Markowitz's Model With Intractable Liabilities

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Certificate of Originality

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__________________________(Signed)

________HOU Danlin________(Name of student)

Dedicate to my parents.

Abstract

This thesis studies robust Markowitz's models with unhedgeable liabilities involved in the final decision. The term "unhedgeable liabilities" refers to the liabilities about which the only things we know are their distributions or a few moments. With the robust idea, the target of the investor is set to minimize the variance of her portfolio in the worst scenario over all possible unhedgeable liabilities that could happen. Because of the time-inconsistent nature of the problem, the classical dynamic programming and stochastic control approaches cannot be directly applied to solve it. Instead, the quantile optimization method is adopted to tackle the problem. Using relaxation method, the optimal solutions to this specific kind of problem are derived in closed-form, and the properties of the mean-variance frontier are fully discussed too.

As we know, this thesis is the first to introduce unhedgeable liabilities into mean-variance formulation, which further generalizes the original mean-variance field and also to some extent draws the model to the real financial world. Since the components of the terminal wealth in our model are based on different markets, a new risk measure is also put forward to avoid the ill-posedness of the problem.

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Contents

List of Notations

$\pi(\cdot)$	the investment portfolio strategy taken by the investor
$X^\pi(t)$	total wealth of the investor corresponding to a portfolio strategy $\pi(\cdot)$ at time t
$S_i(t)$	price of asset i in the market at time t
$r(t)$	interest rate of the riskless asset at time t
$\beta_i(t)$	appreciation rate of risky asset i at time t
$\sigma_{i,j}(t)$	volatility rate of risky asset i with respect to $W^j(t)$
$B(t)$	excess rate of return vector process at time t
$W(t)$	multi-dimensional standard Brownian motion at time t

Chapter 1

Introduction

Since Markowitz has his great work published in 1952, mean-variance portfolio selection model has become the most prevalent and highly acclaimed model in both industry and academic research. The basic idea behind this great model is actually quite simple, which is to improve the performance of the portfolio to the greatest extent, at the same time control the risk the investor will bear under a certain level. Below is a common expression of mean-variance model

$$\min_{X} \quad \mathbf{Var}(X), \tag{1.1}$$
$$\text{subject to} \quad \mathbf{E}[X] = \varpi, \quad X \in \mathcal{A}$$

where $\mathcal{A}$ is the set of all attainable outcomes. In the existing literature, X represents the terminal wealth of the investor's portfolio in common financial markets, which include different kinds of investment tools like stocks, various financial derivatives and bank deposit etc. While X may also consist of some investments based on other "markets" (such as, lottery market or traffic market), no information in the common financial market can help us better understand them. Gambling and car insurance are all typical examples of such investments, which are actually very common in our life but could by no means be hedged or predicted.

Inspired by these observation, we decided to consider the mean-variance portfolio and mean-variance hedging problem with unhedgeable liability from other "markets". In such models, the terminal wealth consists of not only the classical investment outcomes in the common financial market but also some unhedgeable liabilities from other sources. In this thesis, we study Markowitz's mean-variance problem with full-information unhedgeable liability involved in the final decision, model with partial-information unhedgeable liability is briefly discussed in the Conclusions part. In our model, the terminal wealth consists of two parts: an investment outcome X in the financial market and an unhedgeable liability ϑ based on another market. If we followed the classical Markowitz's mean-variance framework, the risk would be measured by the variance of the terminal wealth

$$\mathbf{Var}(X + \vartheta).$$

However, except for the trivial case, this would be ill-posed if ϑ is an uncontrollable liability. In our model, since ϑ and X come from different markets, the relationship between them are unknown, which leads to the ill-posedness of the problem. There have been many other studies on the similar problem where ϑ is not an uncontrollable liability. See, for example, Lim [44] considered to minimize $\mathbf{E}[(X - \vartheta)^2]$ for a given hedgeable liability ϑ. Since the investments in these models are all based on the same market, there are no ill-posedness problems. While this thesis is the first, as we know, to introduce an uncontrollable liability based on different market into the mean-variance framework.

Because the existing variance measure is ill-posed when ϑ is an unhedgeable liability, we need to introduce a new risk measure. In this thesis, we introduce a robust risk measure of variance

$$\max_{Y \sim \vartheta} \mathbf{Var}(X + Y),$$

where the maximum is taken over all the random variables Y that are identically distributed as the unhedgeable liability ϑ. This measure reflects the risk-averse attitude of the investors, they quantify their risk in the worst case. Since this risk measure is a robust version of variance, we call our model robust Markowitz's mean-variance portfolio selection model. Robustness is concerned with the stability of the estimators of parameters from a given model when model misspecification exists; and, in particular, in the presence of outlying observations. An abundance of researches have been conducted on robust control problems. We refer to Toronjadze [62], Goldfarb and Iyengar [26], DeMiguel and Nogales [18], Jin and Zhou [35] for more details in the robust control problems. In these robust models, the investors have updated information on the terminal wealth with time approaching to maturity, and so the dynamic programming approach works. By contrast, in our model, we have no updated information on the unhedgeable liability; and so the existing dynamic programming technique, which heavily depends on such information, cannot be directly applied to tackle such problem. This essential difference makes our problem less tractable and thus a new method needs to be developed to solve it.

In this thesis, we adopt the so-called quantile formulation to rewrite the problem and turn the stochastic control problem into a quantile optimization problem. With the relaxation method initiated by Xu [65], the whole problem can be solved completely and a closed-form solution can be derived at last.

1.1 Background

1.1.1 Classification of Liabilities

Most of existing literature on portfolio selection tend to specify the corresponding portfolio based on a common financial market. The payoff of such portfolio is typically determined by the investor's investment strategy as well as the state of the underlying financial market, that is, a complete specification of all relevant variables describing the financial market over the relevant time horizon. The information about this kind of portfolio will be updated with time approaching to maturity, and that may enable us to hedge its risk. However the composition of the investor's portfolio could be much more complex in the real world. We sometimes encountered some unpredictable gains or losses to our wealth, like picking up 1 million on the street, which will definitely change your financial situation, while by no means you can predict its happening. Take a more common example, many people like gambling, which also has an uncontrollable payoff. Since the payoff of those investments are based on another "market" (for instance, lottery market), even with full information of financial market we cannot gain any understanding of them, and that will make the hedging impossible.

With the aforementioned practical examples, we classify the liabilities in existing literature into the following categories:

Known liabilities: those outcomes whose cash flows are known in the beginning of investment; fixed-income securities such as bonds are outstanding examples of such outcomes;

Hedgeable liabilities: those outcomes that can be perfectly hedged; vanilla European options are examples of such outcomes;

Partially hedgeable liabilities: those outcomes that cannot be perfectly hedged but are still under control; for instance, options that can be super-hedged belong to this category.

Unhedgeable liabilities: those outcomes about which the only things we know are their moments and/or distributions. We call them *(full information) unhedgeable liabilities* if their distributions are known; or *(partial information) unhedgeable liabilities* if not all but only a few moments are known to us. As explained, the outcomes of lottery tickets and insurance contracts are outstanding examples of unhedgeable liabilities.

The outcomes in the first three categories have a common feature that they can be better understood as time approaches to maturity. That means we will obtain more information about these outcomes as time goes by.

In practice, on the other hand, investors often face outcomes that cannot be understood better even if the time is very close to maturity. Such outcomes are essentially different from those outcomes in the first three categories. For example, insurance companies sell millions of insurance contracts every year, and some of these contracts need to be executed by the insurance companies. However, neither the insurer nor the insurant could predict the exact payment from any one of the contracts. Take another good example, lottery ticket holders could not predict their final earnings before the lottery is drawed. We classify these outcomes, which are not known until they occur, as the fourth category for the portfolio, that is uncontrollable liabilities. They are by no means predictable or hedgeable before maturity, which makes them uncontrollable in the original mean-variance framework.

We should point out here that the moments or distributions of unhedgeable liabilities could be time-dependent or time-independent. We call the former time-variant unhedgeable liabilities; and the latter time-invariant unhedgeable liabilities. For example, the total payment of a type of life insurance contracts is time-variant, while the outcome of a lottery ticket is time-invariant.

This thesis only focuses on the robust Markowitz's model with full information time-invariant unhedgeable liabilities, an example with partial information unhedgeable liabilities is also provided in the conclusion part.

1.1.2 Mean-variance Portfolio Selection Problem

Mean-variance problem has been introduced by Markowitz [49, 50] in his seminal work in the middle of the last century. This mean-variance portfolio selection framework has been long recognized as the cornerstone of modern financial mathematics, and also one of the most predominant investment decision rules in financial portfolio selection theory. This framework not only enjoys great and enduring vitality in the research world, but also well received and praised in the industry. Mean-variance portfolio selection framework is concerned with the allocation of wealth among a variety of financial securities so as to achieve a trade-off between the return and the risk at the end of the investment horizon. The return is measured by the expected terminal wealth, while the risk is measured by the variance of the terminal wealth. The crucial essence of this ground-breaking work is to obtain the optimality in portfolio selection by balancing two conflicting objectives, which are maximizing the expected return and minimizing the investment risk.

Markowitz's work has intrigued and inspired many researchers into the mean-variance world. Soon Merton [51] derived the analytical expression of the single-period mean-variance efficient frontier under the assumption that short-selling is allowed and covariance matrix is positive definite. After that, it is natural to consider extending the original single-period setting into multi-period and continuous-time setting. However, the existence of the variance term in the mean-variance formulation makes the corresponding dynamic mean-variance portfolio selection problem nonseparable in the sense of dynamic programming, which brought enormous difficulty in direct extension. To come around this problem, expected utility of the terminal wealth was therefore introduced, which implicitly contained the relationship between risk and return. Afterwards the literature in multi-period and continuous portfolio selection problem has been dominated by the expected utility function maximization method. See, B Dumas and E Luciano [23], AHY Chen, FC Jen, S Zionts [10], EF Fama [24], I Karatzas, JP Lehoczky, SE Shreve [38], Grossman and Z Zhou [28], RC Merton [53]. It took almost 50 years for mathematicians to extend the original formulation in Markowitz[49, 50] into its dynamic counterpart. In 2000, Li and Ng [43] first broke ice in mean-variance field, they derived the analytical optimal portfolio policy and analytical expression of the mean-variance frontier for the discrete-time setting by embedding the original problem in a tractable auxiliary problem. Using the embedding idea and linear-quadratic optimal control theory, continuous-setting problem has been tackled by Zhou and Li [67].

Later, the so-called constrained portfolio selection problem, which essentially renders the market imcomplete, has been widely studied in mean-variance world. Various constraints have been considered for the wealth process, the investment strategies, the parameters, or some or all of them. Stimulated by the comments of Markowitz, Li, Zhou and Lim [47] studied the continuous mean-variance portfolio selection problem with no-shorting constraint using two Riccati equations; Due to the inherent difference between the continuous-time and discrete-time formulation, it took over 10 years for Cui, Gao, Li and Li [15] to solve the discrete-time mean-variance formulation with no-shorting constraint; Zhu, Li and Wang [70] studied risk control over bankruptcy in terms of multi-period mean-variance formulation; Bielecki, Jin, Pliska and Zhou [7] studied the continuous-time mean-variance problem with additional restriction that bankruptcy is prohibited using an extension of risk-neutral approach; Stock processes with random drift and diffusion coefficients are introduced in mean-variance world in Lim and Zhou [46]; Zhou and Yin [68] featured mean-variance formulation in a regime switching market; Sun and Wang [61] characterized a mean-variance formulation consisting of a riskless asset and a risky asset from an incomplete market. In recent years, we have witnessed abundant researches conducted in studying the dynamic mean-variance portfolio problem with various constraints in discrete-time and continuous-time settings. Contributions include, but are not limited to, Duffie and Richardson [22], Li and Ng [43], Zhou and Li [67], Li, Zhou and Lim [47], Goldfarb and Iyengar [26], Hu and Zhou [32], Bielecki, Jin, Pliska and Zhou [6], Labbé and Heunis [41], Cui, Li and Li [14], Czichowsky and Schweizer [17], Heunis [30], Cui, Gao, Li and Li [15], Li and Xu [48], Sun and Wang [61], Zhu, Li, and Wang [69], Xiong and Zhou [64].

In solving the above models, researchers encountered one common difficulty, which is time-inconsistency. Time-inconsistency is a notorious property stems from the non-separability of the variance term in mean-variance formulation, which fails Bellman's principle of optimality and thus generates intractability concerned with dynamic programming. Thus embedding method was put forward by Li and Ng [43], Zhou and Li [67] and a bulk of other literature followed their way. Although the dynamic optimal investment policy could be derived with this method, it requires the investor to deicide her overall optimal policy for the entire time horizon at the initial time. This kind of optimal investment policy is termed as pre-committed dynamic optimal investment policy by Basak and Chabakauri [2], as the investor will find it optimal only when she is able to pre-commit. Due to the time-inconsistency of the mean-variance criteria, the pre-committed investment policy determined at the initial time may not stay optimal for the truncated time horizon, which means the investor may has the incentive to deviate from her initial decision during the investment process. This phenomenon in mean-variance area has been revealed by Zhu, Li and Wang [70], and investigated extensively recently in the following literature. Assuming the investor will decide her consistent policy at any time t in the future, Basak and Chabakauri [2] derived the time-consistent policy by backward induction, which has been further extended by Bjrk, Murgoci and Zhou [7] using state dependent risk aversion and backward time-inconsistent control method. Cui, Li, Wang and Zhou [13] relaxed the original time-consistent concept to “time consistency in efficiency” (TCIE), which relaxed the self-financing restriction to allow withdrawal of money out of the market. On this basis, Cui, Li and Li [14] studied analytically the impact of convex cone-type portfolio constraints on TCIE in a discrete-time market.

All the various progresses above in portfolio selection area have considerably enriched the original mean-variance world. While there is one common similarity shared among these works, that is the terminal wealth in these models is defined as the investment outcome in the common financial market at maturity. While in our model the payoff in another market will be also included in terminal wealth, about which we cannot gain any understanding based on the common financial market. That is the reason we call this kind of payoff unhedgeable liabilities. In this thesis, a new risk measure will be adopted to deal with this new mean-variance problem. The introduction of unhedgeable liabilities into the mean-variance framework is also one of the main contribution of our work.

1.1.3 Mean-variance Hedging Problem

Mean-variance hedging problems started to draw researchers’ attention around 1990, almost 40 years after mean-variance formulation has been raised. In the financial world, hedging is a prevalent and common way of controlling the risk in the investing. A hedge is an investment position intended to offset potential losses or gains that may be incurred by a companion investment. For example, when the investor with a long position in one specific stock wants to reduce her risk, she would step into a short position of correlated future or option, and that behavior is called hedging. Mathematically, mean-variance hedging problems deal with minimizing the distance between the investment outcomes and a given random or deterministic target. The target is typically related to the investor's trading strategies and/or some given processes in the financial market.

Below is the objective function of a specific formulation of mean-variance hedging in Schweizer's *Mean-variance Hedging* in [12]

$$\min \mathbf{E}[|V_T(x,\theta) - H|^2] \quad \text{overall } \theta \in \Theta, \tag{1.2}$$
$$V_t(x,\theta) = x + \int_0^t \theta_u \mathrm{d}S_u,$$

where $V_T(x,\theta)$ is the wealth at time T with initial endowment x and investment policy θ, H is square-integrable F_T-measurable random variable, S_t represents price process of underlying risky asset at time t. Here we are looking for the optimal policy θ to hedge the terminal payoff of H. While the formulation in mean-variance portfolio selection problem is, for some trade off parameter γ

$$\max \ \mathbf{E}[V_T(x,\theta)] - \gamma\, \mathbf{Var}[V_T(x,\theta)] \quad \text{over all } \theta \in \Theta. \tag{1.3}$$

Actually Equation (1.3) can be restated as $m(\theta) - \gamma \mathbf{E}[|V_T(x,\theta) - m(\theta)|^2]$, with the constraint $\mathbf{E}[V_T(x,\theta)] = m(\theta)$. So instead of solving Equation (1.3), we can solve Equation (1.2). The close connection between these two problems could be clearly revealed from their setups.

Mean-variance hedging problems have been widely studied in the literature. Recent contributions on the mean-variance hedging problems include, among many others, Duffie and Richardson [22], Schweizer [60], Lim [45], Schweizer's *Mean-Variance Hedging* in [12], Czichowsky and Schweizer [16], Jeanblanc, Mania, Santacroce, and Schweizer [34]. In Duffie and Richardson [22], a special case of mean-variance hedging problem has been treated, which motivated Schweizer [60] study the continuous hedging problem with a mean-variance objective for general contingent claims following the same method. Lim [45] considered mean-variance hedging in an incomplete market where jump (in a non-predictable way) existed in the underlying asset processes. Jeanblanc, Mania, Santacroce, and Schweizer [34] tackled mean-variance hedging problem via stochastic control in more general semimartingale models which showed explicitly how the backward stochastic differential equations (BSDEs) arise.

After years' development, two main approaches have been adopted to solve mean-variance hedging problem. One of them is to use martingale theory and projection arguments; the other is stochastic control method with BSDE formulation. For the first method, see Rheinlander and Schweizer [59], Pham and Laurent [58], Gourieroux, Laurent and Pham [27], Arai [1]. Among them, Rheinlander and Schweizer [59] solved the problem where asset processes are continuous semimatingale with weighted norm inequalities and the Galtchouk-Kunita-Watanabe decomposition; while in Gourieroux, Laurent and Pham [27] change of numeraire have been adopted instead; as an extension of previous literature, Arai [1] derived the mean-variance hedging policy in terms of discontinuous case. For the second BSDE method, please refer to, Bobrovnytska and Schweizer [8], Lim [44], Li, Zhou and Lim [47]. Bobrovnytska and Schweizer [8] showed how the mean-variance hedging problem can be treated as a linear-quadratic stochastic control problem for continuous semimartingales in a general filtration; Lim [44] studied the continuous version in an incomplete market where parameters could be random, and derived solvability of related Stochastic Riccati Equation (SRE); The main difference between these two approaches is that, martingale theory and projection arguments donot require asset prices to be Ito processes, instead they could belong to a broader class of semi-martingales. While the commonality they shared is, in their frameworks once the underlying processes and the trading strategies are known, so too are the targets. Unlike these papers, the problem tackled in our work could be considered as a distinctive kind of mean-variance hedging problem whose target belongs to unhedgeable liabilities.

1.1.4 Robust Mean-variance Model

There is no doubt that mean-variance portfolio selection is a milestone in financial mathematics, who inspired lots of famous theories like Capital Asset Pricing Model (CAPM). While its success in the theory does not guarantee that its direct application will also enjoy great recognition in the industry. Many practitioners have shied away from the direct application of this model, as Michaud [54] stated. The dominating reason is that, solutions to the mean-variance optimization problem are quite sensitive to the misspecification of parameters; and statistical errors indeed reduce the reliability of the parameters. Unfortunately, the parameters using in the models for portfolio selection, like appreciation rate and volatility rate, are typically estimated from historical data of asset prices using statistical techniques. And statistical errors can barely be avoided while estimating, which leads to the unreliability of the subsequent optimization. Considerable literature on portfolio selection have extensively illustrated this phenomenon, see Chopra and Ziemba [11], Ingersoll [33], Broadie [9] etc. Chopra and Ziemba [11] demonstrated the different influence of the error in mean, variance and covariance separately, and showed this difference in influence will diminish with the decline of risk tolerance. Mean-blur problem which shows the notorious difficulty in estimating the appreciation rate was specifically noted in Ingersoll and Jonathan [33]. Broadie [9] investigated the trade-off between estimation error and stationarity when describing the mean-variance frontier with estimated parameter.

Basically, two directions are regularly adopted to address the unstability of porfolio selection model arising from error-prone parameter estimation. One is to improve the accuracy in the statistical-estimation process as far as possible, which cannot fully rule out the estimation error as we wish. James-Stein estimator was put forward in

Chopra and Ziemba [11] for appreciation parameter. Ledoit and Wolf [56] proposed to estimate the covariance matrix of stock returns by an optimally weighted average of two existing estimators: the sample covariance matrix and single-index covariance matrix. Jorion and Philippe [37] presented Bayes-Stein estimators for uncertainty in the context of a portfolio, which simulation analysis shows should outperform the sample mean and provide significant gains in portfolio selection problems.

Another direction is to reduce the sensitivity of the portfolio model to the input uncertainty. Abundant literature have been dedicated in this area, see Frost and Savarino [25], Klein and Bawa [40], Lakner [42]. However all these techniques are not able to guarantee the performance of the portfolio while reducing the parameter sensitivity. For this reason, robust method has been proposed alternatively in Ben-Tal and Nemirovski [4] for linear programming, also in Ben-Tal and Nemirovski [5] for general convex programming. The techniques introduced in these literature were soon extended into both discrete and continuous portfolio selection problem. For single-period portfolio selection problem, Goldfarb and Iyengar [26] discussed the mean-variance portfolio selection problem with uncertain mean returns, which was reformulated as an second-order cone programs (SOCPs), and it could be solved by interior point algorithms introduced by Nesterov and Nemirovskii [55]. Multi-period robust model was tackled by Ben-Tal and Nemirovski [3] where corresponding uncertainty set is finite. In continuous-time setting, Jin and Zhou [35] studied the robust model in the sense that appreciation parameter are only known to be in a certain convex closed set, and both expected utility and mean-variance framework are considered. The crucial idea of robust method is to optimize the worst situation of the portfolio selection problem, which apparently ensure a certain level of payoff of the portfolio. In our work, min-max formulation was adopted to deal with variance term in continuous time portfolio selection problem with fixed mean term, which significantly reduce the sensitivity of the model to the uncertainty brought by the unhedgeable liability, at the same time guarantee the performance of the portfolio.

1.1.5 Quantile Formulation

A bulk of development has been made in these years in solving the continuous-time portfolio selection problem, especially in expected utility maximization area (mean-variance problem also belongs to this area with its own subtle unique characteristic). Among various methods put forward, there are two approaches enjoyed the greatest prevalence and highest recognition. One is dynamic programming approach, initiated by Merton [53], [52], in which the original problem was transformed into Hamilton-Jacob-Bellman equation (HJB) ; the other is martingale method, developed by Harrison and Kreps [29], which identified the optimal terminal wealth via solving the corresponding static optimization problem. Both of these two directions have attracted and inspired numerous researches, which are gradually unveiling the inner core of the portfolio selection problem to the world.

With further insight of investment models, researchers found classical expected utility model fails to explain many paradoxes in the real practice. Alternative models have therefore been proposed to account for these paradoxes, like cumulative prospect theory (CPT Model), rank-dependent utility theory (RDUT Model, a special case of CPT model), Yarri's dual theory, Lopes' SP/A model and so on. Among them, CPT Model is the most highly regarded one, as it gave convincing explanations of these paradoxes with probability weighting/distortion functions. It distorts the probability cumulative function of the payment, rather than distorting the payment directly with utility functions. Although these theories all have more sound financial logics than classical expected utility theory, how to solve them analytically becomes quite tricky. Probability weighting/distortion functions renders the expectations in these models nonlinear, which means nice properties as time-consistency and convexity no longer exist in the context. That destroys the foundation, on which aforementioned approaches like dynamic programming approach and martingale method based. A generally applicable framework was in great need to cover all these models once and for all. That is the background quantile formulation has been brought forth.

Studying quantile optimization problems, in particular for behavioral finance models such as cumulative prospect theory and rank dependent utility theory, has been an ad hoc research topic in recent years. It is first initiated by Jin and Zhou [36], and extended by He and Zhou [31] to general models with law-invariant preference measures. In He and Zhou [31], they first revealed the commonalities shared by these alternative preference measures, one is law-invariance, the other is all these measures can be rewritten as a distorted mean. Below is the basic form of the reformulated preference measures

$$C(X) := \int_{-\infty}^{\infty} u(X) \ \mathrm{d}[\omega(F_X(x))],$$

where $u(\cdot)$ and $\omega : [0,1] \to [0,1]$ are both nonlinear. Based on these commonalities, they investigated and solved goal-reaching model and Yarris' model with quantile formulation. Later Xia and Zhou [63] took a step forward, they managed to solve an optimal quantile formulation under RDUT model without any monotonicity condition, with the help of calculus of variation and concave envelope techniques. In Xu [65] the relaxation method has independently been introduced to tackle general quantile formulation problems, which is much more easier to follow and also without any monotonicity assumptions. In this thesis, we use the relaxation method developed by Xu [65] to solve our problem. By this method, the mean-variance problem with unhedgeable liability is completely solved and the optimal solution is given in a closed-form.

1.2 Summary of Contributions of the Thesis

The original contributions of this thesis are as follows:

- This thesis is the first to study mean-variance models whose target contains an unhedgeable liability. As we know, our daily investments are not limited to the common financial market, investment decisions based on "another market", like insurance market, may greatly influence our financial conditions. Since these investment outcomes are usually hard to predict based on common financial market, we call them unhedgeable liabilities. The introduction of unhedgeable liabilities extends the generality of the original mean-variance problem, and help draw the model closer to the reality, which at the same time brings us much difficulty in solving the problem.

- It is more tricky to solve the mean-variance model with unhedgeable liabilities. Since the components of our targeted portfolio are based on different information sets, the ill-posedness of the problem arise spontaneously. We first use the idea of robust model to tackle the ill-posedness problem, then the quantile formulation is used to rewrite the whole problem into a more tractable form. With the relaxation method, we solve the model completely with a closed-form solution derived.

- The characteristics of this new mean-variance model are also discussed in the thesis. And the comparison to the mean-variance hedging problem with a deterministic liability is provided.

1.3 Organization of the Thesis

The thesis is structured as follows:

- Chapter 2 focuses on the formulation of our Markowitz's model with unhedgeable liabilities. Firstly, it illustrates the setups of the whole market that this thesis is based on and the assets are invested in. Specific forms of the asset price processes and wealth process are presented. Secondly, the classical Markowitz's mean-variance model is introduced, based on which we promote this new Markowitz's model with the help of robust idea. Finally, we give the full expression of this proposed Markowitz's model with unhedgeable liabilities.

- Chapter 3 is devoted on transformation of our model. After achieving the static form of the original model, we utilize comonotonic theory and Lagrange method to further simplify the model. At last, we successfully translate this stochastic control problem into a quantile optimization problem.

- Chapter 4 presents some of our main theoretic results, and also gives the complete solution of the robust mean-variance portfolio selection problem with unhedgeable liability. The properties of the mean variance frontier are presented too.

- Chapter 5 concludes the whole thesis and plans for the future work. An mean-variance model with partial information unhedgeable liabilities involved is also briefly discussed.

Chapter 2

Problem Formulation

To better explain our model, some basic setups will be first presented in this chapter. A better understanding of this preliminary will be a significant help for us to master the following core content of the thesis. First we will refer the involved notation and some basic features of the market where the assets will be invested. Then we briefly discuss the original mean-variance problem and the reason we introduce unhedgeable liabilities to the former framework. Finally we show the specific model we are introducing in this thesis, and deal with the ill-posedness of this model with robust theory. In a word, this chapter decides the economic environment and foundations which this thesis is based on and introduces the formulation of our specific model.

2.1 Notation

Throughout this thesis, we make use of the following notation:

M^T : the transpose of a matrix or vector M;
$\|M\|$: $\sqrt{\sum_{i,j} m_{ij}^2}$ for a matrix or vector $M = (m_{ij})$;
$\mathbb{R}^m$: m-dimensional real Euclidean space.

Throughout this thesis, let $T > 0$ be a fixed (deterministic) terminal time. Let $(\Omega, \mathcal{F}, \mathbf{P}, \mathbb{F} = (\mathcal{F}_t : t \in [0, T]))$ be a fixed filtered complete probability space on which is defined a standard m-dimensional Brownian motion

$$W(t) \equiv (W^1(t), W^2(t), \cdots, W^m(t))^T, \quad t \in [0, T],$$

with $W(0) = 0$. It is also assumed that $\mathcal{F} = \mathcal{F}_T$ and $\mathcal{F}_t$ is equal to $\sigma\{W(s) : 0 \leqslant s \leqslant t\}$ augmented by all the null sets. Denote the set of all square integrable random variables by

$$L^2_{\mathcal{F}} := \left\{ X \,\middle|\, X \text{ is an } \mathcal{F}\text{-measurable random variable, and } \mathbf{E}[X^2] < +\infty \right\}.$$

Given a Hilbert space $\mathcal{H}$ with the norm $\|\cdot\|_{\mathcal{H}}$, we define a Banach space

$$L^2_{\mathbb{F}}([0, T]; \mathcal{H}) := \left\{ f(\cdot) \,\middle|\, \begin{array}{l} f(\cdot) \text{ is an } \mathbb{F}\text{-adapted, } \mathcal{H}\text{-valued progressively measurable} \\ \text{process on } [0, T] \text{ and } \|f(\cdot)\|_{\mathbb{F}} < +\infty \end{array} \right\}$$

with the norm

$$\|f(\cdot)\|_{\mathbb{F}} := \left(\mathbf{E} \left[\int_0^T \|f(t, \omega)\|^2_{\mathcal{H}} \, \mathrm{d}t \right] \right)^{\frac{1}{2}}.$$

The quantile $Q_X(\cdot)$ of a real-valued random variable X is defined as the right-continuous inverse function of its cumulative distribution function $F_X(\cdot)$, that is,

$$Q_X(t) = \sup\{s \in \mathbb{R} : F_X(s) \leqslant t\}, \quad \forall\, t \in (0, 1),$$

with convention $\sup \emptyset = -\infty$. We call a function a quantile if it is the quantile of a random variable.

2.2 Market Model

Following Karatazas and Shreve [39], we consider a continuous-time arbitrage-free financial market where $m+1$ assets are traded continuously on $[0,T]$. One of the assets is a *bond*, whose price $S_0(\cdot)$ evolves accordingly to an ordinary differential equation

$$\begin{cases} \mathrm{d}S_0(t) = r(t)S_0(t)\,\mathrm{d}t, \quad t\in[0,T], \\ S_0(0) = s_0 > 0, \end{cases}$$

where $r(t)$ is the appreciation rate of the bond at time t. The remaining m assets are *stocks*, and their prices are modeled by a system of stochastic differential equations (SDEs)

$$\begin{cases} \mathrm{d}S_i(t) = S_i(t)\{\beta_i(t)\,\mathrm{d}t + \sum_{j=1}^{m}\sigma_{ij}(t)\,\mathrm{d}W^j(t)\}, \quad t\in[0,T], \\ S_i(0) = s_i > 0, \end{cases}$$

where $\beta_i(t)$ is the appreciation rate of the stock i and $\sigma_{ij}(t)$ is the volatility coefficient of the stock i with respect to W^j at time t. Denote appreciation rate vector process $\beta(t) := (\beta_1(t),\cdots,\beta_m(t))^T$ and the volatility matrix process $\sigma(t) := (\sigma_{ij}(t))_{m\times m}$. We also define the excess return rate vector process

$$B(t) := \beta(t) - r(t)\mathbf{1}, \quad t\in[0,T],$$

where $\mathbf{1} = (1,1,\cdots,1)^T$ is an m-dimensional vector.

We impose the following basic assumptions on the market parameters in this paper:

- The interest rate $r(\cdot)$, the appreciation rate $\beta(\cdot)$, the excess return rate $B(\cdot)$, and the volatility matrix $\sigma(\cdot)$ processes are all uniformly bounded $\mathbb{F}$-progressively measurable stochastic processes on $[0,T]$;

- The process $\sigma(\cdot)$ is non-singular and its inverse process $\sigma(\cdot)^{-1}$ is uniformly bounded on $[0,T]$.

Under these assumptions, we can define the price of risk process

$$\theta(t) := \sigma(t)^{-1}B(t), \quad t \in [0,T].$$

It is easy to see that the process $\theta(\cdot)$ is uniformly bounded on $[0,T]$ and the market is complete.

2.3 Investment Problem

Now let us consider a (female) investor in the market, whose transactions have no influence on the prices of the assets in the market. Suppose she has an initial wealth $x > 0$ to invest in the financial market over the time period $[0,T]$. Denote by $\pi_i(t)$ the total market value of her wealth invested in stock i at time t, $i = 1, \cdots, m$. We assume that short selling is allowed in market so that $\pi_i(t)$ can take negative values. We also assume that the trading of shares takes place continuously in a self-financing fashion (i.e., there is no consumption or income) and the market is frictionless (i.e., the transactions do not incur any fees or costs). We call

$$\pi(\cdot) := (\pi_1(\cdot), \cdots, \pi_m(\cdot))^T \in L^2_{\mathbb{F}}([0,T];\mathbb{R}^m)$$

an admissible portfolio. The investor's total wealth at time $t \geqslant 0$ corresponding to a portfolio $\pi(\cdot)$ is denoted by $X^\pi(t)$. Then the wealth process $X^\pi(\cdot)$ evolves according to an SDE (see, e.g., Karatazas and Shreve [39])

$$\begin{cases} \mathrm{d}X^\pi(t) = [r(t)X^\pi(t) + \pi(t)^T B(t)]\,\mathrm{d}t + \pi(t)^T\sigma(t)\,\mathrm{d}W(t), \quad t \in [0,T], \\ X^\pi(0) = x. \end{cases} \tag{2.1}$$

For any admissible portfolio $\pi(\cdot)$, the above SDE admits a unique solution $X^\pi(\cdot)$ such that $X^\pi(\cdot) \in L^2_{\mathbb{F}}([0,T];\mathbb{R}^m)$. We call $X^\pi(\cdot)$ an admissible wealth process and $(X^\pi(\cdot), \pi(\cdot))$ an admissible pair. We also assume that bankruptcy is allowed so that the wealth process may take negative values. Note that the SDE (2.1) is linear in $X^\pi(\cdot)$ and $\pi(\cdot)$, so the set of all admissible pairs is convex.

Markowitz's mean-variance portfolio selection refers to the problem of, given a favorable mean level ϖ, finding an allowable investment policy $\pi(\cdot)$, (i.e., admissible portfolio), such that the expected terminal wealth $\mathbf{E}[X^\pi(T)]$ is ϖ, while the risk measured by the variance of the terminal wealth $\mathbf{Var}(X^\pi(T))$ is minimized. We state the problem as following.

Definition 2.3.1. *The classical continuous-time Markowitz's mean-variance portfolio selection problem, parameterized by ϖ, is*

$$\begin{aligned} \min_{\pi(\cdot)} \quad & \mathbf{Var}(X^\pi(T)), \\ \text{subject to} \quad & \begin{cases} \mathbf{E}[X^\pi(T)] = \varpi, \\ (X^\pi(\cdot), \pi(\cdot)) \text{ is an admissible pair.} \end{cases} \end{aligned} \tag{2.2}$$

For simplicity, we also call it the classical problem.

Problem (2.2) can be solved by first finding the optimal solution X^* of a static optimization problem

$$\begin{aligned} \min_{X} \quad & \mathbf{Var}(X), \\ \text{subject to} \quad & \mathbf{E}[X] = \varpi, \quad X \in \mathcal{A}, \end{aligned} \tag{2.3}$$

where we denote by $\mathcal{A}$ the set of all the admissible terminal wealth

$$\mathcal{A} = \{X^\pi(T) : (X^\pi(\cdot), \pi(\cdot)) \text{ is an admissible pair}\}. \tag{2.4}$$

The second step is finding an admissible portfolio $\pi(\cdot)$ such that $X^* = X^\pi(T)$. This can be done by the standard theory of backward stochastic differential equation (BSDE) developed by Peng [57]. Please refer to Bielecki, Jin, Pliska and Zhou [6] for more details of this approach.

In the aforementioned Markowitz's mean-variance portfolio selection model, only the investor's wealth in the financial market is considered. In practice, however, an investor may have (or potentially have) gains or losses derived from other sources; for example, people may have gains from lottery tickets; insurance companies may have losses from the execution of car insurance contracts. All of these gains and losses have a common feature: they are by no way predictable, hedgeable, or controllable; all we know about them are their distributions or a few moments. We call them unhedgeable liabilities.[1]

In this thesis, we introduce full information unhedgeable liability into Markowitz's mean-variance portfolio selection model, where the investor has an unhedgeable liability (gain when it is positive, loss when negative) derived from sources other than the financial market at the end of the investment horizon. The investor has no way to predict or hedge (even partially) the unhedgeable liability before it occurs. All she knows about the unhedgeable liability before maturity is its distribution; or equivalently, the quantile. We also assume that the distribution of the unhedgeable liability is time-invariant. This means the information gained by the investor as time approaches maturity has no influence on the estimation of the distribution of the unhedgeable liability. The outcome of a lottery ticket is an outstanding time-invariant unhedgeable liability. That unhedgeable liability is very different from liabilities that may be partially hedged or understood when more information has been gained. As for partial information unhedgeable liabilities, only some of their moments are known to us. An example of partial information unhedgeable liabilities situation will be treated in the conclusion part of the thesis.

We denote the full information unhedgeable liability by ϑ. The classical continuous-time Markowitz's mean-variance problem becomes finding an admissible portfolio $\pi(\cdot)$ such that the expected total terminal wealth $\mathbf{E}[X^{\pi}(T) + \vartheta]$ is ϖ and the risk

[1] If we only know their few moments and not distributions, we call them partial information unhedgeable liabilities. Markowitz's problem with partial information unhedgeable liabilities will be investigated in a forthcoming paper.

measured by the variance of the total terminal wealth

$$\mathbf{Var}(X^{\pi}(T) + \vartheta)$$

is minimized. However, except for in the trivial case, this measure is ill-posed. This is because all we know about the unhedgeable liability ϑ before it occurs is its distribution; we do not have its sample path information and so $\mathbf{Var}(X^{\pi}(T) + \vartheta)$ is not known at all. This makes it is impossible to consider the natural extension of the classical model simply by adding the unhedgeable liability in the terminal wealth.

To overcome this, we have to introduce a new risk measure, which we hope is closely related to the risk measure of variance in Markowitz's model. Observing that the mean-variance investor is risk-averse, we introduce a robust risk measure of variance

$$\max_{Y \sim \vartheta} \mathbf{Var}(X^{\pi}(T) + Y),$$

where the maximum is taken over all the random variables Y that are identically distributed as the unhedgeable liability ϑ, denoted by $Y \sim \vartheta$. We will show that this is a well-defined risk measure later. This new measure reflects the risk-averse attitude of the investor which makes the investor consider the risk in the worst scenario over all the possible outcomes that could happen when making decisions. Because this risk measure is a robust version of variance, we call our model robust Markowitz's mean-variance portfolio selection model.

If the liability ϑ is known, i.e., ϑ is a constant, then

$$\max_{Y \sim \vartheta} \mathbf{Var}(X^{\pi}(T) + Y) = \mathbf{Var}(X^{\pi}(T));$$

and this new risk measure reduces to the classical risk measure of variance, and our model reduces to the classical one. This means our model is an extension of the classical one.

Now we state our model as follows.

Definition 2.3.2. *The continuous-time Robust Markowitz's mean-variance portfolio selection problem with an unhedgeable liability ϑ, parameterized by ϖ, is*

$$\begin{aligned} \min_{\pi(\cdot)} \max_{Y\sim\vartheta} \quad & \mathbf{Var}(X^\pi(T)+Y), \\ \text{subject to} \quad & \begin{cases} \mathbf{E}[X^\pi(T)+\vartheta]=\varpi, \\ (X^\pi(\cdot),\pi(\cdot)) \text{ is an admissible pair.} \end{cases} \end{aligned} \tag{2.5}$$

To ensure the problem is well-posed, we assume that the unhedgeable liability ϑ is square integrable. In other words, we impose the following simple technical assumption throughout the thesis

$$\mathbf{E}[\vartheta^2]=\int_0^1 \mathscr{Q}^2(t)\,\mathrm{d}t<\infty, \tag{2.6}$$

where $\mathscr{Q}(\cdot)$ denotes the quantile of the unhedgeable liability ϑ. This implies that Y is square integrable whenever $Y\sim\vartheta$.

Remark 1. *In this thesis, it is assumed that bankruptcy is allowed. The model can be similarly formulated when bankruptcy is prohibited in the market. Moreover, our argument below to solve the problem still works with slight modifications. Please refer to Bielecki, Jin, Pliska and Zhou [6] for more details. We encourage readers to give the details of the solution.*

Chapter 3

Static Problem and its Quantile Formulation

After achieving the basic formulation of our model, we consider to transform and simplify the model in order to finally solve it analytically and completely. In this chapter, we first turn the original model into a static problem like classical cases; then assisted with Comonotonic Theory and Lagrange Method, we further transform the model; at last we rewrite the whole problem with Quantile Formulation, which makes the model more tractable. This chapter focuses on the transformation of our proposed model, which also provides the fundamental and essential step for us to solve the model in next chapter.

3.1 Static Problem

Similar with the classical case, Problem (2.5) can be solved by first deriving the optimal solution X^* of a static optimization problem

$$\begin{aligned} \min_{X} \max_{Y\sim\vartheta} \quad & \mathbf{Var}(X+Y), \\ \text{subject to} \quad & \mathbf{E}[X+\vartheta] = \varpi, \quad X \in \mathcal{A}; \end{aligned} \tag{3.1}$$

and then finding an admissible portfolio $\pi(\cdot)$ such that $X^* = X^\pi(T)$ holds by the standard theory of BSDE. From now on, we focus on solving Problem (3.1) and leave the second step to the interested readers.

Let us first reformulate Problem (3.1) in a more tractable form. Observing that for any feasible solution X of Problem (3.1) and any $Y \sim \vartheta$, we have

$$\mathbf{Var}(X+Y) = \mathbf{E}[(X+Y)^2] - (\mathbf{E}[X+Y])^2 = \mathbf{E}[(X+Y)^2] - \varpi^2,$$

and

$$\mathbf{E}[X+\vartheta] = \mathbf{E}[X] + \mathbf{E}[\vartheta] = \mathbf{E}[X] + \int_0^1 \mathscr{Q}(t)\,\mathrm{d}t,$$

so Problem (3.1) is equivalent to[1]

$$\begin{aligned} \min_{X} \max_{Y\sim\vartheta} \quad & \mathbf{E}[(X+Y)^2], \\ \text{subject to} \quad & \mathbf{E}[X] = d, \quad X \in \mathcal{A}, \end{aligned} \tag{3.2}$$

where the expected terminal wealth of financial investment is

$$d := \varpi - \int_0^1 \mathscr{Q}(t)\,\mathrm{d}t.$$

To solve Problem (3.2), we need to express the set $\mathcal{A}$ in a more tractable form. The following result is well-known (see, e.g. Bielecki, Jin, Pliska, and Zhou [6]).

[1] In this thesis, problems are called equivalent if they admit the same optimal solution(s).

Lemma 3.1.1. *The set of all the admissible terminal wealth $\mathcal{A}$ can be expressed as*

$$\mathcal{A} = \{X \in L^2_{\mathcal{F}_T} : \mathbf{E}[\rho X] = x\},$$

where

$$\rho := \exp\left(- \int_0^T (r(s) + \frac{1}{2}\|\theta(s)\|^2)\, \mathrm{d}s + \int_0^T \theta(s)^T \, \mathrm{d}W(s) \right).$$

Remark that $\rho > 0$ is not a constant.

Now the static Problem (3.2) is equivalent to

$$\min_{X \in L^2_{\mathcal{F}_T}} \quad J_0(X), \tag{3.3}$$

$$\text{subject to} \quad \mathbf{E}[X] = d, \quad \mathbf{E}[\rho X] = x,$$

where

$$J_0(X) := \max_{Y \sim \vartheta} \mathbf{E}[(X+Y)^2].$$

Let us show some properties of the cost functional $J_0(\cdot)$.

Lemma 3.1.2. *The function $J_0(\cdot)$ is finite and convex on $L^2_{\mathcal{F}_T}$.*

Proof. For any $X \in L^2_{\mathcal{F}_T}$, we have

$$J_0(X) = \max_{Y \sim \vartheta} \mathbf{E}[(X+Y)^2] \leqslant \max_{Y \sim \vartheta} \mathbf{E}[2(X^2+Y^2)]$$
$$= \mathbf{E}[2(X^2+\vartheta^2)] = 2\mathbf{E}[X^2] + 2\int_0^1 \mathscr{Q}^2(t)\, \mathrm{d}t < \infty,$$

where we used the assumption (2.6). Thus the function $J_0(\cdot)$ is finite. For any X_1, $X_2 \in L^2_{\mathcal{F}_T}$ and $\alpha \in (0,1)$, we have by the convexity of square function,

$$J_0(\alpha X_1 + (1-\alpha)X_2) = \max_{Y \sim \vartheta} \mathbf{E}[(\alpha X_1 + (1-\alpha)X_2 + Y)^2]$$
$$\leqslant \max_{Y \sim \vartheta} \mathbf{E}[\alpha(X_1+Y)^2 + (1-\alpha)(X_2+Y)^2]$$
$$\leqslant \alpha \max_{Y \sim \vartheta} \mathbf{E}[(X_1+Y)^2] + (1-\alpha) \max_{Y \sim \vartheta} \mathbf{E}[(X_2+Y)^2]$$
$$= \alpha J_0(X_1) + (1-\alpha) J_0(X_2).$$

Therefore, the function $J_0(\cdot)$ is convex. $\square$

In the subsequent argument, we will use another well-known result frequently, which is often called the Hoeffding-Frechet bounds or Hardy-Littlewood inequality. To better understand the inequality, we will have to introduce the concept of *comonotonicity* first.

Comonotonicity mainly represents the perfect positive dependence between the components of a random variable, and it's widely applied in actuarial science and financial risk management. In Dhaene, Denuit, Goovaerts, Kaas, and Vyncke [20] the definition of comonotonicity has been described as follows.

Definition 3.1.1. *The set $A \subseteq \mathbb{R}^n$ is said to be comonotonic if for any $\underline{x}$ and $\underline{y}$ in A, either $\underline{x} \leq \underline{y}$ or $\underline{x} \geq \underline{y}$ holds.*

Here the notation $\underline{x} \leq \underline{y}$ is used for the componentwise order which is defined by $x_i \leq y_i$ for all $i = 1, 2, ..., n$. Some fundamental characteristics of comonotonicity have been fully explored too.

Theorem 3.1.3. *A random vector $\underline{X} = (X_1, ..., X_n)$ is comonotonic if and only if one of the following equivalent conditions holds:*

(1) $\underline{X}$ has a comonotonic support;

(2) For all $\underline{x} = (x_1, x_2, ..., x_n)$, we have

$$F_{\underline{X}}(\underline{x}) = \min\{F_{X_1}(x_1), F_{X_2}(x_2), ..., F_{X_n}(x_n)\};$$

(3) For $U \sim Uniform(0,1)$, we have

$$\underline{X} \stackrel{d}{=} (F_{X_1}^{-1}(U), F_{X_2}^{-1}(U), ..., F_{X_n}^{-1}(U));$$

(4) There exist a random variable Z and non-decreasing functions $f_i, (i = 1, 2, ..., n)$, such that

$$\underline{X} \stackrel{d}{=} (f_1(Z), f_2(Z), ..., f_n(Z)).$$

Based on the Comonotonicity theory, Hardy-Littlewood inequality has been put forward. Here we give our version of Hardy-Littlewood inequality and also a brief proof of it.

Lemma 3.1.4. *Let $Q_X(\cdot)$ and $Q_Y(\cdot)$ denote the quantiles of $X \in L^2_{\mathcal{F}_T}$ and $Y \in L^2_{\mathcal{F}_T}$, respectively. Then X and Y are comonotonic*[2] *if and only if*

$$\max_{\psi \sim Y} \mathbf{E}[X\psi] = \mathbf{E}[XY] = \int_0^1 Q_X(t) Q_Y(t) \, \mathrm{d}t.$$

Similarly, X and Y are anti-comonotonic if and only if

$$\min_{\psi \sim Y} \mathbf{E}[X\psi] = \mathbf{E}[XY] = \int_0^1 Q_X(t) Q_Y(1-t) \, \mathrm{d}t.$$

Proof. Since we can deal with the positive part and negative part of X and Y separately, here we assume they are both nonnegative without loss of generality, and assume $Y^* \sim Y$ is comonotonic with X. Using Fubini's Theorem, Layer Cake Rep-

[2] See, e.g., [20, 21] for the properties of comonotonic and anti-comonotonic random variables.

resentation and Theorem 3.1.3, for any $\psi \sim Y$, we have

$$\begin{aligned}
\mathbf{E}[X\psi] = \mathbf{E}[XY] &= \mathbf{E}\int_0^{+\infty} \mathbb{1}_{X(\varpi)\geq t}\,\mathrm{d}t \int_0^{+\infty} \mathbb{1}_{Y(\varpi)\geq s}\,\mathrm{d}s \\
&= \int_0^{+\infty}\int_0^{+\infty} \mathbf{E}[\mathbb{1}_{X(\varpi)\geq t}\mathbb{1}_{Y(\varpi)\geq s}]\,\mathrm{d}t\,\mathrm{d}s \\
&= \int_0^{+\infty}\int_0^{+\infty} \mathbf{P}(X(\varpi)\geq t, Y(\varpi)\geq s)\,\mathrm{d}t\,\mathrm{d}s \\
&\leq \int_0^{+\infty}\int_0^{+\infty} \min\{\mathbf{P}(X(\varpi)\geq t), \mathbf{P}(Y(\varpi)\geq s)\}\,\mathrm{d}t\,\mathrm{d}s \\
&= \int_0^{+\infty}\int_0^{+\infty} \min\{\mathbf{P}(X(\varpi)\geq t), \mathbf{P}(Y^*(\varpi)\geq s)\}\,\mathrm{d}t\,\mathrm{d}s \\
&= \int_0^{+\infty}\int_0^{+\infty} \mathbf{P}(X(\varpi)\geq t, Y^*(\varpi)\geq s)\,\mathrm{d}t\,\mathrm{d}s = \mathbf{E}[XY^*].
\end{aligned}$$

From Theorem 3.1.3, we can naturally deduce that

$$\min\{\mathbf{P}(X(\varpi)\geq t), \mathbf{P}(Y(\varpi)\geq s)\} = \mathbf{P}(X(\varpi)\geq t, Y(\varpi)\geq s)$$

for almost everywhere $(t,s) \in \mathbb{R}_+^2$, if and only if X and Y are comonotonic. In order to maximize $\mathbf{E}[X\psi]$, for any $\psi \sim Y$, X and ψ have to be comonotonic. The anti-comonotonic part could also be proved in a similar way. □

Now we are ready to give an explicit expression of $J_0(\cdot)$.

Lemma 3.1.5. *The function $J_0(\cdot)$ is law-invariant*[3] *on $L^2_{\mathcal{F}_T}$ and*

$$J_0(X) = \int_0^1 (Q_X(t) + \mathscr{Q}(t))^2\,\mathrm{d}t, \tag{3.4}$$

where $Q_X(\cdot)$ denotes the quantile of X.

[3] A functional is called law-invariant if it gives the same value for any two identically distributed random variables.

Proof. Observing that for any $X \in L^2_{\mathcal{F}_T}$,

$$J_0(X) = \max_{Y \sim \vartheta} \mathbf{E}[(X+Y)^2] = \max_{Y \sim \vartheta} (\mathbf{E}[X^2] + \mathbf{E}[Y^2] + 2\mathbf{E}[XY])$$
$$= \mathbf{E}[X^2] + \int_0^1 \mathscr{Q}^2(t)\,\mathrm{d}t + 2 \max_{Y \sim \vartheta} \mathbf{E}[XY];$$

and applying Lemma 3.1.4 to the last term, we deduce that

$$J_0(X) = \mathbf{E}[X^2] + \int_0^1 \mathscr{Q}^2(t)\,\mathrm{d}t + 2 \max_{Y \sim \vartheta} \mathbf{E}[XY]$$
$$= \int_0^1 Q_X(t)^2\,\mathrm{d}t + \int_0^1 \mathscr{Q}^2(t)\,\mathrm{d}t + 2\int_0^1 Q_X(t)\mathscr{Q}(t)\,\mathrm{d}t = \int_0^1 (Q_X(t) + \mathscr{Q}(t))^2\,\mathrm{d}t.$$

And this means that $J_0(\cdot)$ is law-invariant. □

Remark 2. *It seems very hard to show the convexity of $J_0(\cdot)$ via the expression* (3.4)*. It is also interesting to investigate whether $J_0(\cdot)$ would be still convex if $\mathscr{Q}(\cdot)$ in* (3.4) *was replaced by a general non monotone function and the financial meaning of such cost functional.*

Because $J_0(\cdot)$ is convex, applying the Lagrange method, Problem (3.3) is equivalent to

$$\min_{X \in L^2_{\mathcal{F}_T}} \quad J_\lambda(X), \tag{3.5}$$
$$\text{subject to} \quad \mathbf{E}[\rho X] = x,$$

for some real scalar λ, where

$$J_\lambda(X) := J_0(X) - \lambda(\mathbf{E}[X] - d) = \max_{Y \sim \vartheta} \quad \mathbf{E}[(X+Y)^2] - \lambda(\mathbf{E}[X] - d)$$
$$= \int_0^1 (Q_X(t) + \mathscr{Q}(t))^2\,\mathrm{d}t - \lambda \int_0^1 Q_X(t)\,\mathrm{d}t + \lambda d. \tag{3.6}$$

Then it follows immediately from the above lemmas that

Lemma 3.1.6. *The function $J_\lambda(\cdot)$ is finite, convex and law-invariant on $L^2_{\mathcal{F}_T}$.*

Let $V_\lambda(x)$ denote the optimal value of Problem (3.5).

Lemma 3.1.7. *The function $V_\lambda(\cdot)$ is finite and convex on $(0, +\infty)$.*

Proof. Since $\frac{x}{\mathbf{E}[\rho]}$ is a feasible solution of Problem (3.5), by definition, we have $V_\lambda(x) \leqslant J_\lambda(\frac{x}{\mathbf{E}[\rho]}) < +\infty$. For any $\alpha \in (0,1)$, $X_1 \in L^2_{\mathcal{F}_T}$ and $X_2 \in L^2_{\mathcal{F}_T}$ such that $\mathbf{E}[\rho X_1] = x_1 > 0$, $\mathbf{E}[\rho X_2] = x_2 > 0$, we have $\mathbf{E}[\rho(\alpha X_1 + (1-\alpha)X_2)] = \alpha x_1 + (1-\alpha)x_2$ and consequently,

$$V_\lambda(\alpha x_1 + (1-\alpha)x_2) \leqslant J_\lambda(\alpha X_1 + (1-\alpha)X_2) \leqslant \alpha J_\lambda(X_1) + (1-\alpha)J_\lambda(X_2),$$

where the last inequality is due to the convexity of $J_\lambda(\cdot)$. Because X_1 and X_2 are arbitrary chosen, the convexity of $V_\lambda(\cdot)$ is thus proved. □

Remark 3. *Because $J_\lambda(\frac{x}{\mathbf{E}[\rho]})$ is a quadratic function in x, $V_\lambda(\cdot)$ is at most quadratic growth.*

If we apply the Lagrange method to solve Problem (3.5), then we need to consider

$$\min_{X \in L^2_{\mathcal{F}_T}} \quad J_\lambda(X) - \mu(\mathbf{E}[\rho X] - x).$$

The sign of μ thus plays an important role in this problem. To determine the sign of μ, we have the following key observation. It is well-known, μ is in fact equal to $V'_\lambda(x)$, so the sign of μ is the same as that of $V'_\lambda(x)$. On the other hand, since $V_\lambda(\cdot)$ is convex, to determine the sign of $V'_\lambda(x)$, it suffices to determine the minimizer of $V_\lambda(\cdot)$. This will be done in the following section.

3.2 Minimizer of $V_\lambda(\cdot)$

We want to determine the minimizer of $V_\lambda(\cdot)$ on $\mathbb{R}$. To this end, consider the unconstraint version of Problem (3.5)

$$\min_{X \in L^2_{\mathcal{F}_T}} \quad J_\lambda(X). \tag{3.7}$$

The optimal value of this problem clearly provides a lower bound for that of Problem (3.5). Let us show that this optimal value is in fact the infimum of $V_\lambda(\cdot)$.

Suppose X^* is an optimal solution of Problem (3.7), we set

$$x^* := \mathbf{E}[\rho X^*]. \tag{3.8}$$

Then X^* is clearly also an optimal solution of problem

$$\min_{X\in L^2_{\mathcal{F}_T}} \quad J_\lambda(X),$$

$$\text{subject to} \quad \mathbf{E}[\rho X] = x^*,$$

whose optimal value is $V_\lambda(x^*)$ by definition. Therefore,

$$V_\lambda(x^*) = J_\lambda(X^*) = \min_{X\in L^2_{\mathcal{F}_T}} \ J_\lambda(X) \leqslant V_\lambda(y), \quad \forall\, y \in \mathbb{R}. \tag{3.9}$$

This means x^* is a minimizer of $V_\lambda(\cdot)$.

Remark 4. *We do not know if the minimizer of Problem* (3.7) *is unique or not. We may take any one if it is not unique, though we will show below that it in fact is unique.*

Because $V_\lambda(\cdot)$ is convex by Lemma 3.1.7, it follows that

Proposition 3.2.1. *The function $V_\lambda(\cdot)$ is increasing on $[x^*, +\infty)$ and decreasing on $(-\infty, x^*]$.*

Remark 5. *We remark that $V_\lambda(\cdot)$ may not be strictly monotonic as x^* may not be unique.*

Now let us focus on Problem (3.7). Recall (3.6),

$$\begin{aligned}\min_{X\in L^2_{\mathcal{F}_T}} J_\lambda(X) &= \int_0^1 (Q_X(t) + \mathscr{Q}(t))^2 \,\mathrm{d}t - \lambda \int_0^1 Q_X(t)\,\mathrm{d}t + \lambda d \\ &= \int_0^1 \left(Q_X(t) + \mathscr{Q}(t) - \tfrac{1}{2}\lambda\right)^2 \mathrm{d}t - \frac{1}{4}\lambda^2 + \lambda \int_0^1 \mathscr{Q}(t)\,\mathrm{d}t + \lambda d. \end{aligned} \tag{3.10}$$

Observe that the objective in above only depends on the quantile of X. We denoted by $\mathcal{Q}$ the set of all quantiles generated by $X \in L^2_{\mathcal{F}_T}$:

$$\mathcal{Q} := \left\{ Q(\cdot) \ : \ Q(\cdot) \text{ is the quantile of } X \in L^2_{\mathcal{F}_T} \right\}.$$

It is not hard to show that $Q(\cdot)$ is the quantile of some $X \in L^2_{\mathcal{F}_T}$ if and only if it is a right-continuous increasing function such that $\int_0^1 Q^2(t)\,\mathrm{d}t < \infty$. Thus we conclude that

$$\mathcal{Q} = \left\{ Q(\cdot) \ : \ Q(\cdot) \text{ is a quantile such that } \int_0^1 Q^2(t)\,\mathrm{d}t < \infty \right\}.$$

We now consider the quantile formulation[4] of Problem (3.10), that is,

$$\min_{Q\in\mathcal{Q}} \int_0^1 \Big(Q(t) + \mathscr{Q}(t) - \tfrac{1}{2}\lambda \Big)^2 \,\mathrm{d}t - \frac{1}{4}\lambda^2 + \lambda \int_0^1 \mathscr{Q}(t)\,\mathrm{d}t + \lambda d. \tag{3.11}$$

Observe that the quantile of any optimal solution of Problem (3.10) solves the above problem. With the help of following lemma, we could derive the solution to Problem (3.11).

Lemma 3.2.2. *Let $f(\cdot)$ be a square integrable increasing function on $[0,1]$ and $c \in [0,1]$. Then*

$$\min_{a\leqslant b} \int_0^c \Big(a + f(t)\Big)^2 \mathrm{d}t + \int_c^1 \Big(b + f(t)\Big)^2 \mathrm{d}t = \min_a \int_0^1 \Big(a + f(t)\Big)^2 \mathrm{d}t.$$

Proof. Denote

$$g(a) = \int_0^c \Big(a + f(t)\Big)^2 \mathrm{d}t, \qquad h(b) = \int_c^1 \Big(b + f(t)\Big)^2 \mathrm{d}t,$$

and the minimizers of g and h are respectively denoted by

$$\overline{a} = -\frac{1}{c}\int_0^c f(t)\,\mathrm{d}t, \qquad \overline{b} = -\frac{1}{1-c}\int_c^1 f(t)\,\mathrm{d}t.$$

[4] See Xu [66] for more about the quantile formulation problem.

We remark that $\overline{a} \geqslant \overline{b}$ as f is increasing. Note both g and h are strictly decreasing on $(-\infty, \overline{a}] \cap (-\infty, \overline{b}] = (-\infty, \overline{b}]$ and strictly increasing on $[\overline{a}, \infty) \cap [\overline{b}, \infty) = [\overline{a}, \infty)$, therefore,

$$g(a) + h(b) > g(\overline{b}) + h(\overline{b}), \quad a \leqslant b < \overline{b},$$

and

$$g(a) + h(b) > g(\overline{a}) + h(\overline{a}), \quad \overline{a} < a \leqslant b.$$

Hence

$$\min_{a \leqslant b} \; g(a) + h(b) = \min_{\overline{b} \leqslant a \leqslant b \leqslant \overline{a}} g(a) + h(b).$$

On $[\overline{b}, \overline{a}]$, we have g is strictly decreasing and h is strictly increasing, therefore,

$$\min_{\overline{b} \leqslant a \leqslant b \leqslant \overline{a}} g(a) + h(b) = \min_{\overline{b} \leqslant a = b \leqslant \overline{a}} g(a) + h(b) = \min_{\overline{b} \leqslant a \leqslant \overline{a}} g(a) + h(a).$$

Thus,

$$\min_{a \leqslant b} g(a) + h(b) = \min_{\overline{b} \leqslant a \leqslant \overline{a}} g(a) + h(a) \geqslant \min_{a} \; g(a) + h(a) = \min_{a} \int_0^1 \Big(a + f(t)\Big)^2 \,\mathrm{d}t.$$

On the other hand,

$$\min_{a \leqslant b} g(a) + h(b) \leqslant \min_{a = b} g(a) + h(b) = \min_{a} \; g(a) + h(a) = \min_{a} \int_0^1 \Big(a + f(t)\Big)^2 \,\mathrm{d}t.$$

The desired result follows. □

Now we turn to solve the Problem (3.11).

Theorem 3.2.3. *The unique optimal solution of Problem* (3.11) *is given by*

$$Q_0(t) := -\int_0^1 \mathscr{Q}(s) \,\mathrm{d}s + \tfrac{1}{2}\lambda, \quad \forall\, t \in (0,1).$$

Here We provide two proofs of this result below.

PROOF OF THEOREM 3.2.3 METHOD 1. Let

$$\widehat{\mathscr{Q}} = \{Q(\cdot) \in \mathscr{Q} \ : \ Q(\cdot) \text{ is continuous}\}.$$

For any $Q \in \widehat{\mathscr{Q}}$, set

$$t_Q = \inf\Big\{t \in (0,1) : Q(t) + \mathscr{Q}(t) - \tfrac{1}{2}\lambda \geqslant 0\Big\}.$$

Because both $Q(t)$ and $\mathscr{Q}(t)$ are increasing, we conclude that

$$Q(t)+\mathscr{Q}(t)-\tfrac{1}{2}\lambda \leqslant Q(t_Q-)+\mathscr{Q}(t)-\tfrac{1}{2}\lambda \leqslant Q(t_Q-)+\mathscr{Q}(t_Q-)-\tfrac{1}{2}\lambda \leqslant 0, \quad \forall\, t \in (0, t_Q), \tag{3.12}$$

and

$$Q(t)+\mathscr{Q}(t)-\tfrac{1}{2}\lambda \geqslant Q(t_Q)+\mathscr{Q}(t)-\tfrac{1}{2}\lambda \geqslant Q(t_Q)+\mathscr{Q}(t_Q)-\tfrac{1}{2}\lambda \geqslant 0, \quad \forall\, t \in [t_Q, 1). \tag{3.13}$$

Since $Q \in \widehat{\mathscr{Q}}$ is continuous, we conclude that $Q(t_Q-) = Q(t_Q)$ and consequently,

$$\Big|Q(t) + \mathscr{Q}(t) - \tfrac{1}{2}\lambda\Big| \geqslant \Big|Q(t_Q) + \mathscr{Q}(t) - \tfrac{1}{2}\lambda\Big|, \quad \forall\, t \in (0,1).$$

Thus

$$\begin{aligned}\int_0^1 \Big(Q(t) + \mathscr{Q}(t) - \tfrac{1}{2}\lambda\Big)^2 \,\mathrm{d}t &\geqslant \int_0^1 \Big(Q(t_Q) + \mathscr{Q}(t) - \tfrac{1}{2}\lambda\Big)^2 \,\mathrm{d}t \\ &\geqslant \min_a \int_0^1 \Big(a + \mathscr{Q}(t) - \tfrac{1}{2}\lambda\Big)^2 \,\mathrm{d}t = \int_0^1 \mathscr{Q}^2(t)\,\mathrm{d}t - \Big(\int_0^1 \mathscr{Q}(t)\,\mathrm{d}t\Big)^2.\end{aligned}$$

Therefore, we proved

$$\min_{Q\in\widehat{\mathscr{Q}}} \int_0^1 \Big(Q(t) + \mathscr{Q}(t) - \tfrac{1}{2}\lambda\Big)^2 \,\mathrm{d}t \geqslant \int_0^1 \mathscr{Q}^2(t)\,\mathrm{d}t - \Big(\int_0^1 \mathscr{Q}(t)\,\mathrm{d}t\Big)^2.$$

Since $\widehat{\mathcal{Q}}$ is a density subset of $\mathcal{Q}$ under the L^2 normal and the mapping

$$Q(\cdot) \mapsto \sqrt{\int_0^1 \Big(Q(t) + \mathscr{Q}(t) - \tfrac{1}{2}\lambda\Big)^2 \,\mathrm{d}t}$$

is continuous under the L^2 normal, we have

$$\min_{Q\in\mathcal{Q}} \int_0^1 \Big(Q(t) + \mathscr{Q}(t) - \tfrac{1}{2}\lambda\Big)^2 \,\mathrm{d}t \geqslant \int_0^1 \mathscr{Q}^2(t)\,\mathrm{d}t - \Big(\int_0^1 \mathscr{Q}(t)\,\mathrm{d}t\Big)^2.$$

We note that the right hand side lower bound is achieved at Q_0. So Q_0 is an optimal solution of Problem (3.11). Clearly, it is also the unique one by the strictly convexity of square function. $\square$

Proof of Theorem 3.2.3 Method 2. For any $Q \in \mathcal{Q}$, by (3.12) and (3.13), we have

$$\Big|Q(t) + \mathscr{Q}(t) - \tfrac{1}{2}\lambda\Big| \geqslant \Big|\widetilde{Q}(t) + \mathscr{Q}(t) - \tfrac{1}{2}\lambda\Big|, \quad \forall\, t \in (0,1),$$

where

$$\widetilde{Q}(t) = \begin{cases} Q(t_Q-), & t \in (0, t_Q); \\ Q(t_Q), & t \in [t_Q, 1). \end{cases}$$

Therefore, observing that $\widetilde{Q}(\cdot)$ is increasing and takes at most two different values, we thus have

$$\begin{aligned}
\int_0^1 \Big(Q(t) + \mathscr{Q}(t) - \tfrac{1}{2}\lambda\Big)^2 \,\mathrm{d}t &\geqslant \int_0^1 \Big(\widetilde{Q}(t) + \mathscr{Q}(t) - \tfrac{1}{2}\lambda\Big)^2 \,\mathrm{d}t \\
&\geqslant \min_{c\in(0,1),\, a\leqslant b} \int_0^1 \Big(a\mathbb{1}_{(0,c)}(t) + b\mathbb{1}_{[c,1)}(t) + \mathscr{Q}(t) - \tfrac{1}{2}\lambda\Big)^2 \,\mathrm{d}t \\
&= \min_{c\in(0,1)} \min_{a\leqslant b} \int_0^c \Big(a + \mathscr{Q}(t) - \tfrac{1}{2}\lambda\Big)^2 \,\mathrm{d}t + \int_c^1 \Big(b + \mathscr{Q}(t) - \tfrac{1}{2}\lambda\Big)^2 \,\mathrm{d}t \\
&= \min_{c\in(0,1)} \min_{a} \int_0^1 \Big(a + \mathscr{Q}(t) - \tfrac{1}{2}\lambda\Big)^2 \,\mathrm{d}t = \int_0^1 \mathscr{Q}^2(t)\,\mathrm{d}t - \Big(\int_0^1 \mathscr{Q}(t)\,\mathrm{d}t\Big)^2,
\end{aligned}$$

where we used Lemma 3.2.2 to get the second last equation. The left is the same as method 1. □

Because the unique solution of Problem (3.11) is a constant, the optimal solution of problem (3.10) (as well as (3.7)) is also unique and is a constant given by

$$X^* = -\int_0^1 \mathscr{Q}(s)\,\mathrm{d}s + \tfrac{1}{2}\lambda.$$

And consequently, the minimizer x^* of $V_\lambda(\cdot)$ on $\mathbb{R}$ is uniquely given by

$$x^* = \mathbf{E}[\rho X^*] = -\mathbf{E}[\rho]\int_0^1 \mathscr{Q}(s)\,\mathrm{d}s + \tfrac{1}{2}\lambda\mathbf{E}[\rho],$$

and the corresponding minimum value is

$$V_\lambda(x^*) = J_\lambda(X^*) = \int_0^1 \mathscr{Q}^2(t)\,\mathrm{d}t - \left(\int_0^1 \mathscr{Q}(t)\,\mathrm{d}t\right)^2 - \frac{1}{4}\lambda^2 + \lambda\int_0^1 \mathscr{Q}(t)\,\mathrm{d}t + \lambda d.$$

Because $V_\lambda(\cdot)$ is a convex function and admits a unique minimizer x^*, we thus conclude that

Corollary 3.2.4. *Let*

$$x^* = -\mathbf{E}[\rho]\int_0^1 \mathscr{Q}(s)\,\mathrm{d}s + \tfrac{1}{2}\lambda\mathbf{E}[\rho]. \tag{3.14}$$

Then the function $V_\lambda(\cdot)$ is strictly increasing on $[x^*, +\infty)$ and strictly decreasing on $(-\infty, x^*]$.

3.3 Quantile Formulation

Because $J_\lambda(\cdot)$ is convex, applying the Lagrange method, we conclude that

Proposition 3.3.1. *Problem* (3.5) *is equivalent to*

$$\min_{X\in L^2_{\mathcal{F}_T}} J_\lambda(X) - \mu(\mathbf{E}[\rho X] - x), \tag{3.15}$$

for some μ satisfying

$$\mu \begin{cases} > 0, & \textit{if } x > x^*; \\ = 0, & \textit{if } x = x^*; \\ < 0, & \textit{if } x < x^*, \end{cases}$$

where

$$x^* = -\mathbf{E}[\rho] \int_0^1 \mathscr{Q}(s)\,\mathrm{d}s + \tfrac{1}{2}\lambda \mathbf{E}[\rho].$$

Proof. It follows from Corollary 3.2.4. □

Recalling (3.6), Problem (3.15) is

$$\min_{X \in L^2_{\mathcal{F}_T}} \int_0^1 (Q_X(t) + \mathscr{Q}(t))^2\,\mathrm{d}t - \lambda \int_0^1 Q_X(t)\,\mathrm{d}t - \mu \mathbf{E}[\rho X] + \lambda d + \mu x, \tag{3.16}$$

and this is equal to

$$\begin{aligned} &\min_{X \in L^2_{\mathcal{F}_T}} \min_{\psi \sim X} \int_0^1 (Q_\psi(t) + \mathscr{Q}(t))^2\,\mathrm{d}t - \lambda \int_0^1 Q_\psi(t)\,\mathrm{d}t - \mu \mathbf{E}[\rho\psi] + \lambda d + \mu x \\ &\quad = \min_{X \in L^2_{\mathcal{F}_T}} \min_{\psi \sim X} \int_0^1 (Q_X(t) + \mathscr{Q}(t))^2\,\mathrm{d}t - \lambda \int_0^1 Q_X(t)\,\mathrm{d}t - \mu \mathbf{E}[\rho\psi] + \lambda d + \mu x \\ &= \min_{X \in L^2_{\mathcal{F}_T}} \left(\int_0^1 (Q_X(t) + \mathscr{Q}(t))^2\,\mathrm{d}t - \lambda \int_0^1 Q_X(t)\,\mathrm{d}t - \max_{\psi \sim X}\ \mu \mathbf{E}[\rho\psi] + \lambda d + \mu x \right). \end{aligned} \tag{3.17}$$

The inner optimization problem is solved by Lemma 3.1.4,

$$\max_{\psi \sim X}\ \mu \mathbf{E}[\rho\psi] = \left\{ \begin{matrix} \mu \int_0^1 Q_X(t) Q_\rho(t)\,\mathrm{d}t, & \text{if } \mu \geqslant 0; \\ \mu \int_0^1 Q_X(t) Q_\rho(1-t)\,\mathrm{d}t, & \text{if } \mu < 0; \end{matrix} \right\} = |\mu| \int_0^1 Q_X(t)\eta(t)\,\mathrm{d}t, \tag{3.18}$$

where

$$\eta(t) := Q_\rho(t)\mathbb{1}_{\{\mu \geqslant 0\}} - Q_\rho(1-t)\mathbb{1}_{\{\mu < 0\}} = Q_\rho(t)\mathbb{1}_{\{x \geqslant x^*\}} - Q_\rho(1-t)\mathbb{1}_{\{x < x^*\}}, \tag{3.19}$$

and the last equation is due to Proposition 3.3.1. And remark that the optimal ψ of (3.18) is comonotonic with ρ if $\mu \geqslant 0$, and anti-comonotonic if $\mu < 0$.

Hence Problem (3.16) reduces to

$$\min_{X \in L^2_{\mathcal{F}_T}} \int_0^1 (Q_X(t) + \mathscr{Q}(t))^2 \,\mathrm{d}t - \lambda \int_0^1 Q_X(t) \,\mathrm{d}t - |\mu| \int_0^1 Q_X(t)\eta(t) \,\mathrm{d}t + \lambda d + \mu x,$$

or

$$\min_{X \in L^2_{\mathcal{F}_T}} \int_0^1 (Q_X(t) - \zeta(t))^2 \,\mathrm{d}t + C_0, \tag{3.20}$$

where

$$\zeta(t) := \frac{1}{2}\lambda + \frac{1}{2}|\mu|\eta(t) - \mathscr{Q}(t), \tag{3.21}$$

and C_0 does not depend on X and is given by

$$C_0 = -\frac{\mu^2}{4} \int_0^1 \eta^2(t) \,\mathrm{d}t - \frac{1}{4}\lambda^2 + \lambda d + \mu x.$$

We remark that the optimal solution of Problem (3.20) is comonotonic with ρ if $\mu \geqslant 0$, and anti-comonotonic if $\mu < 0$.

The quantile formulation of Problem (3.20) is

$$\min_{Q \in \mathcal{Q}} \int_0^1 (Q(t) - \zeta(t))^2 \,\mathrm{d}t + C_0. \tag{3.22}$$

How to solve this quantile problem could be quite tricky. Many methods have been proposed to solve such quantile problems which have been discussed elaborately in our introduction. Among them, calculus of variation method introduced by Xia and Zhou [63] has been favored most nowadays. Since it is still quite hard to follow and has got relatively tight assumptions, we use relaxation method initiated by Xu [65] to solve our model in the next chapter.

Chapter 4

Derivation of Final Result

This is the most important chapter in this thesis, which is devoted to finally solving the whole model. In this chapter, we first introduce the relaxation method to tackle the quantile problem left by the previous chapter. The relaxation method has abandoned the monotonicity assumptions, which enabled its more extensive application in quantile problems. After derivation of the closed-form solutions to the model, we further discussed some basic properties of its mean-variance frontier in the end of this chapter.

4.1 Relaxation Method

In this chapter, we will use the relaxation method introduced by Xu [65] to solve this problem. Remark that if $\overline{Q}(\cdot)$ is an optimal solution of Problem (3.22), then the optimal solution of Problem (3.20) is given by

$$\overline{X} = \begin{cases} \overline{Q}(U), & \text{if } \mu \geqslant 0; \\ \overline{Q}(1-U), & \text{if } \mu < 0, \end{cases}$$

where U is any random variable uniformly distributed on $(0,1)$ and is comonotonic with ρ. Rewrite Problem (3.22) as

$$\begin{aligned} \min_{Q\in\mathcal{Q}} \int_0^1 (Q(t)-\zeta(t))^2 \, \mathrm{d}t + C_0 &= \int_0^1 \Big(Q^2(t) - 2Q(t)\zeta(t)\Big) \, \mathrm{d}t + \int_0^1 \zeta^2(t) \, \mathrm{d}t + C_0 \\ &= \int_0^1 \Big(Q^2(t) + 2Q(t)\varphi'(t)\Big) \, \mathrm{d}t + C_1, \quad (4.1) \end{aligned}$$

where

$$\varphi(t) := \int_t^1 \zeta(s) \, \mathrm{d}s, \tag{4.2}$$

and C_1 does not depend on $Q \in \mathcal{Q}$ and is given by

$$C_1 = \int_0^1 \zeta^2(t) \, \mathrm{d}t + C_0 = \int_0^1 \zeta^2(t) \, \mathrm{d}t - \frac{\mu^2}{4} \int_0^1 \eta^2(t) \, \mathrm{d}t - \frac{1}{4}\lambda^2 + \lambda d + \mu x.$$

We plan to use the relaxation method introduced by Xu [65] to solve Problem (4.1). The key idea is to replace $\varphi(\cdot)$ by some function $\delta(\cdot)$ in Problem (4.1) so that:

1. The new Lagrangian gives a lower bound to that in Problem (4.1);

2. The new problem can be solved by point-wise minimizing the new Lagrangian;

3. There is no gap between the new and old Lagrangians in the point-wise solution.

This approach allows us to solve the problem completely without making any assumptions on the function $\varphi(\cdot)$.

Theorem 4.1.1. *The unique optimal solution of Problem* (3.22) *is*

$$\overline{Q}(t) := -\delta'(t), \quad t \in [0,1],$$

where $\delta(\cdot)$ *is the concave envelope of* $\varphi(\cdot)$ *on* $[0,1]$ *and given by*

$$\delta(t) := \sup_{0\leqslant a\leqslant t\leqslant b\leqslant 1} \frac{(b-t)\varphi(a)+(t-a)\varphi(b)}{b-a}, \quad t \in [0,1].$$

Consequently, the optimal solution of Problem (3.20) *as well as Problem* (3.15) *is given by*

$$\overline{X} = \begin{cases} \overline{Q}(U), & \text{if } \mu \geqslant 0; \\ \overline{Q}(1-U), & \text{if } \mu < 0, \end{cases} \tag{4.3}$$

where U *is any random variable uniformly distributed on* $(0,1)$ *and is comonotonic with* ρ*.*

Proof. Let $\delta(\cdot)$ and $\overline{Q}(\cdot)$ be defined as in the assumption. Then $\delta(\cdot)$ is an absolutely continuous function dominating $\varphi(\cdot)$ on $[0,1]$. Observing $\delta(0) = \varphi(0)$ and $\delta(1) = \varphi(1) = 0$ and using Fubini's theorem,

$$\int_0^1 Q(t)(\varphi'(t)-\delta'(t))\,\mathrm{d}t = \int_0^1 \Big(\delta(t)-\varphi(t)\Big)\,\mathrm{d}Q(t) \geqslant 0, \tag{4.4}$$

for every $Q(\cdot) \in \mathcal{Q}$. The above inequality leads to

$$\begin{aligned} \int_0^1 \Big(Q^2(t)+2Q(t)\varphi'(t)\Big)\,\mathrm{d}t &\geqslant \int_0^1 \Big(Q^2(t)+2Q(t)\delta'(t)\Big)\,\mathrm{d}t \\ &= \int_0^1 \Big(Q(t)+\delta'(t)\Big)^2 - \Big(\delta'(t)\Big)^2\,\mathrm{d}t \geqslant -\int_0^1 \Big(\delta'(t)\Big)^2\,\mathrm{d}t \\ &= \Big(\overline{Q}^2(t)+2\overline{Q}(t)\delta'(t)\Big). \end{aligned} \tag{4.5}$$

To make $\overline{Q}(\cdot)$ an optimal solution of Problem (3.5), it suffices, by (4.5), to have

$$\int_0^1 \Big(\overline{Q}^2(t) + 2\overline{Q}(t)\varphi'(t)\Big)\,\mathrm{d}t = \int_0^1 \Big(\overline{Q}^2(t) + 2\overline{Q}(t)\delta'(t)\Big)\,\mathrm{d}t, \tag{4.6}$$

Observing $\delta(0) = \varphi(0)$ and $\delta(1) = \varphi(1) = 0$ and using Fubini's theorem, the above equality is equivalent to

$$0 = \int_0^1 \delta'(t)\Big(\varphi'(t) - \delta'(t)\Big)\,\mathrm{d}t = \int_0^1 \Big(\delta(t) - \varphi(t)\Big)\,\mathrm{d}\delta'(t).$$

This is true because $\delta'(\cdot)$ is a constant on any sub interval of $\{t \in [0,1] : \delta(t) > \varphi(t)\}$ by the definition of $\delta(\cdot)$. □

4.2 Lagrange Multipliers and Optimal Solution

Observe that Problem (3.15) and Problem (3.3) are equivalent if there exist λ and μ such that

$$\mathbf{E}[\rho\overline{X}] = x, \qquad \mathbf{E}[\overline{X}] = d,$$

where $\overline{X}$ is defined in (4.3). We next to show the existence of such λ and μ.

Observing that

$$\begin{aligned}\int_0^1 \eta(t)\,\mathrm{d}t &= \int_0^1 \Big(Q_\rho(t)\mathbb{1}_{\{\mu\geqslant 0\}} - Q_\rho(1-t)\mathbb{1}_{\{\mu<0\}}\Big)\,\mathrm{d}t\\ &= \int_0^1 Q_\rho(t)\,\mathrm{d}t\mathbb{1}_{\{\mu\geqslant 0\}} - \int_0^1 Q_\rho(1-t)\,\mathrm{d}t\mathbb{1}_{\{\mu<0\}}\\ &= \int_0^1 Q_\rho(t)\,\mathrm{d}t\mathbb{1}_{\{\mu\geqslant 0\}} - \int_0^1 Q_\rho(t)\,\mathrm{d}t\mathbb{1}_{\{\mu<0\}}\\ &= \mathrm{sgn}(\mu)\int_0^1 Q_\rho(t)\,\mathrm{d}t = \mathrm{sgn}(\mu)\mathbf{E}[\rho],\end{aligned} \tag{4.7}$$

where $\mathrm{sgn}(\mu) := \mathbb{1}_{\{\mu\geqslant 0\}} - \mathbb{1}_{\{\mu<0\}}$, so we have

$$\begin{aligned}\mathbf{E}[\overline{X}] &= \int_0^1 \overline{Q}(t)\,\mathrm{d}t = -\int_0^1 \delta'(t)\,\mathrm{d}t = \delta(0)-\delta(1) = \varphi(0)-\varphi(1) = \int_0^1 \zeta(t)\,\mathrm{d}t \\ &= \frac{1}{2}\lambda + \frac{1}{2}|\mu|\int_0^1 \eta(t)\,\mathrm{d}t - \int_0^1 \mathscr{Q}(t)\,\mathrm{d}t = \frac{1}{2}\lambda + \frac{1}{2}|\mu|\mathrm{sgn}(\mu)\mathbf{E}[\rho] - \int_0^1 \mathscr{Q}(t)\,\mathrm{d}t \\ &= \frac{1}{2}\lambda + \frac{1}{2}\mu\mathbf{E}[\rho] - \int_0^1 \mathscr{Q}(t)\,\mathrm{d}t. \quad (4.8)\end{aligned}$$

Hence, $\mathbf{E}[\overline{X}] = d$ if and only if

$$\lambda = 2d + 2\int_0^1 \mathscr{Q}(t)\,\mathrm{d}t - \mu\mathbf{E}[\rho], \tag{4.9}$$

which is henceforth assumed. In this case, by (4.8) and (4.9), we have

$$\delta(0) = \varphi(0) = \varphi(0) - \varphi(1) = d. \tag{4.10}$$

Now our problem reduces to finding μ such that $\mathbf{E}[\rho\overline{X}] = x$ under the condition (4.9).

We need some properties of the function $\delta(\cdot)$ for further discussion; their proofs will be provided accordingly.

Lemma 4.2.1. *The function δ is continuous and increasing with respect to μ on $[0, +\infty)$ and decreasing on $(-\infty, 0]$.*

Proof. We take (4.9) into the definition of φ to get

$$\begin{aligned}
\varphi(t) = \int_t^1 \zeta(s)\,\mathrm{d}s &= \int_t^1 \Big(\frac{1}{2}\lambda + \frac{1}{2}|\mu|\eta(s) - \mathscr{Q}(s)\Big)\,\mathrm{d}s \\
&= \int_t^1 \Big(d + \int_0^1 \mathscr{Q}(r)\,\mathrm{d}r - \frac{1}{2}\mu\mathbf{E}[\rho] + \frac{1}{2}|\mu|\eta(s) - \mathscr{Q}(s)\Big)\,\mathrm{d}s \\
&= (1-t)d + (1-t)\int_0^1 \mathscr{Q}(s)\,\mathrm{d}s - \frac{1}{2}(1-t)\mu\mathbf{E}[\rho] + \frac{1}{2}|\mu|\int_t^1 \eta(s)\,\mathrm{d}s - \int_t^1 \mathscr{Q}(s)\,\mathrm{d}s \\
&= (1-t)d + (1-t)\int_0^1 \mathscr{Q}(s)\,\mathrm{d}s - \frac{1}{2}(1-t)|\mu|\int_0^1 \eta(s)\,\mathrm{d}s + \frac{1}{2}|\mu|\int_t^1 \eta(s)\,\mathrm{d}s - \int_t^1 \mathscr{Q}(s)\,\mathrm{d}s \\
&= (1-t)d + (1-t)\int_0^1 \mathscr{Q}(s)\,\mathrm{d}s + \frac{1}{2}(1-t)|\mu|\Big(\frac{1}{1-t}\int_t^1 \eta(s)\,\mathrm{d}s - \int_0^1 \eta(s)\,\mathrm{d}s\Big) - \int_t^1 \mathscr{Q}(s)\,\mathrm{d}s.
\end{aligned} \tag{4.11}$$

Observe that η is an increasing function, so we have

$$\frac{1}{1-t}\int_t^1 \eta(s)\,\mathrm{d}s - \int_0^1 \eta(s)\,\mathrm{d}s \geqslant 0, \quad \forall\, t \in (0,1).$$

We conclude that, for each fixed $t \in (0,1)$, $\varphi(t)$ is increasing with respect to μ on $[0,+\infty)$ and decreasing on $(-\infty,0]$. Since δ is the concave envelope of φ, it is also increasing with respect to μ on $[0,+\infty)$ and decreasing on $(-\infty,0]$. □

Lemma 4.2.2. *We have*

$$\lim_{|\mu|\to\infty} \delta(t) = +\infty, \quad \forall\, t \in (0,1).$$

Proof. Suppose

$$\frac{1}{1-t}\int_t^1 \eta(s)\,\mathrm{d}s - \int_0^1 \eta(s)\,\mathrm{d}s = 0, \quad \forall\, t \in (0,1),$$

or equivalently,

$$\int_t^1 \eta(s)\,\mathrm{d}s = (1-t)\int_0^1 \eta(s)\,\mathrm{d}s, \quad \forall\, t \in (0,1).$$

Differentiating both sides with respect to t to get that $\eta(\cdot)$ is a constant. This means ρ is a constant, which contradicts our assumption. We conclude that there exists $t_0 \in (0,1)$ such that

$$\frac{1}{1-t_0}\int_{t_0}^{1} \eta(s)\,\mathrm{d}s - \int_0^1 \eta(s)\,\mathrm{d}s > 0;$$

and by (4.11),

$$\lim_{|\mu|\to\infty} \varphi(t_0) = +\infty.$$

Observe that δ is concave and (4.10), so

$$\delta(t) \geqslant \frac{t_0 - t}{t_0}\delta(0) + \frac{t}{t_0}\delta(t_0) \geqslant \frac{t_0 - t}{t_0} d + \frac{t}{t_0}\varphi(t_0), \quad \forall\, t \in [0, t_0];$$

$$\delta(t) \geqslant \frac{1-t}{1-t_0}\delta(t_0) + \frac{t-t_0}{1-t_0}\delta(1) \geqslant \frac{1-t}{1-t_0}\varphi(t_0), \quad \forall\, t \in [t_0, 1].$$

The claims follows immediately. □

Lemma 4.2.3. *We have*

$$\lim_{|\mu|\to 0} \delta(t) = (1-t)d, \quad \forall\, t \in [0,1].$$

In particular, if $\mu = 0$, then

$$\delta(t) = (1-t)d, \quad \forall\, t \in [0,1].$$

Proof. By the continuity of δ in μ, it suffices to show, when $\mu = 0$,

$$\delta(t) = (1-t)d, \quad \forall\, t \in [0,1].$$

Because δ is the smallest concave function dominating φ, it suffices to prove that

$$(1-t)d \geqslant \varphi(t), \quad \forall\, t \in [0,1].$$

In fact, using $\mu = 0$ and (4.11), we have

$$\varphi(t) - (1-t)d = (1-t)\int_0^1 \mathscr{Q}(s)\,\mathrm{d}s - \int_t^1 \mathscr{Q}(s)\,\mathrm{d}s \leqslant 0,$$

where the last inequality is due to that $\mathscr{Q}$ is an increasing function. □

Now we are ready to show that

Proposition 4.2.4. *For any $x > 0$ and d, there exist λ and μ such that $\mathbf{E}[\rho\overline{X}] = x$ and $\mathbf{E}[\overline{X}] = d$. Moreover,*

$$\lambda = 2d + 2\int_0^1 \mathscr{Q}(t)\,\mathrm{d}t - \mu\mathbf{E}[\rho].$$

Proof. It suffices to show that there exists μ such that $\mathbf{E}[\rho\overline{X}] = x$ under the condition $\lambda = 2d + 2\int_0^1 \mathscr{Q}(t)\,\mathrm{d}t - \mu\mathbf{E}[\rho]$.

We consider three different cases. They are corresponding to the lower, middle and upper parts of the mean-variance frontier.

(†) : $d < \frac{1}{\mathbf{E}[\rho]}x$. We will show that there exists $\mu > 0$ such that $\mathbf{E}[\rho\overline{X}] = x$. Observe that if $\mu > 0$

$$\mathbf{E}[\rho\overline{X}] = \int_0^1 \overline{Q}(t)Q_\rho(t)\,\mathrm{d}t = -\int_0^1 \delta'(t)Q_\rho(t)\,\mathrm{d}t = \int_0^1 \delta(t)\,\mathrm{d}Q_\rho(t),$$

where we used Fubini's theorem and $Q_\rho(0) = \delta(1) = 0$. Applying Lemma 4.2.1, Lemma 4.2.2 and Lemma 4.2.3 and using the monotone convergence theorem,

$$\lim_{\mu\to+\infty} \mathbf{E}[\rho\overline{X}] = \lim_{\mu\to+\infty} \int_0^1 \delta(t)\,\mathrm{d}Q_\rho(t) = \int_0^1 \lim_{\mu\to+\infty} \delta(t)\,\mathrm{d}Q_\rho(t) = +\infty,$$

and

$$\lim_{\mu\to0+} \mathbf{E}[\rho\overline{X}] = \lim_{\mu\to0+} \int_0^1 \delta(t)\,\mathrm{d}Q_\rho(t) = \int_0^1 \lim_{\mu\to0+} \delta(t)\,\mathrm{d}Q_\rho(t)$$
$$= \int_0^1 (1-t)d\,\mathrm{d}Q_\rho(t) = d\int_0^1 Q_\rho(t)\,\mathrm{d}t = d\mathbf{E}[\rho] < x,$$

where we used Fubini's theorem to get the second last equality. Therefore, there exists $\mu > 0$ such that $\mathbf{E}[\rho\overline{X}] = x$.

(††) : $d = \frac{1}{\mathbf{E}[\rho]}x$. In this case, we take $\mu = 0$. Then by Lemma 4.2.3,

$$\overline{X} = \overline{Q}(U) = -\delta'(U) = d, \quad \mathbf{E}[\rho\overline{X}] = d\mathbf{E}[\rho] = x.$$

$(\dagger\dagger\dagger): \ d > \frac{1}{\mathbf{E}[\rho]}x$. Observe that if $\mu < 0$,

$$\begin{aligned}\mathbf{E}[\rho\overline{X}] &= \int_0^1 \overline{Q}(t)Q_\rho(1-t)\,\mathrm{d}t = -\int_0^1 \delta'(t)Q_\rho(1-t)\,\mathrm{d}t \\ &= -\int_0^1 \delta'(1-t)Q_\rho(t)\,\mathrm{d}t = \int_0^1 (\delta(1-t)-\delta(0))'Q_\rho(t)\,\mathrm{d}t \\ &= \int_0^1 (\delta(0)-\delta(1-t))\,\mathrm{d}Q_\rho(t) = \int_0^1 (d-\delta(1-t))\,\mathrm{d}Q_\rho(t),\end{aligned}$$

where we used the fact that $Q_\rho(0) = 0$ and Fubini's theorem to get the second last equality. Applying Lemma 4.2.1, Lemma 4.2.2 and Lemma 4.2.3 and using the monotone convergence theorem, we have

$$\begin{aligned}\lim_{\mu\to-\infty} \mathbf{E}[\rho\overline{X}] &= \lim_{\mu\to-\infty}\int_0^1 (d-\delta(1-t))\,\mathrm{d}Q_\rho(t) \\ &= \int_0^1 \lim_{\mu\to-\infty}(d-\delta(1-t))\,\mathrm{d}Q_\rho(t) = -\infty,\end{aligned}$$

and

$$\begin{aligned}\lim_{\mu\to0-} \mathbf{E}[\rho\overline{X}] &= \lim_{\mu\to0-}\int_0^1 (d-\delta(1-t))\,\mathrm{d}Q_\rho(t) \\ &= \int_0^1 \lim_{\mu\to0-}(d-\delta(1-t))\,\mathrm{d}Q_\rho(t) \\ &= \int_0^1 (d-td)\,\mathrm{d}Q_\rho(t) = d\int_0^1 Q_\rho(t)\,\mathrm{d}t = d\mathbf{E}[\rho] > x,\end{aligned}$$

where we used Fubini's theorem to get the last second equality. Therefore, there exists $\mu < 0$ such that $\mathbf{E}[\rho\overline{X}] = x$.

This completes the proof. □

From the proof, we can see that

Corollary 4.2.5. *The implied μ is increasing with respective to x on $\mathbb{R}$.*

This is useful for numerical calculation of the optimal solution and the mean-variance frontier. We also remark that

- When $d < \frac{1}{\mathbf{E}[\rho]}x$, $\overline{X}$ and ρ are comonotonic; the investor is risk-seeking because the initial target d is too low compared with the initial wealth x. This is consistent with the classical model and the optimal portfolio is on the mean-variance frontier but not efficient.

- When $d > \frac{1}{\mathbf{E}[\rho]}x$, $\overline{X}$ and ρ are anti-comonotonic; the investor is risk-averse because the initial target d is too high compared with the initial wealth x. This is also consistent with the classical model and the optimal portfolio is on the efficient frontier.

Putting all of the results obtained thus far together, we conclude that

Theorem 4.2.6. *The optimal solution of Problem* (3.1) *is given by*

$$\overline{X} = \begin{cases} -\delta'(U), & \text{if } d < \frac{1}{\mathbf{E}[\rho]}x; \\ \frac{1}{\mathbf{E}[\rho]}x, & \text{if } d = \frac{1}{\mathbf{E}[\rho]}x; \\ -\delta'(1-U), & \text{if } d > \frac{1}{\mathbf{E}[\rho]}x; \end{cases}$$

where U is any random variable uniformly distributed on $(0,1)$ and is comonotonic with ρ, the function $\delta(\cdot)$ is given by

$$\delta(t) = \sup_{0\leqslant a\leqslant t\leqslant b\leqslant 1} \frac{(b-t)\varphi(a) + (t-a)\varphi(b)}{b-a}, \quad t \in [0,1];$$

the function $\varphi(\cdot)$ *is given by*

$$\varphi(t) = (1-t)d + (1-t)\left(\int_0^1 \mathscr{Q}(t)\,\mathrm{d}t - \frac{1}{1-t}\int_t^1 \mathscr{Q}(s)\,\mathrm{d}s\right)$$

$$- \frac{1}{2}\mu(1-t)\left(\int_0^1 Q_\rho(s)\,\mathrm{d}s - \frac{1}{1-t}\int_t^1 Q_\rho(s)\,\mathrm{d}s\right)\mathbb{1}_{\{d\leqslant\frac{1}{\mathbf{E}[\rho]}x\}}$$

$$- \frac{1}{2}\mu(1-t)\left(\int_0^1 Q_\rho(s)\,\mathrm{d}s - \frac{1}{1-t}\int_0^{1-t} Q_\rho(s)\,\mathrm{d}s\right)\mathbb{1}_{\{d>\frac{1}{\mathbf{E}[\rho]}x\}}, \quad t\in[0,1];$$

and the implied constant μ *exists and is determined by* $\mathbf{E}[\rho\overline{X}] = x$.

Proof. We first notice that $\mu > 0$ is equivalent to $d < \frac{1}{\mathbf{E}[\rho]}x$, and $\mu < 0$ is equivalent to $d > \frac{1}{\mathbf{E}[\rho]}x$. Therefore, by (3.19),

$$\eta(t) = Q_\rho(t)\mathbb{1}_{\{\mu\geqslant 0\}} - Q_\rho(1-t)\mathbb{1}_{\{\mu<0\}} = Q_\rho(t)\mathbb{1}_{\{d\leqslant\frac{1}{\mathbf{E}[\rho]}x\}} - Q_\rho(1-t)\mathbb{1}_{\{d>\frac{1}{\mathbf{E}[\rho]}x\}},$$

and consequently, by (3.21),

$$\zeta(t) = \frac{1}{2}\lambda + \frac{1}{2}|\mu|\eta(t) - \mathscr{Q}(t) = \frac{1}{2}\lambda + \frac{1}{2}\mu Q_\rho(t)\mathbb{1}_{\{d\leqslant\frac{1}{\mathbf{E}[\rho]}x\}} + \frac{1}{2}\mu Q_\rho(1-t)\mathbb{1}_{\{d>\frac{1}{\mathbf{E}[\rho]}x\}} - \mathscr{Q}(t),$$

and by (4.2),

$$
\begin{aligned}
\varphi(t) = \int_t^1 \zeta(s)\,\mathrm{d}s &= \frac{1}{2}\lambda(1-t) + \frac{1}{2}\mu \int_t^1 Q_\rho(s)\,\mathrm{d}s \mathbb{1}_{\{d \leqslant \frac{1}{\mathbf{E}[\rho]}x\}} \\
&\quad + \frac{1}{2}\mu \int_t^1 Q_\rho(1-s)\,\mathrm{d}s \mathbb{1}_{\{d > \frac{1}{\mathbf{E}[\rho]}x\}} - \int_t^1 \mathscr{Q}(s)\,\mathrm{d}s \\
&= \left(d + \int_0^1 \mathscr{Q}(t)\,\mathrm{d}t - \frac{1}{2}\mu\mathbf{E}[\rho]\right)(1-t) + \frac{1}{2}\mu \int_t^1 Q_\rho(s)\,\mathrm{d}s \mathbb{1}_{\{d \leqslant \frac{1}{\mathbf{E}[\rho]}x\}} \\
&\quad + \frac{1}{2}\mu \int_0^{1-t} Q_\rho(s)\,\mathrm{d}s \mathbb{1}_{\{d > \frac{1}{\mathbf{E}[\rho]}x\}} - \int_t^1 \mathscr{Q}(s)\,\mathrm{d}s \\
&= (1-t)d + (1-t)\left(\int_0^1 \mathscr{Q}(t)\,\mathrm{d}t - \frac{1}{1-t}\int_t^1 \mathscr{Q}(s)\,\mathrm{d}s\right) \\
&\quad - \frac{1}{2}\mu(1-t)\left(\int_0^1 Q_\rho(s)\,\mathrm{d}s - \frac{1}{1-t}\int_t^1 Q_\rho(s)\,\mathrm{d}s\right)\mathbb{1}_{\{d \leqslant \frac{1}{\mathbf{E}[\rho]}x\}} \\
&\quad - \frac{1}{2}\mu(1-t)\left(\int_0^1 Q_\rho(s)\,\mathrm{d}s - \frac{1}{1-t}\int_0^{1-t} Q_\rho(s)\,\mathrm{d}s\right)\mathbb{1}_{\{d > \frac{1}{\mathbf{E}[\rho]}x\}}
\end{aligned}
$$

The left is easy to verify. □

Corollary 4.2.7. *Under the same assumption of Theorem 4.2.6, if ϑ is a constant, then $\varphi(t) = \delta(t)$ for all $t \in [0,1]$.*

Proof. Suppose $d \leqslant \frac{1}{\mathbf{E}[\rho]}x$. Then

$$
\begin{aligned}
\varphi(t) &= (1-t)d + (1-t)\left(\int_0^1 \mathscr{Q}(t)\,\mathrm{d}t - \frac{1}{1-t}\int_t^1 \mathscr{Q}(s)\,\mathrm{d}s\right) \\
&\quad - \frac{1}{2}\mu(1-t)\left(\int_0^1 Q_\rho(s)\,\mathrm{d}s - \frac{1}{1-t}\int_t^1 Q_\rho(s)\,\mathrm{d}s\right) \\
&= (1-t)d + (1-t)\int_0^1 \mathscr{Q}(t)\,\mathrm{d}t - \int_t^1 \mathscr{Q}(s)\,\mathrm{d}s - \frac{1}{2}\mu(1-t)\int_0^1 Q_\rho(s)\,\mathrm{d}s + \frac{1}{2}\mu\int_t^1 Q_\rho(s)\,\mathrm{d}s,
\end{aligned}
$$

and thus,

$$
\varphi'(t) = -d - \int_0^1 \mathscr{Q}(t)\,\mathrm{d}t + \mathscr{Q}(t) + \frac{1}{2}\mu\int_0^1 Q_\rho(s)\,\mathrm{d}s - \frac{1}{2}\mu Q_\rho(t). \tag{4.12}
$$

If ϑ is a constant, so is $\mathscr{Q}(\cdot)$; and consequently,

$$\varphi'(t) = -d + \frac{1}{2}\mu \int_0^1 Q_\rho(s)\,\mathrm{d}s - \frac{1}{2}\mu Q_\rho(t)$$

is decreasing as $\mu \geqslant 0$. This means $\varphi(\cdot)$ is concave, so it coincides with its concave envelope $\delta(\cdot)$. The case $d > \frac{1}{\mathbf{E}[\rho]}x$ can be treated similarly. □

If the unhedgeable liability is a constant, our model reduces to the classical one, and so does the optimal solution.

If unhedgeable liability is not a constant, we may see that $\delta(\cdot)$ and $\varphi(\cdot)$ are different and this tells us the distribution of the optimal investment outcome is different from the classical result.

4.3 Mean-variance Frontier

In this part, we study the properties of the mean-variance frontier of the model. Similar to the classical model, the mean-variance frontier of the robust model is defined by

$$\left\{\left(\max_{Y\sim\vartheta}\sqrt{\mathbf{Var}(\overline{X}+Y)}, \mathbf{E}[\overline{X}+\vartheta]\right) : \varpi \in \mathbb{R}\right\},$$

where $\overline{X}$ is defined in Theorem 4.2.6 and the target ϖ is taken over $\mathbb{R}$.

Theorem 4.3.1. *If*

$$\frac{1}{\mathbf{Var}(\rho)}\left(x - \varpi\mathbf{E}[\rho] + \int_0^1 \mathscr{Q}(t)Q_\rho(1-t)\,\mathrm{d}t\right) \leqslant -\sup_{t\in(0,1)} \frac{\mathscr{Q}'(t)}{Q'_\rho(1-t)},$$

then the corresponding mean-variance frontier is given by

$$\left(\frac{1}{\sqrt{\mathbf{Var}(\rho)}}\left(\varpi\mathbf{E}[\rho] - x - \int_0^1 \mathscr{Q}(t)Q_\rho(1-t)\,\mathrm{d}t\right), \varpi\right).$$

If

$$\frac{1}{\mathbf{Var}(\rho)}\left(x-\varpi\mathbf{E}[\rho]+\int_0^1 \mathscr{Q}(t)Q_\rho(t)\,\mathrm{d}t\right) \geqslant \sup_{t\in(0,1)}\frac{\mathscr{Q}'(t)}{Q'_\rho(t)},$$

then it is given by

$$\left(\frac{1}{\sqrt{\mathbf{Var}(\rho)}}\left(x-\varpi\mathbf{E}[\rho]+\int_0^1 \mathscr{Q}(t)Q_\rho(t)\,\mathrm{d}t\right),\varpi\right).$$

Proof. Suppose

$$\frac{1}{\mathbf{Var}(\rho)}\left(x-\varpi\mathbf{E}[\rho]+\int_0^1 \mathscr{Q}(t)Q_\rho(t)\,\mathrm{d}t\right) \geqslant \sup_{t\in(0,1)}\frac{\mathscr{Q}'(t)}{Q'_\rho(t)}.$$

We will show

$$\mu := \frac{2}{\mathbf{Var}(\rho)}\left(x-\varpi\mathbf{E}[\rho]+\int_0^1 \mathscr{Q}(t)Q_\rho(t)\,\mathrm{d}t\right)$$

satisfies the requirement in Theorem 4.2.6. Observing $\mu \geqslant 0$, so $d \leqslant \frac{1}{\mathbf{E}[\rho]}x$. By (4.12),

$$\varphi'(t) = -d - \int_0^1 \mathscr{Q}(t)\,\mathrm{d}t + \mathscr{Q}(t) + \frac{1}{2}\mu\int_0^1 Q_\rho(s)\,\mathrm{d}s - \frac{1}{2}\mu Q_\rho(t)$$

and consequently,

$$\varphi''(t) = \mathscr{Q}'(t) - \frac{1}{2}\mu Q'_\rho(t) \leqslant 0.$$

This means $\varphi(\cdot)$ is concave, so $\varphi(\cdot) = \delta(\cdot)$. Thus,

$$
\begin{aligned}
\mathbf{E}[\rho\overline{X}] &= \int_0^1 \overline{Q}(t)Q_\rho(t)\,\mathrm{d}t = -\int_0^1 \delta'(t)Q_\rho(t)\,\mathrm{d}t = -\int_0^1 \varphi'(t)Q_\rho(t)\,\mathrm{d}t \\
&= -\int_0^1 \left(-d - \int_0^1 \mathscr{Q}(t)\,\mathrm{d}t + \mathscr{Q}(t) + \frac{1}{2}\mu\int_0^1 Q_\rho(s)\,\mathrm{d}s - \frac{1}{2}\mu Q_\rho(t)\right) Q_\rho(t)\,\mathrm{d}t \\
&= d\int_0^1 Q_\rho(t)\,\mathrm{d}t + \int_0^1 \mathscr{Q}(t)\,\mathrm{d}t \int_0^1 Q_\rho(t)\,\mathrm{d}t - \int_0^1 \mathscr{Q}(t)Q_\rho(t)\,\mathrm{d}t \\
&\quad - \frac{1}{2}\mu\left(\int_0^1 Q_\rho(t)\,\mathrm{d}t\right)^2 + \frac{1}{2}\mu\int_0^1 Q_\rho^2(t)\,\mathrm{d}t \\
&= d\mathbf{E}[\rho] + \mathbf{E}[\rho]\mathbf{E}[\vartheta] - \int_0^1 \mathscr{Q}(t)Q_\rho(t)\,\mathrm{d}t + \frac{1}{2}\mu\,\mathbf{Var}(\rho) \\
&= \varpi\mathbf{E}[\rho] - \int_0^1 \mathscr{Q}(t)Q_\rho(t)\,\mathrm{d}t + \frac{1}{2}\mu\,\mathbf{Var}(\rho) = x.
\end{aligned}
$$

Observing that

$$
\varphi'(t) - \mathscr{Q}(t) = -d - \int_0^1 \mathscr{Q}(t)\,\mathrm{d}t + \frac{1}{2}\mu\int_0^1 Q_\rho(s)\,\mathrm{d}s - \frac{1}{2}\mu Q_\rho(t) = A - \frac{1}{2}\mu Q_\rho(t),
$$

where A is independent of t. Therefore,

$$\max_{Y\sim\vartheta} \mathbf{Var}(\overline{X}+Y) = \max_{Y\sim\vartheta} \mathbf{E}[(\overline{X}+Y)^2] - (\mathbf{E}[\overline{X}+Y])^2 = \max_{Y\sim\vartheta} \mathbf{E}[(\overline{X}+Y)^2] - \varpi^2$$

$$= J_0(\overline{X}) - \left(\int_0^1 Q_{\overline{X}}(t) + \mathscr{Q}(t)\,\mathrm{d}t\right)^2 = \int_0^1 (Q_{\overline{X}}(t)+\mathscr{Q}(t))^2\,\mathrm{d}t - \left(\int_0^1 Q_{\overline{X}}(t) + \mathscr{Q}(t)\,\mathrm{d}t\right)^2$$

$$= \int_0^1 (-\delta'(t) + \mathscr{Q}(t))^2\,\mathrm{d}t - \left(\int_0^1 -\delta'(t) + \mathscr{Q}(t)\,\mathrm{d}t\right)^2$$

$$= \int_0^1 (-\varphi'(t) + \mathscr{Q}(t))^2\,\mathrm{d}t - \left(\int_0^1 -\varphi'(t) + \mathscr{Q}(t)\,\mathrm{d}t\right)^2$$

$$= \int_0^1 (-A + \frac{1}{2}\mu Q_\rho(t))^2\,\mathrm{d}t - \left(\int_0^1 -A + \frac{1}{2}\mu Q_\rho(t)\,\mathrm{d}t\right)^2$$

$$= \frac{1}{4}\mu^2 \int_0^1 Q_\rho(t)^2\,\mathrm{d}t - \frac{1}{4}\mu^2 \left(\int_0^1 Q_\rho(t)\,\mathrm{d}t\right)^2$$

$$= \frac{1}{4}\mu^2\,\mathbf{Var}(\rho) = \frac{1}{\mathbf{Var}(\rho)} \left(x - \varpi\mathbf{E}[\rho] + \int_0^1 \mathscr{Q}(t)Q_\rho(t)\,\mathrm{d}t\right)^2.$$

This finishes the proof. The other case can be treated in a similar way. □

The financial meaning of this results is as follows. When the investor sets up an extreme high (or low) target level of the expected total terminal wealth, the unhedgeable liability would not have significant effect on the trading strategy. By contrast, if the target level is not too aggressive, then the unhedgeable liability would fundamentally change the trading strategy of the investor. This is seen from the following remark.

Remark 6. *We see that* $\varphi''(t) = \mathscr{Q}'(t) - \frac{1}{2}\mu Q'_\rho(t) > 0$ *as* $\mu \to 0+$ *from the proof of Theorem 4.3.1. This is also true when* $\mu \to 0-$*. This means* φ *is not concave when* μ *is close to zero, or equivalently,* d *is close to* $\frac{1}{\mathbf{E}[\rho]}x$*. In other words, the corresponding part of the mean-variance frontier is not linear.*

In particular, we have the following result for the Black-Scholes market.

Corollary 4.3.2. *If ρ is log-normal distributed and $\mathscr{Q}'(t)$ is bounded.*[1] *Then the mean-variance frontier is linear for big and small ϖ.*

Proof. It suffices to show $Q'_\rho(t)$ is uniformly bounded away from zero. In fact,

$$Q'_\rho(F_\rho(t)) = \frac{1}{F'_\rho(t)} = t\sqrt{2\pi\,\mathbf{Var}(\rho)}e^{\frac{1}{2\,\mathbf{Var}(\rho)}(\ln(t)-\mathbf{E}[\rho])^2}$$

$$= \sqrt{2\pi\,\mathbf{Var}(\rho)}e^{\frac{1}{2\,\mathbf{Var}(\rho)}(\ln(t)-\mathbf{E}[\rho])^2+\ln(t)} \geqslant \sqrt{2\pi\,\mathbf{Var}(\rho)}e^{\mathbf{E}[\rho]-\frac{1}{2}\,\mathbf{Var}(\rho)}.$$

The right hand side is a positive constant, so the claim follows. □

[1] Remark that $\mathscr{Q}'(t)$ is upper bounded if ϑ is uniformly distributed.

Chapter 5

Conclusions and Future Work

This chapter draws conclusions on the thesis, and points out some possible research directions related to the work done in this thesis.

5.1 Conclusions

The results obtained in this thesis can be used to find the best allocation of unhedgeable liabilities taken by the investor; for example, insurance companies need to consider the number of insurance contracts to sell so as to achieve a trade-off between the insurance premiums received from selling these contracts and the risk derived from execution of these contracts.

In the previous Markowitz's model, problems involving investments from other markets will be hard to treat. Although this kind of portfolio is quite normal in the real world, the relevant literature is quite scarce. To fill this gap, the further extension of the original Markowitz's model is in great need. That's the basic motivation of us to start the study of Markowitz's model with unhedgeable liabilities.

As we mentioned in the Introduction, unhedgeable liabilities could be classified into two categories, one is *full information unhedgeable liabilities*, another is *partial information unhedgeable liabilities*. Since only the problem with full information unhedgeable liabilities has been elaborately discussed in previous parts, here we will introduce a specific kind of partial information unhedgeable liability into our robust mean-variance framework. The only thing we know about this partial information unhedgeable liability is its mean and variance. Below is the original form of this problem

$$\begin{aligned} \min_X \max_Y \quad & \mathbf{Var}(X+Y), \\ \text{subject to} \quad & \mathbf{E}[X+Y] = \varpi, \quad X \in \mathcal{A}, \\ & \mathbf{E}[Y] = a, \quad \mathbf{Var}(Y) = b. \end{aligned}$$

After some simple transformation, the problem is turned into following form

$$\begin{aligned} \min_X \max_Y \quad & \{\mathbf{Var}(X) + b + 2\rho_{X,Y}\sqrt{b\,\mathbf{Var}(X)}\}, \\ \text{subject to} \quad & \mathbf{E}[X] = \varpi - a, \quad X \in \mathcal{A}, \end{aligned}$$

where $\rho_{X,Y}$ represents the correlation coefficient between X and Y. To maximize the above cost function with respect to Y, we need to find an optimal Y^* so that $\rho_{X,Y} = 1$ holds. Naturally, Y^* has to satisfy that $Y^* = \lambda X + k$. Take expectation and variance respectively on both sides, it is easy to deduce that

$$\begin{aligned} \lambda =& \sqrt{\frac{b}{\mathbf{Var}(X)}}, \\ k =& a - \sqrt{\frac{b}{\mathbf{Var}(X)}}(\varpi - a). \end{aligned}$$

Since the relationship between X and Y is clear now, we could derive Y as soon as X is known. The current form of the problem is as following

$$\begin{aligned} \min_X \quad & \{\mathbf{Var}(X) + b + 2\sqrt{b\,\mathbf{Var}(X)}\}, \\ \text{subject to} \quad & \mathbf{E}[X] = \varpi - a, \quad X \in \mathcal{A}. \end{aligned}$$

This problem is clearly equivalent to the classical Markowitz's model as below

$$\begin{aligned} \min_X \quad & \mathbf{Var}(X), \\ \text{subject to} \quad & \mathbf{E}[X] = \varpi', \quad X \in \mathcal{A}. \end{aligned}$$

5.2 Future Work

In this thesis, we have considered Markowitz's model with a full information unhedgeable liability, and also provided a simple example of partial information unhedgeable liability situation. In the future, we will carry on research on Markowitz's model with partial information unhedgeable liabilities. The utility indifference pricing models for full and partial information unhedgeable liabilities will be studied too.

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