ are anti-multimodular in $(\boldsymbol{x}, d^w_t, \boldsymbol{d^r}, d^r_{t+L})$ and $(\boldsymbol{x}, \boldsymbol{d^r})$, respectively.*

(ii) d_t^{w*} *and* d_{t+L}^{r*} *are nonincreasing in* $(\boldsymbol{x}, \boldsymbol{d^r})$. *Moreover,*

$$-1 \leq \triangle_{d_t^r} d_t^{w*} \leq \triangle_{x_{t-1}} d_t^{w*} \leq \cdots \leq \triangle_{x_{t-\mu+1}} d_t^{w*} \leq 0 \tag{4.3}$$

$$-1 \leq \triangle_{d_{t+1}^r} d_t^{w*} \leq \triangle_{d_{t+2}^r} d_t^{w*} \leq \cdots \leq \triangle_{d_{t+L-1}^r} d_t^{w*} \leq 0 \tag{4.4}$$

$$-1 \leq \triangle_{d_{t+L-1}^r} d_{t+L}^{r*} \leq \triangle_{d_{t+L-2}^r} d_{t+L}^{r*} \cdots \leq \triangle_{d_t^r} d_{t+L}^r \leq \cdots \triangle_{x_{t-\mu+1}} d_{t+L}^{r*} \leq 0 \tag{4.5}$$

Proof. (i) We prove the anti-multimodularity of $g_t(\boldsymbol{x}, d_t^w, \boldsymbol{d^r}, d_{t+L}^r)$ and $f_t(\boldsymbol{x}, \boldsymbol{d^r})$ by induction on t. Note that $f_{T+1}(\boldsymbol{x}, \boldsymbol{d^r}) = 0$ is an anti-multimodular function. Suppose that $f_{t+1}(\boldsymbol{x}_+, \boldsymbol{d}^r_+)$ is anti-multimodular in $(\boldsymbol{x}_+, \boldsymbol{d}^r_+)$. From Lemma 4.3, $R_t^w(z_t, d_t^w)$ is anti-multimodular in (z_t, d_t^w) and recall that $z_t = \sum_{k=1}^{\mu-1} x_{t-k} + d_t^r$, thus $R_t^w(z_t, d_t^w)$ is anti-multimodular in $(\boldsymbol{x}, d_t^w)$ according to Lemma 4.1 (d). Similarly, $R_t^r(z_{t,L}, d_{t+L}^r)$ is anti-multimodular in $(\boldsymbol{x}, d_t^w, \boldsymbol{d^r}, d_{t+L}^r)$. In addition, expectation operation preserves anti-multimodularity from Lemma 4.1 (c) thus the third term is anti-multimodular. For the last term in Eq.(4.1), from system dynamics and Lemma 4.1 (d), $f_{t+1}(\boldsymbol{x}_+, \boldsymbol{d}^r_+)$ is anti-multimodular in $(\boldsymbol{x}, d_t^w, \boldsymbol{d^r}, d_{t+L}^r)$. Thus, $g_t(\boldsymbol{x}, d_t^w, \boldsymbol{d^r}, d_{t+L}^r)$ is the sum of anti-multimodular functions and therefore is anti-multimodular in $(\boldsymbol{x}, d_t^w, \boldsymbol{d^r}, d_{t+L}^r)$. The constraint set $A(d)$ of Eq.(4.2) is a polyhedron satisfying (P1). Thus, the anti-multimodularity of $f_t(\boldsymbol{x}, \boldsymbol{d^r})$ follows Lemma 4.1 (f).

(ii) Next, we reformulate the problem in period t with two maximization steps. We maximize $g_t(\boldsymbol{x}, d_t^w, \boldsymbol{d^r}, d_{t+L}^r)$ sequentially. The reformulated problem is defined as follows.

$$\tilde{g}_t(\boldsymbol{x}, \boldsymbol{d^r}, d_{t+L}^r) = \max_{d_t^w \in A(d_t^w)} g_t(\boldsymbol{x}, d_t^w, \boldsymbol{d^r}, d_{t+L}^r)$$

$$f_t(\boldsymbol{x}, \boldsymbol{d^r}) = \max_{d_{t+L}^r \in A(d_{t+L}^r)} \tilde{g}_t(\boldsymbol{x}, \boldsymbol{d^r}, d_{t+L}^r)$$

As $g_t(\boldsymbol{x}, d_t^w, \boldsymbol{d^r}, d_{t+L}^r)$ is anti-multimodular, $\tilde{g}_t(\boldsymbol{x}, \boldsymbol{d^r}, d_{t+L}^r)$ is anti-multimodular from Lemma 4.1 (f). And from Lemma 4.2, we know that d_t^{w*} is nonincreasing in $(\boldsymbol{x}, \boldsymbol{d^r})$, and the inequalities (4.3) (4.4) hold. The inequality (4.5) is proved similarly.

□

Theorem 4.1 partially characterizes the optimal properties of the two decision variables and indicates that the optimal demands d_t^w and d_{t+L}^r are nonincreasing in the number of containers leased out in the last $(\mu - 1)$ periods and the number of containers reserved for the next $L-1$ periods. This is intuitive when lead time is less than the lease duration, the optimal walk-in demand of period t directly affects the availability of the following reserved demand from period $t+1$ to period $t+L-1$. Thus the more containers leased out during the last $(\mu - 1)$ periods or booked during the next $L - 1$ periods, the less available containers for walk-in customer at the current period. Moreover, the theorem suggests that the optimal policy has bounded and monotone sensitivity. The optimal walk-in demand is most sensitive to the number of leased containers and reserved containers near period t as these directly affect the available number of containers to rent. The optimal reserved demand is most sensitive to the latest booking because the pricing of reserved demand is actually a kind of advance sell and the latest booking affects the available number of reserved containers at period $t + L$. The inequalities about the optimal walk-in/reserved demand also confirm that the decision variables are economic substitutes of the leased quantity of last $\mu - 1$ periods and reserved demand in the following $L - 1$ periods (system state). The optimal walk-in and reserved prices accordingly are nondecreasing in $(\boldsymbol{x}, \boldsymbol{d^r})$.

4.4 Myopic Pricing Policy

Due to the curse of dimensionality, it is computationally intractable to compute the optimal state-dependent policy in a problem with multimodularity structure.

We propose a myopic cycle-based pricing policy to maximize the expected profit. The myopic policy dynamically breaks the finite horizon into serval rental cycle and determines the appropriate first rental cycle and each periods price as a function of inventory level.

The finite time horizon can be divided into two types of intervals: interior interval and boundary interval. The interior interval is the time interval where the total number of containers on hire is less than the capacity and the boundary interval refers to the interval in which the total number of containers rent out equals to the capacity. The time that the rental process entry from interior interval to boundary interval is called entry point. We concentrate on the case that the finite capacity is insufficient to meet the demands of all customers, that is, there is at least one entry point in the rental process. In the stochastic leasing system, demand uncertainty may result in several entry points. In particular, we focus on the first entry point since it is easy to adjust the prices once the first entry point is settled. Next, we address the two types of intervals respectively.

4.4.1 Boundary interval

In a boundary interval with the first known entry point t_e, the available number of containers at each period $t > t_e$ is the number of containers rent out at period $t - \mu$. From period $t_e + 1$, the μ period rental cycle is $\boldsymbol{x} = \{x_{1 \vee t_e - \mu + 1}, \cdots, x_{t_e \vee \mu}\}$. The boundary interval indicates that there is no left inventory and the new allocated containers can not exceed the number of returned containers. Thus, if $t > t_e$, the profit of single period $t + L$ can be expressed as follows:

$$
\begin{aligned}
& \max_{d^w_{t+L} + d^r_{t+L} = x_{t'}} R_{t+L}(z_{t,L}, d^w_{t+L}) + \alpha^{-L} R^r_t(z_{t,L}, d^r_{t+L}) \\
= & \max_{0 \le d^r_{t+L} \le x_{t'}} R_{t+L}(C - x_{t'}, x_{t'} - d^r_{t+L}) + \alpha^{-L} R_t(C - x_{t'}, d^r_{t+L})
\end{aligned}
$$

where $t' = (t + L - t_e) \bmod \mu$. Let $\hat{d}^r_{t+L}$ be the myopic reserved demand to maximize the single period profit. Then it is easy to derive that

$$\hat{d}^r_{t+L} = \arg\max_{0 \le d^r_{t+L} \le x_{t'}} \{P^w(x_{t'} - d^r_{t+L})(x_{t'} - d^r_{t+L}) + \alpha^{-L} P^r(d^r_{t+L}) d^r_{t+L}\},$$

$$\hat{d}^w_t = x_{t'} - \hat{d}^r_t.$$

4.4.2 Interior interval

4.4.2.1 First interior interval

In this part, we consider the pricing and allocation policy to maximize the expected profit in the first interior interval. Given the evolution of rental process, rented containers are returned sequentially after the first entry point. Thus, in order to achieve the maximum expected profit, it is fairly reasonable to maximize the expected profit of the first rental cycle in the myopic pricing policy. The decision variables of the rental company are the first entry point t_e and the expected demands d^w_t for $1 \le t \le t_e$ and d^r_{t+L} for $1 \le t \le t_e - L$. The first entry point is determined by the total rental demand x_t (walk-in and reserved demands) of each period in the interior interval. Besides, if x_t is chosen, it is easy to calculate the optimal portion of reserved and walk-in demands just as in the boundary interval. Therefore, the myopic pricing problem downgrades into selecting x_t and x_{t+L} based on the inventory level z_t and $z_{t,L}$ within the cycle length t_e. The myopic expected profit $\hat{f}_t(z_t, z_{t,L})$ is defined as follows.

$$\hat{g}_t(z_t, z_{t,L}, t_e) = \max_{0 \le x_t, x_{t+L} \le C} R_t(z_t, x_t) + R_t(z_{t,L}, x_{t+L}) - H(z_t, d^w_t) + \alpha \mathbb{E} \hat{f}_{t+1}(z_{t+1}, z_{t+1,L}) \tag{4.6}$$

$$\hat{f}_t(z_t, z_{t,L}) = \max_{1 \le t_e \le T - \mu} \frac{\hat{g}_t(z_t, z_{t,L}, t_e)}{t_e \vee \mu} \tag{4.7}$$

$\hat{f}_t(z_t, z_{t,L})$ computes the average expected profit over the first rental cycle. The rationale behind maximizing the average expected profit rather than the total expected profit over t_e is as follows. When the cycle length t_e is longer than μ, containers rent out in the first μ periods will be returned in order, if the myopic objective is to maximize the total expected profit, one extreme case is that when the horizon ends, the total capacity still fails to be allocated out all once, then such an allocation policy definitely is not an optimal policy.

Solution Algorithm. To solve the above optimization problem, we first fix the value of t_e, solve the inner maximization problem (4.6), and enumerate all possible values of t_e, finally choose the maximum $\hat{f}_t(z_t, z_{t,L})$. The decision variables of $\hat{g}_t(z_t, z_{t,L}, t_e)$ in the first L periods are actually d_t^w and x_{t+L}. While the decision variables of $\hat{g}_t(z_t, z_{t,L}, t_e)$ in the following periods are x_{t+L} for $L < t \leq t_e - L$, since x_t for $L < t \leq t_e$ has already been settled down in period $t - L$. Thus, (4.6) can be classified into the two subproblems.

$$\hat{g}_t(z_t, t_e) = \max_{0 \leq d_t^w \leq C - z_t} R_t^w(z_t, d_t^w) - H(z_t, d_t^w) + \alpha \mathbb{E} \hat{f}_{t+1}(z_{t+1}), \quad 1 \leq t \leq L \tag{4.8}$$

$$\hat{g}_t(z_{t,L}, t_e) = \max_{0 \leq x_{t+L} \leq C - z_{t,L}} R_t(z_{t,L}, x_{t+L}) - H(z_{t+L}, d_{t+L}^r) + \alpha \mathbb{E} \hat{f}_{t+1}(z_{t+1,L}), 1 \leq t \leq t_e - L \tag{4.9}$$

Myopic expected walk-in demand of the first L periods

Under the fixed rental cycle t_e, let $\hat{d}_t^w(z_t, t_e)$ be the myopic expected walk-in demand that maximizes the walk-in revenue of first L periods in the rental cycle t_e.

$$\max_{\hat{d}_t^w(z_t, t_e) \in D_t^w} \{R_t(z_t, \hat{d}_t^w(z_t, t_e)) - \hat{H}(z_t, t_e, \hat{d}_t^w(z_t, t_e))\},$$

where

$$\hat{H}(z_t, t_e, \hat{d}_t^w(z_t, t_e)) = \frac{1-\alpha^{t_e - t}}{1-\alpha} h(C - z_t - \hat{d}_t^w(z_t, t_e))^+ b\hat{d}_t^w(z_t, t_e)\alpha^{2t_e - i - \mu}.$$

$\hat{H}(z_t, t_e, \hat{d}_t^w(z_t, t_e))$ denotes the possible holding and penalty cost associated with the walk-in demand $\hat{d}_t^w(z_t, t_e)$ in the first rental cycle. Considering that t_e is the last period in the first rental cycle to consume all the capacity, the first term is the possible holding cost and the second term is the maximum penalty cost caused by the walk-in demand of current period. Note that the exponent of the second term $2t_e - i - \mu$ functions as the regulator of the walk-in demand. When $t_e > \mu$, $2t_e - i - \mu$ decreases the possible penalty cost of walk-in demand so that the capacity could be depleted as early as possible; on the other hand, when $t_e < \mu$, $2t_e - i - \mu$ increase the possible penalty cost in order to slow down the consuming rate of capacity. In a word, $2t_e - i - \mu$ is the regulator which enables the entry point approach the rental duration.

Myopic rental demand of reserved periods $t + L$ for $1 < t \leq t_e - L$

In this case, we adopt an improved version of (*Federgruen and Heching*, 1999) to reduce system states based on the modified accounting scheme. *Federgruen and Heching* (1999) investigate a pricing and inventory control problem under demand uncertainty in finite and infinite horizon settings. They prove the existence of a list-price and base-stock policy and develop an efficient heuristic which the price is fixed in the lead time. In our problem, the basic idea is that rent out the same amount x_{t+L} for the remaining $t_e - t - L$ periods and $(C - z_{t,L} - (t_e - t)x_{t+L})^+$ in period t_e, and choose the expected inventory level to maximize $\hat{g}_t(z_{t,L}, t_e)$.

$$\max_{0 \le x_{t+L}(z_{t,L}) \le C - z_{t,L}} \{\sum_{k=t}^{t_e-1} \alpha^{k-1}(R_k(z_k, x_{t+L}(z_{t,L})) - h[C - z_k - (k-t)x_{t+L}(z_{t,L})]^+$$
$$- b[(k-t)x_{t+L}(z_{t,L}) + z_k - C]^+)$$
$$+ \alpha^{t_e-1} R_{t_e}(z_{t_e}, C - z_{t,L} - (t_e - t)x_{t+L}(z_{t,L}))\}$$

Once the rental demand of period $t + L$ is determined, the reserved demand d^r_{t+L} is

$$\hat{d}^r_{t+L}(z_{t,L}) = \operatorname*{arg\,max}_{0 \le d^r_{t+L} \le x_{t+L}(z_{t,L})} \{P^w(x_{t+L}(z_{t,L}) - d^r_{t+L})(x_{t+L} - d^r_{t+L}) + \alpha^{-L} P^r_t(d^r_{t+L}) d^r_{t+L}\}.$$

The walk-in demand of period t for $L + 1 \le t \le t_e$ is the rental demand x_t minus the realization $\hat{d}^r_t$.

4.4.2.2 Other interior intervals

When the time point lies in the interior interval after t_e due to demand uncertainty, the main objective is to rent out the remaining inventory as early as possible. The only difference in the myopic pricing policy between the interior interval after t_e and boundary interval is the computation of the walk-in demand.

$$\hat{d}^w_t(z_t) = C - z_t - \hat{d}^r_t.$$
$$\hat{d}^r_{t+L} = \operatorname*{arg\,max}_{0 \le d^r_{t+L} \le x_{t'}} \{P^w(x_{t'} - d^r_{t+L})(x_{t'} - d^r_{t+L}) + \alpha^{-L} P^r(d^r_{t+L}) d^r_{t+L}\},$$
$$\text{where } t' = (t + L - t_e) \mod \mu.$$

4.5 Different Lease Durations

In this section, we relax the assumption that walk-in customers and reserved customers share the same lease duration. Let μ^w and μ^r be the lease duration of walk-in customers and reserved customers, respectively, and $\triangle\mu = \mu^w - \mu^r$. The demand distribution and cost parameters are the same as in Section 4.3. The basic thought of the system state is combining the reserved demand with the associated walk-in demand sharing the same returned date together. The values of lead time L, lease durations μ^w and μ^r affects the order of system state.

4.5.1 $\triangle\mu \geq L$

When $\triangle\mu > L$, the system state becomes to

$$\begin{aligned}\boldsymbol{x} =&(x_{t-\mu^w+1}, \cdots, x_{t-\triangle\mu-1}, x_{t-\triangle\mu}, \cdots, x_{t-\triangle\mu+L-1})\\ =&(x_{t-\mu^w+1}, \cdots, x_{t-\triangle\mu-1}, d^w_{t-\triangle\mu} + d^r_t, \cdots, d^w_{t-\triangle\mu+L-1} + d^r_{t+L-1}),\end{aligned}$$

and $\boldsymbol{d^w} = (d^w_{t-\triangle\mu+L}, \cdots, d^w_{t-1})$. The system dynamics are

$$(\boldsymbol{x_+}, \boldsymbol{d^w_+}) = (x_{t-\mu^w+2}, \cdots, x_{t-\triangle\mu+L-1}, d^w_{t-\triangle\mu+L} + d^r_{t+L}, \cdots, d^w_t).$$

The number of leased containers at period $t+l$, $z_{t,l}$, in this case becomes to $z_{t,l} = \sum_{j=l+1}^{\mu^r+L-1} x_{t-\mu^w+j} + \sum_{k=(\mu^r+L)\vee(l+1)}^{\mu^w-1} d^w_{t-\mu^w+k}$ for $0 \leq l < \mu^w$.

The capacity constraints for d^w_t and d^r_{t+L} are given as follows.

$$\begin{aligned}A(d) = \{(\boldsymbol{x}, d^r_{t+L}, \boldsymbol{d^w}, d^w_t) | d^r_{t+L} \in D^r, d^r_{t+L} + z_{t,L} \leq C,\\ d^w_t \in D^w, d^w_t + z_{t,l} \leq C, \text{ for } 1 \leq l \leq \mu^w - 1\}\end{aligned}$$

The problem in this case can be expressed as the following stochastic dynamic programming formulation.

$$g_t(\boldsymbol{x}, d^r_{t+L}, \boldsymbol{d^w}, d^w_t) = R_t(z_t, d^w_t) + R^r_t(z_{t,L}, d^r_{t+L}) - H(z_t, d^w_t) + \alpha \mathbb{E} f_{t+1}(\boldsymbol{x_+}, \boldsymbol{d^w_+})$$

$$f_t(\boldsymbol{x}, \boldsymbol{d^w}) = \max_{d^r_{t+L}, d^w_t \in A(d)} g_t(\boldsymbol{x}, d^r_{t+L}, \boldsymbol{d^w}, d^w_t)$$

Corollary 7.1 partially characterizes the optimal policy with bounded sensitivity when $\triangle\mu \geq L$.

Corollary 4.1. *(i) In the case with $\triangle\mu \geq L$, the functions $g_t(\boldsymbol{x}, d^r_{t+L}, \boldsymbol{d^w}, d^w_t)$ and $f_t(\boldsymbol{x}, \boldsymbol{d^w})$ are anti-multimodular in $(\boldsymbol{x}, d^r_{t+L}, \boldsymbol{d^w}, d^w_t)$ and $(\boldsymbol{x}, \boldsymbol{d^w})$, respectively for each $t \in \{1, \cdots, T\}$.*

(ii) d^{w*}_t and d^{r*}_{t+L} are nonincreasing in $(\boldsymbol{x}, \boldsymbol{d^w})$. Moreover,

$$-1 \leq \triangle_{x_{t-\triangle\mu+L-1}} d^{r*}_{t+L} \leq \triangle_{x_{t-\triangle\mu+L-2}} d^{r*}_{t+L} \leq \cdots \leq \triangle_{x_{t-\mu^w+1}} d^{r*}_{t+L} \leq 0$$

$$-1 \leq \triangle_{d^w_{t-\triangle\mu+L}} d^{r*}_{t+L} \leq \triangle_{d^w_{t-\triangle\mu+L+1}} d^{r*}_{t+L} \leq \cdots \leq \triangle_{d^w_{t-1}} d^{r*}_{t+L} \leq 0$$

$$-1 \leq \triangle_{d^w_{t-1}} d^{w*}_t \leq \triangle_{d^w_{t-2}} d^{w*}_t \cdots \leq \triangle_{d^w_{t-\triangle\mu+L}} d^{w*}_t \leq \cdots \triangle_{x_{t-\mu^w+1}} d^{w*}_t \leq 0$$

Different from the former case, d^{w*}_t in this case is most sensitive to the last walk-in demand d^w_{t-1} and most insensitive to the leased amount $x_{t-\mu^w+1}$ which is about to return in the next period. While d^r_{t+L} is most sensitive to $x_{t-\triangle\mu+L-1}$ and $d^w_{t-\triangle\mu+L}$ and most insensitive to $x_{t-\mu^w+1}$ and d^w_{t-1}. The reason behind this is that when $\triangle\mu > L$, the system state is represented by moving the vector of reserved demands by $\triangle\mu$ periods earlier, so d^r_{t+L} is most sensitive to the leased amount $d^w_{t-\triangle\mu+L-1} + d^r_{t+L-1}$ and $d^w_{t-\triangle\mu+L}$, and d^{w*}_t is most sensitive to the recent walk-in demand.

4.5.2 $0 < \triangle\mu < L$

In these two cases, the system state becomes to

$$\begin{aligned}\boldsymbol{x} =& (x_{t-\mu^w+1}, \cdots, x_{t-\triangle\mu-1}, x_{t-\triangle\mu}, \cdots, x_{t-1}) \\ =& (x_{t-\mu^w+1}, \cdots, x_{t-\triangle\mu-1}, d^w_{t-\triangle\mu} + d^r_t, \cdots, d^w_{t-1} + d^r_{t+\triangle\mu-1}),\end{aligned}$$

and $\boldsymbol{d^r} = (d^r_{t+\triangle\mu}, \cdots, d^r_{L-1})$. The system dynamics are

$$(\boldsymbol{x_+}, \boldsymbol{d^r_+}) = (x_{t-\mu^w+2}, \cdots, x_{t-1}, d^w_t + d^r_{t+\triangle\mu}, d^r_{t+\triangle\mu+1} \cdots, d^r_{t+L}).$$

In this case, $z_{t,l}$ becomes to

$$z_{t,l} = \begin{cases} \sum_{j=l+1}^{\mu^w-1} x_{t-\mu^w+j} + \sum_{k=0}^{l} d^r_{t+k}, & 0 \leq l < L \wedge \mu^w \\ \sum_{j=L+1}^{\mu^w-1} x_{t-\mu^w+j} + \sum_{k=(L-\mu^r+1)^+}^{L-1} d^r_{t+k}, & l = L \end{cases}$$

Note that the inventory level at period $t + L$, $z_{t,L}$, is different from the former cases. The capacity constraints for d^w_t and d^r_{t+L} are given as follows.

$$\begin{aligned}A(d) = \{(\boldsymbol{x}, d^w_t, \boldsymbol{d^r}, d^r_{t+L}) | d^w_t \in D^w, d^w_t + z_{t,l} \leq C, \text{ for } 1 \leq l < L \wedge \mu^w, \\ d^r_{t+L} \in D^r, d^r_{t+L} + z_{t,L} \leq C\};\end{aligned}$$

The objective functions are the same as Eq.(4.1,4.2). Corollary 7.2 partially characterizes the optimal policy with bounded sensitivity when $0 < \triangle\mu < L$.

Corollary 4.2. *(i) In the cases with $0 < \triangle\mu < L$, the functions $g_t(\boldsymbol{x}, d^w_t, \boldsymbol{d^r}, d^r_{t+L})$ and $f_t(\boldsymbol{x}, \boldsymbol{d^r})$ are anti-multimodular in $(\boldsymbol{x}, d^w_t, \boldsymbol{d^r}, d^r_{t+L})$ and $(\boldsymbol{x}, \boldsymbol{d^r})$, respectively for each $t \in \{1, \cdots, T\}$.*

(ii) d_t^{w*} *and* d_{t+L}^{r*} *are nonincreasing in* $(\boldsymbol{x}, \boldsymbol{d^r})$*. Moreover,*

$$
\begin{aligned}
&-1 \leq \triangle_{x_{t-1}} d_t^{w*} \leq \triangle_{x_{t-2}} d_t^{w*} \leq \cdots \leq \triangle_{d^w_{t-\mu^w+1}} d_t^{w*} \leq 0 \\
&-1 \leq \triangle_{d^r_{t+\triangle\mu}} d_t^{w*} \leq \triangle_{d^r_{t+\triangle\mu+1}} d_t^{w*} \leq \cdots \leq \triangle_{d^r_{t+L-1}} d_t^{w*} \leq 0 \\
&-1 \leq \triangle_{d^r_{t+L-1}} d_{t+L}^{r*} \leq \triangle_{d^r_{t+L-2}} d_{t+L}^{r*} \cdots \leq \triangle_{d^r_t} d^r_{t+L} \leq \cdots \triangle_{x_{t-\mu^w+1}} d_{t+L}^{r*} \leq 0
\end{aligned}
$$

In this case, d_t^{w*} is most sensitive to the leased containers of last period x_{t-1} and the reserved demand of period $t + \triangle\mu$. The positivity of $\triangle\mu$ results in moving the vector $\boldsymbol{d^r}$ by $\triangle\mu$ periods earlier in the system state. d_{t+L}^{r*} is most sensitive to the recent reserved demand and insensitive to the oldest leased amount at period $t - \mu^w + 1$.

4.5.3 $\triangle\mu < 0$

The system state is slightly different from the system state in Section 4.3. Let $\boldsymbol{x} = (x^w_{t-\mu^w+1}, \cdots, x^w_{t-1}) = (d^w_{t-\mu^w+1}, \cdots, d^w_{t-1})$ and $\boldsymbol{d^r} = (d^r_t, d^r_{t+1}, \cdots, d^r_{t+L-1})$. Due to the longer lease duration of reserved customers, it should keep d^w_t as one dimension in the system state. The system state of period t can be represented by $(\boldsymbol{x}, \boldsymbol{d^r})$ and the system dynamics are

$$(\boldsymbol{x}_+, \boldsymbol{d}^r_+) = (d^w_{t-\mu^w+2}, \cdots, d^w_{t-1}, d^w_t, d^r_t, d^r_{t+1}, \cdots, d^r_{t+L-1}, d^r_{t+L}).$$

In this case, $z_{t,l}$ becomes to

$$
z_{t,l} = \begin{cases} \sum_{j=l+1}^{\mu^w-1} x_{t-\mu^w+j} + \sum_{k=0}^{l} d^r_{t+k}, & 0 \leq l < L \wedge \mu^w \\ \sum_{j=L+1}^{\mu^w-1} x_{t-\mu^w+j} + \sum_{k=(L-\mu^r+1)^+}^{L-1} d^r_{t+k}, & l = L \end{cases}
$$

The capacity constraints and the objective functions are the same as in Case $0 < \triangle\mu < L$. Corollary 7.3 partially characterizes the optimal policy with bounded sensitivity when $\triangle\mu < 0$.

Corollary 4.3. *(i) In the case with $\triangle\mu < 0$, the functions $g_t(\boldsymbol{x}, d_t^w, \boldsymbol{d^r}, d_{t+L}^r)$ and $f_t(\boldsymbol{x}, \boldsymbol{d^r})$ are anti-multimodular in $(\boldsymbol{x}, d_t^w, \boldsymbol{d^r}, d_{t+L}^r)$ and $(\boldsymbol{x}, \boldsymbol{d^r})$, respectively for each $t \in \{1, \cdots, T\}$.*

(ii) d_t^{w*} and d_{t+L}^{r*} are nonincreasing in $(\boldsymbol{x}, \boldsymbol{d^r})$. Moreover,

$$
\begin{aligned}
&-1 \leq \triangle_{d_{t-1}^w} d_t^{w*} \leq \triangle_{d_{t-2}^w} d_t^{w*} \leq \cdots \leq \triangle_{d_{t-\mu^w+1}^w} d_t^{w*} \leq 0 \\
&-1 \leq \triangle_{d_t^r} d_t^{w*} \leq \triangle_{d_{t+1}^r} d_t^{w*} \leq \cdots \leq \triangle_{d_{t+L-1}^r} d_t^{w*} \leq 0 \\
&-1 \leq \triangle_{d_{t+L-1}^r} d_{t+L}^{r*} \leq \triangle_{d_{t+L-2}^r} d_{t+L}^{r*} \cdots \leq \triangle_{d_t^r} d_{t+L}^r \leq \cdots \triangle_{d_{t-\mu^w+1}^w} d_{t+L}^{r*} \leq 0
\end{aligned}
$$

In this case, d_t^{w*} is most sensitive to the walk-in demand of last period d_{t-1}^w and the reserved demand of current period d_t^r instead of d_t^r and d_{t+1}^r in Section 4.3. The longer lease duration μ^r actually postpone the vector $\boldsymbol{d^r}$ by $\triangle\mu$ periods later in the system state. d_{t+L}^{r*} is most sensitive to the recent reserved demand and insensitive to the oldest walk-in demand.

In a word, when $\triangle\mu < 0$, the vector $\boldsymbol{d^r}$ moves $\triangle\mu$ periods later in the system state; while when $\triangle\mu > 0$, the system state $\boldsymbol{d^r}$ moves $\triangle\mu$ periods earlier in the system state. Especially when $\triangle\mu \geq L$, the reserved demand vector is drown in the past walk-in demand vector and d_t^w is the last returned amount in the system state on the point of current period t.

4.6 Summary

This part studies the dynamic pricing problem of a container leasing company with two customer types, reserved customers and walk-in customers. We show that the objective function is anti-multimodular and the optimal pricing is nonincreasing in the system state with bounded sensitivity. Further, we propose an effective heuristic to obtain myopic pricing policies to the dynamic pricing problem. Last, we partially characterize the optimal policies under different lease durations of two customer types.

CHAPTER 5

Conclusion

This dissertation addresses three monopolist's dynamic pricing problems based on the characterization of customers in the context of container leasing industry. First, customers are categorized by their hire time preference or hire quantity preference. We apply the mechanism design method to derive the optimal closed-form solution for customers with hire time preference. The capacity constraints and dynamic arrivals have different effects for customers with same/different hire time preference(s). Second, customers with unit capacity request are grouped by the lead time. We utilize the continuous-time Markov decision process to analyze the problem and derive the optimal allocation and pricing policy. There exists a state dependent rationing and nondecreasing posted pricing policy. Last we classify customers with multiple units of capacity request by lead time. Employing the concept of anti-multimodularity, we partially characterize the optimal policies under same/different lease duration(s) and show that the optimal policies are nonincreasing in the system state and have bounded and monotone sensitivity.

One future research direction is including the competition effect into the model. The competition contains peer competition and downstream competition. Peer competition refers to the small number of major leasing companies. The container leasing industry is an oligopoly and such situation remains 20 or 30 years. Competition among peer leasing firms relies on many factors such as lease rate, lease term, availability, drop-off restrictions and repair provisions and customer service. Downstream competition relates to the decision by shipping lines to buy their containers rather than leasing containers. Shippers may choose to purchase containers considering world trade and economic growth, the price of new containers, fluctuations in interest rates and etc. Competition is another crucial element to be incorporated in the pricing determination.

The capacity extension could also be included in the model. In the current stage, we assume that capacity is fixed in the planning horizon. In practice, the leasing firm purchases new containers from manufacturers quarterly according to the demand, supply and the average age of current fleet. Besides the expansion of new containers, it is also possible to transship some idle containers from low-demand areas to high-demand areas.

Another future research direction for dynamic nonlinear pricing problem is the multidimensional screening problem under contemporaneous and dynamic arrivals. In the operating lease practice, master lease provides a master framework pursuant to which lessees can lease containers on an as-needed basis based on the price. It would be interesting to further study how the hire quantity and hire time influence each other through the pricing decision and the effect of capacity constraint and dynamic arrivals over time.

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