MEAN-FIELD FORMULATION FOR MULTI-PERIOD ASSET-LIABILITY MEAN-VARIANCE PORTFOLIO SELECTION WITH CASH FLOW

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Certificate of Originality

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______________________________(Signed)

__________LIU WEI__________(Name of student)

Dedicate to my parents.

Abstract

This thesis introduces a mean-field formulation to investigate the multi-period mean-variance model with cash flow, liability and uncertain exit time. As this model cannot be decomposed by a stage-wise backward recursion stage by stage on the basis of dynamic programming, it is a nonseparable problem. This thesis devotes to resolving this nonseparability as well as searching analytical optimal solutions and numerical example.

On the one hand, the original bi-objective mean-variance problem can be transformed into a single-objective one by putting weights on the mean and variance. In substitution of the parameterized method, a mean-field formulation is employed to tackle various optimal multi-period mean-variance policy problems with cash flow and uncertain exit time, respectively. As a matter of fact, parameterized method and embedding technique cannot work smoothly when these constraints are considered. We will illuminate the efficiency and accuracy of mean-field formulation when models are not separable in dynamic programming. By taking expectation of the constraints with some calculations, in the language of optimal control, the state space and the control space will be enlarged, then the objective function becomes separable enabling us to use dynamic programming to solve this problem in expanded spaces. An analytical form of optimal policy and efficient frontier are also derived in this thesis.

On the other hand, we take into account the liability on mean-variance model.

Since in dynamic mean-variance problems, the optimal portfolio policy is always linear with current wealth and liability. Therefore, we employ the mean-field method and derive analytical optimal policies whose results are more explicit and accurate compared with the solution from embedding technique. During the whole derivation, the relationship among investment, cash flow and liability plays an important role. We investigate several cases such correlated or uncorrelated return rates at the same period, and we also illustrate the differences as well as the effects on optimal strategies theoretically and numerically.

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Contents

Notation

w_t	: the value of the assets that the manager controls at time t
l_t	: the value of the liability that the manager controls at time t
v_t^i	: the investment at the start of the t period in the i-th risky asset
n_t	: the risk-free return rate of the investment during the t period
m_t^i	: the return rate of the i-th risky investment during the t period
R_t^i	: the excess return rate of the i-th investment during the t period
c_t	: the random cash flow of the investor during the t period
q_t	: the random rate of the liability during the t period
I_n	: the n-dimensional identity matrix

Chapter 1

Introduction

1.1 Background of Mean-Variance Model

Most of us own a portfolio of assets, which may include real assets, such as a house, a car, or a laptop, and financial assets, such as stocks and bonds. Portfolio selection is a tool concerning the pursuit of the most desirable group of funds, which is vital in the process of gathering wealth. Rational investors prefer a higher expected return as well as a lower risk. However, the portfolio with maximum expected return is not always the one with lowest risk. Mean-variance portfolio selection refers to the design of optimal portfolios balancing the gain with the risk, which are in expression of expectation and variance of final return, respectively. For the sake of tracing out the efficient frontier for this bi-objective optimization problem, a typical method is to put weights on both mean and variance so as to convert the original problem into a single-objective one.

The mean-variance model, which is introduced by Markowitz, has provided a fundamental basis for designing the optimal strategy balancing the trade-off between return and risk. The principles introduced in Markowitz (1952) are still at the core of many modern approaches for asset allocation, investment analysis and risk management. In recent years, research on mean-variance portfolio selection problems has been well developed. Li and Ng (2000) extended Markowitz's model in single period to dynamic version and derived analytical solution by the embedding technique. Costa and Nabholz (2007) generalized the results of Li and Ng (2000) for the case where the intermediate variances and expected values of the portfolio are also considered in the performance criterion and/or constraints. Zhou and Li (2000) introduced a stochastic linear quadratic control as a general framework to study the mean-variance problem in a continuous-time setting and also derived analytical optimal policy and explicit expression of efficient frontier. Li et al. (2002) developed it to a constrained one in which short-selling is disallowed. Yin and Zhou (2004) studied a discrete-time problem where the market regime switches among finite market states and also revealed their relationship with the continuous-time counterparts. Based on local mean-variance efficiency, the portfolio selection problem has been developed by a time-consistent formulation by Czichowsky (2013) for both discrete and continuous time cases. Cui et al. (2014) presented a mean-field method to deal with the multi-period mean-variance optimal policy problem and also get the analytical optimal strategies and efficient frontiers. Pang et al. (2014) discussed a continuous mean-variance problem under partial information by dynamic programming approach through analyzing the process of filtering and wealth evolution.

Asset-liability management is a financial tool for an investor that sets out to maximize their wealth. The aim of asset-liability management is to reduce risks as well as to increase returns and it has been used successfully for banks, pension funds, insurance companies and wise individuals. A judicious investment considers assets and liabilities simultaneously. A financial institution taking liabilities into account can operate more soundly and lucratively. Krouse (1970) noticed that many mean-variance models concentrated only on assets and with little or no effort being directed to the liabilities. The mean-variance framework of asset-liability management was first investigated in a single time period by Sharpe and Tint (1990). Leippold et al. (2004) derived the closed form optimal policies and mean-variance frontiers under exogenous and endogenous liabilities using a geometric approach; Chiu and Li (2006) analyzed asset-liability management in continuous-time period by employing stochastic optimal control theory; Xie et al. (2008) considered the situation where the market is incomplete. Chen and Yang (2011) studied the case with regime switching; Zeng and Li (2011) investigated the models under benchmark and mean-variance criteria in a jump diffusion market. Wu and Li (2012) considered regime switching and cash flow together in the model.

An important assumption is that the exact exit time is deterministic, which means that time horizon of the investment is deterministic in the portfolio selection models. However, it is more realistic for the original investment schedule to be changed or abandoned because of some accidents or unexpected events such as resignation, sudden illness, huge consumption etc. Thus, in the real world, it is better to adjust the constraint with an uncertain exit time assumption so that the investment horizon is undetermined. Under a pure deterministic investment condition, Yaari (1965) considered a case where the death date of an individual can not be determined and discussed the problem of consumption. The work was extended to a multi-period version with the setting of uncertain exit time and only one risky asset by Hakansson (1969). Merton (1971) use an independent Poisson process to define the uncertain time and introduced the constraint into the problem. Li and Xie (2010) investigated the optimal mean-variance portfolio strategy under the setting of an uncertain and continuous time horizon. Yi et al. (2008) incorporated an uncertain exit time into the multi-period mean-variance model. The constraints of regime switching and uncertain exit time were introduced by Wu and Li (2011) Yao et al. (2013) extended the model and added the uncontrolled cash flow and uncertain time horizon.

Most studies above are under a circumstance that the asset and liability are independent. In fact, the returns of risky assets or liability always exhibit some dependency in different time periods. Correlated returns are necessary and meaningful to be considered in the mean-variance portfolio selection. Since the model becomes difficult to solve, there are a few works in the literature. Balvers and Mitchell (1997) was the first to derive an explicit solution to the dynamic portfolio problem when the returns are autocorrelated by a normal ARMA(1,1) process. Xu and Li (2008) investigated a dynamic portfolio selection in a market with one risky asset and one risk-free asset, and Zhang and Li (2012) extended it to the case with uncertain exit time. Gao and Li (2014) considered the capital market consisting of all risky assets. By an embedding technique, last three derived analytical optimal strategies.

The following is a general summary of the analytical solution of the mean-variance optimal policy problem. Markowitz (1952) proposed the single period model, and the analytic solution was given by Merton (1972) under the situation with allowed short-selling and positive definite covariance matrix. However, it is more delicate to consider multi-period and continuous-time cases. In order to use dynamic programming to tackle this problem, the following principle of optimality should be satisfied: regarding the optimal policy, no matter what the initial decision or initial state is, the remaining decision should constitute the optimal strategies with the state rooting in the first decision (See Bellman (2010)). That is to say, the objective function should be decomposed by a backward recursion when the problem is separable. Nevertheless, the variance of asset in the M-V model is inconsistent with the smoothing property, i.e.,

$$\mathrm{Var}\big(\mathrm{Var}(\cdot|\mathcal{F}_j)|\mathcal{F}_k\big) \neq \mathrm{Var}(\cdot|\mathcal{F}_k), \quad \forall\ j > k,$$

the information set at period k is denoted as $\mathcal{F}_k$. Therefore, the multi-period mean-variance problem is un-separable in the sense of dynamic programming. Thus all traditional methods are invalid to derive the optimal stochastic control solution. There are three main methods to deal with this problem involving embedding technique, parameterized method, and mean-field formulation.

Let us first review the embedding technique by Li and Ng (2000) in detail which is widely used to solve the noseparability (See Leippold et al. (2004), Chiu and Li (2006), Yi et al. (2008), Li and Xie (2010), Zhang and Li (2012), Yao et al. (2013) etc.). We assume there are n kinds of risky assets with given return $m_t = [m_t^1, \cdots, m_t^n]'$ and one risk-free asset with given return n_t in the financial market. An investor wants to allocate his/her wealth within a time horizon T for obtaining the highest return with lowest risk, and he/she plans to take part in the financial market with initial asset w_0 at time 0. Li and Ng (2000) formulated the classical multi-period mean-variance portfolio selection model as follows :

$$
\begin{aligned}
\min \quad & \omega \mathrm{Var}[w_T] - \mathbb{E}[w_T], \\
\text{s.t.} \quad & w_{t+1} = \sum_{i=1}^{n} m_t^i v_t^i + n_t \left(w_t - \sum_{i=1}^{n} v_t^i \right) \\
& \qquad = R_t' v_t + n_t w_t, \\
& \qquad t = 0, 1, 2, \cdots, T-1.
\end{aligned} \tag{1.1}
$$

The total asset at time t is denoted as w_t, and v_t^i denote the ith risky investment. Trade-off parameter $\omega > 0$ on behalf of the risk aversion degree. By dynamic programming, it is difficult to solve this problem. Therefore, an embedding scheme is adopted and a family of auxiliary problems were applied to solve the model.

$$
\begin{aligned}
\mathcal{A}(\omega, \lambda) \quad \min \;\; & \mathbb{E}[\omega w_T^2 - \lambda w_T], \\
\text{s.t.} \;\; & w_{t+1} = n_t w_t + R_t' v_t.
\end{aligned}
$$

therefore we transform the problem to a separable linear-quadratic stochastic control (LQSC) formulation and solve it by employing dynamic programming.

We briefly summarize the parameterized method to overcome the nonseparability. An auxiliary variable d was introduced by Li et al. (2002) with the constraint $\mathbb{E}(w_T) = d$ which denoted the expected asset in the terminal period. The following slightly modified and equivalent version of (1.1) was given as (the no-shorting constraint is omitted here),

$$\begin{aligned}(MV(d)) \quad & \min_{\pi} \operatorname{Var}[w_T] = \mathbb{E}[(w_T - d)^2], \\ & \text{s.t. } \mathbb{E}[w_T] = d, \\ & \qquad w_{t+1} = n_t w_t + R_t' v_t, \\ & \qquad t = 0, 1, 2, ..., T-1.\end{aligned}$$

By employing the Lagrangian relaxation and introducing the Lagrangian multiplier λ, we transform the problem to the following LQSC formulation,

$$\begin{aligned}(L(\lambda)) \quad \min \quad & \mathbb{E}(w_T - d)^2 - \lambda \mathbb{E}(w_T - d), \\ \text{s.t.} \quad & w_{t+1} = n_t w_t + R_t' v_t, \\ & t = 0, 1, 2, ..., T-1.\end{aligned}$$

Therefore, the optimal strategies can be derived by maximizing the dual function $L(\lambda)$. Set $\gamma = d + \lambda/2$, we can rewrite the Lagrangian problem $(L(\lambda))$ as the following LQSC problem,

$$\begin{aligned}(MVH(\gamma)) \quad & \min \mathbb{E}(w_T - \gamma)^2, \\ & \text{s.t. } w_{t+1} = n_t w_t + R_t' v_t, \\ & \qquad t = 0, 1, 2, ..., T-1.\end{aligned}$$

which is a special mean-variance hedging problem. Under a quadratic objective function, the investor can hedge the target γ by his/her portfolio. $(MVH(\gamma))$ has been well investigated and can be solved by different methods such as LQSC theory (see Li et al. (2002)), martingale/convex duality theory (see Schweizer et al. (1996), Xia and Yan (2006)) and sequential regression method (see Černỳ and Kallsen (2009)).

Another method is the mean-field formulation approach developed by Cui et al. (2014). The *mean-field* method can solve the problem where both the objective functional and the dynamic system involve their expectations and state processes.

The theory of mean-field method has been well investigated with widespread application during the past few years. Typically, Yong (2013) proposed the mean-field LQ control problems; Li-Zhou-Lim Li et al. (2002), Li-Zhou Li and Zhou (2006), Fu-Lari-Lavassani-Li Fu et al. (2010) investigated the application on financial area through the theory. It is demonstrated that the mean-field method improves the quality of optimal policy compared with some existing results in the previous literature, and figures a new way in tackling the stochastic control problems with non-separability such as multi-period mean-variance model. Even though there were lots of applications appeared in previous literature, the mean-field method still has largely unexplored area and application.

1.2 Contributions and Organization

In this thesis, we study mean-variance portfolio selection with cash flow under the framework of asset-liability management. The main difficulty in solving this problem is the nonseparability. As mentioned above, most multi-period mean-variance models derive the analytical optimal policies based on the embedding technique. One of the prominent features of the embedding technique is that it builds a bridge between multi-period portfolio selection problems and standard stochastic control models. Embedding scheme is indeed an efficient way to deal with problems with nonseparable property. However, it is prone to involve inefficient and complicated calculation during the derivation of the optimal strategies and efficient frontiers by embedding since an auxiliary problem should be built and a long list of notation should be established, especially when adding some constraints such as asset-liability management, uncertain exit time and risk control over bankruptcy and/or serial correlated returns. We resort to exploring new methods to solve the optimal multi-period mean-variance policy with cash flow efficiently.

In Chapter 2 we present a brief introduction of multi-period mean-variance portfolio selection problem. Some useful lemmas also will be given to proof the theorem in the following chapters.

Chapter 3 tackles the optimal multi-period mean-variance policy problem with cash flow by using the mean-field method addressed in Cui et al. (2014). We can enlarge the state space and control space by taking expectation on both sides of the constraints as well as some calculations. The objective function becomes separable enabling us to tackle the problem by dynamic programming in the expanded spaces. Under the construction of the new formulation, we can turn the original problem which is nonseparable into a stochastic linear quadratic control problem which is solvable. There is a property that the optimal strategy is always linear to the current wealth and cash flow. Thus, we can derive the analytical optimal strategies and give the optimal expectation of the final surplus in analytical form. Compare the results with those from the embedding method, we see that these results are more accurate and explicit. Another important thing is that the relationship of returns is vital in the whole derivation. We first assume that there is no correlation between the assets and cash flow in the same time period, then we investigated the case with correlated setting. Numerical examples are given to shed light on the optimal strategies in this chapter.

Chapter 4 is devoted to tackling the multi-period mean-variance problem by mean-field formulation method, adding the constraints of cash flow and uncertain exit time. It is showed that compared with embedding method, mean-field method is simpler in calculation (see Yi et al. (2008)). When the constraints of cash flow and uncertain exit time are considered, neither of the parameterized method and the embedding technique can work smoothly. We emphasize that mean-field formulation is more efficient and accurate when those models are not separable in dynamic programming. In the third section, we introduce mean-field formulation and use it to deal with the nonseparability of a simple multi-period mean-variance problem without uncertainty. Then we employed the mean-field formulation to solve the M-V problem when the exit time is indeterminate. We directly assume that the assets and cash flow are correlated, and show the efficiency and accuracy by presenting numerical examples. When the terminal exit time is deterministic, results from the mean-field formulation are the same with that from parameterized method, which also presents the accuracy of these two methods.

Chapter 5 deals with the multi-period asset-liability mean-variance portfolio selection problem with cash flow. We directly assume the assets and cash flow are correlated and derive strictly the optimal strategies of the mean-variance model with uncorrelation of assets and liability as well as the solution with correlation of assets and liability respectively. The effect of control over liability is showed theoretically and numerically. When the liability control is left out and the correlation between asset and cash flow are consistent with the model in Chapter 3, the results are also the same as it.

Chapter 6 resolves the problem of Chapter 5 with uncertain exit time. Mean-field formulation is proved to be also efficient when we take all the additional condition mentioned above into account. We directly concern the case that the assets and cash flow are correlated. The model becomes much more complex but it is always the case in real financial market. We prove that the similar results hold when different constraints are added. In other words, the results in this Chapter can be reduced to chapter 5 when the investor exit time is deterministic.

The whole thesis deals with the multi-period mean-variance asset-liability portfolio selection problem with different constraints, such as cash flow, uncertain exit time, correlated returns between asset and cash flow by employing the mean-field formulation. We can also consider other situations such as regime switching or time consistent problems. Chapter 7 provide the conclusion and future work of the thesis.

Chapter 2

Preliminary

In chapter 2, we introduce the basic construction of multi-period mean-variance problem with liability. Some characteristics are introduced in order to deriving the analytical solution of optimal strategy.

2.1 Basic Formulation

We assume the financial market that have one liability, one risk-free capital and n risky capitals within a time horizon T. Let n_t represent the deterministic return of the risk-free capital, $m_t = [m_{t,1}, m_{t,2}, \cdots, m_{t,n}]'$ be the vector of random returns of n risky capitals, and q_t is represented as random return of the liability at period t. The investor will join the financial market at the beginning of time period 0 and propose to quit the investment at time T. Let w_0 denote the initial wealth, and l_0 denote the initial liability. At the beginning of every time period t between 0 and T, he/she can reallocates his/her portfolio selection in order to maximize the expect return as well as minimize the risk.

At different time periods t, the random variable q_t and the random vector $m_t = [m_t^1, \cdots, m_t^n]'$ are assumed to be statistically independent and are defined from the probability space $(\Omega, \mathcal{F}, P)$.

The only information about q_t and m_t are their first two moment. We further define their covariance matrix is positive definite. Thus,

$$\mathrm{Cov}\left(\begin{pmatrix} m_t \\ q_t \end{pmatrix}\right) = \mathbb{E}\left[\begin{pmatrix} m_t \\ q_t \end{pmatrix}\begin{pmatrix} m_t' & q_t \end{pmatrix}\right] - \mathbb{E}\left[\begin{pmatrix} m_t \\ q_t \end{pmatrix}\right]\mathbb{E}\left[\begin{pmatrix} m_t' & q_t \end{pmatrix}\right] \succ 0.$$

Due to $\mathrm{Cov}[(m_t, q_t)'] \succ 0$, the following matrix is also positive definite for $t = 0, 1, \cdots, \cdots, T-1$

$$\begin{pmatrix} n_t^2 & n_t\mathbb{E}[m_t'] & n_t\mathbb{E}[q_t] \\ n_t\mathbb{E}[m_t] & \mathbb{E}[m_t m_t'] & \mathbb{E}[m_t q_t] \\ n_t\mathbb{E}[q_t] & \mathbb{E}[q_t m_t'] & \mathbb{E}[q_t^2] \end{pmatrix} \succ 0.$$

Let $R_t = (R_t^1, \cdots, R_t^n)'$ represents the excess return vector of risky assets which is equal to $(e_t^1 - n_t, \cdots, e_t^n - n_t)'$. According to the upper assumptions, we get

$$\begin{aligned} &\begin{pmatrix} n_t^2 & n_t\mathbb{E}[R_t'] & n_t\mathbb{E}[q_t] \\ n_t\mathbb{E}[R_t] & \mathbb{E}[R_t R_t'] & \mathbb{E}[R_t q_t] \\ n_t\mathbb{E}[q_t] & \mathbb{E}[q_t R_t'] & \mathbb{E}[q_t^2] \end{pmatrix} \\ =&\begin{pmatrix} 1 & \mathbf{0}' & 0 \\ -\mathbf{1} & I & \mathbf{0} \\ 0 & \mathbf{0}' & 1 \end{pmatrix}\begin{pmatrix} n_t^2 & n_t\mathbb{E}[m_t'] & n_t\mathbb{E}[q_t] \\ n_t\mathbb{E}[m_t] & \mathbb{E}[m_t m_t'] & \mathbb{E}[m_t q_t] \\ n_t\mathbb{E}[q_t] & \mathbb{E}[q_t m_t'] & \mathbb{E}[q_t^2] \end{pmatrix}\begin{pmatrix} 1 & -\mathbf{1}' & 0 \\ \mathbf{0} & I & \mathbf{0} \\ 0 & \mathbf{0}' & 1 \end{pmatrix} \\ \succ& 0, \end{aligned}$$

where I denotes the $n \times n$ identity matrix, $\mathbf{0}$ and $\mathbf{1}$ denote the n-dimensional all-zero and all-one vectors respectively, which imply,

$$\begin{aligned} \mathbb{E}[R_t R_t'] &\succ 0, \\ \mathbb{E}[q_t^2] - \mathbb{E}[q_t R_t']\mathbb{E}^{-1}[R_t R_t']\mathbb{E}[R_t q_t] &> 0, \\ n_t^2(1 - \mathbb{E}[R_t']\mathbb{E}^{-1}[R_t R_t']\mathbb{E}[R_t]) &> 0. \end{aligned}$$

Therefore $0 < \mathbb{E}[R_t']\mathbb{E}^{-1}[R_t R_t']\mathbb{E}[R_t] < 1$. In order to express the equation more concisely, we define the following symbol

$$B_t \triangleq \mathbb{E}[R_t']\mathbb{E}^{-1}[R_tR_t']\mathbb{E}[R_t],$$

$$\widehat{B}_t \triangleq \mathbb{E}[c_tR_t']\mathbb{E}^{-1}[R_tR_t']\mathbb{E}[R_t],$$

$$\widetilde{B}_t \triangleq \mathbb{E}[c_tR_t']\mathbb{E}^{-1}[R_tR_t']\mathbb{E}[c_tR_t],$$

$$\overline{B}_t \triangleq \mathbb{E}[q_tR_t']\mathbb{E}^{-1}[R_tR_t']\mathbb{E}[R_t],$$

$$B'_t \triangleq \mathbb{E}[q_tR_t']\mathbb{E}^{-1}[R_tR_t']\mathbb{E}[R_tq_t],$$

$$\overrightarrow{B}_t \triangleq \mathbb{E}[c_tR_t']\mathbb{E}^{-1}[R_tR_t']\mathbb{E}[R_tq_t].$$

If there is no correlation between the return rates and cash flow during each period, we have

$$\widehat{B}_t = \mathbb{E}[c_t]B_t \quad \text{and} \quad \widetilde{B}_t = (\mathbb{E}[c_t])^2B_t.$$

If there is no correlation between the return rates and liability during each period, we have

$$\overline{B}_t = \mathbb{E}[q_t]B_t \quad \text{and} \quad B'_t = (\mathbb{E}[q_t])^2B_t.$$

At the beginning of time period t, the wealth and liability of the investor are denoted by w_t and l_t respectively. Therefore, the surplus is denoted by $w_t - l_t$. If v_t^i is the money invested in the i-th risky asset for $i = 1, 2, \cdots, n$ at period t, thus $w_t - \sum_{i=1}^n v_t^i$ can be defined as risk-free investment. In this thesis we suppose the liability is exogenous. In other words, the investor's strategies cannot effect the liability because of its uncontrollability.

$\mathcal{F}_t = \sigma(R_0, R_1, \cdots, R_{t-1}, c_0, c_1, \cdots, c_{t-1}, q_0, q_1, \cdots, q_{t-1})$ represents the information set at the start of time period t for $t = 1, 2, \cdots, T-1$, and $\mathcal{F}_0$ represents the trivial σ-algebra over Ω . Thus, $\mathbb{E}[\cdot|\mathcal{F}_0]$ is equal to the unconditional expectation $\mathbb{E}[\cdot]$. We denote $\pi_t = (\pi_t^1,\ \pi_t^2,\ \cdots,\ \pi_t^n)' \in \mathcal{F}_t$ which means all admissible investment strategies are $\mathcal{F}_t$-adapted Markov controls and $\mathcal{F}_t = \sigma(w_t, l_t)$. Moreover, if v_t and R_t are independent, $\{w_t, l_t\}$ is an adapted Markovian process.

If there is no liability ($q_t = l_t = 0$) in this portfolio selection model. That is to say, there are just one risk-free asset and n kinds of risky assets in the financial market. Then the information set is denoted by $\mathcal{F}_t = \sigma(R_0, R_1, \cdots, R_{t-1})$ at the beginning of period t, and covariance matrix is also positive definite

$$\mathrm{Cov}\,(m_t) = \mathbb{E}[m_t m_t'] - \mathbb{E}[m_t]\mathbb{E}[m_t'] = \begin{bmatrix} \sigma_{t,11} & \cdots & \sigma_{t,1n} \\ \vdots & \ddots & \vdots \\ \sigma_{t,1n} & \cdots & \sigma_{t,nn} \end{bmatrix} \succ 0.$$

We can also get the similar result as the first case, i.e., $0 < \mathbb{E}[R_t']\mathbb{E}^{-1}[R_t R_t']\mathbb{E}[R_t] < 1$.

Chapter 3

Mean-Field Method for Optimal Multi-Period Mean-Variance Policy with Cash Flow

We study the multi-period portfolio selection problem with cash flow under the framework of mean-variance model by using the mean-field method in this chapter. The model is formulated in minimizing the variance with an indeterministic expected return. Due to the variance term, the model is non-separable and we need to tackle it by introducing the mean-field formulation. Compared to the embedding technique proposed by Li and Ng (2000) and the parameterized method developed by Li et al. (2002), the mean-field method can result in the optimal policy more easily and directly to the whole process. We first reduce the case when the return rates between assets and cash flow are uncorrelated, then we extend it to the correlated setting. We derive the analytical form of optimal strategies as well as the optimal value in expression for the initial problem. Finally, the numerical examples in both cases are given to illustrate the results established in this work.

3.1 Mean-Field Formulation

Similar with the classical multi-period mean-variance portfolio selection model which we have mentioned in Chapter 1, we transform the bi-objective optimization problem to a single-objective one. Thus,

$$\begin{cases} \min \quad \mathrm{Var}(w_T) - \lambda \mathbb{E}[w_T], \\ \text{s.t.} \quad w_{t+1} = n_t \Big(w_t - \sum_{i=1}^n v_t^i \Big) + \sum_{i=1}^n m_t^i v_t^i + c_t \\ \qquad\qquad\;\; = n_t w_t + R_t' v_t + c_t, \\ \qquad\qquad\; t = 0, 1, \cdots, T-1. \end{cases} \tag{3.1}$$

Here λ represents the trade-off parameter. Because of smooth property is no longer hold on the variance, we cannot solve this multi-period mean-variance model by using dynamic programming directly. In order to tackle this difficulty, we employ the mean-field formulation approach proposed by Cui et al. (2014) in this section. A mean-field type method transform the problem to another formulation where either the dynamic system or the objective functional involves their expectations and state processes .

Thus, we construct the mean-field formulation for problem (3.1). Firstly, according to the independence between R_t and v_t, the dynamics equations of the wealth expectation can be derived as

$$\begin{cases} \mathbb{E}[w_{t+1}] = n_t \mathbb{E}[w_t] + \mathbb{E}[R_t'] \mathbb{E}[v_t] + \mathbb{E}[c_t], \\ \mathbb{E}[w_0] = w_0. \end{cases} \tag{3.2}$$

where $t = 0, 1, \cdots, T-1$. Combining the dynamics equation (3.1) and the expectation equations (3.2), we have the following

$$\begin{cases} w_{t+1} - \mathbb{E}[w_{t+1}] = n_t\big(w_t - \mathbb{E}[w_t]\big) + R_t' v_t - \mathbb{E}[R_t'] \mathbb{E}[v_t] + (c_t - \mathbb{E}[c_t]) \\ \qquad\qquad\qquad\;\; = n_t\big(w_t - \mathbb{E}[w_t]\big) + R_t'\big(v_t - \mathbb{E}[v_t]\big) + \big(R_t' - \mathbb{E}[R_t']\big)\mathbb{E}[v_t] \\ \qquad\qquad\qquad\quad\;\; +(c_t - \mathbb{E}[c_t]), \\ w_0 - \mathbb{E}[w_0] = 0. \end{cases} \tag{3.3}$$

By doing this kind of transformation, we enlarge the control space (v_t) and state space (w_t) into $(\mathbb{E}[v_t], v_t - \mathbb{E}[v_t])$ and $(\mathbb{E}[w_t], w_t - \mathbb{E}[w_t])$, respectively.

Secondly, even though the new control vectors $v_t - \mathbb{E}[v_t]$ and $\mathbb{E}[v_t]$ are independently determined at time t, they also should satisfy the following equation

$$\mathbb{E}(v_t - \mathbb{E}[v_t]) = \mathbf{0}.$$

Furthermore, we confine v_t and w_t are $\mathcal{F}_t$-measurable. $\mathbb{E}[v_t]$ and $\mathbb{E}[w_t]$ are $\mathcal{F}_0$-measurable. Therefore, we have $v_t - \mathbb{E}[v_t]$ and $w_t - \mathbb{E}[w_t]$ are $\mathcal{F}_t$-measurable while $\mathcal{F}_t = \sigma(\mathbb{E}[w_t], w_t - \mathbb{E}[w_t])$.

It is necessary to emphasize that we cannot observe the state $\mathbb{E}[w_{t+1}]$ directly in the financial market. Actually, we computed $\mathbb{E}[w_{t+1}]$ according to the dynamic equation (3.2) after we chose the $\mathbb{E}[v_t]$. Moreover, we observed the wealth level w_{t+1} at the period $t+1$ then derived the $\mathcal{F}_t$-measurable state $w_{t+1} - \mathbb{E}[w_{t+1}]$. The control system $v_t - \mathbb{E}[v_t]$ and $\mathbb{E}[v_t]$ are consistent satisfying the constraint $\mathbb{E}(v_t - \mathbb{E}[v_t]) = \mathbf{0}$.

Based on the above construction, we can equivalently reformulate problem (3.1) into a linear quadratic optimal stochastic control problem in mean-field type.

$$\begin{cases} \min & \mathbb{E}\left[(w_T - \mathbb{E}[w_T])^2\right] - \lambda \mathbb{E}[w_T], \\ \text{s.t.} & \mathbb{E}[w_t] \text{ satisfies dynamic equation (3.2)}, \\ & w_t - \mathbb{E}[w_t] \text{ consist with equation (3.3)}, \\ & \mathbb{E}(v_t - \mathbb{E}[v_t]) = \mathbf{0}, \\ & t = 0, 1, \cdots, T-1. \end{cases} \tag{3.4}$$

We are able to solve the multi-period mean-variance model with cash flow in this mean-field formulation by dynamic programming. However, during the solution process, we should pay attention to the imposed control vector $v_t - \mathbb{E}[v_t]$.

3.2 The Optimal Strategy with Uncorrelation between Return Rates and Cash Flow

3.2.1 The Dynamic Programming and Optimal Strategy

We assume the return rates between assets and cash flow are uncorrelated at every period in this subsection, i.e., R_t and c_t are independent with each other during time period t.

Theorem 3.1. *The best strategy of (3.4) under the constraint of uncorrelation is derived as*

$$\begin{aligned}
\pi_t^* - \mathbb{E}[\pi_t^*] &= -n_t(w_t - \mathbb{E}[w_t])\mathbb{E}^{-1}[R_t R_t']\mathbb{E}[(R_t)], \\
\mathbb{E}[\pi_t^*] &= \Big(\mathbb{E}[R_t R_t'] - \mathbb{E}[R_t]\mathbb{E}[R_t']\Big)^{-1}\Big(\frac{\lambda\eta_{t+1}}{2\beta_{t+1}}\mathbb{E}[R_t]\Big) \\
&= \frac{\lambda\eta_{t+1}}{2\beta_{t+1}}\frac{\mathbb{E}^{-1}[R_t R_t']\mathbb{E}[R_t]}{1 - B_t}
\end{aligned} \tag{3.5}$$

Proof. The dynamic programming approach is employed to prove the theorem. Given the information set $\mathcal{F}_t$ at time period t, we define the following cost-to-go functional of problem (3.4)

$$J_t(\mathbb{E}[w_t], w_t - \mathbb{E}[w_t]) = \min_{v_t} \mathbb{E}\big[J_{t+1}(\mathbb{E}[w_{t+1}], w_{t+1} - \mathbb{E}[w_{t+1}])\big|\mathcal{F}_t\big],$$

with the boundary condition $J_T(\mathbb{E}[w_T], w_T - \mathbb{E}[w_T]) = (w_T - \mathbb{E}[w_T])^2 - \lambda\mathbb{E}[w_T]$.

We begin from stage $T-1$. Thus the conditional expectation at $t = T-1$ is

$$\begin{aligned}
& \mathbb{E}\big[J_T(\mathbb{E}[w_T], w_T - \mathbb{E}[w_T])|\mathcal{F}_{T-1}\big] \\
= \;& \mathbb{E}\big[(w_T - \mathbb{E}[w_T])^2 - \lambda\mathbb{E}[w_T]|\mathcal{F}_{T-1}\big] \\
= \;& \mathbb{E}\Big[\Big(n_{T-1}(w_{T-1} - \mathbb{E}[w_{T-1}]) + R_{T-1}'(v_{T-1} - \mathbb{E}[v_{T-1}]) + (R_{T-1}' - \mathbb{E}[R_{T-1}'])\mathbb{E}[v_{T-1}] \\
& +(c_{T-1} - \mathbb{E}[c_{T-1}])\Big)^2 - \lambda\Big(n_{T-1}\mathbb{E}[w_{T-1}] + \mathbb{E}[R_{T-1}']\mathbb{E}[v_{T-1}] + \mathbb{E}[c_{T-1}]\Big)\Big|\mathcal{F}_{T-1}\Big]
\end{aligned} \tag{3.6}$$

where

$$
\begin{aligned}
& \Big(n_{T-1}(w_{T-1}-\mathbb{E}[w_{T-1}]) + R'_{T-1}(v_{T-1}-\mathbb{E}[v_{T-1}]) + (R'_{T-1}-\mathbb{E}[R'_{T-1}])\mathbb{E}[v_{T-1}] \\
& +(c_{T-1}-\mathbb{E}[c_{T-1}])\Big)^2 \\
= \;& \big(n_{T-1}(w_{T-1}-\mathbb{E}[w_{T-1}])\big)^2 + \big(R'_{T-1}(v_{T-1}-\mathbb{E}[v_{T-1}])\big)^2 \\
& +\big((R'_{T-1}-\mathbb{E}[R'_{T-1}])\mathbb{E}[v_{T-1}]\big)^2 + \big(c_{T-1}-\mathbb{E}[c_{T-1}]\big)^2 \\
& +2\big(n_{T-1}(w_{T-1}-\mathbb{E}[w_{T-1}])\big)\big(R'_{T-1}(v_{T-1}-\mathbb{E}[v_{T-1}])\big) \\
& +2\big(n_{T-1}(w_{T-1}-\mathbb{E}[w_{T-1}])\big)\big((R'_{T-1}-\mathbb{E}[R'_{T-1}])\mathbb{E}[v_{T-1}]\big) \\
& +2\big(n_{T-1}(w_{T-1}-\mathbb{E}[w_{T-1}])\big)\big(c_{T-1}-\mathbb{E}[c_{T-1}]\big) \\
& +2\big(v_{T-1}-\mathbb{E}[v_{T-1}]\big)'(R_{T-1})\big(R'_{T-1}-\mathbb{E}[R'_{T-1}])\mathbb{E}[v_{T-1}] \\
& +2R'_{T-1}(v_{T-1}-\mathbb{E}[v_{T-1}])(c_{T-1}-\mathbb{E}[c_{T-1}]) \\
& +2\big(R'_{T-1}-\mathbb{E}[R'_{T-1}]\big)\mathbb{E}[v_{T-1}]\big(c_{T-1}-\mathbb{E}[c_{T-1}]\big).
\end{aligned}
\tag{3.7}
$$

Since $w_t^i-\mathbb{E}[w_t^i]$, $\mathbb{E}[w_t^i]$, $v_t^i-\mathbb{E}[v_t^i]$, $\mathbb{E}[v_t^i]$, are $\mathcal{F}_t$-measurable, we get

$$
\begin{aligned}
& \mathbb{E}\Big[\big(n_{T-1}(w_{T-1}-\mathbb{E}[w_{T-1}])\big)^2\Big|\mathcal{F}_{T-1}\Big] = (n_{T-1})^2\big(w_{T-1}-\mathbb{E}[w_{T-1}]\big)^2, \\
& \mathbb{E}\Big[\Big(R'_{T-1}(v_{T-1}-\mathbb{E}[v_{T-1}])\Big)^2\Big|\mathcal{F}_{T-1}\Big] \\
& = \big(v_{T-1}-\mathbb{E}[v_{T-1}]\big)'\mathbb{E}[R_{T-1}R'_{T-1}]\big(v_{T-1}-\mathbb{E}[v_{T-1}]\big), \\
& \mathbb{E}\Big[\Big((R'_{T-1}-\mathbb{E}[R'_{T-1}])\mathbb{E}[v_{T-1}]\Big)^2\Big|\mathcal{F}_{T-1}\Big] \\
& = \mathbb{E}[(v_{T-1})']\Big(\mathbb{E}[R_{T-1}R'_{T-1}]-\mathbb{E}[R_{T-1}]\mathbb{E}[R'_{T-1}]\Big)\mathbb{E}[v_{T-1}], \\
& \mathbb{E}\Big[\big(c_{T-1}-\mathbb{E}[c_{T-1}]\big)^2\Big|\mathcal{F}_{T-1}\Big] = \mathbb{E}[c_{T-1}^2]-\mathbb{E}[c_{T-1}]^2, \\
& \mathbb{E}\Big[2n_{T-1}(w_{T-1}-\mathbb{E}[w_{T-1}])R'_{T-1}(v_{T-1}-\mathbb{E}[v_{T-1}])\Big|\mathcal{F}_{T-1}\Big] \\
& = 2n_{T-1}\big(w_{T-1}-\mathbb{E}[w_{T-1}]\big)\mathbb{E}[R'_{T-1}]\big(v_{T-1}-\mathbb{E}[v_{T-1}]\big),
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbb{E}\Big[2n_{T-1}\big(w_{T-1}-\mathbb{E}[w_{T-1}]\big)\big(R'_{T-1}-\mathbb{E}[R'_{T-1}]\big)\mathbb{E}[v_{T-1}]\Big|\mathcal{F}_{T-1}\Big]=0,\\
&\mathbb{E}\Big[2n_{T-1}(w_{T-1}-\mathbb{E}[w_{T-1}])(c_{T-1}-\mathbb{E}[c_{T-1}])\Big|\mathcal{F}_{T-1}\Big]=0,\\
&\mathbb{E}\Big[2\big(v_{T-1}-\mathbb{E}[v_{T-1}]\big)'(R_{T-1})\Big(R'_{T-1}-\mathbb{E}[R'_{T-1}]\Big)\mathbb{E}[v_{T-1}]\Big|\mathcal{F}_{T-1}\Big]\\
&=2\big(v_{T-1}-\mathbb{E}[v_{T-1}]\big)'\Big(\mathbb{E}[R_{T-1}R'_{T-1}]-\mathbb{E}[R_{T-1}]\mathbb{E}[R'_{T-1}]\Big)\mathbb{E}[v_{T-1}],\\
&\mathbb{E}\Big[2R'_{T-1}(v_{T-1}-\mathbb{E}[v_{T-1}])(c_{T-1}-\mathbb{E}[c_{T-1}])\Big|\mathcal{F}_{T-1}\Big]\\
&=2\big(\mathbb{E}[c_{T-1}R'_{T-1}]-\mathbb{E}[c_{T-1}]\mathbb{E}[R'_{T-1}\big)(v_{T-1}-\mathbb{E}[v_{T-1}]),\\
&\mathbb{E}\Big[2\big(R'_{T-1}-\mathbb{E}[R'_{T-1}]\big)\mathbb{E}[v_{T-1}]\big(c_{T-1}-\mathbb{E}[c_{T-1}]\big)\Big|\mathcal{F}_{T-1}\Big]\\
&=2\Big(\mathbb{E}[c_{T-1}R'_{T-1}]-\mathbb{E}[c_{T-1}]\mathbb{E}[R'_{T-1}]\Big)\mathbb{E}[v_{T-1}]=0.
\end{aligned}
$$

Note that $\mathbb{E}\big[v_t^i-\mathbb{E}[v_t^i]|\mathcal{F}_0\big]=0$, which implies

$$
\begin{aligned}
&\mathbb{E}\Big[\mathbb{E}\Big[2\big(v_{T-1}-\mathbb{E}[v_{T-1}]\big)'(R_{T-1})\Big(R'_{T-1}-\mathbb{E}[R'_{T-1}]\Big)\mathbb{E}[v_{T-1}]\Big|\mathcal{F}_{T-1}\Big]\Big|\mathcal{F}_0\Big]\\
&=\mathbb{E}\Big[2\big(v_{T-1}-\mathbb{E}[v_{T-1}]\big)'\Big(\mathbb{E}[R_{T-1}R'_{T-1}]-\mathbb{E}[R_{T-1}]\mathbb{E}[R'_{T-1}]\Big)\mathbb{E}[v_{T-1}]\Big|\mathcal{F}_0\Big]=0,\\
&\mathbb{E}\Big[\mathbb{E}\Big[2R'_{T-1}(v_{T-1}-\mathbb{E}[v_{T-1}])(c_{T-1}-\mathbb{E}[c_{T-1}])\Big|\mathcal{F}_{T-1}\Big]\Big|\mathcal{F}_0\Big]\\
&=\mathbb{E}\Big[2\big(\mathbb{E}[c_{T-1}R'_{T-1}]-\mathbb{E}[c_{T-1}]\mathbb{E}[R'_{T-1}\big)(v_{T-1}-\mathbb{E}[v_{T-1}])\Big|\mathcal{F}_0\Big]=0.
\end{aligned}
$$

Therefore, we can reduce (3.6) into

$$
\begin{aligned}
&\mathbb{E}\big[J_T(\mathbb{E}[w_T],w_T-\mathbb{E}[w_T])|\mathcal{F}_{T-1}\big]\\
=\;&\mathbb{E}\big[(w_T-\mathbb{E}[w_T])^2-\lambda\mathbb{E}[w_T]|\mathcal{F}_{T-1}\big]\\
=\;&\mathbb{E}\Big[\Big(n_{T-1}(w_{T-1}-\mathbb{E}[w_{T-1}])+R'_{T-1}(v_{T-1}-\mathbb{E}[v_{T-1}])+(R'_{T-1}-\mathbb{E}[R'_{T-1}])\mathbb{E}[v_{T-1}]\\
&+(c_{T-1}-\mathbb{E}[c_{T-1}])\Big)^2-\lambda\Big(n_{T-1}\mathbb{E}[w_{T-1}]+\mathbb{E}[R'_{T-1}]\mathbb{E}[v_{T-1}]+\mathbb{E}[c_{T-1}]\Big)\Big|\mathcal{F}_{T-1}\Big]\\
=\;&(n_{T-1})^2\big(w_{T-1}-\mathbb{E}[w_{T-1}]\big)^2+\big(v_{T-1}-\mathbb{E}[v_{T-1}]\big)'\mathbb{E}[R_{T-1}R'_{T-1}]\big(v_{T-1}-\mathbb{E}[v_{T-1}]\big)\\
&+\mathbb{E}[(v_{T-1})']\Big(\mathbb{E}[R_{T-1}R'_{T-1}]-\mathbb{E}[R_{T-1}]\mathbb{E}[R'_{T-1}]\Big)\mathbb{E}[v_{T-1}]+\mathbb{E}[(c_{T-1})^2]-\mathbb{E}[c_{T-1}]^2\\
&+2n_{T-1}\big(w_{T-1}-\mathbb{E}[w_{T-1}]\big)\mathbb{E}[R'_{T-1}]\big(v_{T-1}-\mathbb{E}[v_{T-1}]\big)\\
&-\lambda n_{T-1}\mathbb{E}[w_{T-1}]-\lambda\mathbb{E}[R'_{T-1}]\mathbb{E}[v_{T-1}]-\lambda\mathbb{E}[c_{T-1}].
\end{aligned}
\tag{3.8}
$$

The optimal strategies at period $T-1$ can be derived by minimizing the above equation corresponding to v_{T-1}

$$\begin{aligned} v_{T-1}-\mathbb{E}[v_{T-1}] &= -n_{T-1}(w_{T-1}-\mathbb{E}[w_{T-1}])\mathbb{E}^{-1}[R_{T-1}R'_{T-1}]\mathbb{E}[(R_{T-1})], \\ \mathbb{E}[v_{T-1}] &= \Big(\mathbb{E}[R_{T-1}R'_{T-1}]-\mathbb{E}[R_{T-1}]\mathbb{E}[R'_{T-1}]\Big)^{-1}\frac{\lambda}{2}\mathbb{E}[R_{T-1}] \\ &= \frac{\lambda}{2}\frac{\mathbb{E}^{-1}[R_{T-1}R'_{T-1}]\mathbb{E}[R_{T-1}]}{1-B_{T-1}}, \end{aligned} \tag{3.9}$$

where

$$B_{T-1} = \mathbb{E}[R'_{T-1}]\mathbb{E}^{-1}[R_{T-1}R'_{T-1}]\mathbb{E}[R_{T-1}]. \tag{3.10}$$

Substituting the optimal strategies back to (3.8), we obtain

$$\begin{aligned} & J_{T-1}(\mathbb{E}[w_{T-1}], w_{T-1}-\mathbb{E}[w_{T-1}]) \\ = \; & \mathbb{E}\big[J_T(\mathbb{E}[w_T], w_T-\mathbb{E}[w_T])|\mathcal{F}_{T-1}\big] \\ = \; & (n_{T-1})^2\big(w_{T-1}-\mathbb{E}[w_{T-1}]\big)^2+(\mathbb{E}[(c_{T-1})^2]-\mathbb{E}[c_{T-1}]^2)-\lambda n_{T-1}\mathbb{E}[w_{T-1}] \\ & -\lambda\mathbb{E}[c_{T-1}]-(n_{T-1})^2B_{T-1}\big(w_{T-1}-\mathbb{E}[w_{T-1}]\big)^2-\Big(\frac{\lambda}{2}\frac{\mathbb{E}[R'_{T-1}]\mathbb{E}^{-1}[R_{T-1}R'_{T-1}]}{1-B_{T-1}}\Big) \\ & \times\Big(\mathbb{E}[R_{T-1}R'_{T-1}]-\mathbb{E}[R_{T-1}]\mathbb{E}[R'_{T-1}]\Big)\Big(\frac{\lambda}{2}\frac{\mathbb{E}^{-1}[R_{T-1}R'_{T-1}]\mathbb{E}[R_{T-1}]}{1-B_{T-1}}\Big) \\ = \; & (n_{T-1})^2(1-B_{T-1})\big(w_{T-1}-\mathbb{E}[w_{T-1}]\big)^2+(\mathbb{E}[c^2_{T-1}]-\mathbb{E}[c_{T-1}]^2) \\ & -\lambda n_{T-1}\mathbb{E}[w_{T-1}]-\lambda\mathbb{E}[c_{T-1}]-\frac{\lambda^2}{4}\frac{B_{T-1}}{1-B_{T-1}} \\ = \; & \beta_{T-1}\big(w_{T-1}-\mathbb{E}[w_{T-1}]\big)^2-\lambda\eta_{T-1}\mathbb{E}[w_{T-1}]+\Delta_{T-1}, \end{aligned} \tag{3.11}$$

where

$$\begin{aligned} \beta_{T-1} = \; & (n_{T-1})^2(1-B_{T-1}), \\ \eta_{T-1} = \; & n_{T-1}, \\ \Delta_{T-1} = \; & \mathbb{E}[c^2_{T-1}]-\mathbb{E}[c_{T-1}]^2-\lambda\mathbb{E}[c_{T-1}]-\frac{\lambda^2}{4}\frac{B_{T-1}}{1-B_{T-1}}. \end{aligned} \tag{3.12}$$

Repeating the process at time $T-2$, we have

$$\begin{aligned}
& J_{T-2}(\mathbb{E}[w_{T-2}], w_{T-2}-\mathbb{E}[w_{T-2}]) \\
= \; & \mathbb{E}\big[J_{T-1}(\mathbb{E}[w_{T-1}], w_{T-1}-\mathbb{E}[w_{T-1}])|\mathcal{F}_{T-2}\big] \\
= \; & \mathbb{E}\big[\beta_{T-1}\big(w_{T-1}-\mathbb{E}[w_{T-1}]\big)^2 - \lambda\eta_{T-1}\mathbb{E}[w_{T-1}] + \Delta_{T-1}|\mathcal{F}_{T-2}\big] \\
= \; & \beta_{T-1}(n_{T-2})^2\big(w_{T-2}-\mathbb{E}[w_{T-2}]\big)^2 + \beta_{T-1}\big(v_{T-2}-\mathbb{E}[v_{T-2}]\big)'\mathbb{E}[R_{T-2}R'_{T-2}] \\
& \times\big(v_{T-2}-\mathbb{E}[v_{T-2}]\big) + \beta_{T-1}\mathbb{E}[(v_{T-2})']\Big(\mathbb{E}[R_{T-2}R'_{T-2}] - \mathbb{E}[R_{T-2}]\mathbb{E}[R'_{T-2}]\Big)\mathbb{E}[v_{T-2}] \\
& +\beta_{T-1}\mathbb{E}[c^2_{T-2}] - \beta_{T-1}\mathbb{E}[c_{T-2}]^2 + 2\beta_{T-1}n_{T-2}\big(w_{T-2}-\mathbb{E}[w_{T-2}]\big)\mathbb{E}[R'_{T-2}] \\
& \times\big(v_{T-2}-\mathbb{E}[v_{T-2}]\big) - \lambda\eta_{T-1}n_{T-2}\mathbb{E}[w_{T-2}] - \lambda\eta_{T-1}\mathbb{E}[R'_{T-2}]\mathbb{E}[v_{T-2}] \\
& -\lambda\eta_{T-1}\mathbb{E}[c_{T-2}] + \Delta_{T-1}.
\end{aligned} \tag{3.13}$$

The optimal strategies at period $T-2$ can be derived by minimizing the above equation with respect to v_{T-2}

$$\begin{aligned}
v_{T-2}-\mathbb{E}[v_{T-2}] &= -n_{T-2}(w_{T-2}-\mathbb{E}[w_{T-2}])\mathbb{E}^{-1}[R_{T-2}R'_{T-2}]\mathbb{E}[(R_{T-2})], \\
\mathbb{E}[v_{T-2}] &= \Big(\mathbb{E}[R_{T-2}R'_{T-2}] - \mathbb{E}[R_{T-2}]\mathbb{E}[R'_{T-2}]\Big)^{-1}\Big(\frac{\lambda\eta_{T-1}}{2\beta_{T-1}}\mathbb{E}[R_{T-2}]\Big) \\
&= \frac{\lambda\eta_{T-1}}{2\beta_{T-1}}\frac{\mathbb{E}^{-1}[R_{T-2}R'_{T-2}]\mathbb{E}[R_{T-2}]}{1-B_{T-2}}.
\end{aligned} \tag{3.14}$$

Substituting the optimal strategies back to (3.13), we obtain

$$\begin{aligned}
& J_{T-2}(\mathbb{E}[w_{T-2}], w_{T-2}-\mathbb{E}[w_{T-2}]) \\
= \; & \beta_{T-1}(n_{T-2})^2(1-B_{T-2})\big(w_{T-2}-\mathbb{E}[w_{T-2}]\big)^2 + \beta_{T-1}(\mathbb{E}[c^2_{T-2}] - \mathbb{E}[c_{T-2}]^2) \\
& -\lambda\eta_{T-1}n_{T-2}\mathbb{E}[w_{T-2}] - \lambda\eta_{T-1}\mathbb{E}[c_{T-2}] - \beta_{T-1}\Big(\frac{\lambda^2\eta^2_{T-1}}{4\beta^2_{T-1}}\frac{B_{T-2}}{1-B_{T-2}}\Big) + \Delta_{T-1} \\
= \; & \beta_{T-2}\big(w_{T-2}-\mathbb{E}[w_{T-2}]\big)^2 - \lambda\eta_{T-2}\mathbb{E}[w_{T-2}] + \Delta_{T-2},
\end{aligned} \tag{3.15}$$

where

$$\begin{aligned}
\beta_{T-2} = \quad & \beta_{T-1}(n_{T-2})^2(1-B_{T-2}), \\
\eta_{T-2} = \quad & \eta_{T-1}n_{T-2}, \\
\Delta_{T-2} = \quad & \Delta_{T-1} + \beta_{T-1}\mathbb{E}[c_{T-2}^2] - \beta_{T-1}\mathbb{E}[c_{T-2}]^2 - \lambda\eta_{T-1}\mathbb{E}[c_{T-2}] \\
& -\frac{\lambda^2\eta_{T-1}^2}{4\beta_{T-1}}\frac{B_{T-2}}{1-B_{T-2}}.
\end{aligned} \tag{3.16}$$

Assume that the following equation (3.17) holds at time $t+1$, then we can prove it according to mathematical induction.

$$J_{t+1}(\mathbb{E}[w_{t+1}], w_{t+1}-\mathbb{E}[w_{t+1}]) = \beta_{t+1}\big(w_{t+1}-\mathbb{E}[w_{t+1}]\big)^2 - \lambda\eta_{t+1}\mathbb{E}[w_{t+1}] + \Delta_{t+1}, \tag{3.17}$$

where

$$\begin{aligned}
\beta_{t+1} = \quad & \beta_{t+2}(n_{t+1})^2(1-B_{t+1}), \\
\eta_{t+1} = \quad & \eta_{t+2}n_{t+1}, \\
\Delta_{t+1} = \quad & \Delta_{t+2} + \beta_{t+2}\mathbb{E}[c_{t+1}^2] - \beta_{t+2}\mathbb{E}[c_{t+1}]^2 - \lambda\eta_{t+2}\mathbb{E}[c_{t+1}] - \frac{\lambda^2\eta_{t+2}^2}{4\beta_{t+2}}\frac{B_{t+1}}{1-B_{t+1}},
\end{aligned} \tag{3.18}$$

and

$$\beta_T = 1, \qquad \eta_T = 1, \qquad \Delta_T = 0. \tag{3.19}$$

According to the equation (3.3) and (3.17), we derive the result at time t.

$$\begin{aligned}
& J_t(\mathbb{E}[w_t], w_t-\mathbb{E}[w_t]) \\
= \quad & \mathbb{E}\big[J_{t+1}(\mathbb{E}[w_{t+1}], w_{t+1}-\mathbb{E}[w_{t+1}])|\mathcal{F}_t\big] \\
= \quad & \mathbb{E}\big[\beta_{t+1}\big(w_{t+1}-\mathbb{E}[w_{t+1}]\big)^2 - \lambda\eta_{t+1}\mathbb{E}[w_{t+1}] + \Delta_{t+1}|\mathcal{F}_t\big] \\
= \quad & \beta_{t+1}(n_t)^2\big(w_t-\mathbb{E}[w_t]\big)^2 + \beta_{t+1}\big(v_t-\mathbb{E}[v_t]\big)'\mathbb{E}[R_tR_t']\big(v_t-\mathbb{E}[v_t]\big) \\
& +\beta_{t+1}\mathbb{E}[(v_t)']\Big(\mathbb{E}[R_tR_t'] - \mathbb{E}[R_t]\mathbb{E}[R_t']\Big)\mathbb{E}[v_t] + \beta_{t+1}\mathbb{E}[c_t^2] - \beta_{t+1}\mathbb{E}[c_t]^2 \\
& +2\beta_{t+1}n_t\big(w_t-\mathbb{E}[w_t]\big)\mathbb{E}[R_t']\big(v_t-\mathbb{E}[v_t]\big) - \lambda\eta_{t+1}n_t\mathbb{E}[w_t] - \lambda\eta_{t+1}\mathbb{E}[R_t']\mathbb{E}[v_t] \\
& -\lambda\eta_{t+1}\mathbb{E}[c_t] + \Delta_{t+1}.
\end{aligned} \tag{3.20}$$

The optimal strategies at period t can be derived by minimizing the above equation corresponding to v_t

$$\begin{aligned} v_t - \mathbb{E}[v_t] &= -n_t(w_t - \mathbb{E}[w_t])\mathbb{E}^{-1}[R_tR_t']\mathbb{E}[(R_t)], \\ \mathbb{E}[v_t] &= \Big(\mathbb{E}[R_tR_t'] - \mathbb{E}[R_t]\mathbb{E}[R_t']\Big)^{-1}\Big(\frac{\lambda\eta_{t+1}}{2\beta_{t+1}}\mathbb{E}[R_t]\Big) \\ &= \frac{\lambda\eta_{t+1}}{2\beta_{t+1}}\frac{\mathbb{E}^{-1}[R_tR_t']\mathbb{E}[R_t]}{1-B_t}. \end{aligned} \tag{3.21}$$

Substituting the optimal strategies back to (3.20), we obtain

$$\begin{aligned} & J_t(\mathbb{E}[w_t], w_t - \mathbb{E}[w_t]) \\ = \;& \beta_{t+1}(n_t)^2(1-B_t)\big(w_t - \mathbb{E}[w_t]\big)^2 + \beta_{t+1}(\mathbb{E}[(c_t)^2] - \mathbb{E}[c_t]^2) \\ & -\lambda\eta_{t+1}n_t\mathbb{E}[w_t] - \lambda\eta_{t+1}\mathbb{E}[c_t] - \beta_{t+1}\Big(\frac{\lambda^2\eta_{t+1}^2}{4\beta_{t+1}^2}\frac{B_t}{1-B_t}\Big) \\ & -\beta_{t+1}\Big(\big(\mathbb{E}[c_t] + \frac{\lambda\eta_{t+1}}{2\beta_{t+1}} - \widehat{B}_t\big)^2\frac{B_t}{1-B_t} + \Delta_{t+1} \\ = \;& \beta_t\big(w_t - \mathbb{E}[w_t]\big)^2 - \lambda\eta_t\mathbb{E}[w_t] + \Delta_t, \end{aligned} \tag{3.22}$$

where

$$\begin{aligned} \beta_t = \;& \beta_{t+1}(n_t)^2(1-B_t), \\ \eta_t = \;& \eta_{t+1}n_t, \\ \Delta_t = \;& \Delta_{t+1} + \beta_{t+1}\mathbb{E}[c_t^2] - \beta_{t+1}\mathbb{E}[c_t]^2 - \lambda\eta_{t+1}\mathbb{E}[c_t] - \frac{\lambda^2\eta_{t+1}^2}{4\beta_{t+1}}\frac{B_t}{1-B_t}. \end{aligned} \tag{3.23}$$

Substituting $\mathbb{E}[v_t^*]$ to the dynamics eqation in (3.2) yields

$$\mathbb{E}[w_{t+1}] = n_t\mathbb{E}[w_t] + \frac{\lambda\eta_{t+1}}{2\beta_{t+1}}\frac{B_t}{1-B_t} + \mathbb{E}[c_t], \tag{3.24}$$

which implies

$$\mathbb{E}[w_t] = w_0\prod_{k=0}^{t-1}n_k + \sum_{j=0}^{t-1}\big(\frac{\lambda\eta_{j+1}}{2\beta_{j+1}}\frac{B_j}{1-B_j} + \mathbb{E}[c_j]\big)\prod_{l=j+1}^{t-1}n_l. \tag{3.25}$$

Typically, we know the optimal value of problem (3.1) is equal to J_0. Thus,

$$\begin{aligned}\mathrm{Var}(w_T) = &\ J_0(\mathbb{E}[w_0], w_0 - \mathbb{E}[w_0]) + \lambda\mathbb{E}[w_T] \\ = &\ -\lambda\eta_0 w_0 + \Delta_0 + \lambda\Big(w_0 \prod_{k=0}^{T-1} n_k + \sum_{j=0}^{T-1} \big(\frac{\lambda\eta_{j+1}}{2\beta_{j+1}} \frac{B_j}{1-B_j} + \mathbb{E}[c_j]\big) \prod_{l=j+1}^{T-1} n_l\Big).\end{aligned} \tag{3.26}$$

3.2.2 Numerical Examples

According to the data provided in Elton et al. (2009), We investigate the example of a pension fund consisting of S&P 500 (SP), the index of Emerging Market (EM) and small Stock (MS) of U.S market. Moreover, we consider a bank account and cash flow in the model. Table 3.1 gives the expected return, standard deviation and the correlation coefficients between the three assets and cash flow. The annual risk free return rate is setted as 5% ($n_t = 1.05$).

Table 3.1: Data for assets and cashflow without correlation

	SP	EM	MS	cashflow
Expected return	14%	16%	17%	1
Standard deviation	18.5%	30%	24%	20%
Correlation coefficient				
SP	1	0.64	0.79	0
EM	0.64	1	0.75	0
MS	0.79	0.75	1	0
cashflow	0	0	0	1

Thus,

$$\mathbb{E}[R_t] = \begin{pmatrix} 0.09 \\ 0.11 \\ 0.12 \end{pmatrix}, \ \mathrm{Cov}(R_t) = \begin{pmatrix} 0.0342 & 0.0355 & 0.0351 \\ 0.0355 & 0.0900 & 0.0540 \\ 0.0351 & 0.0540 & 0.0576 \end{pmatrix},$$

$$\mathbb{E}[R_t R_t'] = \begin{pmatrix} 0.0423 & 0.0454 & 0.0459 \\ 0.0454 & 0.1021 & 0.0672 \\ 0.0459 & 0.0672 & 0.0690 \end{pmatrix}.$$

The correlation coefficient of the i-th asset and the cashflow is defined as $\rho = (\rho_1, \rho_2, \rho_3)$, where

$$\rho_i = 0.$$

This means the return rates between assets and the cashflow are uncorrelated. Thus, we have

$$\mathbb{E}[c_t R_t^i] = \mathbb{E}[c_t]\mathbb{E}[R_t^i],$$

$$\mathbb{E}[c_t^2] = \mathbb{E}[c_t]^2 + \mathrm{Var}(c_t).$$

Hence,

$$\begin{aligned}
\mathrm{Cov}\left(\begin{pmatrix} R_t \\ c_t \end{pmatrix}\right) &= \begin{pmatrix} \mathrm{Cov}(R_t) & \mathrm{Cov}(c_t, R_t) \\ \mathrm{Cov}(c_t, R_t') & \mathrm{Var}(c_t) \end{pmatrix} \\
&= \begin{pmatrix} 0.0342 & 0.0355 & 0.0351 & 0 \\ 0.0355 & 0.0900 & 0.0540 & 0 \\ 0.0351 & 0.0540 & 0.0576 & 0 \\ 0 & 0 & 0 & 0.0400 \end{pmatrix} \\
&\succ 0.
\end{aligned}$$

In order to make the equations more clearly, we define the notation as follow,

$$Y_1 = \mathbb{E}^{-1}[R_t R_t']\mathbb{E}[R_t] = \begin{pmatrix} 1.0589 \\ -0.1196 \\ 1.1033 \end{pmatrix}, \quad Y_2 = \mathbb{E}^{-1}[R_t R_t']\mathbb{E}[c_t R_t] = \begin{pmatrix} 1.0589 \\ -0.1196 \\ 1.1033 \end{pmatrix}.$$

$$B_t = 0.2190. \tag{3.27}$$

We assume the market period $t = 5$. Thus we have the optimal expected value of asset in different time period.

$$\mathbb{E}[w] = (\mathbb{E}[w_1], \mathbb{E}[w_2], \mathbb{E}[w_3], \mathbb{E}[w_4], \mathbb{E}[w_5]),$$

are given by

$$\mathbb{E}[w] = (4.4452, 5.9109, 7.4072, 8.9431, 10.5268).$$

We suppose the initial wealth $w_0 = 3$, and trade-off parameter $\lambda = 1$, then the optimal strategy is derived as follows,

$$v_0^* = -1.05(w_0 - 4.3103)Y_1 - Y_2,$$

$$v_1^* = -1.05(w_1 - 5.5259)Y_1 - Y_2,$$

$$v_2^* = -1.05(w_2 - 6.8022)Y_1 - Y_2,$$

$$v_3^* = -1.05(w_3 - 8.1423)Y_1 - Y_2,$$

$$v_4^* = -1.05(w_4 - 9.5494)Y_1 - Y_2.$$

The optimal final variance is $\mathrm{Var}(w_5) = 1.9294$.

3.3 The Optimal Strategy with Correlation between Return Rates and Cash Flow

3.3.1 The Dynamic Programming and Optimal Strategy

We assume the return rates between assets and cash flow are correlated during each period t in this subsection.

$$\begin{cases} \min & \mathrm{Var}(w_T) - \lambda\mathbb{E}[w_T], \\ \text{s.t.} & w_{t+1} = n_t w_t + R_t' v_t + c_t, \end{cases} \tag{3.28}$$

where

$$w_{t+1} = n_t w_t + R_t' v_t + c_t, \tag{3.29}$$

$$\mathbb{E}[w_{t+1}] = n_t\mathbb{E}[w_t] + \mathbb{E}[R_t']\mathbb{E}[v_t] + \mathbb{E}[c_t], \tag{3.30}$$

$$w_{t+1} - \mathbb{E}[w_{t+1}] = n_t(w_t - \mathbb{E}[w_t]) + R_t'(v_t - \mathbb{E}[v_t]) + (R_t' - \mathbb{E}[R_t'])\mathbb{E}[v_t] + (c_t - \mathbb{E}[c_t]). \tag{3.31}$$

We define the cost-go-functional as

$$J_t(\mathbb{E}[w_t, w_t - \mathbb{E}[w_t]) = \min_{v_t} \mathbb{E}\big[J_{t+1}(\mathbb{E}[w_{t+1}], w_{t+1} - \mathbb{E}[w_{t+1}])\big|\mathcal{F}_t\big],$$

with the boundary condition $J_T(\mathbb{E}[w_T], w_T - \mathbb{E}[w_T]) = (w_T - \mathbb{E}[w_T])^2 - \lambda\mathbb{E}[w_T]$. Thus,

$$\begin{aligned}
& \mathbb{E}\big[J_T(\mathbb{E}[w_T], w_T - \mathbb{E}[w_T])|\mathcal{F}_{T-1}\big] \\
= \;& \mathbb{E}\big[(w_T - \mathbb{E}[w_T])^2 - \lambda\mathbb{E}[w_T]|\mathcal{F}_{T-1}\big] \\
= \;& \mathbb{E}\Big[\Big(n_{T-1}(w_{T-1} - \mathbb{E}[w_{T-1}]) + R'_{T-1}(v_{T-1} - \mathbb{E}[v_{T-1}]) + (R'_{T-1} - \mathbb{E}[R'_{T-1}])\mathbb{E}[v_{T-1}] \\
& +(c_{T-1} - \mathbb{E}[c_{T-1}])\Big)^2 - \lambda\Big(n_{T-1}\mathbb{E}[w_{T-1}] + \mathbb{E}[R'_{T-1}]\mathbb{E}[v_{T-1}] + \mathbb{E}[c_{T-1}]\Big)\Big|\mathcal{F}_{T-1}\Big],
\end{aligned} \tag{3.32}$$

where

$$\begin{aligned}
& \Big(n_{T-1}(w_{T-1} - \mathbb{E}[w_{T-1}]) + R'_{T-1}(v_{T-1} - \mathbb{E}[v_{T-1}]) + (R'_{T-1} - \mathbb{E}[R'_{T-1}])\mathbb{E}[v_{T-1}] \\
& +(c_{T-1} - \mathbb{E}[c_{T-1}])\Big)^2 \\
= \;& \big(n_{T-1}(w_{T-1} - \mathbb{E}[w_{T-1}])\big)^2 + \big(R'_{T-1}(v_{T-1} - \mathbb{E}[v_{T-1}])\big)^2 \\
& +\big((R'_{T-1} - \mathbb{E}[R'_{T-1}])\mathbb{E}[v_{T-1}]\big)^2 + \big(c_{T-1} - \mathbb{E}[c_{T-1}]\big)^2 \\
& +2\big(n_{T-1}(w_{T-1} - \mathbb{E}[w_{T-1}])\big)\big(R'_{T-1}(v_{T-1} - \mathbb{E}[v_{T-1}])\big) \\
& +2\big(n_{T-1}(w_{T-1} - \mathbb{E}[w_{T-1}])\big)\big((R'_{T-1} - \mathbb{E}[R'_{T-1}])\mathbb{E}[v_{T-1}]\big) \\
& +2\big(n_{T-1}(w_{T-1} - \mathbb{E}[w_{T-1}])\big)\big(c_{T-1} - \mathbb{E}[c_{T-1}]\big) \\
& +2\big(v_{T-1} - \mathbb{E}[v_{T-1}]\big)'(R_{T-1})\big(R'_{T-1} - \mathbb{E}[R'_{T-1}])\mathbb{E}[v_{T-1}]\big) \\
& +2R'_{T-1}(v_{T-1} - \mathbb{E}[v_{T-1}])(c_{T-1} - \mathbb{E}[c_{T-1}]) \\
& +2\big(R'_{T-1} - \mathbb{E}[R'_{T-1}]\big)\mathbb{E}[v_{T-1}]\big(c_{T-1} - \mathbb{E}[c_{T-1}]\big)
\end{aligned} \tag{3.33}$$

Since $w_t^i - \mathbb{E}[w_t^i]$, $\mathbb{E}[w_t^i]$, $v_t^i - \mathbb{E}[v_t^i]$, $\mathbb{E}[v_t^i]$ are $\mathcal{F}_t$-measurable, we get

$$
\begin{aligned}
&\mathbb{E}\Big[\big(n_{T-1}(w_{T-1} - \mathbb{E}[w_{T-1}])\big)^2\Big|\mathcal{F}_{T-1}\Big] = (n_{T-1})^2\big(w_{T-1} - \mathbb{E}[w_{T-1}]\big)^2,\\
&\mathbb{E}\Big[\Big(R'_{T-1}(v_{T-1} - \mathbb{E}[v_{T-1}])\Big)^2\Big|\mathcal{F}_{T-1}\Big]\\
&= \big(v_{T-1} - \mathbb{E}[v_{T-1}]\big)'\mathbb{E}[R_{T-1}R'_{T-1}]\big(v_{T-1} - \mathbb{E}[v_{T-1}]\big),\\
&\mathbb{E}\Big[\Big((R'_{T-1} - \mathbb{E}[R'_{T-1}])\mathbb{E}[v_{T-1}]\Big)^2\Big|\mathcal{F}_{T-1}\Big]\\
&= \mathbb{E}[(v_{T-1})']\Big(\mathbb{E}[R_{T-1}R'_{T-1}] - \mathbb{E}[R_{T-1}]\mathbb{E}[R'_{T-1}]\Big)\mathbb{E}[v_{T-1}],\\
&\mathbb{E}\Big[\big(c_{T-1} - \mathbb{E}[c_{T-1}]\big)^2\Big|\mathcal{F}_{T-1}\Big] = \mathbb{E}[(c_{T-1})^2] - \mathbb{E}[c_{T-1}]^2,\\
&\mathbb{E}\Big[2n_{T-1}(w_{T-1} - \mathbb{E}[w_{T-1}])R'_{T-1}(v_{T-1} - \mathbb{E}[v_{T-1}])\Big|\mathcal{F}_{T-1}\Big]\\
&= 2n_{T-1}\big(w_{T-1} - \mathbb{E}[w_{T-1}]\big)\mathbb{E}[R'_{T-1}]\big(v_{T-1} - \mathbb{E}[v_{T-1}]\big),\\
&\mathbb{E}\Big[2n_{T-1}\big(w_{T-1} - \mathbb{E}[w_{T-1}]\big)\big(R'_{T-1} - \mathbb{E}[R'_{T-1}]\big)\mathbb{E}[v_{T-1}]\Big|\mathcal{F}_{T-1}\Big] = 0,\\
&\mathbb{E}\Big[2n_{T-1}(w_{T-1} - \mathbb{E}[w_{T-1}])(c_{T-1} - \mathbb{E}[c_{T-1}])\Big|\mathcal{F}_{T-1}\Big] = 0,\\
&\mathbb{E}\Big[2\big(v_{T-1} - \mathbb{E}[v_{T-1}]\big)'(R_{T-1})\Big(R'_{T-1} - \mathbb{E}[R'_{T-1}]\Big)\mathbb{E}[v_{T-1}]\Big|\mathcal{F}_{T-1}\Big]\\
&= 2\big(v_{T-1} - \mathbb{E}[v_{T-1}]\big)'\Big(\mathbb{E}[R_{T-1}R'_{T-1}] - \mathbb{E}[R_{T-1}]\mathbb{E}[R'_{T-1}]\Big)\mathbb{E}[v_{T-1}],\\
&\mathbb{E}\Big[2R'_{T-1}(v_{T-1} - \mathbb{E}[v_{T-1}])(c_{T-1} - \mathbb{E}[c_{T-1}])\Big|\mathcal{F}_{T-1}\Big]\\
&= 2\big(\mathbb{E}[c_{T-1}R'_{T-1}] - \mathbb{E}[c_{T-1}]\mathbb{E}[R'_{T-1})\big)(v_{T-1} - \mathbb{E}[v_{T-1}]),\\
&\mathbb{E}\Big[2\big(R'_{T-1} - \mathbb{E}[R'_{T-1}]\big)\mathbb{E}[v_{T-1}]\big(c_{T-1} - \mathbb{E}[c_{T-1}]\big)\Big|\mathcal{F}_{T-1}\Big]\\
&= 2\Big(\mathbb{E}[c_{T-1}R'_{T-1}] - \mathbb{E}[c_{T-1}]\mathbb{E}[R'_{T-1}]\Big)\mathbb{E}[v_{T-1}].
\end{aligned}
$$

Note that $\mathbb{E}\big[v_t^i - \mathbb{E}[v_t^i]|\mathcal{F}_0\big] = 0$, which implies

$$
\begin{aligned}
&\mathbb{E}\Big[\mathbb{E}\Big[2\big(v_{T-1} - \mathbb{E}[v_{T-1}]\big)'(R_{T-1})\Big(R'_{T-1} - \mathbb{E}[R'_{T-1}]\Big)\mathbb{E}[v_{T-1}]\Big|\mathcal{F}_{T-1}\Big]\Big|\mathcal{F}_0\Big]\\
&= \mathbb{E}\Big[2\big(v_{T-1} - \mathbb{E}[v_{T-1}]\big)'\Big(\mathbb{E}[R_{T-1}R'_{T-1}] - \mathbb{E}[R_{T-1}]\mathbb{E}[R'_{T-1}]\Big)\mathbb{E}[v_{T-1}]\Big|\mathcal{F}_0\Big] = 0,\\
&\mathbb{E}\Big[\mathbb{E}\Big[2R'_{T-1}(v_{T-1} - \mathbb{E}[v_{T-1}])(c_{T-1} - \mathbb{E}[c_{T-1}])\Big|\mathcal{F}_{T-1}\Big]\Big|\mathcal{F}_0\Big]\\
&= \mathbb{E}\Big[2\big(\mathbb{E}[c_{T-1}R'_{T-1}] - \mathbb{E}[c_{T-1}]\mathbb{E}[R'_{T-1})\big)(v_{T-1} - \mathbb{E}[v_{T-1}])\Big|\mathcal{F}_0\Big] = 0.
\end{aligned}
$$

Therefore, we can reduce (3.32) into

$$
\begin{aligned}
& \mathbb{E}\big[J_T(\mathbb{E}[w_T], w_T - \mathbb{E}[w_T])|\mathcal{F}_{T-1}\big] \\
= \;& \mathbb{E}\big[(w_T - \mathbb{E}[w_T])^2 - \lambda\mathbb{E}[w_T]|\mathcal{F}_{T-1}\big] \\
= \;& \mathbb{E}\Big[\Big(n_{T-1}(w_{T-1} - \mathbb{E}[w_{T-1}]) + R'_{T-1}(v_{T-1} - \mathbb{E}[v_{T-1}]) + (R'_{T-1} - \mathbb{E}[R'_{T-1}])\mathbb{E}[v_{T-1}] \\
& +(c_{T-1} - \mathbb{E}[c_{T-1}])\Big)^2 - \lambda\Big(n_{T-1}\mathbb{E}[w_{T-1}] + \mathbb{E}[R'_{T-1}]\mathbb{E}[v_{T-1}] + \lambda\mathbb{E}[c_{T-1}]\Big)\Big|\mathcal{F}_{T-1}\Big] \\
= \;& (n_{T-1})^2\big(w_{T-1} - \mathbb{E}[w_{T-1}]\big)^2 + \big(v_{T-1} - \mathbb{E}[v_{T-1}]\big)'\mathbb{E}[R_{T-1}R'_{T-1}]\big(v_{T-1} - \mathbb{E}[v_{T-1}]\big) \\
& +\mathbb{E}[(v_{T-1})']\Big(\mathbb{E}[R_{T-1}R'_{T-1}] - \mathbb{E}[R_{T-1}]\mathbb{E}[R'_{T-1}]\Big)\mathbb{E}[v_{T-1}] + \mathbb{E}[(c_{T-1})^2] - \mathbb{E}[c_{T-1}]^2 \\
& +2n_{T-1}\big(w_{T-1} - \mathbb{E}[w_{T-1}]\big)\mathbb{E}[R'_{T-1}]\big(v_{T-1} - \mathbb{E}[v_{T-1}]\big) \\
& +2\Big(\mathbb{E}[c_{T-1}R'_{T-1}] - \mathbb{E}[c_{T-1}]\mathbb{E}[R'_{T-1}]\Big)\mathbb{E}[v_{T-1}] - \lambda n_{T-1}\mathbb{E}[w_{T-1}] - \lambda\mathbb{E}[R'_{T-1}]\mathbb{E}[v_{T-1}] \\
& -\lambda\mathbb{E}[c_{T-1}].
\end{aligned}
\tag{3.34}
$$

The optimal strategies at period $T-1$ can be derived by minimizing the above equation corresponding to v_{T-1}

$$
\begin{aligned}
v_{T-1} - \mathbb{E}[v_{T-1}] &= -n_{T-1}(w_{T-1} - \mathbb{E}[w_{T-1}])\mathbb{E}^{-1}[R_{T-1}R'_{T-1}]\mathbb{E}[(R_{T-1})], \\
\mathbb{E}[v_{T-1}] &= -\Big(\mathbb{E}[R_{T-1}R'_{T-1}] - \mathbb{E}[R_{T-1}]\mathbb{E}[R'_{T-1}]\Big)^{-1} \\
& \quad \times\Big(\mathbb{E}[c_{T-1}R_{T-1}] - \mathbb{E}[c_{T-1}]\mathbb{E}[R_{T-1}] - \frac{\lambda}{2}\mathbb{E}[R_{T-1}]\Big) \\
&= \Big(\mathbb{E}[c_{T-1}] + \frac{\lambda}{2} - \widehat{B}_{T-1}\Big)\frac{\mathbb{E}^{-1}[R_{T-1}R'_{T-1}]\mathbb{E}[R_{T-1}]}{1 - B_{T-1}} \\
& \quad -\mathbb{E}^{-1}[R_{T-1}R'_{T-1}]\mathbb{E}[c_{T-1}R_{T-1}],
\end{aligned}
\tag{3.35}
$$

where

$$
\begin{aligned}
B_{T-1} = \;& \mathbb{E}[R'_{T-1}]\mathbb{E}^{-1}[R_{T-1}R'_{T-1}]\mathbb{E}[R_{T-1}], \\
\widehat{B}_{T-1} = \;& \mathbb{E}[c_{T-1}R'_{T-1}]\mathbb{E}^{-1}[R_{T-1}R'_{T-1}]\mathbb{E}[R_{T-1}], \\
\widetilde{B}_{T-1} = \;& \mathbb{E}[c_{T-1}R'_{T-1}]\mathbb{E}^{-1}[R_{T-1}R'_{T-1}]\mathbb{E}[c_{T-1}R_{T-1}].
\end{aligned}
\tag{3.36}
$$

Substituting the optimal strategies back to (3.34), we obtain

$$
\begin{aligned}
& J_{T-1}(\mathbb{E}[w_{T-1}], w_{T-1}-\mathbb{E}[w_{T-1}]) \\
= \;& (n_{T-1})^2\big(w_{T-1}-\mathbb{E}[w_{T-1}]\big)^2 + (\mathbb{E}[(c_{T-1})^2]-\mathbb{E}[c_{T-1}]^2) - \lambda n_{T-1}\mathbb{E}[w_{T-1}] \\
& -\lambda\mathbb{E}[c_{T-1}] - (n_{T-1})^2 B_{T-1}\big(w_{T-1}-\mathbb{E}[w_{T-1}]\big)^2 \\
& -\Big(\big(\mathbb{E}[c_{T-1}]+\frac{\lambda}{2}-\widehat{B}_{T-1}\big)\frac{\mathbb{E}[R'_{T-1}]\mathbb{E}^{-1}[R_{T-1}R'_{T-1}]}{1-B_{T-1}} \\
& -\mathbb{E}[c_{T-1}R'_{T-1}]\mathbb{E}^{-1}[R_{T-1}R'_{T-1}]\Big)\Big(\mathbb{E}[R_{T-1}R'_{T-1}]-\mathbb{E}[R_{T-1}]\mathbb{E}[R'_{T-1}]\Big) \\
& \times\Big(\big(\mathbb{E}[c_{T-1}]+\frac{\lambda}{2}-\widehat{B}_{T-1}\big)\frac{\mathbb{E}^{-1}[R_{T-1}R'_{T-1}]\mathbb{E}[R_{T-1}]}{1-B_{T-1}} \qquad (3.37)\\
& -\mathbb{E}^{-1}[R_{T-1}R'_{T-1}]\mathbb{E}[c_{T-1}R_{T-1}]\Big) \\
= \;& (n_{T-1})^2(1-B_{T-1})\big(w_{T-1}-\mathbb{E}[w_{T-1}]\big)^2 + (\mathbb{E}[(c_{T-1})^2]-\mathbb{E}[c_{T-1}]^2) \\
& -\lambda n_{T-1}\mathbb{E}[w_{T-1}] - \lambda\mathbb{E}[c_{T-1}] - \Big(\big(\mathbb{E}[c_{T-1}]+\frac{\lambda}{2}-\widehat{B}_{T-1}\big)^2\frac{B_{T-1}}{1-B_{T-1}} \\
& -2\big(\mathbb{E}[c_{T-1}]+\frac{\lambda}{2}-\widehat{B}_{T-1}\big)\widehat{B}_{T-1}+\widetilde{B}_{T-1}-\widehat{B}^2_{T-1}\Big) \\
= \;& \beta_{T-1}\big(w_{T-1}-\mathbb{E}[w_{T-1}]\big)^2 - \lambda\eta_{T-1}\mathbb{E}[w_{T-1}] + \Delta_{T-1},
\end{aligned}
$$

where

$$
\begin{aligned}
\beta_{T-1} = \;& (n_{T-1})^2(1-B_{T-1}), \\
\eta_{T-1} = \;& n_{T-1}, \\
\Delta_{T-1} = \;& \mathbb{E}[(c_{T-1})^2]-\mathbb{E}[c_{T-1}]^2-\lambda\mathbb{E}[c_{T-1}] - \Big(\big(\mathbb{E}[c_{T-1}]+\frac{\lambda}{2}-\widehat{B}_{T-1}\big)^2\frac{B_{T-1}}{1-B_{T-1}} \\
& -2\big(\mathbb{E}[c_{T-1}]+\frac{\lambda}{2}-\widehat{B}_{T-1}\big)\widehat{B}_{T-1}+\widetilde{B}_{T-1}-\widehat{B}^2_{T-1}\Big).
\end{aligned}
\qquad (3.38)
$$

Repeating the process at time $T-2$, we have

$$
\begin{aligned}
& J_{T-2}(\mathbb{E}[w_{T-2}], w_{T-2}-\mathbb{E}[w_{T-2}]) \\
= \;& \mathbb{E}\big[J_{T-1}(\mathbb{E}[w_{T-1}], w_{T-1}-\mathbb{E}[w_{T-1}])|\mathcal{F}_{T-2}\big] \\
= \;& \mathbb{E}\big[\beta_{T-1}\big(w_{T-1}-\mathbb{E}[w_{T-1}]\big)^2-\lambda\eta_{T-1}\mathbb{E}[w_{T-1}]+\Delta_{T-1}|\mathcal{F}_{T-2}\big] \\
= \;& \beta_{T-1}(n_{T-2})^2\big(w_{T-2}-\mathbb{E}[w_{T-2}]\big)^2+\beta_{T-1}\big(v_{T-2}-\mathbb{E}[v_{T-2}]\big)'\mathbb{E}[R_{T-2}R'_{T-2}] \\
& \times\big(v_{T-2}-\mathbb{E}[v_{T-2}]\big)+\beta_{T-1}\mathbb{E}[(v_{T-2})']\big(\mathbb{E}[R_{T-2}R'_{T-2}]-\mathbb{E}[R_{T-2}]\mathbb{E}[R'_{T-2}]\big) \\
& \times\mathbb{E}[v_{T-2}]+\beta_{T-1}\mathbb{E}[(c_{T-2})^2]-\beta_{T-1}\mathbb{E}[c_{T-2}]^2 \\
& +2\beta_{T-1}n_{T-2}\big(w_{T-2}-\mathbb{E}[w_{T-2}]\big)\mathbb{E}[R'_{T-2}]\big(v_{T-2}-\mathbb{E}[v_{T-2}]\big) \\
& +2\beta_{T-1}\big(\mathbb{E}[c_{T-2}R'_{T-2}]-\mathbb{E}[c_{T-2}]\mathbb{E}[R'_{T-2}]\big)\mathbb{E}[v_{T-2}] \\
& -\lambda\eta_{T-1}n_{T-2}\mathbb{E}[w_{T-2}]-\lambda\eta_{T-1}\mathbb{E}[R'_{T-2}]\mathbb{E}[v_{T-2}]-\lambda\eta_{T-1}\mathbb{E}[c_{T-2}]+\Delta_{T-1}.
\end{aligned}
\tag{3.39}
$$

The optimal strategies at period $T-2$ can be derived from the above equation corresponding to v_{T-2}

$$
\begin{aligned}
v_{T-2}-\mathbb{E}[v_{T-2}] &= -n_{T-2}(w_{T-2}-\mathbb{E}[w_{T-2}])\mathbb{E}^{-1}[R_{T-2}R'_{T-2}]\mathbb{E}[(R_{T-2})], \\
\mathbb{E}[v_{T-2}] &= -\Big(\mathbb{E}[R_{T-2}R'_{T-2}]-\mathbb{E}[R_{T-2}]\mathbb{E}[R'_{T-2}]\Big)^{-1} \\
& \qquad \times\Big(\mathbb{E}[c_{T-2}R_{T-2}]-\mathbb{E}[c_{T-2}]\mathbb{E}[R_{T-2}]-\frac{\lambda\eta_{T-1}}{2\beta_{T-1}}\mathbb{E}[R_{T-2}]\Big) \\
& = \Big(\mathbb{E}[c_{T-2}]+\frac{\lambda\eta_{T-1}}{2\beta_{T-1}}-\widehat{B}_{T-2}\Big)\frac{\mathbb{E}^{-1}[R_{T-2}R'_{T-2}]\mathbb{E}[R_{T-2}]}{1-B_{T-2}} \\
& \qquad -\mathbb{E}^{-1}[R_{T-2}R'_{T-2}]\mathbb{E}[c_{T-2}R_{T-2}].
\end{aligned}
\tag{3.40}
$$

Substituting the optimal strategies back to (3.39), we obtain

$$\begin{aligned}
& J_{T-2}(\mathbb{E}[w_{T-2}], w_{T-2} - \mathbb{E}[w_{T-2}]) \\
= \; & \beta_{T-1}(n_{T-2})^2(1 - B_{T-2})\big(w_{T-2} - \mathbb{E}[w_{T-2}]\big)^2 + \beta_{T-1}(\mathbb{E}[(c_{T-2})^2] - \mathbb{E}[c_{T-2}]^2) \\
& -\lambda\eta_{T-1}n_{T-2}\mathbb{E}[w_{T-2}] - \lambda\eta_{T-1}\mathbb{E}[c_{T-2}] \\
& -\beta_{T-1}\Big(\big(\mathbb{E}[c_{T-2}] + \frac{\lambda\eta_{T-1}}{2\beta_{T-1}} - \widehat{B}_{T-2}\big)^2 \frac{B_{T-2}}{1 - B_{T-2}} \\
& -2\big(\mathbb{E}[c_{T-2}] + \frac{\lambda\eta_{T-1}}{2\beta_{T-1}} - \widehat{B}_{T-2}\big)\widehat{B}_{T-2} + \widetilde{B}_{T-2} - \widehat{B}^2_{T-2}\Big) + \Delta_{T-1} \\
= \; & \beta_{T-2}\big(w_{T-2} - \mathbb{E}[w_{T-2}]\big)^2 - \lambda\eta_{T-2}\mathbb{E}[w_{T-2}] + \Delta_{T-2},
\end{aligned} \tag{3.41}$$

where

$$\begin{aligned}
\beta_{T-2} = \; & \beta_{T-1}(n_{T-2})^2(1 - B_{T-2}), \\
\eta_{T-2} = \; & \eta_{T-1}n_{T-2}, \\
\Delta_{T-2} = \; & \Delta_{T-1} + \beta_{T-1}\mathbb{E}[(c_{T-2})^2] - \beta_{T-1}\mathbb{E}[c_{T-2}]^2 - \lambda\eta_{T-1}\mathbb{E}[c_{T-2}] \\
& -\beta_{T-1}\Big(\big(\mathbb{E}[c_{T-2}] + \frac{\lambda\eta_{T-1}}{2\beta_{T-1}} - \widehat{B}_{T-2}\big)^2 \frac{B_{T-2}}{1 - B_{T-2}} \\
& -2\big(\mathbb{E}[c_{T-2}] + \frac{\lambda\eta_{T-1}}{2\beta_{T-1}} - \widehat{B}_{T-2}\big)\widehat{B}_{T-2} + \widetilde{B}_{T-2} - \widehat{B}^2_{T-2}\Big).
\end{aligned} \tag{3.42}$$

Assume that the following equation (3.43) holds at time $t+1$, we prove it according to the mathematical induction.

$$J_{t+1}(\mathbb{E}[w_{t+1}], w_{t+1} - \mathbb{E}[w_{t+1}]) = \beta_{t+1}\big(w_{t+1} - \mathbb{E}[w_{t+1}]\big)^2 - \lambda\eta_{t+1}\mathbb{E}[w_{t+1}] + \Delta_{t+1}, \tag{3.43}$$

where

$$\begin{aligned}
\beta_{t+1} = \; & \beta_{t+2}(n_{t+1})^2(1 - B_{t+1}), \\
\eta_{t+1} = \; & \eta_{t+2}n_{t+1}, \\
\Delta_{t+1} = \; & \Delta_{t+2} + \beta_{t+2}\mathbb{E}[(c_{t+1})^2] - \beta_{t+2}\mathbb{E}[c_{t+1}]^2 - \lambda\eta_{t+2}\mathbb{E}[c_{t+1}] \\
& -\beta_{t+2}\Big(\big(\mathbb{E}[c_{t+1}] + \frac{\lambda\eta_{t+2}}{2\beta_{t+2}} - \widehat{B}_{t+1}\big)^2 \frac{B_{t+1}}{1 - B_{t+1}} \\
& -2\big(\mathbb{E}[c_{t+1}] + \frac{\lambda\eta_{t+2}}{2\beta_{t+2}} - \widehat{B}_{t+1}\big)\widehat{B}_{t+1} + \widetilde{B}_{t+1} - \widehat{B}^2_{t+1}\Big),
\end{aligned} \tag{3.44}$$

and

$$\beta_T = 1, \qquad \eta_T = 1, \qquad \Delta_T = 0. \tag{3.45}$$

According to equations (3.43) and (3.44), we derive the result at time t.

$$\begin{aligned}
& J_t(\mathbb{E}[w_t], w_t - \mathbb{E}[w_t]) \\
= \ & \mathbb{E}\big[J_{t+1}(\mathbb{E}[w_{t+1}], w_{t+1} - \mathbb{E}[w_{t+1}])|\mathcal{F}_t\big] \\
= \ & \mathbb{E}\big[\beta_{t+1}\big(w_{t+1} - \mathbb{E}[w_{t+1}]\big)^2 - \lambda\eta_{t+1}\mathbb{E}[w_{t+1}] + \Delta_{t+1}|\mathcal{F}_t\big] \\
= \ & \beta_{t+1}(n_t)^2\big(w_t - \mathbb{E}[w_t]\big)^2 + \beta_{t+1}\big(v_t - \mathbb{E}[v_t]\big)'\mathbb{E}[R_tR_t']\big(v_t - \mathbb{E}[v_t]\big) \\
& +\beta_{t+1}\mathbb{E}[(v_t)']\Big(\mathbb{E}[R_tR_t'] - \mathbb{E}[R_t]\mathbb{E}[R_t']\Big)\mathbb{E}[v_t] + \beta_{t+1}\mathbb{E}[(c_t)^2] - \beta_{t+1}\mathbb{E}[c_t]^2 \\
& +2\beta_{t+1}n_t\big(w_t - \mathbb{E}[w_t]\big)\mathbb{E}[R_t']\big(v_t - \mathbb{E}[v_t]\big) \\
& +2\beta_{t+1}\Big(\mathbb{E}[c_tR_t'] - \mathbb{E}[c_t]\mathbb{E}[R_t']\Big)\mathbb{E}[v_t] \\
& -\lambda\eta_{t+1}n_t\mathbb{E}[w_t] - \lambda\eta_{t+1}\mathbb{E}[R_t']\mathbb{E}[v_t] - \lambda\eta_{t+1}\mathbb{E}[c_t] + \Delta_{t+1}.
\end{aligned} \tag{3.46}$$

The optimal strategies at period t can be derived from the above equation corresponding to v_t

$$\begin{aligned}
v_t - \mathbb{E}[v_t] &= -n_t(w_t - \mathbb{E}[w_t])\mathbb{E}^{-1}[R_tR_t']\mathbb{E}[(R_t)], \\
\mathbb{E}[v_t] &= -\Big(\mathbb{E}[R_tR_t'] - \mathbb{E}[R_t]\mathbb{E}[R_t']\Big)^{-1}\Big(\mathbb{E}[c_tR_t] - \mathbb{E}[c_t]\mathbb{E}[R_t] - \frac{\lambda\eta_{t+1}}{2\beta_{t+1}}\mathbb{E}[R_t]\Big) \\
&= \Big(\mathbb{E}[c_t] + \frac{\lambda\eta_{t+1}}{2\beta_{t+1}} - \widehat{B}_t\Big)\frac{\mathbb{E}^{-1}[R_tR_t']\mathbb{E}[R_t]}{1 - B_t} - \mathbb{E}^{-1}[R_tR_t']\mathbb{E}[c_tR_t].
\end{aligned} \tag{3.47}$$

Substituting the optimal strategies back to (3.46), we obtain

$$\begin{aligned}
& J_t(\mathbb{E}[w_t], w_t - \mathbb{E}[w_t]) \\
= \ & \beta_{t+1}(n_t)^2(1 - B_t)\big(w_t - \mathbb{E}[w_t]\big)^2 + \beta_{t+1}(\mathbb{E}[(c_t)^2] - \mathbb{E}[c_t]^2) \\
& -\lambda\eta_{t+1}n_t\mathbb{E}[w_t] - \lambda\eta_{t+1}\mathbb{E}[c_t] \\
& -\beta_{t+1}\Big(\big(\mathbb{E}[c_t] + \frac{\lambda\eta_{t+1}}{2\beta_{t+1}} - \widehat{B}_t\big)^2\frac{B_t}{1 - B_t} - 2\big(\mathbb{E}[c_t] + \frac{\lambda\eta_{t+1}}{2\beta_{t+1}} - \widehat{B}_t\big)\widehat{B}_t + \widetilde{B}_t - \widehat{B}_t^2\Big) \\
& +\Delta_{t+1} \\
= \ & \beta_t\big(w_t - \mathbb{E}[w_t]\big)^2 - \lambda\eta_t\mathbb{E}[w_t] + \Delta_t,
\end{aligned} \tag{3.48}$$

where

$$\begin{aligned}
\beta_t = &\ \beta_{t+1}(n_t)^2(1-B_t), \\
\eta_t = &\ \eta_{t+1}n_t, \\
\Delta_t = &\ \Delta_{t+1} + \beta_{t+1}\mathbb{E}[(c_t)^2] - \beta_{t+1}\mathbb{E}[c_t]^2 - \lambda\eta_{t+1}\mathbb{E}[c_t] \\
&-\beta_{t+1}\Big(\big(\mathbb{E}[c_t] + \frac{\lambda\eta_{t+1}}{2\beta_{t+1}} - \widehat{B}_t\big)^2\frac{B_t}{1-B_t} - 2\big(\mathbb{E}[c_t] + \frac{\lambda\eta_{t+1}}{2\beta_{t+1}} - \widehat{B}_t\big)\widehat{B}_t \\
&+\widetilde{B}_t - \widehat{B}_t^2\Big).
\end{aligned} \tag{3.49}$$

Substituting $\mathbb{E}[v_t^*]$ to the equation (3.30) we have

$$\mathbb{E}[w_{t+1}] = n_t\mathbb{E}[w_t] + \big(\mathbb{E}[c_t] + \frac{\lambda\eta_{t+1}}{2\beta_{t+1}} - \widehat{B}_t\big)\frac{B_t}{1-B_t} - \widehat{B}_t + \mathbb{E}[c_t]. \tag{3.50}$$

Therefore,

$$\mathbb{E}[w_t] = w_0\prod_{k=0}^{t-1}n_k + \sum_{j=0}^{t-1}\Big(\big(\mathbb{E}[c_j] + \frac{\lambda\eta_{j+1}}{2\beta_{j+1}} - \widehat{B}_j\big)\frac{B_j}{1-B_j} - \widehat{B}_j + \mathbb{E}[c_j]\Big)\prod_{l=j+1}^{t-1}n_l. \tag{3.51}$$

Typically, we know the optimal value of (3.28) is equal to J_0. Thus,

$$\begin{aligned}
\mathrm{Var}(w_T) = &\ J_0(\mathbb{E}[w_0], w_0 - \mathbb{E}[w_0]) + \lambda\mathbb{E}[w_T] \\
= &\ -\lambda\eta_0 w_0 + \Delta_0 + \lambda\Bigg(w_0\prod_{k=0}^{T-1}n_k + \sum_{j=0}^{T-1}\Big(\big(\mathbb{E}[c_j] + \frac{\lambda\eta_{j+1}}{2\beta_{j+1}} - \widehat{B}_j\big)\frac{B_j}{1-B_j} \\
&-\widehat{B}_j + \mathbb{E}[c_j]\Big)\prod_{l=j+1}^{T-1}n_l\Bigg).
\end{aligned} \tag{3.52}$$

3.3.2 Numerical Examples

Similar to the uncorrelation case, table 3.2 gives the expected return, standard deviation and the correlation coefficients between the three assets and cash flow. We also assume the risk-free return rate as 5% ($n_t = 1.05$).

Table 3.2: Data for assets and cashflow with correlation

	SP	EM	MS	cashflow
Expected return	14%	16%	17%	1
Standard deviation	18.5%	30%	24%	20%
Correlation coefficient				
SP	1	0.64	0.79	ρ_1
EM	0.64	1	0.75	ρ_2
MS	0.79	0.75	1	ρ_3
cashflow	ρ_1	ρ_2	ρ_3	1

Thus,

$$\mathbb{E}[R_t] = \begin{pmatrix} 0.09 \\ 0.11 \\ 0.12 \end{pmatrix}, \ \mathrm{Cov}(R_t) = \begin{pmatrix} 0.0342 & 0.0355 & 0.0351 \\ 0.0355 & 0.0900 & 0.0540 \\ 0.0351 & 0.0540 & 0.0576 \end{pmatrix},$$

$$\mathbb{E}[R_t R_t'] = \begin{pmatrix} 0.0423 & 0.0454 & 0.0459 \\ 0.0454 & 0.1021 & 0.0672 \\ 0.0459 & 0.0672 & 0.0720 \end{pmatrix}.$$

We define the correlation coefficient of the cash flow and i-th asset as $\rho = (\rho_1, \rho_2, \rho_3)$, where

$$\rho_i = \frac{\mathrm{Cov}(c_t, R_t^i)}{\sqrt{\mathrm{Var}(c_t)}\sqrt{\mathrm{Var}(R_t^i)}},$$

which means return rates between assets and the cashflow are correlated. Thus, we have

$$\mathbb{E}[c_t R_t^i] = \mathbb{E}[c_t]\mathbb{E}[R_t^i] + \rho_i \sqrt{\mathrm{Var}(c_t)}\sqrt{\mathrm{Var}(R_t^i)},$$

$$\mathbb{E}[c_t^2] = \mathbb{E}[c_t]^2 + \mathrm{Var}(c_t).$$

Assume that

$$\rho = (\rho_1, \rho_2, \rho_3) = (-0.25, 0.5, 0.25).$$

Thus,

$$\begin{aligned}
\mathrm{Cov}\left(\begin{pmatrix} R_t \\ c_t \end{pmatrix}\right) &= \begin{pmatrix} \mathrm{Cov}(R_t) & \mathrm{Cov}(c_t, R_t) \\ \mathrm{Cov}(c_t, R_t') & \mathrm{Var}(c_t) \end{pmatrix} \\
&= \begin{pmatrix} 0.0342 & 0.0355 & 0.0351 & -0.0092 \\ 0.0355 & 0.0900 & 0.0540 & 0.0300 \\ 0.0351 & 0.0540 & 0.0576 & 0.0120 \\ -0.0092 & 0.0300 & 0.0120 & 0.0400 \end{pmatrix} \\
&\succ 0.
\end{aligned}$$

Employing the above formula of $\mathbb{E}[c_t R_t^i]$, we have $\mathbb{E}[c_t R_t] = (0.0898, 0.1510, 0.1440)'$. Moreover, in order to simplify the equations, we define the following notation

$$Y_1 = \mathbb{E}^{-1}[R_t R_t']\mathbb{E}[R_t] = \begin{pmatrix} 1.0589 \\ -0.1196 \\ 1.1033 \end{pmatrix}, \quad Y_2 = \mathbb{E}^{-1}[R_t R_t']\mathbb{E}[c_t R_t] = \begin{pmatrix} -0.3451 \\ 0.4490 \\ 1.6342 \end{pmatrix}.$$

$$B_t = 0.2145, \qquad \widehat{B}_t = 0.2144, \qquad \widetilde{B}_t = \;\; 0.2507. \tag{3.53}$$

Suppose the investor will exit the financial market at time $t = 5$. Thus we have the optimal expected value of asset in different time period

$$\mathbb{E}[w] = (\mathbb{E}[w_1], \mathbb{E}[w_2], \mathbb{E}[w_3], \mathbb{E}[w_4], \mathbb{E}[w_5])$$

is given by

$$\mathbb{E}[w] = (4.4453, 5.9111, 7.4075, 8.9436, 10.5275).$$

We suppose the initial wealth $w_0 = 3$, and trade-off parameter $\lambda = 1$. Then we substitute the data to the optimal strategy equation,

$$\begin{aligned}
v_0^* &= -1.05(w_0 - 5.2628)Y_1 - Y_2, \\
v_1^* &= -1.05(w_1 - 6.4785)Y_1 - Y_2, \\
v_2^* &= -1.05(w_2 - 7.7549)Y_1 - Y_2, \\
v_3^* &= -1.05(w_3 - 9.0951)Y_1 - Y_2, \\
v_4^* &= -1.05(w_4 - 10.5024)Y_1 - Y_2.
\end{aligned}$$

The variance of the final optimal wealth levels is $\mathrm{Var}(w_5) = 0.6002$.

3.4 Conclusion

Using the mean-field method, the state variable transformation technique and the dynamic programming approach, we obtain in this chapter the closed-form expressions for the optimal investment strategy of our multi-period mean-variance portfolio selection problem with cash flow. Compared with previous literatures, our method is simpler yet more efficient, and the result is more concise and powerful.

Chapter 4

Mean-Variance Portfolio Selection with Cash Flow under an Uncertain Exit Time

Most investors realize that they never know exactly the time exiting the market. That is due to many factors which can affect the exit time, for example, the price movement of risky assets, securities markets behavior, exogenous huge consumption such as purchasing a house or an accident. Therefore, it is more realistic to adjust the restrictive constraint to an uncertain exit time assumption that the investment horizon is undetermined. Many papers (see Yi et al. (2008); Li and Xie (2010); Wu and Li (2011); Zhang and Li (2012)) concerned with multi-period mean- variance model and derived analytical solutions for their problems. The main difficulty of the model is the non-separability induced by the variance term. There are several methods to conquer it, such as the embedding technique proposed by Li and Ng (2000), the parameterized method developed by Li et al. (2002), and the mean-field formulation presented by Cui et al. (2014) and etc. In fact, when the investor exits the capital market with an uncertain time, the first two methods do not work smoothly and efficiently. In this chapter, we focus on the mean-field method to tackle the mean-variance model with cash flow and the time horizon of investment is uncertain. We drive the analytical optimal strategies and present numerical examples to show the efficiency and accuracy by employing the mean-field formulation. The results are much more explicit and accurate compared with the similar model solved by the embedding technique. We directly introduce the mean-field formulation to solve an uncertain exit model with correlation return rates between the assets and cash flow. The results can reduce to those derived in Chapter 3 if we fix the expected return and the exit time to the terminal, which suggests further that our methods make sense.

4.1 The Model

The investor plans to optimize the portfolio selection during time period T. However, the investment might be forced to be changed or abandoned at an uncertain time τ before T because of some accidents or unexpected events such as sudden resignation, serious illness, huge consumption and etc.

We define the τ as an exogenous random variable. The probability mass function is defined as $\tilde{p}_t = \Pr\{\tau = t\}$. Thus, investor will quit the financial market eventually at time $T \wedge \tau = \min\{T, \tau\}$. We have the following probability mass function

$$p_t \triangleq \Pr\{T \wedge \tau = t\} = \begin{cases} \tilde{p}_t, & t = 1, 2, \cdots, T-1, \\ 1 - \sum_{j=1}^{T-1} \tilde{p}_j, & t = T. \end{cases}$$

In order to derive the optimal strategy $v_t^* = [(v_t^1)^*, (v_t^2)^*, \cdots, (v_t^n)^*]'$, we introduce the following multi-period mean-variance model with cash flow and uncertain exit time.

$$\begin{cases} \min & \mathrm{Var}^{(\tau)}(w_{T\wedge\tau}) - \lambda \mathbb{E}^{(\tau)}[w_{T\wedge\tau}], \\ \text{s.t.} & w_{t+1} = \sum_{i=1}^{n} m_t^i v_t^i + \left(w_t - \sum_{i=1}^{n} v_t^i\right) n_t + c_t \\ & \qquad = n_t w_t + R_t' v_t + c_t, \quad t = 0, 1, \cdots, T-1, \end{cases} \tag{4.1}$$

where λ represents the trade-off parameter. We define $\mathbb{E}^{(\tau)}[w_{T\wedge\tau}]$ and $\mathrm{Var}^{(\tau)}(w_{T\wedge\tau})$

as follows,

$$\mathbb{E}^{(\tau)}[w_{T\wedge\tau}] \triangleq \sum_{t=1}^{T} \mathbb{E}\big[w_{T\wedge\tau}\big|T\wedge\tau=t\big]\Pr\{T\wedge\tau=t\} = \sum_{t=1}^{T}\mathbb{E}[w_t]p_t,$$

$$\mathrm{Var}^{(\tau)}(w_{T\wedge\tau}) \triangleq \sum_{t=1}^{T} \mathrm{Var}\big(w_{T\wedge\tau}\big|T\wedge\tau=t\big)\Pr\{T\wedge\tau=t\} = \sum_{t=1}^{T}\mathrm{Var}(w_t)p_t,$$

Thus we can rewrite the model as follows,

$$\begin{cases} \min & \displaystyle\sum_{t=1}^{T} p_t\Big\{\mathrm{Var}(w_t) - \lambda\mathbb{E}[w_t]\Big\}, \\ \text{s.t.} & w_{t+1} = n_t w_t + R_t' v_t + c_t. \end{cases} \tag{4.2}$$

4.2 The Mean-Field Formulation

Similar with the construction we mentioned in section 3.1, we have the following dynamic equations

$$\begin{cases} \mathbb{E}[w_{t+1}] = & n_t\mathbb{E}[w_t] + \mathbb{E}[R_t']\mathbb{E}[v_t] + \mathbb{E}[c_t], \\ \mathbb{E}[w_0] = & w_0, \end{cases} \tag{4.3}$$

$$\begin{cases} w_{t+1} - \mathbb{E}[w_{t+1}] = & n_t\big(w_t - \mathbb{E}[w_t]\big) + R_t'v_t - \mathbb{E}[R_t']\mathbb{E}[v_t] + (c_t - \mathbb{E}[c_t]) \\ = & n_t\big(w_t - \mathbb{E}[w_t]\big) + R_t'\big(v_t - \mathbb{E}[v_t]\big) + \big(R_t' - \mathbb{E}[R_t']\big)\mathbb{E}[v_t] \\ & +(c_t - \mathbb{E}[c_t]), \\ w_0 - \mathbb{E}[w_0] = & 0, \end{cases} \tag{4.4}$$

where $t = 0, 1, \cdots, T-1$. Therefore, we can equivalently reformulate problem (4.2) to a linear quadratic optimal stochastic control problem in a mean-field type.

$$\begin{cases} \min & \displaystyle\sum_{t=1}^{T} p_t\Big\{\mathbb{E}\big[(w_t - \mathbb{E}[w_t])^2\big] - \lambda\mathbb{E}[w_t]\Big\}, \\ \text{s.t.} & \mathbb{E}(v_t - \mathbb{E}[v_t]) = \mathbf{0}, \\ & \mathbb{E}[w_t] \text{ satisfies dynamic equation (4.3)}, \\ & w_t - \mathbb{E}[w_t] \text{ satisfies dynamic equation (4.4)}, \\ & t = 0, 1, \cdots, T-1. \end{cases} \tag{4.5}$$

We are able to solve the multi-period mean-variance model with cash flow and uncertain exit time in this mean-filed formulation by dynamic programming. However, during the solution process, we should pay attention to the imposed control vector $v_t - \mathbb{E}[v_t]$.

4.3 The Optimal Strategy with Correlation between Return Rates and Cash Flow under Uncertain Exit Time

We derive the optimal strategy of problem (4.5) by employing the dynamic programming in this section. Before we get the main results, one useful lemma is introduced as follows.

Lemma 4.1. *If the matrix $\mathbb{E}[R_t R_t'] - \mathbb{E}[R_t]\mathbb{E}[R_t']$ is invertible, then*

$$\Big(\mathbb{E}[R_t R_t'] - \mathbb{E}[R_t]\mathbb{E}[R_t']\Big)^{-1}\mathbb{E}[R_t] = \frac{1}{1-B_t}\mathbb{E}^{-1}[R_t R_t']\mathbb{E}[R_t].$$

Proof. Applying Sherman-Morrison formula (lemma **??**) directly gives rise to the result. □

Remark 4.1. *Lemma 4.1 is the same as the one in Cui et al. (2014). In order to keep the solution procedure intact, we present them here again.*

Theorem 4.1. *We derived the optimal portfolio section of problem (4.5) as follows*

$$\begin{aligned} v_t - \mathbb{E}[v_t] &= -n_t(w_t - \mathbb{E}[w_t])\mathbb{E}^{-1}[R_t R_t']\mathbb{E}[(R_t)], \\ \mathbb{E}[v_t] &= -\Big(\mathbb{E}[R_t R_t'] - \mathbb{E}[R_t]\mathbb{E}[R_t']\Big)^{-1}\Big(\mathbb{E}[c_t R_t] - \mathbb{E}[c_t]\mathbb{E}[R_t] - \frac{\lambda\eta_{t+1}}{2\beta_{t+1}}\mathbb{E}[R_t]\Big) \\ &= \Big(\mathbb{E}[c_t] + \frac{\lambda\eta_{t+1}}{2\beta_{t+1}} - \widehat{B}_t\Big)\frac{\mathbb{E}^{-1}[R_t R_t']\mathbb{E}[R_t]}{1-B_t} - \mathbb{E}^{-1}[R_t R_t']\mathbb{E}[c_t R_t], \end{aligned} \tag{4.6}$$

and the optimal expected wealth value is

$$\mathbb{E}[w_t] = w_0 \prod_{k=0}^{t-1} n_k + \sum_{j=0}^{t-1} \Big(\big(\mathbb{E}[c_j] + \frac{\lambda \eta_{j+1}}{2\beta_{j+1}} - \widehat{B}_j \big) \frac{B_j}{1 - B_j} - \widehat{B}_j + \mathbb{E}[c_j] \Big) \prod_{l=j+1}^{t-1} n_l, \quad (4.7)$$

for $t = 1, 2, \cdots, T$.

Proof. Given an information set $\mathcal{F}_t = \sigma\big(\mathbb{E}[w_t], w_t - \mathbb{E}[w_t]\big)$, we prove it by employing the backward recursion, and define the cost-go-functional as follows

$$\begin{aligned} J_t(\mathbb{E}[w_t], w_t - \mathbb{E}[w_t]) = \min_{v_t} \mathbb{E}\big[J_{t+1}(\mathbb{E}[w_{t+1}], w_{t+1} - \mathbb{E}[w_{t+1}]) \big| \mathcal{F}_t \big] \\ + p_t(w_t - \mathbb{E}[w_t])^2 - p_t \lambda \mathbb{E}[w_t]. \end{aligned} \quad (4.8)$$

with the boundary condition

$$J_T(\mathbb{E}[w_T], w_T - \mathbb{E}[w_T]) = p_T \Big\{ \mathbb{E}\big[(w_T - \mathbb{E}[w_T])^2 \big] - \lambda \mathbb{E}[w_T] \Big\}$$

and $\prod_{\emptyset}(\cdot) = 1$, $\sum_{\emptyset}(\cdot) = 0$ in this section.

Thus, we begin from the $T-1$ state

$$\begin{aligned} & \mathbb{E}\big[J_T(\mathbb{E}[w_T], w_T - \mathbb{E}[w_T]) | \mathcal{F}_{T-1} \big] \\ = \;& p_T \mathbb{E}\big[(w_T - \mathbb{E}[w_T])^2 - \lambda \mathbb{E}[w_T] | \mathcal{F}_{T-1} \big] \\ = \;& p_T \mathbb{E}\Big[\Big(n_{T-1}(w_{T-1} - \mathbb{E}[w_{T-1}]) + R'_{T-1}(v_{T-1} - \mathbb{E}[v_{T-1}]) \\ & + (R'_{T-1} - \mathbb{E}[R'_{T-1}]) \mathbb{E}[v_{T-1}] + (c_{T-1} - \mathbb{E}[c_{T-1}]) \Big)^2 \\ & - \lambda \Big(n_{T-1} \mathbb{E}[w_{T-1}] + \mathbb{E}[R'_{T-1}] \mathbb{E}[v_{T-1}] + \mathbb{E}[c_{T-1}] \Big) \Big| \mathcal{F}_{T-1} \Big], \end{aligned} \quad (4.9)$$

where

$$
\begin{aligned}
& \Big(n_{T-1}(w_{T-1}-\mathbb{E}[w_{T-1}]) + R'_{T-1}(v_{T-1}-\mathbb{E}[v_{T-1}]) + (R'_{T-1}-\mathbb{E}[R'_{T-1}])\mathbb{E}[v_{T-1}] \\
& +(c_{T-1}-\mathbb{E}[c_{T-1}])\Big)^2 \\
= \; & \big(n_{T-1}(w_{T-1}-\mathbb{E}[w_{T-1}])\big)^2 + \big(R'_{T-1}(v_{T-1}-\mathbb{E}[v_{T-1}])\big)^2 \\
& +\big((R'_{T-1}-\mathbb{E}[R'_{T-1}])\mathbb{E}[v_{T-1}]\big)^2 + \big(c_{T-1}-\mathbb{E}[c_{T-1}]\big)^2 \\
& +2\big(n_{T-1}(w_{T-1}-\mathbb{E}[w_{T-1}])\big)\big(R'_{T-1}(v_{T-1}-\mathbb{E}[v_{T-1}])\big) \\
& +2\big(n_{T-1}(w_{T-1}-\mathbb{E}[w_{T-1}])\big)\big((R'_{T-1}-\mathbb{E}[R'_{T-1}])\mathbb{E}[v_{T-1}]\big) \\
& +2\big(n_{T-1}(w_{T-1}-\mathbb{E}[w_{T-1}])\big)\big(c_{T-1}-\mathbb{E}[c_{T-1}]\big) \\
& +2\big(v_{T-1}-\mathbb{E}[v_{T-1}]\big)'(R_{T-1})\big(R'_{T-1}-\mathbb{E}[R'_{T-1}])\mathbb{E}[v_{T-1}] \\
& +2R'_{T-1}(v_{T-1}-\mathbb{E}[v_{T-1}])(c_{T-1}-\mathbb{E}[c_{T-1}]) \\
& +2\big(R'_{T-1}-\mathbb{E}[R'_{T-1}]\big)\mathbb{E}[v_{T-1}]\big(c_{T-1}-\mathbb{E}[c_{T-1}]\big).
\end{aligned}
\tag{4.10}
$$

Since $w_t^i - \mathbb{E}[w_t^i]$, $\mathbb{E}[w_t^i]$, $v_t^i - \mathbb{E}[v_t^i]$, $\mathbb{E}[v_t^i]$, are $\mathcal{F}_t$-measurable,

$$
\begin{aligned}
& \mathbb{E}\Big[\big(n_{T-1}(w_{T-1}-\mathbb{E}[w_{T-1}])\big)^2\Big|\mathcal{F}_{T-1}\Big] = (n_{T-1})^2\big(w_{T-1}-\mathbb{E}[w_{T-1}]\big)^2, \\
& \mathbb{E}\Big[\Big(R'_{T-1}(v_{T-1}-\mathbb{E}[v_{T-1}])\Big)^2\Big|\mathcal{F}_{T-1}\Big] \\
& = \big(v_{T-1}-\mathbb{E}[v_{T-1}]\big)'\mathbb{E}[R_{T-1}R'_{T-1}]\big(v_{T-1}-\mathbb{E}[v_{T-1}]\big), \\
& \mathbb{E}\Big[\Big((R'_{T-1}-\mathbb{E}[R'_{T-1}])\mathbb{E}[v_{T-1}]\Big)^2\Big|\mathcal{F}_{T-1}\Big] \\
& = \mathbb{E}[(v_{T-1})']\Big(\mathbb{E}[R_{T-1}R'_{T-1}] - \mathbb{E}[R_{T-1}]\mathbb{E}[R'_{T-1}]\Big)\mathbb{E}[v_{T-1}],
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbb{E}\Big[\big(c_{T-1}-\mathbb{E}[c_{T-1}]\big)^2\Big|\mathcal{F}_{T-1}\Big]=\mathbb{E}[c_{T-1}^2]-\mathbb{E}[c_{T-1}]^2,\\
&\mathbb{E}\Big[2n_{T-1}(w_{T-1}-\mathbb{E}[w_{T-1}])R'_{T-1}(v_{T-1}-\mathbb{E}[v_{T-1}])\Big|\mathcal{F}_{T-1}\Big]\\
&=2n_{T-1}\big(w_{T-1}-\mathbb{E}[w_{T-1}]\big)\mathbb{E}[R'_{T-1}]\big(v_{T-1}-\mathbb{E}[v_{T-1}]\big),\\
&\mathbb{E}\Big[2n_{T-1}\big(w_{T-1}-\mathbb{E}[w_{T-1}]\big)\big(R'_{T-1}-\mathbb{E}[R'_{T-1}]\big)\mathbb{E}[v_{T-1}]\Big|\mathcal{F}_{T-1}\Big]=0,\\
&\mathbb{E}\Big[2n_{T-1}(w_{T-1}-\mathbb{E}[w_{T-1}])(c_{T-1}-\mathbb{E}[c_{T-1}])\Big|\mathcal{F}_{T-1}\Big]=0,\\
&\mathbb{E}\Big[2\big(v_{T-1}-\mathbb{E}[v_{T-1}]\big)'(R_{T-1})\Big(R'_{T-1}-\mathbb{E}[R'_{T-1}]\Big)\mathbb{E}[v_{T-1}]\Big|\mathcal{F}_{T-1}\Big]\\
&=2\big(v_{T-1}-\mathbb{E}[v_{T-1}]\big)'\Big(\mathbb{E}[R_{T-1}R'_{T-1}]-\mathbb{E}[R_{T-1}]\mathbb{E}[R'_{T-1}]\Big)\mathbb{E}[v_{T-1}],\\
&\mathbb{E}\Big[2R'_{T-1}(v_{T-1}-\mathbb{E}[v_{T-1}])(c_{T-1}-\mathbb{E}[c_{T-1}])\Big|\mathcal{F}_{T-1}\Big]\\
&=2\big(\mathbb{E}[c_{T-1}R'_{T-1}]-\mathbb{E}[c_{T-1}]\mathbb{E}[R'_{T-1}\big)(v_{T-1}-\mathbb{E}[v_{T-1}]),\\
&\mathbb{E}\Big[2\big(R'_{T-1}-\mathbb{E}[R'_{T-1}]\big)\mathbb{E}[v_{T-1}]\big(c_{T-1}-\mathbb{E}[c_{T-1}]\big)\Big|\mathcal{F}_{T-1}\Big]\\
&=2\Big(\mathbb{E}[c_{T-1}R'_{T-1}]-\mathbb{E}[c_{T-1}]\mathbb{E}[R'_{T-1}]\Big)\mathbb{E}[v_{T-1}].
\end{aligned}
$$

Note that $\mathbb{E}\big[v_t^i-\mathbb{E}[v_t^i]|\mathcal{F}_0\big]=0$, which implies

$$
\begin{aligned}
&\mathbb{E}\Big[\mathbb{E}\Big[2\big(v_{T-1}-\mathbb{E}[v_{T-1}]\big)'(R_{T-1})\Big(R'_{T-1}-\mathbb{E}[R'_{T-1}]\Big)\mathbb{E}[v_{T-1}]\Big|\mathcal{F}_{T-1}\Big]\Big|\mathcal{F}_0\Big]\\
&=\mathbb{E}\Big[2\big(v_{T-1}-\mathbb{E}[v_{T-1}]\big)'\Big(\mathbb{E}[R_{T-1}R'_{T-1}]-\mathbb{E}[R_{T-1}]\mathbb{E}[R'_{T-1}]\Big)\mathbb{E}[v_{T-1}]\Big|\mathcal{F}_0\Big]=0,\\
&\mathbb{E}\Big[\mathbb{E}\Big[2R'_{T-1}(v_{T-1}-\mathbb{E}[v_{T-1}])(c_{T-1}-\mathbb{E}[c_{T-1}])\Big|\mathcal{F}_{T-1}\Big]\Big|\mathcal{F}_0\Big]\\
&=\mathbb{E}\Big[2\big(\mathbb{E}[c_{T-1}R'_{T-1}]-\mathbb{E}[c_{T-1}]\mathbb{E}[R'_{T-1}\big)(v_{T-1}-\mathbb{E}[v_{T-1}])\Big|\mathcal{F}_0\Big]=0.
\end{aligned}
$$

Therefore, we can reduce (4.9) into

$$\begin{aligned}
& \mathbb{E}\big[(w_T - \mathbb{E}[w_T])^2 - \lambda\mathbb{E}[w_T]|\mathcal{F}_{T-1}\big] \\
= \ & \mathbb{E}\Big[\Big(n_{T-1}(w_{T-1} - \mathbb{E}[w_{T-1}]) + R'_{T-1}(v_{T-1} - \mathbb{E}[v_{T-1}]) + (R'_{T-1} - \mathbb{E}[R'_{T-1}])\mathbb{E}[v_{T-1}] \\
& +(c_{T-1} - \mathbb{E}[c_{T-1}])\Big)^2 - \lambda\Big(n_{T-1}\mathbb{E}[w_{T-1}] + \mathbb{E}[R'_{T-1}]\mathbb{E}[v_{T-1}] + \mathbb{E}[c_{T-1}]\Big)\Big|\mathcal{F}_{T-1}\Big] \\
= \ & (n_{T-1})^2\big(w_{T-1} - \mathbb{E}[w_{T-1}]\big)^2 + \big(v_{T-1} - \mathbb{E}[v_{T-1}]\big)'\mathbb{E}[R_{T-1}R'_{T-1}]\big(v_{T-1} - \mathbb{E}[v_{T-1}]\big) \\
& +\mathbb{E}[(v_{T-1})']\Big(\mathbb{E}[R_{T-1}R'_{T-1}] - \mathbb{E}[R_{T-1}]\mathbb{E}[R'_{T-1}]\Big)\mathbb{E}[v_{T-1}] + \mathbb{E}[c^2_{T-1}] - \mathbb{E}[c_{T-1}]^2 \\
& +2n_{T-1}\big(w_{T-1} - \mathbb{E}[w_{T-1}]\big)\mathbb{E}[R'_{T-1}]\big(v_{T-1} - \mathbb{E}[v_{T-1}]\big) \\
& +2\Big(\mathbb{E}[c_{T-1}R'_{T-1}] - \mathbb{E}[c_{T-1}]\mathbb{E}[R'_{T-1}]\Big)\mathbb{E}[v_{T-1}] \\
& -\lambda n_{T-1}\mathbb{E}[w_{T-1}] - \lambda\mathbb{E}[R'_{T-1}]\mathbb{E}[v_{T-1}] - \lambda\mathbb{E}[c_{T-1}].
\end{aligned} \tag{4.11}$$

The optimal strategies at period $T-1$ can be derived by minimizing the above equation with respect to v_{T-1}

$$\begin{aligned}
v_{T-1} - \mathbb{E}[v_{T-1}] &= -n_{T-1}(w_{T-1} - \mathbb{E}[w_{T-1}])\mathbb{E}^{-1}[R_{T-1}R'_{T-1}]\mathbb{E}[(R_{T-1})], \\
\mathbb{E}[v_{T-1}] &= -\Big(\mathbb{E}[R_{T-1}R'_{T-1}] - \mathbb{E}[R_{T-1}]\mathbb{E}[R'_{T-1}]\Big)^{-1} \\
&\quad \times\Big(\mathbb{E}[c_{T-1}R_{T-1}] - \mathbb{E}[c_{T-1}]\mathbb{E}[R_{T-1}] - \frac{\lambda}{2}\mathbb{E}[R_{T-1}]\Big) \\
&= \Big(\mathbb{E}[c_{T-1}] + \frac{\lambda}{2} - \widehat{B}_{T-1}\Big)\frac{\mathbb{E}^{-1}[R_{T-1}R'_{T-1}]\mathbb{E}[R_{T-1}]}{1 - B_{T-1}} \\
&\quad -\mathbb{E}^{-1}[R_{T-1}R'_{T-1}]\mathbb{E}[c_{T-1}R_{T-1}],
\end{aligned} \tag{4.12}$$

where

$$\begin{aligned}
B_{T-1} = \ & \mathbb{E}[R'_{T-1}]\mathbb{E}^{-1}[R_{T-1}R'_{T-1}]\mathbb{E}[R_{T-1}], \\
\widehat{B}_{T-1} = \ & \mathbb{E}[c_{T-1}R'_{T-1}]\mathbb{E}^{-1}[R_{T-1}R'_{T-1}]\mathbb{E}[R_{T-1}], \\
\widetilde{B}_{T-1} = \ & \mathbb{E}[c_{T-1}R'_{T-1}]\mathbb{E}^{-1}[R_{T-1}R'_{T-1}]\mathbb{E}[c_{T-1}R_{T-1}].
\end{aligned} \tag{4.13}$$

Substituting the optimal strategies back to (4.11), we obtain

$$\begin{aligned}
& J_{T-1}(\mathbb{E}[w_{T-1}], w_{T-1}-\mathbb{E}[w_{T-1}]) \\
= \; & \mathbb{E}\big[J_T(\mathbb{E}[w_T], w_T-\mathbb{E}[w_T])|\mathcal{F}_{T-1}\big] + p_{T-1}(w_{T-1}-\mathbb{E}[w_{T-1}])^2 - p_{T-1}\lambda\mathbb{E}[w_{T-1}] \\
= \; & p_T(n_{T-1})^2\big(w_{T-1}-\mathbb{E}[w_{T-1}]\big)^2 + p_T(\mathbb{E}[c_{T-1}^2]-\mathbb{E}[c_{T-1}]^2) - p_T\lambda n_{T-1}\mathbb{E}[w_{T-1}] \\
& -p_T\lambda\mathbb{E}[c_{T-1}] - p_T(n_{T-1})^2 B_{T-1}\big(w_{T-1}-\mathbb{E}[w_{T-1}]\big)^2 \\
& -p_T\Big(\big(\mathbb{E}[c_{T-1}]+\frac{\lambda}{2}-\widehat{B}_{T-1}\big)\frac{\mathbb{E}[R'_{T-1}]\mathbb{E}^{-1}[R_{T-1}R'_{T-1}]}{1-B_{T-1}} \\
& -\mathbb{E}[c_{T-1}R'_{T-1}]\mathbb{E}^{-1}[R_{T-1}R'_{T-1}]\Big)\Big(\mathbb{E}[R_{T-1}R'_{T-1}]-\mathbb{E}[R_{T-1}]\mathbb{E}[R'_{T-1}]\Big) \\
& \times\Big(\big(\mathbb{E}[c_{T-1}]+\frac{\lambda}{2}-\widehat{B}_{T-1}\big)\frac{\mathbb{E}^{-1}[R_{T-1}R'_{T-1}]\mathbb{E}[R_{T-1}]}{1-B_{T-1}} \\
& -\mathbb{E}^{-1}[R_{T-1}R'_{T-1}]\mathbb{E}[c_{T-1}R_{T-1}]\Big) + p_{T-1}(w_{T-1}-\mathbb{E}[w_{T-1}])^2 - p_{T-1}\lambda\mathbb{E}[w_{T-1}] \\
= \; & \big(p_T(n_{T-1})^2(1-B_{T-1})+p_{T-1}\big)\big(w_{T-1}-\mathbb{E}[w_{T-1}]\big)^2 + p_T(\mathbb{E}[c_{T-1}^2]-\mathbb{E}[c_{T-1}]^2) \\
& -\lambda(p_T n_{T-1}+p_{T-1})\mathbb{E}[w_{T-1}] - p_T\lambda\mathbb{E}[c_{T-1}] \\
& -p_T\Big(\big(\mathbb{E}[c_{T-1}]+\frac{\lambda}{2}-\widehat{B}_{T-1}\big)^2\frac{B_{T-1}}{1-B_{T-1}} - 2\big(\mathbb{E}[c_{T-1}]+\frac{\lambda}{2}-\widehat{B}_{T-1}\big)\widehat{B}_{T-1} \\
& +\widetilde{B}_{T-1}-\widehat{B}_{T-1}^2\Big) \\
= \; & \beta_{T-1}\big(w_{T-1}-\mathbb{E}[w_{T-1}]\big)^2 - \lambda\eta_{T-1}\mathbb{E}[w_{T-1}] + \Delta_{T-1},
\end{aligned} \tag{4.14}$$

where

$$\begin{aligned}
\beta_{T-1} = \; & p_T(n_{T-1})^2(1-B_{T-1})+p_{T-1}, \\
\eta_{T-1} = \; & p_T n_{T-1}+p_{T-1}, \\
\Delta_{T-1} = \; & p_T\mathbb{E}[c_{T-1}^2] - p_T\mathbb{E}[c_{T-1}]^2 - p_T\lambda\mathbb{E}[c_{T-1}] \\
& -p_T\Big(\big(\mathbb{E}[c_{T-1}]+\frac{\lambda}{2}-\widehat{B}_{T-1}\big)^2\frac{B_{T-1}}{1-B_{T-1}} - 2\big(\mathbb{E}[c_{T-1}]+\frac{\lambda}{2}-\widehat{B}_{T-1}\big)\widehat{B}_{T-1} \\
& +\widetilde{B}_{T-1}-\widehat{B}_{T-1}^2\Big).
\end{aligned} \tag{4.15}$$

Repeating the process at time $T-2$, we have

$$
\begin{aligned}
& J_{T-2}(\mathbb{E}[w_{T-2}], w_{T-2}-\mathbb{E}[w_{T-2}]) \\
= \;& \mathbb{E}\big[J_{T-1}(\mathbb{E}[w_{T-1}], w_{T-1}-\mathbb{E}[w_{T-1}])|\mathcal{F}_{T-2}\big] + p_{T-2}(w_{T-2}-\mathbb{E}[w_{T-2}])^2 - p_{T-2}\lambda\mathbb{E}[w_{T-2}] \\
= \;& \mathbb{E}\big[\beta_{T-1}\big(w_{T-1}-\mathbb{E}[w_{T-1}]\big)^2 - \lambda\eta_{T-1}\mathbb{E}[w_{T-1}] + \Delta_{T-1}|\mathcal{F}_{T-2}\big] \\
& + p_{T-2}(w_{T-2}-\mathbb{E}[w_{T-2}])^2 - p_{T-2}\lambda\mathbb{E}[w_{T-2}] \\
= \;& \beta_{T-1}(n_{T-2})^2\big(w_{T-2}-\mathbb{E}[w_{T-2}]\big)^2 + \beta_{T-1}\big(v_{T-2}-\mathbb{E}[v_{T-2}]\big)'\mathbb{E}[R_{T-2}R'_{T-2}] \\
& \times\big(v_{T-2}-\mathbb{E}[v_{T-2}]\big) + \beta_{T-1}\mathbb{E}[(v_{T-2})']\Big(\mathbb{E}[R_{T-2}R'_{T-2}] - \mathbb{E}[R_{T-2}]\mathbb{E}[R'_{T-2}]\Big)\mathbb{E}[v_{T-2}] \\
& + \beta_{T-1}\mathbb{E}[c^2_{T-2}] - \beta_{T-1}\mathbb{E}[c_{T-2}]^2 + 2\beta_{T-1}n_{T-2}\big(w_{T-2}-\mathbb{E}[w_{T-2}]\big)\mathbb{E}[R'_{T-2}] \\
& \times\big(v_{T-2}-\mathbb{E}[v_{T-2}]\big) + 2\beta_{T-1}\Big(\mathbb{E}[c_{T-2}R'_{T-2}] - \mathbb{E}[c_{T-2}]\mathbb{E}[R'_{T-2}]\Big)\mathbb{E}[v_{T-2}] \\
& - \lambda\eta_{T-1}n_{T-2}\mathbb{E}[w_{T-2}] - \lambda\eta_{T-1}\mathbb{E}[R'_{T-2}]\mathbb{E}[v_{T-2}] - \lambda\eta_{T-1}\mathbb{E}[c_{T-2}] + \Delta_{T-1} \\
& + p_{T-2}(w_{T-2}-\mathbb{E}[w_{T-2}])^2 - p_{T-2}\lambda\mathbb{E}[w_{T-2}].
\end{aligned}
\tag{4.16}
$$

The optimal strategies at period $T-2$ can be derived from the above equation corresponding to v_{T-2}

$$
\begin{aligned}
v_{T-2}-\mathbb{E}[v_{T-2}] &= -n_{T-2}(w_{T-2}-\mathbb{E}[w_{T-2}])\mathbb{E}^{-1}[R_{T-2}R'_{T-2}]\mathbb{E}[(R_{T-2})], \\
\mathbb{E}[v_{T-2}] &= -\Big(\mathbb{E}[R_{T-2}R'_{T-2}] - \mathbb{E}[R_{T-2}]\mathbb{E}[R'_{T-2}]\Big)^{-1} \\
& \qquad \times\Big(\mathbb{E}[c_{T-2}R_{T-2}] - \mathbb{E}[c_{T-2}]\mathbb{E}[R_{T-2}] - \frac{\lambda\eta_{T-1}}{2\beta_{T-1}}\mathbb{E}[R_{T-2}]\Big) \\
& = \Big(\mathbb{E}[c_{T-2}] + \frac{\lambda\eta_{T-1}}{2\beta_{T-1}} - \widehat{B}_{T-2}\Big)\frac{\mathbb{E}^{-1}[R_{T-2}R'_{T-2}]\mathbb{E}[R_{T-2}]}{1-B_{T-2}} \\
& \quad -\mathbb{E}^{-1}[R_{T-2}R'_{T-2}]\mathbb{E}[c_{T-2}R_{T-2}].
\end{aligned}
\tag{4.17}
$$

Substituting the optimal strategies back to (4.16), we obtain

$$\begin{aligned}
& J_{T-2}(\mathbb{E}[w_{t-2}], w_{t-2}-\mathbb{E}[w_{t-2}]) \\
= \; & \beta_{T-1}(n_{T-2})^2(1-B_{T-2})\big(w_{T-2}-\mathbb{E}[w_{T-2}]\big)^2+\beta_{T-1}(\mathbb{E}[c_{T-2}^2]-\mathbb{E}[c_{T-2}]^2) \\
& -\lambda\eta_{T-1}n_{T-2}\mathbb{E}[w_{T-2}]-\lambda\eta_{T-1}\mathbb{E}[c_{T-2}] \\
& -\beta_{T-1}\Big(\big(\mathbb{E}[c_{T-2}]+\frac{\lambda\eta_{T-1}}{2\beta_{T-1}}-\widehat{B}_{T-2}\big)^2\frac{B_{T-2}}{1-B_{T-2}} \\
& -2\big(\mathbb{E}[c_{T-2}]+\frac{\lambda\eta_{T-1}}{2\beta_{T-1}}-\widehat{B}_{T-2}\big)\widehat{B}_{T-2}+\widetilde{B}_{T-2}-\widehat{B}_{T-2}^2\Big) \\
& +\Delta_{T-1}+p_{T-2}(w_{T-2}-\mathbb{E}[w_{T-2}])^2-p_{T-2}\lambda\mathbb{E}[w_{T-2}] \\
= \; & \beta_{T-2}\big(w_{T-2}-\mathbb{E}[w_{T-2}]\big)^2-\lambda\eta_{T-2}\mathbb{E}[w_{T-2}]+\Delta_{T-2},
\end{aligned} \tag{4.18}$$

where

$$\begin{aligned}
\beta_{T-2} = \; & \beta_{T-1}(n_{T-2})^2(1-B_{T-2})+p_{T-2}, \\
\eta_{T-2} = \; & \eta_{T-1}n_{T-2}+p_{T-2}, \\
\Delta_{T-2} = \; & \Delta_{T-1}+\beta_{T-1}\mathbb{E}[c_{T-2}^2]-\beta_{T-1}\mathbb{E}[c_{T-2}]^2-\lambda\eta_{T-1}\mathbb{E}[c_{T-2}] \\
& -\beta_{T-1}\Big(\big(\mathbb{E}[c_{T-2}]+\frac{\lambda\eta_{T-1}}{2\beta_{T-1}}-\widehat{B}_{T-2}\big)^2\frac{B_{T-2}}{1-B_{T-2}} \\
& -2\big(\mathbb{E}[c_{T-2}]+\frac{\lambda\eta_{T-1}}{2\beta_{T-1}}-\widehat{B}_{T-2}\big)\widehat{B}_{T-2}+\widetilde{B}_{T-2}-\widehat{B}_{T-2}^2\Big).
\end{aligned} \tag{4.19}$$

Assume that the following equation (4.20) holds at time $t+1$, we prove it according to the mathematical induction.

$$J_{t+1}(\mathbb{E}[w_{t+1}], w_{t+1}-\mathbb{E}[w_{t+1}]) = \beta_{t+1}\big(w_{t+1}-\mathbb{E}[w_{t+1}]\big)^2-\lambda\eta_{t+1}\mathbb{E}[w_{t+1}]+\Delta_{t+1}, \tag{4.20}$$

where

$$\begin{aligned}
\beta_{t+1} = \; & \beta_{t+2}(n_{t+1})^2(1-B_{t+1})+p_{t+1}, \\
\eta_{t+1} = \; & \eta_{t+2}n_{t+1}+p_{t+1}, \\
\Delta_{t+1} = \; & \Delta_{t+2}+\beta_{t+2}\mathbb{E}[c_{t+1}^2]-\beta_{t+2}\mathbb{E}[c_{t+1}]^2-\lambda\eta_{t+2}\mathbb{E}[c_{t+1}] \\
& -\beta_{t+2}\Big(\big(\mathbb{E}[c_{t+1}]+\frac{\lambda\eta_{t+2}}{2\beta_{t+2}}-\widehat{B}_{t+1}\big)^2\frac{B_{t+1}}{1-B_{t+1}} \\
& -2\big(\mathbb{E}[c_{t+1}]+\frac{\lambda\eta_{t+2}}{2\beta_{t+2}}-\widehat{B}_{t+1}\big)\widehat{B}_{t+1}+\widetilde{B}_{t+1}-\widehat{B}_{t+1}^2\Big).
\end{aligned} \tag{4.21}$$

and

$$\beta_T = p_T, \qquad \eta_T = p_T, \qquad \Delta_T = 0. \tag{4.22}$$

According to the equation (4.20) and (4.22), we derive the result at time t.

$$\begin{aligned}
& J_t(\mathbb{E}[w_t], w_t - \mathbb{E}[w_t]) \\
= \;& \mathbb{E}\big[J_{t+1}(\mathbb{E}[w_{t+1}], w_{t+1} - \mathbb{E}[w_{t+1}])|\mathcal{F}_t\big] + p_t(w_t - \mathbb{E}[w_t])^2 - p_t\lambda\mathbb{E}[w_t] \\
= \;& \mathbb{E}\big[\beta_{t+1}\big(w_{t+1} - \mathbb{E}[w_{t+1}]\big)^2 - \lambda\eta_{t+1}\mathbb{E}[w_{t+1}] + \Delta_{t+1}|\mathcal{F}_t\big] + p_t(w_t - \mathbb{E}[w_t])^2 - p_t\lambda\mathbb{E}[w_t] \\
= \;& \beta_{t+1}(n_t)^2\big(w_t - \mathbb{E}[w_t]\big)^2 + \beta_{t+1}\big(v_t - \mathbb{E}[v_t]\big)'\mathbb{E}[R_tR_t']\big(v_t - \mathbb{E}[v_t]\big) \\
& +\beta_{t+1}\mathbb{E}[(v_t)']\Big(\mathbb{E}[R_tR_t'] - \mathbb{E}[R_t]\mathbb{E}[R_t']\Big)\mathbb{E}[v_t] + \beta_{t+1}\mathbb{E}[c_t^2] - \beta_{t+1}\mathbb{E}[c_t]^2 \\
& +2\beta_{t+1}n_t\big(w_t - \mathbb{E}[w_t]\big)\mathbb{E}[R_t']\big(v_t - \mathbb{E}[v_t]\big) \\
& +2\beta_{t+1}\Big(\mathbb{E}[c_tR_t'] - \mathbb{E}[c_t]\mathbb{E}[R_t']\Big)\mathbb{E}[v_t] \\
& -\lambda\eta_{t+1}n_t\mathbb{E}[w_t] - \lambda\eta_{t+1}\mathbb{E}[R_t']\mathbb{E}[v_t] - \lambda\eta_{t+1}\mathbb{E}[c_t] + \Delta_{t+1} \\
& +p_t(w_t - \mathbb{E}[w_t])^2 - p_t\lambda\mathbb{E}[w_t].
\end{aligned} \tag{4.23}$$

The optimal strategies at period t can be derived from above equation corresponding to t

$$\begin{aligned}
v_t - \mathbb{E}[v_t] &= -n_t(w_t - \mathbb{E}[w_t])\mathbb{E}^{-1}[R_tR_t']\mathbb{E}[(R_t)], \\
\mathbb{E}[v_t] &= -\Big(\mathbb{E}[R_tR_t'] - \mathbb{E}[R_t]\mathbb{E}[R_t']\Big)^{-1}\Big(\mathbb{E}[c_tR_t] - \mathbb{E}[c_t]\mathbb{E}[R_t] - \frac{\lambda\eta_{t+1}}{2\beta_{t+1}}\mathbb{E}[R_t]\Big) \\
&= \Big(\mathbb{E}[c_t] + \frac{\lambda\eta_{t+1}}{2\beta_{t+1}} - \widehat{B}_t\Big)\frac{\mathbb{E}^{-1}[R_tR_t']\mathbb{E}[R_t]}{1 - B_t} - \mathbb{E}^{-1}[R_tR_t']\mathbb{E}[c_tR_t].
\end{aligned} \tag{4.24}$$

Substituting the optimal strategies back to (4.23), we obtain

$$\begin{aligned}
& J_t(\mathbb{E}[w_t], w_t - \mathbb{E}[w_t]) \\
= \; & \big(\beta_{t+1}(n_t)^2(1-B_t) + p_t\big)\big(w_t - \mathbb{E}[w_t]\big)^2 + \beta_{t+1}(\mathbb{E}[c_t^2] - \mathbb{E}[c_t]^2) \\
& -\lambda(\eta_{t+1}n_t + p_t)\mathbb{E}[w_t] - \lambda\eta_{t+1}\mathbb{E}[c_t] - \beta_{t+1}\Big(\big(\mathbb{E}[c_t] + \frac{\lambda\eta_{t+1}}{2\beta_{t+1}} - \widehat{B}_t\big)^2 \frac{B_t}{1-B_t} \\
& -2\big(\mathbb{E}[c_t] + \frac{\lambda\eta_{t+1}}{2\beta_{t+1}} - \widehat{B}_t\big)\widehat{B}_t + \widetilde{B}_t - \widehat{B}_t^2\Big) + \Delta_{t+1} \\
= \; & \beta_t\big(w_t - \mathbb{E}[w_t]\big)^2 - \lambda\eta_t\mathbb{E}[w_t] + \Delta_t,
\end{aligned} \tag{4.25}$$

where

$$\begin{aligned}
\beta_t = \; & \beta_{t+1}(n_t)^2(1-B_t) + p_t, \\
\eta_t = \; & \eta_{t+1}n_t + p_t, \\
\Delta_t = \; & \Delta_{t+1} + \beta_{t+1}\mathbb{E}[c_t^2] - \beta_{t+1}\mathbb{E}[c_t]^2 - \lambda\eta_{t+1}\mathbb{E}[c_t] \\
& -\beta_{t+1}\Big(\big(\mathbb{E}[c_t] + \frac{\lambda\eta_{t+1}}{2\beta_{t+1}} - \widehat{B}_t\big)^2\frac{B_t}{1-B_t} - 2\big(\mathbb{E}[c_t] + \frac{\lambda\eta_{t+1}}{2\beta_{t+1}} - \widehat{B}_t\big)\widehat{B}_t \\
& +\widetilde{B}_t - \widehat{B}_t^2\Big).
\end{aligned} \tag{4.26}$$

Substituting $\mathbb{E}[v_t^*]$ to the dynamics eqation in (4.3) yields

$$\mathbb{E}[w_{t+1}] = n_t\mathbb{E}[w_t] + \big(\mathbb{E}[c_t] + \frac{\lambda\eta_{t+1}}{2\beta_{t+1}} - \widehat{B}_t\big)\frac{B_t}{1-B_t} - \widehat{B}_t + \mathbb{E}[c_t]. \tag{4.27}$$

Therefore,

$$\mathbb{E}[w_t] = w_0\prod_{k=0}^{t-1} n_k + \sum_{j=0}^{t-1}\Big(\big(\mathbb{E}[c_j] + \frac{\lambda\eta_{j+1}}{2\beta_{j+1}} - \widehat{B}_j\big)\frac{B_j}{1-B_j} - \widehat{B}_j + \mathbb{E}[c_j]\Big)\prod_{l=j+1}^{t-1} n_l. \tag{4.28}$$

Typically, we know the optimal value of (4.5) is equal to J_0. Thus,

$$
\begin{aligned}
& \sum_{t=1}^{T} p_t \mathrm{Var}(w_t) \\
= \;& J_0(\mathbb{E}[w_0], w_0 - \mathbb{E}[w_0]) + \sum_{t=1}^{T} p_t \lambda \mathbb{E}[w_t] \\
= \;& -\lambda\eta_0 w_0 + \Delta_0 + \sum_{t=1}^{T} p_t \lambda \Bigg(w_0 \prod_{k=0}^{t-1} n_k + \sum_{j=0}^{t-1} \bigg(\Big(\mathbb{E}[c_j] + \frac{\lambda\eta_{j+1}}{2\beta_{j+1}} - \widehat{B}_j \Big) \frac{B_j}{1-B_j} \\
& -\widehat{B}_j + \mathbb{E}[c_j] \bigg) \prod_{l=j+1}^{t-1} n_l \Bigg).
\end{aligned}
\tag{4.29}
$$

□

The optimal strategy obtained in Theorem 4.1 will keep the same as the result established in section 3.3 if the exit time is determined by T.

4.4 Numerical Example

Example 4.1. Consider the example in Section 3. Here we ignore the case of uncorrelation, i.e., the asset return rates and cash flow are correlated.

Table 4.1: Data for assets and cashflow example

	SP	EM	MS	cashflow
Expected return	14%	16%	17%	1
Standard deviation	18.5%	30%	24%	20%
Correlation coefficient				
SP	1	0.64	0.79	ρ_1
EM	0.64	1	0.75	ρ_2
MS	0.79	0.75	1	ρ_3
cashflow	ρ_1	ρ_2	ρ_3	1

Thus,

$$\mathbb{E}[R_t] = \begin{pmatrix} 0.09 \\ 0.11 \\ 0.12 \end{pmatrix}, \ \mathrm{Cov}(R_t) = \begin{pmatrix} 0.0342 & 0.0355 & 0.0351 \\ 0.0355 & 0.0900 & 0.0540 \\ 0.0351 & 0.0540 & 0.0576 \end{pmatrix},$$

$$\mathbb{E}[R_t R_t'] = \begin{pmatrix} 0.0423 & 0.0454 & 0.0459 \\ 0.0454 & 0.1021 & 0.0672 \\ 0.0459 & 0.0672 & 0.0720 \end{pmatrix}.$$

The correlation coefficient between cash flow and i-th asset are defined as $\rho = (\rho_1, \rho_2, \rho_3)$,

$$\rho_i = \frac{\mathrm{Cov}(c_t, R_t^i)}{\sqrt{\mathrm{Var}(c_t)}\sqrt{\mathrm{Var}(R_t^i)}}.$$

Therefore, we have the following eqation

$$\mathbb{E}[c_t R_t^i] = \mathbb{E}[c_t]\mathbb{E}[R_t^i] + \rho_i \sqrt{\mathrm{Var}(c_t)}\sqrt{\mathrm{Var}(R_t^i)},$$

$$\mathbb{E}[c_t^2] = \mathbb{E}[c_t]^2 + \mathrm{Var}(c_t).$$

Assume that

$$\rho = (\rho_1, \rho_2, \rho_3) = (-0.25, 0.5, 0.25).$$

Thus,

$$\begin{aligned}
\mathrm{Cov}\left(\begin{pmatrix} R_t \\ c_t \end{pmatrix}\right) &= \begin{pmatrix} \mathrm{Cov}(R_t) & \mathrm{Cov}(c_t, R_t) \\ \mathrm{Cov}(c_t, R_t') & \mathrm{Var}(c_t) \end{pmatrix} \\
&= \begin{pmatrix} 0.0342 & 0.0355 & 0.0351 & -0.0092 \\ 0.0355 & 0.0900 & 0.0540 & 0.0300 \\ 0.0351 & 0.0540 & 0.0576 & 0.0120 \\ -0.0092 & 0.0300 & 0.0120 & 0.0400 \end{pmatrix} \\
&\succ 0.
\end{aligned}$$

Employing the above formula of $\mathbb{E}[c_t R_t^i]$, we have $\mathbb{E}[c_t R_t] = (0.0898, 0.1510, 0.1440)'$. Moreover, in order to simplify the equations, we define the following notation

$$Y_1 = \mathbb{E}^{-1}[R_t R_t']\mathbb{E}[R_t] = \begin{pmatrix} 1.0589 \\ -0.1196 \\ 1.1033 \end{pmatrix}, \quad Y_2 = \mathbb{E}^{-1}[R_t R_t']\mathbb{E}[c_t R_t] = \begin{pmatrix} -0.3451 \\ 0.4490 \\ 1.6342 \end{pmatrix}.$$

$$B_{T-1} = 0.2145, \qquad \widehat{B}_{T-1} = 0.2144, \qquad \widetilde{B}_{T-1} = \ \ 0.2507. \tag{4.30}$$

The probability mass function of an exit time τ is

$$(p_1, p_2, p_3, p_4, p_5) = (0.10, 0.15, 0.2, 0.25, 0.3),$$

respectively, for $t = 1, 2, 3, 4, 5$. Thus we can derive the optimal expected value as

$$\mathbb{E}[w] = (\mathbb{E}[w_1], \mathbb{E}[w_2], \mathbb{E}[w_3], \mathbb{E}[w_4], \mathbb{E}[w_5]),$$

is given by

$$\mathbb{E}[w] = (4.3676, 5.7763, 7.2341, 8.7473, 10.3214).$$

We suppose the initial wealth $w_0 = 3$, and trade-off parameter $\lambda = 1$. Thus

$$\begin{aligned}
v_0^* &= -1.05(w_0 - 4.9178)Y_1 - Y_2,\\
v_1^* &= -1.05(w_1 - 6.1647)Y_1 - Y_2,\\
v_2^* &= -1.05(w_2 - 7.4785)Y_1 - Y_2,\\
v_3^* &= -1.05(w_3 - 8.8585)Y_1 - Y_2,\\
v_4^* &= -1.05(w_4 - 10.3061)Y_1 - Y_2.
\end{aligned}$$

The mean and variance of the final optimal value are $\mathbb{E}^{(\tau)}(w_{5\wedge\tau}) = 8.0333$ and $\mathrm{Var}^{(\tau)}(w_{5\wedge\tau}) = 1.7700$, respectively.

Chapter 5

Multi-Period Mean-Variance Asset-Liability Management with Cash Flow

We investigate the multi-period asset-liability mean-variance portfolio selection with cash flow in this chapter. Based on the framework of mean-variance model with mean-field formulation in Chapter 3, we directly assume that the return rate between assets and cash flow are correlated. Then we investigate the cases where the return rates between assets and liability are uncorrelated as well as correlated. We derive the analytical form of optimal strategies and optimal value in expression for the new model. Finally, the numerical examples for both cases are given to illustrate the results.

5.1 Mean-Field Formulation

The goal of the portfolio selection problem with liability and cash flow is investigating the optimal strategy, $v_t^* = [(v_t^1)^*,\, (v_t^2)^*,\, \cdots,\, (v_t^n)^*]'$, which can be derived from the following model,

$$\begin{cases} \min & \mathrm{Var}(w_T) - \lambda \mathbb{E}[w_T], \\ \text{s.t.} & w_{t+1} = n_t w_t + R_t' v_t + c_t, \\ & l_{t+1} = q_t l_t, \quad \text{for } t = 0, 1, \cdots, T-1. \end{cases} \tag{5.1}$$

Here λ represents the risk aversion. Thus, we construct the mean-field type of model (5.1). According to the independence between R_t and v_t, q_t and l_t, the dynamic equations of the expectation of the wealth and liability are presented as follows

$$\begin{cases} \mathbb{E}[w_{t+1}] = n_t\mathbb{E}[w_t] + \mathbb{E}[R_t']\mathbb{E}[v_t] + \mathbb{E}[c_t], \\ \quad \mathbb{E}[w_0] = w_0, \end{cases}$$

$$\begin{cases} \mathbb{E}[l_{t+1}] = \mathbb{E}[q_t]\mathbb{E}[l_t], \\ \quad \mathbb{E}[l_0] = l_0, \end{cases} \tag{5.2}$$

where $t = 0, 1, \cdots, T-1$. Combining the equations (5.1) and (5.2) we have

$$\begin{cases} w_{t+1} - \mathbb{E}[w_{t+1}] = n_t\big(w_t - \mathbb{E}[w_t]\big) + R_t'v_t - \mathbb{E}[R_t']\mathbb{E}[v_t] + (c_t - \mathbb{E}[c_t]) \\ \qquad\qquad = n_t\big(w_t - \mathbb{E}[w_t]\big) + R_t'\big(v_t - \mathbb{E}[v_t]\big) + \big(R_t' - \mathbb{E}[R_t']\big)\mathbb{E}[v_t] \\ \qquad\qquad\quad +(c_t - \mathbb{E}[c_t]), \\ w_0 - \mathbb{E}[w_0] = 0, \end{cases} \tag{5.3}$$

$$\begin{cases} l_{t+1} - \mathbb{E}[l_{t+1}] = q_t(l_t - \mathbb{E}[l_t]) + (q_t - \mathbb{E}[q_t])\mathbb{E}[l_t], \\ \quad l_0 - \mathbb{E}[l_0] = 0, \end{cases} \tag{5.4}$$

where $t = 0, 1, \cdots, T-1$. By conducting this kind of transformation, we enlarge the state-space (w_t, l_t) and control space (v_t) into $(\mathbb{E}[w_t], w_t - \mathbb{E}[w_t], \mathbb{E}[l_t], l_t - \mathbb{E}[l_t])$ and $(\mathbb{E}[v_t], v_t - \mathbb{E}[v_t])$, respectively.

Based on the above construction, we can equivalently reformulate problem (5.1) to a linear quadratic optimal stochastic control problem in mean-field type.

$$\begin{cases} \min \quad \mathbb{E}\big[(w_T - \mathbb{E}[w_T])^2\big] - \lambda\mathbb{E}[w_T], \\ \text{s.t.} \quad \mathbb{E}[w_t] \text{ satisfies dynamic equation (5.2)}, \\ \qquad \mathbb{E}[l_t] \text{ satisfies dynamic equation (5.2)}, \\ \qquad w_t - \mathbb{E}[w_t] \text{ satisfies (5.3)}, \\ \qquad l_t - \mathbb{E}[l_t] \text{ satisfies (5.4)}, \\ \qquad \mathbb{E}(v_t - \mathbb{E}[v_t]) = \mathbf{0}, \\ \qquad \mathbb{E}(l_t - \mathbb{E}[l_t]) = \mathbf{0}, \quad t = 0, 1, \cdots, T-1. \end{cases} \tag{5.5}$$

We are able to solve the mean-variance asset-liability portfolio with cash flow in this mean-field formulation by dynamic programming.

5.2 The Optimal Strategy with Uncorrelation

5.2.1 The Dynamic Programming and Optimal Strategy

On the one hand, we directly assume the return rates of assets and cash flow are correlated during each period t. On the other hand, in this subsection we assume return rates between assets and liability are uncorrelated during each period t, i.e., R_t and q_t are independent.

Theorem 5.1. *Optimal strategy of problem (5.5) under the constraint of uncorrelation is derived as*

$$\begin{aligned}
v_{T-1} - \mathbb{E}[v_{T-1}] =& -n_{T-1}(w_{T-1} - \mathbb{E}[w_{T-1}])\mathbb{E}^{-1}[R_{T-1}R'_{T-1}]\mathbb{E}[R_{T-1}] \\
&+(l_{T-1} - \mathbb{E}[l_{T-1}])\mathbb{E}^{-1}[R_{T-1}R'_{T-1}]\mathbb{E}[q_{T-1}R_{T-1}], \\
\mathbb{E}[v_{T-1}] =& -\Big(\mathbb{E}[R_{T-1}R'_{T-1}] - \mathbb{E}[R_{T-1}]\mathbb{E}[R'_{T-1}]\Big)^{-1} \\
&\times\Big(\mathbb{E}[c_{T-1}R_{T-1}] - \mathbb{E}[c_{T-1}]\mathbb{E}[R_{T-1}] - \frac{\lambda}{2}\mathbb{E}[R_{T-1}]\Big) \\
=& \Big(\mathbb{E}[c_{T-1}] + \frac{\lambda}{2} - \widehat{B}_{T-1}\Big)\frac{\mathbb{E}^{-1}[R_{T-1}R'_{T-1}]\mathbb{E}[R_{T-1}]}{1 - B_{T-1}} \\
&-\mathbb{E}^{-1}[R_{T-1}R'_{T-1}]\mathbb{E}[c_{T-1}R_{T-1}].
\end{aligned} \tag{5.6}$$

Proof. The dynamic programming approach is employed to prove the theorem. Given the information set $\mathcal{F}_t$ at time period t, we have the following cost-to-go functional of problem (5.5)

$$\begin{aligned}
&J_t(\mathbb{E}[w_t], w_t - \mathbb{E}[w_t], \mathbb{E}[l_t], l_t - \mathbb{E}[l_t]) \\
=& \min_{v_t} \mathbb{E}\big[J_{t+1}(\mathbb{E}[w_{t+1}], w_{t+1} - \mathbb{E}[w_{t+1}], \mathbb{E}[l_{t+1}], l_{t+1} - \mathbb{E}[l_{t+1}])\big|\mathcal{F}_t\big],
\end{aligned}$$

with the boundary condition

$J_T(\mathbb{E}[w_T, w_T - \mathbb{E}[w_T], \mathbb{E}[l_T], l_T - \mathbb{E}[l_T]) = (w_T - l_T - \mathbb{E}[w_T - l_T])^2 - \lambda\mathbb{E}[w_T - l_T].$

We begin from stage $T-1$. Thus the conditional expectation at $t=T-1$ is

$$
\begin{aligned}
& \mathbb{E}\big[J_T(\mathbb{E}[w_T], w_T-\mathbb{E}[w_T], \mathbb{E}[l_T], l_T-\mathbb{E}[l_T])|\mathcal{F}_{T-1}\big] \\
= \; & \mathbb{E}\big[(w_T-l_T-\mathbb{E}[w_T-l_T])^2-\lambda\mathbb{E}[w_T-l_T]|\mathcal{F}_{T-1}\big] \\
= \; & \mathbb{E}\Big[\Big(n_{T-1}(w_{T-1}-\mathbb{E}[w_{T-1}])+R'_{T-1}(v_{T-1}-\mathbb{E}[v_{T-1}])+(R'_{T-1}-\mathbb{E}[R'_{T-1}])\mathbb{E}[v_{T-1}] \\
& +(c_{T-1}-\mathbb{E}[c_{T-1}])-q_{T-1}(l_{T-1}-\mathbb{E}[l_{T-1}])-(q_{T-1}-\mathbb{E}[q_{T-1}])\mathbb{E}[l_{T-1}]\Big)^2 \\
& -\lambda\Big(n_{T-1}\mathbb{E}[w_{T-1}]+\mathbb{E}[R'_{T-1}]\mathbb{E}[v_{T-1}]+\mathbb{E}[c_{T-1}]-\mathbb{E}[q_{T-1}]\mathbb{E}[l_{T-1}]\Big)\Big|\mathcal{F}_{T-1}\Big],
\end{aligned}
\tag{5.7}
$$

where

$$
\begin{aligned}
& \Big(n_{T-1}(w_{T-1}-\mathbb{E}[w_{T-1}])+R'_{T-1}(v_{T-1}-\mathbb{E}[v_{T-1}])+(R'_{T-1}-\mathbb{E}[R'_{T-1}])\mathbb{E}[v_{T-1}] \\
& \quad +(c_{T-1}-\mathbb{E}[c_{T-1}])-q_{T-1}(l_{T-1}-\mathbb{E}[l_{T-1}])-(q_{T-1}-\mathbb{E}[q_{T-1}])\mathbb{E}[l_{T-1}]\Big)^2 \\
= \; & \big(n_{T-1}(w_{T-1}-\mathbb{E}[w_{T-1}])\big)^2+\big(R'_{T-1}(v_{T-1}-\mathbb{E}[v_{T-1}])\big)^2 \\
& +\big((R'_{T-1}-\mathbb{E}[R'_{T-1}])\mathbb{E}[v_{T-1}]\big)^2+\big(c_{T-1}-\mathbb{E}[c_{T-1}]\big)^2+\big(q_{T-1}(l_{T-1}-\mathbb{E}[l_{T-1}])\big)^2 \\
& +\big((q_{T-1}-\mathbb{E}[q_{T-1}])\mathbb{E}[l_{T-1}]\big)^2+2n_{T-1}(w_{T-1}-\mathbb{E}[w_{T-1}])R'_{T-1}(v_{T-1}-\mathbb{E}[v_{T-1}]) \\
& +2n_{T-1}(w_{T-1}-\mathbb{E}[w_{T-1}])(R'_{T-1}-\mathbb{E}[R'_{T-1}])\mathbb{E}[v_{T-1}] \\
& +2n_{T-1}(w_{T-1}-\mathbb{E}[w_{T-1}])(c_{T-1}-\mathbb{E}[c_{T-1}]) \\
& -2n_{T-1}(w_{T-1}-\mathbb{E}[w_{T-1}])q_{T-1}(l_{T-1}-\mathbb{E}[l_{T-1}]) \\
& -2n_{T-1}(w_{T-1}-\mathbb{E}[w_{T-1}])(q_{T-1}-\mathbb{E}[q_{T-1}])\mathbb{E}[l_{T-1}] \\
& +2(v_{T-1}-\mathbb{E}[v_{T-1}])'R_{T-1}(R'_{T-1}-\mathbb{E}[R'_{T-1}])\mathbb{E}[v_{T-1}] \\
& +2R'_{T-1}(v_{T-1}-\mathbb{E}[v_{T-1}])(c_{T-1}-\mathbb{E}[c_{T-1}]) \\
& -2R'_{T-1}(v_{T-1}-\mathbb{E}[v_{T-1}])q_{T-1}(l_{T-1}-\mathbb{E}[l_{T-1}]) \\
& -2R'_{T-1}(v_{T-1}-\mathbb{E}[v_{T-1}])(q_{T-1}-\mathbb{E}[q_{T-1}])\mathbb{E}[l_{T-1}] \\
& +2(R'_{T-1}-\mathbb{E}[R'_{T-1}])\mathbb{E}[v_{T-1}](c_{T-1}-\mathbb{E}[c_{T-1}]) \\
& -2(R'_{T-1}-\mathbb{E}[R'_{T-1}])\mathbb{E}[v_{T-1}]q_{T-1}(l_{T-1}-\mathbb{E}[l_{T-1}]) \\
& -2(R'_{T-1}-\mathbb{E}[R'_{T-1}])\mathbb{E}[v_{T-1}](q_{T-1}-\mathbb{E}[q_{T-1}])\mathbb{E}[l_{T-1}] \\
& -2(c_{T-1}-\mathbb{E}[c_{T-1}])q_{T-1}(l_{T-1}-\mathbb{E}[l_{T-1}]) \\
& -2(c_{T-1}-\mathbb{E}[c_{T-1}])(q_{T-1}-\mathbb{E}[q_{T-1}])\mathbb{E}[l_{T-1}] \\
& +2q_{T-1}(l_{T-1}-\mathbb{E}[l_{T-1}])(q_{T-1}-\mathbb{E}[q_{T-1}])\mathbb{E}[l_{T-1}].
\end{aligned}
\tag{5.8}
$$

Since $w_t^i - \mathbb{E}[w_t^i]$, $\mathbb{E}[w_t^i]$, $v_t^i - \mathbb{E}[v_t^i]$, $\mathbb{E}[v_t^i]$, are $\mathcal{F}_t$-measurable,

$$
\begin{aligned}
&\mathbb{E}\Big[\big(n_{T-1}(w_{T-1} - \mathbb{E}[w_{T-1}])\big)^2\Big|\mathcal{F}_{T-1}\Big] = (n_{T-1})^2\big(w_{T-1} - \mathbb{E}[w_{T-1}]\big)^2,\\
&\mathbb{E}\Big[\Big(R'_{T-1}(v_{T-1} - \mathbb{E}[v_{T-1}])\Big)^2\Big|\mathcal{F}_{T-1}\Big]\\
&= \big(v_{T-1} - \mathbb{E}[v_{T-1}]\big)'\mathbb{E}[R_{T-1}R'_{T-1}]\big(v_{T-1} - \mathbb{E}[v_{T-1}]\big),\\
&\mathbb{E}\Big[\Big((R'_{T-1} - \mathbb{E}[R'_{T-1}])\mathbb{E}[v_{T-1}]\Big)^2\Big|\mathcal{F}_{T-1}\Big]\\
&= \mathbb{E}[(v_{T-1})']\Big(\mathbb{E}[R_{T-1}R'_{T-1}] - \mathbb{E}[R_{T-1}]\mathbb{E}[R'_{T-1}]\Big)\mathbb{E}[v_{T-1}],\\
&\mathbb{E}\Big[\big(c_{T-1} - \mathbb{E}[c_{T-1}]\big)^2\Big|\mathcal{F}_{T-1}\Big] = \mathbb{E}[c_{T-1}^2] - \mathbb{E}[c_{T-1}]^2,\\
&\mathbb{E}\Big[\big(q_{T-1}(l_{T-1} - \mathbb{E}[l_{T-1}])\big)^2\Big|\mathcal{F}_{T-1}\Big] = \mathbb{E}[q_{T-1}^2](l_{T-1} - \mathbb{E}[l_{T-1}])^2,\\
&\mathbb{E}\Big[\big((q_{T-1} - \mathbb{E}[q_{T-1}])\mathbb{E}[l_{T-1}]\big)^2\Big|\mathcal{F}_{T-1}\Big] = \Big(\mathbb{E}[q_{T-1}^2] - \mathbb{E}[q_{T-1}]^2\Big)\mathbb{E}[l_{T-1}]^2,\\
&\mathbb{E}\Big[2n_{T-1}(w_{T-1} - \mathbb{E}[w_{T-1}])R'_{T-1}(v_{T-1} - \mathbb{E}[v_{T-1}])\Big|\mathcal{F}_{T-1}\Big]\\
&= 2n_{T-1}\big(w_{T-1} - \mathbb{E}[w_{T-1}]\big)\mathbb{E}[R'_{T-1}]\big(v_{T-1} - \mathbb{E}[v_{T-1}]\big),\\
&\mathbb{E}\Big[2n_{T-1}\big(w_{T-1} - \mathbb{E}[w_{T-1}]\big)\big(R'_{T-1} - \mathbb{E}[R'_{T-1}]\big)\mathbb{E}[v_{T-1}]\Big|\mathcal{F}_{T-1}\Big] = 0,\\
&\mathbb{E}\Big[2n_{T-1}(w_{T-1} - \mathbb{E}[w_{T-1}])(c_{T-1} - \mathbb{E}[c_{T-1}])\Big|\mathcal{F}_{T-1}\Big] = 0,\\
&\mathbb{E}\Big[2n_{T-1}(w_{T-1} - \mathbb{E}[w_{T-1}])q_{T-1}(l_{T-1} - \mathbb{E}[l_{T-1}])\Big|\mathcal{F}_{T-1}\Big]\\
&= 2n_{T-1}\mathbb{E}[q_{T-1}](w_{T-1} - \mathbb{E}[w_{T-1}])(l_{T-1} - \mathbb{E}[l_{T-1}]),\\
&\mathbb{E}\Big[2n_{T-1}(w_{T-1} - \mathbb{E}[w_{T-1}])(q_{T-1} - \mathbb{E}[q_{T-1}])\mathbb{E}[l_{T-1}]\Big|\mathcal{F}_{T-1}\Big] = 0,\\
&\mathbb{E}\Big[2\big(v_{T-1} - \mathbb{E}[v_{T-1}]\big)'(R_{T-1})\Big(R'_{T-1} - \mathbb{E}[R'_{T-1}]\Big)\mathbb{E}[v_{T-1}]\Big|\mathcal{F}_{T-1}\Big] = 0,\\
&\mathbb{E}\Big[2R'_{T-1}(v_{T-1} - \mathbb{E}[v_{T-1}])(c_{T-1} - \mathbb{E}[c_{T-1}])\Big|\mathcal{F}_{T-1}\Big]\\
&= 2\big(\mathbb{E}[c_{T-1}R'_{T-1}] - \mathbb{E}[c_{T-1}]\mathbb{E}[R'_{T-1}]\big)(v_{T-1} - \mathbb{E}[v_{T-1}]) = 0,\\
&\mathbb{E}\Big[2R'_{T-1}(v_{T-1} - \mathbb{E}[v_{T-1}])q_{T-1}(l_{T-1} - \mathbb{E}[l_{T-1}])\Big|\mathcal{F}_{T-1}\Big]\\
&= 2\mathbb{E}[q_{T-1}R'_{T-1}](v_{T-1} - \mathbb{E}[v_{T-1}])(l_{T-1} - \mathbb{E}[l_{T-1}]),\\
&\mathbb{E}\Big[2R'_{T-1}(v_{T-1} - \mathbb{E}[v_{T-1}])(q_{T-1} - \mathbb{E}[q_{T-1}])\mathbb{E}[l_{T-1}]\Big|\mathcal{F}_{T-1}\Big]\\
&= 2\big(\mathbb{E}[q_{T-1}R'_{T-1}] - \mathbb{E}[q_{T-1}]\mathbb{E}[R'_{T-1}]\big)(v_{T-1} - \mathbb{E}[v_{T-1}])\mathbb{E}[l_{T-1}] = 0,\\
&\mathbb{E}\Big[2\big(R'_{T-1} - \mathbb{E}[R'_{T-1}]\big)\mathbb{E}[v_{T-1}]\big(c_{T-1} - \mathbb{E}[c_{T-1}]\big)\Big|\mathcal{F}_{T-1}\Big]\\
&= 2\Big(\mathbb{E}[c_{T-1}R'_{T-1}] - \mathbb{E}[c_{T-1}]\mathbb{E}[R'_{T-1}]\Big)\mathbb{E}[v_{T-1}],\\
&\mathbb{E}\Big[2(R'_{T-1} - \mathbb{E}[R'_{T-1}])\mathbb{E}[v_{T-1}]q_{T-1}(l_{T-1} - \mathbb{E}[l_{T-1}])\Big|\mathcal{F}_{T-1}\Big] = 0,
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbb{E}\Big[2(R'_{T-1}-\mathbb{E}[R'_{T-1}])\mathbb{E}[v_{T-1}](q_{T-1}-\mathbb{E}[q_{T-1}])\mathbb{E}[l_{T-1}]\Big|\mathcal{F}_{T-1}\Big]\\
&=2\big(\mathbb{E}[q_{T-1}R'_{T-1}]-\mathbb{E}[q_{T-1}]\mathbb{E}[R'_{T-1}]\big)\mathbb{E}[v_{T-1}]\mathbb{E}[l_{T-1}]=0.\\
&\mathbb{E}\Big[2(c_{T-1}-\mathbb{E}[c_{T-1}])q_{T-1}(l_{T-1}-\mathbb{E}[l_{T-1}])\Big|\mathcal{F}_{T-1}\Big]\\
&=2\big(\mathbb{E}[q_{T-1}c_{T-1}]-\mathbb{E}[q_{T-1}]\mathbb{E}[c_{T-1}]\big)(l_{T-1}-\mathbb{E}[l_{T-1}])=0,\\
&\mathbb{E}\Big[2(c_{T-1}-\mathbb{E}[c_{T-1}])(q_{T-1}-\mathbb{E}[q_{T-1}])\mathbb{E}[l_{T-1}]\Big|\mathcal{F}_{T-1}\Big]\\
&=2\big(\mathbb{E}[q_{T-1}c_{T-1}]-\mathbb{E}[q_{T-1}]\mathbb{E}[c_{T-1}]\big)\mathbb{E}[l_{T-1}]=0,\\
&\mathbb{E}\Big[2q_{T-1}(l_{T-1}-\mathbb{E}[l_{T-1}])(q_{T-1}-\mathbb{E}[q_{T-1}])\mathbb{E}[l_{T-1}]\Big|\mathcal{F}_{T-1}\Big]\\
&=2\big(\mathbb{E}[q^2_{T-1}]-\mathbb{E}[q_{T-1}]^2\big)(l_{T-1}-\mathbb{E}[l_{T-1}])\mathbb{E}[l_{T-1}]=0.
\end{aligned}
$$

Therefore, we can reduce (5.7) into

$$
\begin{aligned}
&\mathbb{E}\big[J_T(\mathbb{E}[w_T],w_T-\mathbb{E}[w_T],\mathbb{E}[l_T],l_T-\mathbb{E}[l_T])|\mathcal{F}_{T-1}\big]\\
=\;&\mathbb{E}\big[(w_T-l_T-\mathbb{E}[w_T-l_T])^2-\lambda\mathbb{E}[w_T-l_T]|\mathcal{F}_{T-1}\big]\\
=\;&\mathbb{E}\Big[\Big(n_{T-1}(w_{T-1}-\mathbb{E}[w_{T-1}])+R'_{T-1}(v_{T-1}-\mathbb{E}[v_{T-1}])+(R'_{T-1}-\mathbb{E}[R'_{T-1}])\mathbb{E}[v_{T-1}]\\
&+(c_{T-1}-\mathbb{E}[c_{T-1}])-q_{T-1}(l_{T-1}-\mathbb{E}[l_{T-1}])-(q_{T-1}-\mathbb{E}[q_{T-1}])\mathbb{E}[l_{T-1}]\Big)^2\\
&-\lambda\Big(n_{T-1}\mathbb{E}[w_{T-1}]+\mathbb{E}[R'_{T-1}]\mathbb{E}[v_{T-1}]+\mathbb{E}[c_{T-1}]-\mathbb{E}[q_{T-1}]\mathbb{E}[l_{T-1}]\Big)\Big|\mathcal{F}_{T-1}\Big]\\
=\;&(n_{T-1})^2\big(w_{T-1}-\mathbb{E}[w_{T-1}]\big)^2+\big(v_{T-1}-\mathbb{E}[v_{T-1}]\big)'\mathbb{E}[R_{T-1}R'_{T-1}]\big(v_{T-1}-\mathbb{E}[v_{T-1}]\big)\\
&+\mathbb{E}[(v_{T-1})']\Big(\mathbb{E}[R_{T-1}R'_{T-1}]-\mathbb{E}[R_{T-1}]\mathbb{E}[R'_{T-1}]\Big)\mathbb{E}[v_{T-1}]+\mathbb{E}[(c_{T-1})^2]-\mathbb{E}[c_{T-1}]^2\\
&+\mathbb{E}[q^2_{T-1}](l_{T-1}-\mathbb{E}[l_{T-1}])^2+\Big(\mathbb{E}[q^2_{T-1}]-\mathbb{E}[q_{T-1}]^2\Big)\mathbb{E}[l_{T-1}]^2\\
&+2n_{T-1}\big(w_{T-1}-\mathbb{E}[w_{T-1}]\big)\mathbb{E}[R'_{T-1}]\big(v_{T-1}-\mathbb{E}[v_{T-1}]\big)\\
&-2n_{T-1}\mathbb{E}[q_{T-1}](w_{T-1}-\mathbb{E}[w_{T-1}])(l_{T-1}-\mathbb{E}[l_{T-1}])\\
&-2\mathbb{E}[q_{T-1}R'_{T-1}](v_{T-1}-\mathbb{E}[v_{T-1}])(l_{T-1}-\mathbb{E}[l_{T-1}])\\
&+2\Big(\mathbb{E}[c_{T-1}R'_{T-1}]-\mathbb{E}[c_{T-1}]\mathbb{E}[R'_{T-1}]\Big)\mathbb{E}[v_{T-1}]\\
&-\lambda n_{T-1}\mathbb{E}[w_{T-1}]-\lambda\mathbb{E}[R'_{T-1}]\mathbb{E}[v_{T-1}]-\lambda\mathbb{E}[c_{T-1}]+\lambda\mathbb{E}[q_{T-1}]\mathbb{E}[l_{T-1}].
\end{aligned}
\tag{5.9}
$$

The optimal strategies at period $T-1$ can be derived from the above equation

corresponding to v_{T-1}

$$
\begin{aligned}
v_{T-1} - \mathbb{E}[v_{T-1}] =& -n_{T-1}(w_{T-1} - \mathbb{E}[w_{T-1}])\mathbb{E}^{-1}[R_{T-1}R'_{T-1}]\mathbb{E}[R_{T-1}] \\
& +(l_{T-1} - \mathbb{E}[l_{T-1}])\mathbb{E}^{-1}[R_{T-1}R'_{T-1}]\mathbb{E}[q_{T-1}R_{T-1}], \\
\mathbb{E}[v_{T-1}] =& -\Big(\mathbb{E}[R_{T-1}R'_{T-1}] - \mathbb{E}[R_{T-1}]\mathbb{E}[R'_{T-1}]\Big)^{-1} \\
& \times\Big(\mathbb{E}[c_{T-1}R_{T-1}] - \mathbb{E}[c_{T-1}]\mathbb{E}[R_{T-1}] - \frac{\lambda}{2}\mathbb{E}[R_{T-1}]\Big) \\
=& \Big(\mathbb{E}[c_{T-1}] + \frac{\lambda}{2} - \widehat{B}_{T-1}\Big)\frac{\mathbb{E}^{-1}[R_{T-1}R'_{T-1}]\mathbb{E}[R_{T-1}]}{1 - B_{T-1}} \\
& -\mathbb{E}^{-1}[R_{T-1}R'_{T-1}]\mathbb{E}[c_{T-1}R_{T-1}]
\end{aligned}
\tag{5.10}
$$

where

$$
\begin{aligned}
B_{T-1} =& \ \mathbb{E}[R'_{T-1}]\mathbb{E}^{-1}[R_{T-1}R'_{T-1}]\mathbb{E}[R_{T-1}], \\
\widehat{B}_{T-1} =& \ \mathbb{E}[c_{T-1}R'_{T-1}]\mathbb{E}^{-1}[R_{T-1}R'_{T-1}]\mathbb{E}[R_{T-1}], \\
\widetilde{B}_{T-1} =& \ \mathbb{E}[c_{T-1}R'_{T-1}]\mathbb{E}^{-1}[R_{T-1}R'_{T-1}]\mathbb{E}[c_{T-1}R_{T-1}], \\
\overline{B}_{T-1} =& \ \mathbb{E}[q_{T-1}R'_{T-1}]\mathbb{E}^{-1}[R_{T-1}R'_{T-1}]\mathbb{E}[R_{T-1}], \\
B'{}_{T-1} =& \ \mathbb{E}[q_{T-1}R'_{T-1}]\mathbb{E}^{-1}[R_{T-1}R'_{T-1}]\mathbb{E}[R_{T-1}q_{T-1}].
\end{aligned}
\tag{5.11}
$$

Substituting the optimal strategies back to (5.9), we obtain

$$
\begin{aligned}
& J_{T-1}(\mathbb{E}[w_{T-1}], w_{T-1}-\mathbb{E}[w_{T-1}], \mathbb{E}[l_{T-1}], l_{T-1}-\mathbb{E}[l_{T-1}]) \\
= \;& (n_{T-1})^2\big(w_{T-1}-\mathbb{E}[w_{T-1}]\big)^2 + (\mathbb{E}[(c_{T-1})^2]-\mathbb{E}[c_{T-1}]^2) \\
& +\mathbb{E}[q_{T-1}^2](l_{T-1}-\mathbb{E}[l_{T-1}])^2 + \Big(\mathbb{E}[q_{T-1}^2]-\mathbb{E}[q_{T-1}]^2\Big)\mathbb{E}[l_{T-1}]^2 \\
& -2n_{T-1}\mathbb{E}[q_{T-1}](w_{T-1}-\mathbb{E}[w_{T-1}])(l_{T-1}-\mathbb{E}[l_{T-1}]) - \lambda n_{T-1}\mathbb{E}[w_{T-1}] - \lambda\mathbb{E}[c_{T-1}] \\
& +\lambda\mathbb{E}[q_{T-1}]\mathbb{E}[l_{T-1}] - (n_{T-1})^2 B_{T-1}(w_{T-1}-\mathbb{E}[w_{T-1}])^2 - B'_{T-1}(l_{T-1}-\mathbb{E}[l_{T-1}])^2 \\
& +2n_{T-1}\overline{B}_{T-1}(w_{T-1}-\mathbb{E}[w_{T-1}])(l_{T-1}-\mathbb{E}[l_{T-1}]) \\
& -\Big(\big(\mathbb{E}[c_{T-1}]+\frac{\lambda}{2}-\widehat{B}_{T-1}\big)\frac{\mathbb{E}[R'_{T-1}]\mathbb{E}^{-1}[R_{T-1}R'_{T-1}]}{1-B_{T-1}} \\
& -\mathbb{E}[c_{T-1}R'_{T-1}]\mathbb{E}^{-1}[R_{T-1}R'_{T-1}]\Big)\Big(\mathbb{E}[R_{T-1}R'_{T-1}]-\mathbb{E}[R_{T-1}]\mathbb{E}[R'_{T-1}]\Big) \\
& \times\Big(\big(\mathbb{E}[c_{T-1}]+\frac{\lambda}{2}-\widehat{B}_{T-1}\big)\frac{\mathbb{E}^{-1}[R_{T-1}R'_{T-1}]\mathbb{E}[R_{T-1}]}{1-B_{T-1}} \\
& -\mathbb{E}^{-1}[R_{T-1}R'_{T-1}]\mathbb{E}[c_{T-1}R_{T-1}]\Big) \\
= \;& (n_{T-1})^2(1-B_{T-1})\big(w_{T-1}-\mathbb{E}[w_{T-1}]\big)^2 + (\mathbb{E}[(c_{T-1})^2]-\mathbb{E}[c_{T-1}]^2) \\
& +\big(\mathbb{E}[q_{T-1}^2]-B'_{T-1}\big)(l_{T-1}-\mathbb{E}[l_{T-1}])^2 \\
& -2\big(n_{T-1}\mathbb{E}[q_{T-1}]-n_{T-1}\overline{B}_{T-1}\big)(w_{T-1}-\mathbb{E}[w_{T-1}])(l_{T-1}-\mathbb{E}[l_{T-1}]) \\
& +\Big(\mathbb{E}[q_{T-1}^2]-\mathbb{E}[q_{T-1}]^2\Big)\mathbb{E}[l_{T-1}]^2 - \lambda n_{T-1}\mathbb{E}[w_{T-1}] - \lambda\mathbb{E}[c_{T-1}] + \lambda\mathbb{E}[q_{T-1}]\mathbb{E}[l_{T-1}] \\
& -\Big(\big(\mathbb{E}[c_{T-1}]+\frac{\lambda}{2}-\widehat{B}_{T-1}\big)^2\frac{B_{T-1}}{1-B_{T-1}} - 2\big(\mathbb{E}[c_{T-1}]+\frac{\lambda}{2}-\widehat{B}_{T-1}\big)\widehat{B}_{T-1} \\
& +\widetilde{B}_{T-1}-\widehat{B}_{T-1}^2\Big) \\
= \;& \xi_{T-1}\big(w_{T-1}-\mathbb{E}[w_{T-1}]\big)^2 - 2\eta_{T-1}(w_{T-1}-\mathbb{E}[w_{T-1}])(l_{T-1}-\mathbb{E}[l_{T-1}]) \\
& +\epsilon_{T-1}(l_{T-1}-\mathbb{E}[l_{T-1}])^2 - \lambda\zeta_{T-1}\mathbb{E}[w_{T-1}] + \lambda\theta_{T-1}\mathbb{E}[l_{T-1}] + \delta_{T-1}\mathbb{E}[l_{T-1}]^2 + \Delta_{T-1},
\end{aligned}
\tag{5.12}
$$

where

$$
\begin{aligned}
\xi_{T-1} = & \quad (n_{T-1})^2(1-B_{T-1}), \\
\eta_{T-1} = & \quad n_{T-1}\mathbb{E}[q_{T-1}] - n_{T-1}\overline{B}_{T-1}, \\
\epsilon_{T-1} = & \quad \mathbb{E}[q_{T-1}^2] - B'_{T-1}, \\
\zeta_{T-1} = & \quad n_{T-1}, \\
\theta_{T-1} = & \quad \mathbb{E}[q_{T-1}], \\
\delta_{T-1} = & \quad \mathbb{E}[q_{T-1}^2] - \mathbb{E}[q_{T-1}]^2, \\
\Delta_{T-1} = & \quad \mathbb{E}[(c_{T-1})^2] - \mathbb{E}[c_{T-1}]^2 - \lambda\mathbb{E}[c_{T-1}] - \Big(\big(\mathbb{E}[c_{T-1}] + \frac{\lambda}{2} - \widehat{B}_{T-1}\big)^2 \frac{B_{T-1}}{1-B_{T-1}} \\
& \quad -2\big(\mathbb{E}[c_{T-1}] + \frac{\lambda}{2} - \widehat{B}_{T-1}\big)\widehat{B}_{T-1} + \widetilde{B}_{T-1} - \widehat{B}_{T-1}^2\Big).
\end{aligned}
\tag{5.13}
$$

Repeating the process at time $T-2$, we have

$$
\begin{aligned}
& J_{T-2}(\mathbb{E}[w_{T-2}], w_{T-2} - \mathbb{E}[w_{T-2}], \mathbb{E}[l_{T-2}], l_{T-2} - \mathbb{E}[l_{T-2}]) \\
= \quad & \mathbb{E}\big[J_{T-1}(\mathbb{E}[w_{T-1}], w_{T-1} - \mathbb{E}[w_{T-1}], \mathbb{E}[l_{T-1}], l_{T-1} - \mathbb{E}[l_{T-1}])|\mathcal{F}_{T-2}\big] \\
= \quad & \mathbb{E}\big[\xi_{T-1}\big(w_{T-1} - \mathbb{E}[w_{T-1}]\big)^2 - 2\eta_{T-1}(w_{T-1} - \mathbb{E}[w_{T-1}])(l_{T-1} - \mathbb{E}[l_{T-1}]) \\
& +\epsilon_{T-1}(l_{T-1} - \mathbb{E}[l_{T-1}])^2 - \lambda\zeta_{T-1}\mathbb{E}[w_{T-1}] + \lambda\theta_{T-1}\mathbb{E}[l_{T-1}] + \delta_{T-1}\mathbb{E}[l_{T-1}]^2 \\
& +\Delta_{T-1}|\mathcal{F}_{T-2}\big] \\
= \quad & \mathbb{E}\Big[\xi_{T-1}\Big[n_{T-2}(w_{T-2} - \mathbb{E}[w_{T-2}]) + R'_{T-2}(v_{T-2} - \mathbb{E}[v_{T-2}]) \\
& +(R'_{T-2} - \mathbb{E}[R'_{T-2}])\mathbb{E}[v_{T-2}] + (c_{T-2} - \mathbb{E}[c_{T-2}])\Big]^2 \\
& -2\eta_{T-1}\Big[q_{T-2}(l_{T-2} - \mathbb{E}[l_{T-2}]) + (q_{T-2} - \mathbb{E}[q_{T-2}])\mathbb{E}[l_{T-2}]\Big] \\
& \times\Big[n_{T-2}(w_{T-2} - \mathbb{E}[w_{T-2}]) + R'_{T-2}(v_{T-2} - \mathbb{E}[v_{T-2}]) \\
& +(R'_{T-2} - \mathbb{E}[R'_{T-2}])\mathbb{E}[v_{T-2}] + (c_{T-2} - \mathbb{E}[c_{T-2}])\Big] \\
& +\epsilon_{T-1}\big[q_{T-2}(l_{T-2} - \mathbb{E}[l_{T-2}]) + (q_{T-2} - \mathbb{E}[q_{T-2}])\mathbb{E}[l_{T-2}]\big]^2 \\
& -\lambda\zeta_{T-1}\big(n_{T-2}\mathbb{E}[w_{T-2}] + \mathbb{E}[R'_{T-2}]\mathbb{E}[v_{T-2}] + \mathbb{E}[c_{T-2}]\big) \\
& +\lambda\theta_{T-1}\mathbb{E}[q_{T-2}]\mathbb{E}[l_{T-2}] + \delta_{T-1}(\mathbb{E}[q_{T-2}]\mathbb{E}[l_{T-2}])^2 + \Delta_{T-1}\Big|\mathcal{F}_{T-2}\Big]
\end{aligned}
$$

Thus,

$$
\begin{aligned}
& J_{T-2}(\mathbb{E}[w_{T-2}], w_{T-2}-\mathbb{E}[w_{T-2}], \mathbb{E}[l_{T-2}], l_{T-2}-\mathbb{E}[l_{T-2}]) \\
= \;& \xi_{T-1}\Big[n_{T-2}^2\big(w_{T-2}-\mathbb{E}[w_{T-2}]\big)^2+\big(v_{T-2}-\mathbb{E}[v_{T-2}]\big)'\mathbb{E}[R_{T-2}R_{T-2}']\big(v_{T-2}-\mathbb{E}[v_{T-2}]\big) \\
& +2n_{T-2}\big(w_{T-2}-\mathbb{E}[w_{T-2}]\big)\mathbb{E}[R_{T-2}']\big(v_{T-2}-\mathbb{E}[v_{T-2}]\big) \\
& +\mathbb{E}[v_{T-2}']\big(\mathbb{E}[R_{T-2}R_{T-2}']-\mathbb{E}[R_{T-2}]\mathbb{E}[R_{T-2}']\big)\mathbb{E}[v_{T-2}] \\
& +(\mathbb{E}[(c_{T-2})^2]-\mathbb{E}[c_{T-2}]^2)+2\Big(\mathbb{E}[c_{T-2}R_{T-2}']-\mathbb{E}[c_{T-2}]\mathbb{E}[R_{T-2}']\Big)\mathbb{E}[v_{T-2}]\Big] \\
& -2\eta_{T-1}\Big[n_{T-2}\mathbb{E}[q_{T-2}]\big(l_{T-2}-\mathbb{E}[l_{T-2}]\big)\big(w_{T-2}-\mathbb{E}[w_{T-2}]\big) \\
& +\mathbb{E}[q_{T-2}R_{T-2}']\big(l_{T-2}-\mathbb{E}[l_{T-2}]\big)\big(v_{T-2}-\mathbb{E}[v_{T-2}]\big)\Big] \\
& +\epsilon_{T-1}\Big[\mathbb{E}[q_{T-2}^2](l_{T-2}-\mathbb{E}[l_{T-2}])^2+(\mathbb{E}[q_{T-2}^2]-(\mathbb{E}[q_{T-2}])^2)(\mathbb{E}[l_{T-2}])^2\Big] \\
& -\lambda\zeta_{T-1}\big(n_{T-2}\mathbb{E}[w_{T-2}]+\mathbb{E}[R_{T-2}']\mathbb{E}[v_{T-2}]+\mathbb{E}[c_{T-2}]\big) \\
& +\lambda\theta_{T-1}\mathbb{E}[q_{T-2}]\mathbb{E}[l_{T-2}]+\delta_{T-1}(\mathbb{E}[q_{T-2}]\mathbb{E}[l_{T-2}])^2+\Delta_{T-1}.
\end{aligned} \tag{5.14}
$$

The optimal strategies at period $T-2$ can be derived from the above equation corresponding to v_{T-2}

$$
\begin{aligned}
v_{T-2}-\mathbb{E}[v_{T-2}] =& -n_{T-2}(w_{T-2}-\mathbb{E}[w_{T-2}])\mathbb{E}^{-1}[R_{T-2}R_{T-2}']\mathbb{E}[(R_{T-2})] \\
& +\eta_{T-1}\xi_{T-1}^{-1}\big(l_{T-2}-\mathbb{E}[l_{T-2}]\big)\mathbb{E}^{-1}[R_{T-2}R_{T-2}']\mathbb{E}[q_{T-2}R_{T-2}], \\
\mathbb{E}[v_{T-2}] =& -\Big(\mathbb{E}[R_{T-2}R_{T-2}']-\mathbb{E}[R_{T-2}]\mathbb{E}[R_{T-2}']\Big)^{-1} \\
& \times\Big(\mathbb{E}[c_{T-2}R_{T-2}]-\mathbb{E}[c_{T-2}]\mathbb{E}[R_{T-2}]-\frac{\lambda\zeta_{T-1}}{2\xi_{T-1}}\mathbb{E}[R_{T-2}]\Big) \\
=& \Big(\mathbb{E}[c_{T-2}]+\frac{\lambda\zeta_{T-1}}{2\xi_{T-1}}-\widehat{B}_{T-2}\Big)\frac{\mathbb{E}^{-1}[R_{T-2}R_{T-2}']\mathbb{E}[R_{T-2}]}{1-B_{T-2}} \\
& -\mathbb{E}^{-1}[R_{T-2}R_{T-2}']\mathbb{E}[c_{T-2}R_{T-2}].
\end{aligned} \tag{5.15}
$$

Substituting the optimal strategies back to (5.14), we obtain

$$
\begin{aligned}
& J_{T-2}(\mathbb{E}[w_{T-2}], w_{T-2}-\mathbb{E}[w_{T-2}], \mathbb{E}[l_{T-2}], l_{T-2}-\mathbb{E}[l_{T-2}]) \\
= \; & \xi_{T-1}(n_{T-2})^2\big(w_{T-2}-\mathbb{E}[w_{T-2}]\big)^2 + \xi_{T-1}(\mathbb{E}[(c_{T-2})^2]-\mathbb{E}[c_{T-2}]^2) \\
& +\epsilon_{T-1}\mathbb{E}[q_{T-2}^2](l_{T-2}-\mathbb{E}[l_{T-2}])^2 + \epsilon_{T-1}\Big(\mathbb{E}[q_{T-2}^2]-\mathbb{E}[q_{T-2}]^2\Big)\mathbb{E}[l_{T-2}]^2 \\
& -2\eta_{T-1}n_{T-2}\mathbb{E}[q_{T-2}](w_{T-2}-\mathbb{E}[w_{T-2}])(l_{T-2}-\mathbb{E}[l_{T-2}]) \\
& -\lambda\zeta_{T-1}n_{T-2}\mathbb{E}[w_{T-2}] - \lambda\zeta_{T-1}\mathbb{E}[c_{T-2}] + \lambda\theta_{T-1}\mathbb{E}[q_{T-2}]\mathbb{E}[l_{T-2}] \\
& +\delta_{T-1}(\mathbb{E}[q_{T-2}]\mathbb{E}[l_{T-2}])^2 + \Delta_{T-1} - \xi_{T-1}(n_{T-2})^2B_{T-2}(w_{T-2}-\mathbb{E}[w_{T-2}])^2 \\
& -\eta_{T-1}^2\xi_{T-1}^{-1}B'_{T-2}(l_{T-2}-\mathbb{E}[l_{T-2}])^2 \\
& +2\eta_{T-1}n_{T-2}\overline{B}_{T-2}(w_{T-2}-\mathbb{E}[w_{T-2}])(l_{T-2}-\mathbb{E}[l_{T-2}]) \\
& -\xi_{T-1}\Big(\big(\mathbb{E}[c_{T-2}]+\frac{\lambda\zeta_{T-1}}{2\xi_{T-1}}-\widehat{B}_{T-2}\big)\frac{\mathbb{E}[R'_{T-2}]\mathbb{E}^{-1}[R_{T-2}R'_{T-2}]}{1-B_{T-2}} \\
& -\mathbb{E}[c_{T-2}R'_{T-2}]\mathbb{E}^{-1}[R_{T-2}R'_{T-2}]\Big)\Big(\mathbb{E}[R_{T-2}R'_{T-2}]-\mathbb{E}[R_{T-2}]\mathbb{E}[R'_{T-2}]\Big) \\
& \times\Big(\big(\mathbb{E}[c_{T-2}]+\frac{\lambda\zeta_{T-1}}{2\xi_{T-1}}-\widehat{B}_{T-2}\big)\frac{\mathbb{E}^{-1}[R_{T-2}R'_{T-2}]\mathbb{E}[R_{T-2}]}{1-B_{T-2}} \\
& -\mathbb{E}^{-1}[R_{T-2}R'_{T-2}]\mathbb{E}[c_{T-2}R_{T-2}]\Big) \\
= \; & \xi_{T-1}(n_{T-2})^2(1-B_{T-2})\big(w_{T-2}-\mathbb{E}[w_{T-2}]\big)^2 + \xi_{T-1}(\mathbb{E}[(c_{T-2})^2]-\mathbb{E}[c_{T-2}]^2) \\
& +\big(\epsilon_{T-1}\mathbb{E}[q_{T-2}^2]-\eta_{T-1}^2\xi_{T-1}^{-1}B'_{T-2}\big)(l_{T-2}-\mathbb{E}[l_{T-2}])^2 \\
& -2\eta_{T-1}\big(n_{T-2}\mathbb{E}[q_{T-2}]-n_{T-2}\overline{B}_{T-2}\big)(w_{T-2}-\mathbb{E}[w_{T-2}])(l_{T-2}-\mathbb{E}[l_{T-2}]) \\
& +\epsilon_{T-1}\Big(\mathbb{E}[q_{T-2}^2]-\mathbb{E}[q_{T-2}]^2\Big)\mathbb{E}[l_{T-2}]^2 - \lambda\zeta_{T-1}n_{T-2}\mathbb{E}[w_{T-2}] - \lambda\zeta_{T-1}\mathbb{E}[c_{T-2}] \\
& +\lambda\theta_{T-1}\mathbb{E}[q_{T-2}]\mathbb{E}[l_{T-2}] + \delta_{T-1}(\mathbb{E}[q_{T-2}]\mathbb{E}[l_{T-2}])^2 + \Delta_{T-1} \\
& -\xi_{T-1}\Big(\big(\mathbb{E}[c_{T-2}]+\frac{\lambda\zeta_{T-1}}{2\xi_{T-1}}-\widehat{B}_{T-2}\big)^2\frac{B_{T-2}}{1-B_{T-2}} \\
& -2\big(\mathbb{E}[c_{T-2}]+\frac{\lambda\zeta_{T-1}}{2\xi_{T-1}}-\widehat{B}_{T-2}\big)\widehat{B}_{T-2}+\widetilde{B}_{T-2}-\widehat{B}_{T-2}^2\Big) \\
= \; & \xi_{T-2}\big(w_{T-2}-\mathbb{E}[w_{T-2}]\big)^2 - 2\eta_{T-2}(w_{T-2}-\mathbb{E}[w_{T-2}])(l_{T-2}-\mathbb{E}[l_{T-2}]) \\
& +\epsilon_{T-2}(l_{T-2}-\mathbb{E}[l_{T-2}])^2 - \lambda\zeta_{T-2}\mathbb{E}[w_{T-2}] + \lambda\theta_{T-2}\mathbb{E}[l_{T-2}] + \delta_{T-2}\mathbb{E}[l_{T-2}]^2 + \Delta_{T-2},
\end{aligned}
\tag{5.16}
$$

where

$$\begin{aligned}
\xi_{T-2} = & \ \xi_{T-1}(n_{T-2})^2(1-B_{T-2}), \\
\eta_{T-2} = & \ \eta_{T-1}\big(n_{T-2}\mathbb{E}[q_{T-2}] - n_{T-2}\overline{B}_{T-2}\big), \\
\epsilon_{T-2} = & \ \epsilon_{T-1}\mathbb{E}[q_{T-2}^2] - \eta_{T-1}^2\xi_{T-1}^{-1}B'_{T-2}, \\
\zeta_{T-2} = & \ \zeta_{T-1}n_{T-2}, \\
\theta_{T-2} = & \ \theta_{T-1}\mathbb{E}[q_{T-2}], \\
\delta_{T-2} = & \ \epsilon_{T-1}\Big(\mathbb{E}[q_{T-2}^2] - \mathbb{E}[q_{T-2}]^2\Big) + \delta_{T-1}(\mathbb{E}[q_{T-2}])^2, \\
\Delta_{T-2} = & \ \Delta_{T-1} + \xi_{T-1}\mathbb{E}[(c_{T-2})^2] - \xi_{T-1}\mathbb{E}[c_{T-2}]^2 - \lambda\zeta_{T-1}\mathbb{E}[c_{T-2}] \\
& -\xi_{T-1}\Big(\big(\mathbb{E}[c_{T-2}] + \frac{\lambda\zeta_{T-1}}{2\xi_{T-1}} - \widehat{B}_{T-2}\big)^2\frac{B_{T-2}}{1-B_{T-2}} \\
& -2\big(\mathbb{E}[c_{T-2}] + \frac{\lambda\zeta_{T-1}}{2\xi_{T-1}} - \widehat{B}_{T-2}\big)\widehat{B}_{T-2} + \widetilde{B}_{T-2} - \widehat{B}_{T-2}^2\Big).
\end{aligned} \tag{5.17}$$

Assume that the following equation (5.18) holds at time $t+1$, we prove it according to the mathematical induction.

$$\begin{aligned}
& J_{t+1}(\mathbb{E}[w_{t+1}], w_{t+1} - \mathbb{E}[w_{t+1}], \mathbb{E}[l_{t+1}], l_{t+1} - \mathbb{E}[l_{t+1}]) \\
= & \ \xi_{t+1}\big(w_{t+1} - \mathbb{E}[w_{t+1}]\big)^2 - 2\eta_{t+1}(w_{t+1} - \mathbb{E}[w_{t+1}])(l_{t+1} - \mathbb{E}[l_{t+1}]) \\
& +\epsilon_{t+1}(l_{t+1} - \mathbb{E}[l_{t+1}])^2 - \lambda\zeta_{t+1}\mathbb{E}[w_{t+1}] + \lambda\theta_{t+1}\mathbb{E}[l_{t+1}] + \delta_{t+1}\mathbb{E}[l_{t+1}]^2 + \Delta_{t+1},
\end{aligned} \tag{5.18}$$

where

$$\begin{aligned}
\xi_{t+1} = & \ \xi_{t+2}(n_{t+1})^2(1-B_{t+1}), \\
\eta_{t+1} = & \ \eta_{t+2}\big(n_{t+1}\mathbb{E}[q_{t+1}] - n_{t+1}\overline{B}_{t+1}\big), \\
\epsilon_{t+1} = & \ \epsilon_{t+2}\mathbb{E}[q_{t+1}^2] - \eta_{t+2}^2\xi_{t+2}^{-1}B'_{t+1}, \\
\zeta_{t+1} = & \ \zeta_{t+2}n_{t+1}, \\
\theta_{t+1} = & \ \theta_{t+2}\mathbb{E}[q_{t+1}], \\
\delta_{t+1} = & \ \epsilon_{t+2}\Big(\mathbb{E}[q_{t+1}^2] - \mathbb{E}[q_{t+1}]^2\Big) + \delta_{t+2}(\mathbb{E}[q_{t+1}])^2, \\
\Delta_{t+1} = & \ \Delta_{t+2} + \xi_{t+2}\mathbb{E}[(c_{t+1})^2] - \xi_{t+2}\mathbb{E}[c_{t+1}]^2 - \lambda\zeta_{t+2}\mathbb{E}[c_{t+1}] \\
& -\xi_{t+2}\Big(\big(\mathbb{E}[c_{t+1}] + \frac{\lambda\zeta_{t+2}}{2\xi_{t+2}} - \widehat{B}_{t+1}\big)^2\frac{B_{t+1}}{1-B_{t+1}} \\
& -2\big(\mathbb{E}[c_{t+1}] + \frac{\lambda\zeta_{t+2}}{2\xi_{t+2}} - \widehat{B}_{t+1}\big)\widehat{B}_{t+1} + \widetilde{B}_{t+1} - \widehat{B}_{t+1}^2\Big),
\end{aligned} \tag{5.19}$$

and

$$\xi_T = 1,\ \eta_T = 1,\ \epsilon_T = 1,\ \zeta_T = 1,\ \theta_T = 1,\ \delta_T = 0,\ \Delta_T = 0.$$

According to equation (5.18) and (5.19), we derive the result at time t.

$$\begin{aligned}
& J_t(\mathbb{E}[w_t], w_t - \mathbb{E}[w_t], \mathbb{E}[l_t], l_t - \mathbb{E}[l_t]) \\
= \ & \mathbb{E}\big[J_{t+1}(\mathbb{E}[w_{t+1}], w_{t+1} - \mathbb{E}[w_{t+1}], \mathbb{E}[l_{t+1}], l_{t+1} - \mathbb{E}[l_{t+1}])|\mathcal{F}_t\big] \\
= \ & \mathbb{E}\big[\xi_{t+1}\big(w_{t+1} - \mathbb{E}[w_{t+1}]\big)^2 - 2\eta_{t+1}(w_{t+1} - \mathbb{E}[w_{t+1}])(l_{t+1} - \mathbb{E}[l_{t+1}]) \\
& + \epsilon_{t+1}(l_{t+1} - \mathbb{E}[l_{t+1}])^2 - \lambda\zeta_{t+1}\mathbb{E}[w_{t+1}] + \lambda\theta_{t+1}\mathbb{E}[l_{t+1}] + \delta_{t+1}\mathbb{E}[l_{t+1}]^2 + \Delta_{t+1}|\mathcal{F}_t\big] \\
= \ & \mathbb{E}\Bigg[\xi_{t+1}\Big[n_t(w_t - \mathbb{E}[w_t]) + R_t'(v_t - \mathbb{E}[v_t]) + (R_t' - \mathbb{E}[R_t'])\mathbb{E}[v_t] + (c_t - \mathbb{E}[c_t])\Big]^2 \\
& - 2\eta_{t+1}\Big[q_t(l_t - \mathbb{E}[l_t]) + (q_t - \mathbb{E}[q_t])\mathbb{E}[l_t]\Big]\Big[n_t(w_t - \mathbb{E}[w_t]) + R_t'(v_t - \mathbb{E}[v_t]) \\
& + (R_t' - \mathbb{E}[R_t'])\mathbb{E}[v_t] + (c_t - \mathbb{E}[c_t])\Big] + \epsilon_{t+1}\big[q_t(l_t - \mathbb{E}[l_t]) + (q_t - \mathbb{E}[q_t])\mathbb{E}[l_t]\big]^2 \\
& - \lambda\zeta_{t+1}\big(n_t\mathbb{E}[w_t] + \mathbb{E}[R_t']\mathbb{E}[v_t] + \mathbb{E}[c_t]\big) + \lambda\theta_{t+1}\mathbb{E}[q_t]\mathbb{E}[l_t] + \delta_{t+1}(\mathbb{E}[q_t]\mathbb{E}[l_t])^2 \\
& + \Delta_{t+1}\Bigg|\mathcal{F}_t\Bigg] \\
= \ & \xi_{t+1}\Big[n_t^2\big(w_t - \mathbb{E}[w_t]\big)^2 + \big(v_t - \mathbb{E}[v_t]\big)'\mathbb{E}[R_tR_t']\big(v_t - \mathbb{E}[v_t]\big) \\
& + 2n_t\big(w_t - \mathbb{E}[w_t]\big)\mathbb{E}[R_t']\big(v_t - \mathbb{E}[v_t]\big) + \mathbb{E}[v_t']\big(\mathbb{E}[R_tR_t'] - \mathbb{E}[R_t]\mathbb{E}[R_t']\big)\mathbb{E}[v_t] \\
& + (\mathbb{E}[(c_t)^2] - \mathbb{E}[c_t]^2) + 2\Big(\mathbb{E}[c_tR_t'] - \mathbb{E}[c_t]\mathbb{E}[R_t']\Big)\mathbb{E}[v_t]\Big] \\
& - 2\eta_{t+1}\Big[n_t\mathbb{E}[q_t]\big(l_t - \mathbb{E}[l_t]\big)\big(w_t - \mathbb{E}[w_t]\big) + \mathbb{E}[q_tR_t']\big(l_t - \mathbb{E}[l_t]\big)\big(v_t - \mathbb{E}[v_t]\big)\Big] \\
& + \epsilon_{t+1}\Big[\mathbb{E}[q_t^2](l_t - \mathbb{E}[l_t])^2 + (\mathbb{E}[q_t^2] - (\mathbb{E}[q_t])^2)(\mathbb{E}[l_t])^2\Big] \\
& - \lambda\zeta_{t+1}\big(n_t\mathbb{E}[w_t] + \mathbb{E}[R_t']\mathbb{E}[v_t] + \mathbb{E}[c_t]\big) + \lambda\theta_{t+1}\mathbb{E}[q_t]\mathbb{E}[l_t] + \delta_{t+1}(\mathbb{E}[q_t]\mathbb{E}[l_t])^2 + \Delta_{t+1}.
\end{aligned} \tag{5.20}$$

The optimal strategies at period t can be derived from the above equation corresponding to t

$$
\begin{aligned}
v_t - \mathbb{E}[v_t] &= -n_t(w_t - \mathbb{E}[w_t])\mathbb{E}^{-1}[R_t R_t']\mathbb{E}[(R_t)] \\
&\qquad + \eta_{t+1}\xi_{t+1}^{-1}\big(l_t - \mathbb{E}[l_t]\big)\mathbb{E}^{-1}[R_t R_t']\mathbb{E}[q_t R_t], \\
\mathbb{E}[v_t] &= -\Big(\mathbb{E}[R_t R_t'] - \mathbb{E}[R_t]\mathbb{E}[R_t']\Big)^{-1}\Big(\mathbb{E}[c_t R_t] - \mathbb{E}[c_t]\mathbb{E}[R_t] - \frac{\lambda\zeta_{t+1}}{2\xi_{t+1}}\mathbb{E}[R_t]\Big) \\
&= \Big(\mathbb{E}[c_t] + \frac{\lambda\zeta_{t+1}}{2\xi_{t+1}} - \widehat{B}_t\Big)\frac{\mathbb{E}^{-1}[R_t R_t']\mathbb{E}[R_t]}{1 - B_t} - \mathbb{E}^{-1}[R_t R_t']\mathbb{E}[c_t R_t].
\end{aligned}
\tag{5.21}
$$

Substituting the optimal strategies back to (5.20), we obtain

$$
\begin{aligned}
& J_t(\mathbb{E}[w_t], w_t - \mathbb{E}[w_t], \mathbb{E}[l_t], l_t - \mathbb{E}[l_t]) \\
= \;& \xi_{t+1}(n_t)^2\big(w_t - \mathbb{E}[w_t]\big)^2 + \xi_{t+1}(\mathbb{E}[(c_t)^2] - \mathbb{E}[c_t]^2) \\
& + \epsilon_{t+1}\mathbb{E}[q_t^2](l_t - \mathbb{E}[l_t])^2 + \epsilon_{t+1}\Big(\mathbb{E}[q_t^2] - \mathbb{E}[q_t]^2\Big)\mathbb{E}[l_t]^2 \\
& - 2\eta_{t+1} n_t \mathbb{E}[q_t](w_t - \mathbb{E}[w_t])(l_t - \mathbb{E}[l_t]) \\
& - \lambda\zeta_{t+1} n_t \mathbb{E}[w_t] - \lambda\zeta_{t+1}\mathbb{E}[c_t] + \lambda\theta_{t+1}\mathbb{E}[q_t]\mathbb{E}[l_t] + \delta_{t+1}(\mathbb{E}[q_t]\mathbb{E}[l_t])^2 + \Delta_{t+1} \\
& - \xi_{t+1}(n_t)^2 B_t (w_t - \mathbb{E}[w_t])^2 - \eta_{t+1}^2\xi_{t+1}^{-1} B_t'(l_t - \mathbb{E}[l_t])^2 \\
& + 2\eta_{t+1} n_t \overline{B}_t (w_t - \mathbb{E}[w_t])(l_t - \mathbb{E}[l_t]) \\
& - \xi_{t+1}\Big(\big(\mathbb{E}[c_t] + \frac{\lambda\zeta_{t+1}}{2\xi_{t+1}} - \widehat{B}_t\big)\frac{\mathbb{E}[R_t']\mathbb{E}^{-1}[R_t R_t']}{1 - B_t} - \mathbb{E}[c_t R_t']\mathbb{E}^{-1}[R_t R_t']\Big) \\
& \times \Big(\mathbb{E}[R_t R_t'] - \mathbb{E}[R_t]\mathbb{E}[R_t']\Big) \\
& \times \Big(\big(\mathbb{E}[c_t] + \frac{\lambda\zeta_{t+1}}{2\xi_{t+1}} - \widehat{B}_t\big)\frac{\mathbb{E}^{-1}[R_t R_t']\mathbb{E}[R_t]}{1 - B_t} - \mathbb{E}^{-1}[R_t R_t']\mathbb{E}[c_t R_t]\Big) \\
= \;& \xi_{t+1}(n_t)^2(1 - B_t)\big(w_t - \mathbb{E}[w_t]\big)^2 + \xi_{t+1}(\mathbb{E}[(c_t)^2] - \mathbb{E}[c_t]^2) \\
& + \big(\epsilon_{t+1}\mathbb{E}[q_t^2] - \eta_{t+1}^2\xi_{t+1}^{-1}B_t'\big)(l_t - \mathbb{E}[l_t])^2 \\
& - 2\eta_{t+1}\big(n_t\mathbb{E}[q_t] - n_t\overline{B}_t\big)(w_t - \mathbb{E}[w_t])(l_t - \mathbb{E}[l_t]) \\
& + \epsilon_{t+1}\Big(\mathbb{E}[q_t^2] - \mathbb{E}[q_t]^2\Big)\mathbb{E}[l_t]^2 - \lambda\zeta_{t+1} n_t\mathbb{E}[w_t] - \lambda\zeta_{t+1}\mathbb{E}[c_t] + \lambda\theta_{t+1}\mathbb{E}[q_t]\mathbb{E}[l_t] \\
& + \delta_{t+1}(\mathbb{E}[q_t]\mathbb{E}[l_t])^2 + \Delta_{t+1} \\
& - \xi_{t+1}\Big(\big(\mathbb{E}[c_t] + \frac{\lambda\zeta_{t+1}}{2\xi_{t+1}} - \widehat{B}_t\big)^2\frac{B_t}{1 - B_t} - 2\big(\mathbb{E}[c_t] + \frac{\lambda\zeta_{t+1}}{2\xi_{t+1}} - \widehat{B}_t\big)\widehat{B}_t + \widetilde{B}_t - \widehat{B}_t^2\Big).
\end{aligned}
\tag{5.22}
$$

Thus,

$$\begin{aligned}
& J_t(\mathbb{E}[w_t], w_t - \mathbb{E}[w_t], \mathbb{E}[l_t], l_t - \mathbb{E}[l_t]) \\
= \; & \xi_t\big(w_t - \mathbb{E}[w_t]\big)^2 - 2\eta_t(w_t - \mathbb{E}[w_t])(l_t - \mathbb{E}[l_t]) + \epsilon_t(l_t - \mathbb{E}[l_t])^2 \\
& -\lambda\zeta_t\mathbb{E}[w_t] + \lambda\theta_t\mathbb{E}[l_t] + \delta_t\mathbb{E}[l_t]^2 + \Delta_t,
\end{aligned}$$

where

$$\begin{aligned}
\xi_t = \; & \xi_{t+1}(n_t)^2(1 - B_t), \\
\eta_t = \; & \eta_{t+1}\big(n_t\mathbb{E}[q_t] - n_t\overline{B}_t\big), \\
\epsilon_t = \; & \epsilon_{t+1}\mathbb{E}[q_t^2] - \eta_{t+1}^2\xi_{t+1}^{-1}B_t', \\
\zeta_t = \; & \zeta_{t+1}n_t, \\
\theta_t = \; & \theta_{t+1}\mathbb{E}[q_t], \\
\delta_t = \; & \epsilon_{t+1}\Big(\mathbb{E}[q_t^2] - \mathbb{E}[q_t]^2\Big) + \delta_{t+1}(\mathbb{E}[q_t])^2, \\
\Delta_t = \; & \Delta_{t+1} + \xi_{t+1}\mathbb{E}[(c_t)^2] - \xi_{t+1}\mathbb{E}[c_t]^2 - \lambda\zeta_{t+1}\mathbb{E}[c_t] \\
& -\xi_{t+1}\Big(\big(\mathbb{E}[c_t] + \frac{\lambda\zeta_{t+1}}{2\xi_{t+1}} - \widehat{B}_t\big)^2\frac{B_t}{1 - B_t} - 2\big(\mathbb{E}[c_t] + \frac{\lambda\zeta_{t+1}}{2\xi_{t+1}} - \widehat{B}_t\big)\widehat{B}_t + \widetilde{B}_t - \widehat{B}_t^2\Big).
\end{aligned} \tag{5.23}$$

Substituting $\mathbb{E}[v_t^*]$ to the dynamics eqation in (5.2) yields

$$\mathbb{E}[w_{t+1}] = n_t\mathbb{E}[w_t] + \big(\mathbb{E}[c_t] + \frac{\lambda\zeta_{t+1}}{2\xi_{t+1}} - \widehat{B}_t\big)\frac{B_t}{1 - B_t} - \widehat{B}_t + \mathbb{E}[c_t]. \tag{5.24}$$

Therefore,

$$\mathbb{E}[w_t] = w_0\prod_{k=0}^{t-1}n_k + \sum_{j=0}^{t-1}\Big(\big(\mathbb{E}[c_j] + \frac{\lambda\zeta_{j+1}}{2\xi_{j+1}} - \widehat{B}_j\big)\frac{B_j}{1 - B_j} - \widehat{B}_j + \mathbb{E}[c_j]\Big)\prod_{l=j+1}^{t-1}n_l. \tag{5.25}$$

Typically, we know the optimal value of problem (3.1) is equal to J_0. Thus,

$$
\begin{aligned}
& \mathrm{Var}(w_T) \\
= \; & J_0(\mathbb{E}[w_0], w_0 - \mathbb{E}[w_0], \mathbb{E}[l_0], l_0 - \mathbb{E}[l_0]) + \lambda \mathbb{E}[w_T] \\
= \; & -\lambda\zeta_0 w_0 + \lambda\theta_0 l_0 + \delta_0 l_0^2 + \Delta_0 + \lambda\Bigg(w_0 \prod_{k=0}^{T-1} n_k + \sum_{j=0}^{T-1} \Bigg(\Big(\mathbb{E}[c_j] + \frac{\lambda\zeta_{j+1}}{2\xi_{j+1}} - \widehat{B}_j \Big) \frac{B_j}{1-B_j} \\
& -\widehat{B}_j + \mathbb{E}[c_j] \Bigg) \prod_{l=j+1}^{T-1} n_l \Bigg).
\end{aligned}
\tag{5.26}
$$

5.2.2 Numerical Examples

Table 5.1 gives the expected return, standard deviation and the correlation coefficients among the assets, cash flow and liability. We also assume the risk-free return rate as 5% ($n_t = 1.05$).

Table 5.1: Data for assets, cashflow and liability

	SP	EM	MS	cashflow	liability
Expected return	14%	16%	17%	1	10%
Standard deviation	18.5%	30%	24%	20%	20%
Correlation coefficient					
SP	1	0.64	0.79	ρ_1	0
EM	0.64	1	0.75	ρ_2	0
MS	0.79	0.75	1	ρ_3	0
cashflow	ρ_1	ρ_2	ρ_3	1	0
liability	0	0	0	0	1

Thus,

$$
\mathbb{E}[R_t] = \begin{pmatrix} 0.09 \\ 0.11 \\ 0.12 \end{pmatrix}, \; \mathrm{Cov}(R_t) = \begin{pmatrix} 0.0342 & 0.0355 & 0.0351 \\ 0.0355 & 0.0900 & 0.0540 \\ 0.0351 & 0.0540 & 0.0576 \end{pmatrix},
$$

$$
\mathbb{E}[R_t R_t'] = \begin{pmatrix} 0.0423 & 0.0454 & 0.0459 \\ 0.0454 & 0.1021 & 0.0672 \\ 0.0459 & 0.0672 & 0.0720 \end{pmatrix}.
$$

The correlation coefficient between cash flow and i-th asset are defined as $\rho = (\rho_1, \rho_2, \rho_3)$,

$$\rho_i = \frac{\mathrm{Cov}(c_t, R_t^i)}{\sqrt{\mathrm{Var}(c_t)}\sqrt{\mathrm{Var}(R_t^i)}}$$

Therefore, we have the following eqation

$$\mathbb{E}[c_t R_t^i] = \mathbb{E}[c_t]\mathbb{E}[R_t^i] + \rho_i \sqrt{\mathrm{Var}(c_t)}\sqrt{\mathrm{Var}(R_t^i)},$$

$$\mathbb{E}[c_t^2] = \mathbb{E}[c_t]^2 + \mathrm{Var}(c_t), \quad \mathbb{E}[c_t q_t] = \mathbb{E}[c_t]\mathbb{E}[q_t], \quad \mathbb{E}[q_t R_t] = \mathbb{E}[R_t]\mathbb{E}[q_t].$$

Assume that $\rho = (\rho_1, \rho_2, \rho_3) = (-0.25, 0.5, 0.25)$. Thus,

$$\begin{aligned}\mathrm{Cov}\left(\begin{pmatrix} R_t \\ c_t \end{pmatrix}\right) &= \begin{pmatrix} \mathrm{Cov}(R_t) & \mathrm{Cov}(c_t, R_t) \\ \mathrm{Cov}(c_t, R_t') & \mathrm{Var}(c_t) \end{pmatrix} \\ &= \begin{pmatrix} 0.0342 & 0.0355 & 0.0351 & -0.0092 \\ 0.0355 & 0.0900 & 0.0540 & 0.0300 \\ 0.0351 & 0.0540 & 0.0576 & 0.0120 \\ -0.0092 & 0.0300 & 0.0120 & 0.0400 \end{pmatrix} \\ &\succ 0.\end{aligned}$$

In order to make the equations more clearly, we define some notations as follows

$$Y_1 = \mathbb{E}^{-1}[R_t R_t']\mathbb{E}[R_t] = \begin{pmatrix} 1.0589 \\ -0.1196 \\ 1.1033 \end{pmatrix},$$

$$Y_2 = \mathbb{E}^{-1}[R_t R_t']\mathbb{E}[c_t R_t] = \begin{pmatrix} -0.3490 \\ 0.4493 \\ 1.6365 \end{pmatrix},$$

$$Y_3 = \mathbb{E}^{-1}[R_t R_t']\mathbb{E}[q_t R_t] = \begin{pmatrix} 0.1059 \\ -0.0120 \\ 0.1103 \end{pmatrix}.$$

$$B_t = 0.2145,\ \widehat{B}_t = 0.2144,\ \widetilde{B}_t = 0.2507,\ \overline{B}_t = 0.0215,\ B'_t = 0.0021. \tag{5.27}$$

We assume the investor will exit the market at period $t = 5$. Therefore we define the optimal expected value as

$$\mathbb{E}[w] = (\mathbb{E}[w_1], \mathbb{E}[w_2], \mathbb{E}[w_3], \mathbb{E}[w_4], \mathbb{E}[w_5])$$

Substituting the data, we have

$$\mathbb{E}[w] = (4.4454, 5.9112, 7.4078, 8.9439, 10.5279).$$

The initial wealth is $w_0 = 3$ while initial liability $l_0 = 1$ and trade-off parameter $\lambda = 1$. Thus,

$$v_0^* = -1.05(w_0 - 5.2629)Y_1 - Y_2 + 0.000075Y_3l_0, ,$$

$$v_1^* = -1.05(w_1 - 6.4786)Y_1 - Y_2 + 0.0008Y_3l_1,$$

$$v_2^* = -1.05(w_2 - 7.7551)Y_1 - Y_2 + 0.0085Y_3l_2,$$

$$v_3^* = -1.05(w_3 - 9.0954)Y_1 - Y_2 + 0.0920Y_3l_3,$$

$$v_4^* = -1.05(w_4 - 10.5027)Y_1 - Y_2 + 1.0000Y_3l_4.$$

The variance of the final optimal wealth levels is $\mathrm{Var}(w_5) = 0.5996$.

5.3 The Optimal Strategy with Correlation

5.3.1 The Dynamic Programming and Optimal Strategy

On the one hand, we directly assume the return rates of assets and cash flow are correlated during each period t. On the other hand, in this subsection we assume return rates between liability and assets are correlated during each time t, i.e., q_t and R_t are not independent.

Thus, we investigate the following model by employing the mean-field method

$$\begin{cases} \min & \mathrm{Var}(w_T - l_T) - \lambda\mathbb{E}[w_T - l_T], \\ \text{s.t.} & w_{t+1} = n_t w_t + R_t' v_t + c_t, \\ & l_{t+1} = q_t l_t, \quad \text{for } t = 0, 1, \cdots, T-1, \end{cases} \tag{5.28}$$

where

$$w_{t+1} = n_t w_t + R_t' v_t + c_t, \tag{5.29}$$

$$\mathbb{E}[w_{t+1}] = n_t\mathbb{E}[w_t] + \mathbb{E}[R_t']\mathbb{E}[v_t] + \mathbb{E}[c_t], \tag{5.30}$$

$$\mathbb{E}[l_{t+1}] = \mathbb{E}[q_t]\mathbb{E}[l_t], \tag{5.31}$$

$$w_{t+1} - \mathbb{E}[w_{t+1}] = n_t(w_t - \mathbb{E}[w_t]) + R'_t(v_t - \mathbb{E}[v_t]) + (R'_t - \mathbb{E}[R'_t])\mathbb{E}[v_t] + (c_t - \mathbb{E}[c_t]), \tag{5.32}$$

$$l_{t+1} - \mathbb{E}[l_{t+1}] = q_t(l_t - \mathbb{E}[l_t]) + (q_t - \mathbb{E}[q_t])\mathbb{E}[l_t]. \tag{5.33}$$

Given the information set $\mathcal{F}_t$ at time period t, we have the following cost-to-go functional of problem (5.5)

$$\begin{aligned} & J_t(\mathbb{E}[w_t], w_t - \mathbb{E}[w_t], \mathbb{E}[l_t], l_t - \mathbb{E}[l_t]) \\ & = \min_{v_t} \mathbb{E}\big[J_{t+1}(\mathbb{E}[w_{t+1}], w_{t+1} - \mathbb{E}[w_{t+1}], \mathbb{E}[l_{t+1}], l_{t+1} - \mathbb{E}[l_{t+1}])\big|\mathcal{F}_t\big], \end{aligned}$$

with the boundary condition

$J_T(\mathbb{E}[w_T], w_T - \mathbb{E}[w_T], \mathbb{E}[l_T], l_T - \mathbb{E}[l_T]) = (w_T - l_T - \mathbb{E}[w_T - l_T])^2 - \lambda\mathbb{E}[w_T - l_T]$. Thus,

$$\begin{aligned} & \mathbb{E}\big[J_T(\mathbb{E}[w_T], w_T - \mathbb{E}[w_T], \mathbb{E}[l_T], l_T - \mathbb{E}[l_T])|\mathcal{F}_{T-1}\big] \\ = \; & \mathbb{E}\big[(w_T - l_T - \mathbb{E}[w_T - l_T])^2 - \lambda\mathbb{E}[w_T - l_T]|\mathcal{F}_{T-1}\big] \\ = \; & \mathbb{E}\Big[\Big(n_{T-1}(w_{T-1} - \mathbb{E}[w_{T-1}]) + R'_{T-1}(v_{T-1} - \mathbb{E}[v_{T-1}]) + (R'_{T-1} - \mathbb{E}[R'_{T-1}])\mathbb{E}[v_{T-1}] \\ & +(c_{T-1} - \mathbb{E}[c_{T-1}]) - q_{T-1}(l_{T-1} - \mathbb{E}[l_{T-1}]) - (q_{T-1} - \mathbb{E}[q_{T-1}])\mathbb{E}[l_{T-1}]\Big)^2 \\ & -\lambda\Big(n_{T-1}\mathbb{E}[w_{T-1}] + \mathbb{E}[R'_{T-1}]\mathbb{E}[v_{T-1}] + \mathbb{E}[c_{T-1}] - \mathbb{E}[q_{T-1}]\mathbb{E}[l_{T-1}]\Big)\Big|\mathcal{F}_{T-1}\Big], \end{aligned} \tag{5.34}$$

where

$$
\begin{aligned}
&\Big(n_{T-1}(w_{T-1}-\mathbb{E}[w_{T-1}])+R'_{T-1}(v_{T-1}-\mathbb{E}[v_{T-1}])+(R'_{T-1}-\mathbb{E}[R'_{T-1}])\mathbb{E}[v_{T-1}]\\
&\quad+(c_{T-1}-\mathbb{E}[c_{T-1}])-q_{T-1}(l_{T-1}-\mathbb{E}[l_{T-1}])-(q_{T-1}-\mathbb{E}[q_{T-1}])\mathbb{E}[l_{T-1}]\Big)^2\\
=\;&\big(n_{T-1}(w_{T-1}-\mathbb{E}[w_{T-1}])\big)^2+\big(R'_{T-1}(v_{T-1}-\mathbb{E}[v_{T-1}])\big)^2\\
&+\big((R'_{T-1}-\mathbb{E}[R'_{T-1}])\mathbb{E}[v_{T-1}]\big)^2+\big(c_{T-1}-\mathbb{E}[c_{T-1}]\big)^2+\big(q_{T-1}(l_{T-1}-\mathbb{E}[l_{T-1}])\big)^2\\
&+\big((q_{T-1}-\mathbb{E}[q_{T-1}])\mathbb{E}[l_{T-1}]\big)^2+2n_{T-1}(w_{T-1}-\mathbb{E}[w_{T-1}])R'_{T-1}(v_{T-1}-\mathbb{E}[v_{T-1}])\\
&+2n_{T-1}(w_{T-1}-\mathbb{E}[w_{T-1}])(R'_{T-1}-\mathbb{E}[R'_{T-1}])\mathbb{E}[v_{T-1}]\\
&+2n_{T-1}(w_{T-1}-\mathbb{E}[w_{T-1}])(c_{T-1}-\mathbb{E}[c_{T-1}])\\
&-2n_{T-1}(w_{T-1}-\mathbb{E}[w_{T-1}])q_{T-1}(l_{T-1}-\mathbb{E}[l_{T-1}])\\
&-2n_{T-1}(w_{T-1}-\mathbb{E}[w_{T-1}])(q_{T-1}-\mathbb{E}[q_{T-1}])\mathbb{E}[l_{T-1}]\\
&+2(v_{T-1}-\mathbb{E}[v_{T-1}])'R_{T-1}(R'_{T-1}-\mathbb{E}[R'_{T-1}])\mathbb{E}[v_{T-1}]\\
&+2R'_{T-1}(v_{T-1}-\mathbb{E}[v_{T-1}])(c_{T-1}-\mathbb{E}[c_{T-1}])\\
&-2R'_{T-1}(v_{T-1}-\mathbb{E}[v_{T-1}])q_{T-1}(l_{T-1}-\mathbb{E}[l_{T-1}])\\
&-2R'_{T-1}(v_{T-1}-\mathbb{E}[v_{T-1}])(q_{T-1}-\mathbb{E}[q_{T-1}])\mathbb{E}[l_{T-1}]\\
&+2(R'_{T-1}-\mathbb{E}[R'_{T-1}])\mathbb{E}[v_{T-1}](c_{T-1}-\mathbb{E}[c_{T-1}])\\
&-2(R'_{T-1}-\mathbb{E}[R'_{T-1}])\mathbb{E}[v_{T-1}]q_{T-1}(l_{T-1}-\mathbb{E}[l_{T-1}])\\
&-2(R'_{T-1}-\mathbb{E}[R'_{T-1}])\mathbb{E}[v_{T-1}](q_{T-1}-\mathbb{E}[q_{T-1}])\mathbb{E}[l_{T-1}]\\
&-2(c_{T-1}-\mathbb{E}[c_{T-1}])q_{T-1}(l_{T-1}-\mathbb{E}[l_{T-1}])\\
&-2(c_{T-1}-\mathbb{E}[c_{T-1}])(q_{T-1}-\mathbb{E}[q_{T-1}])\mathbb{E}[l_{T-1}]\\
&+2q_{T-1}(l_{T-1}-\mathbb{E}[l_{T-1}])(q_{T-1}-\mathbb{E}[q_{T-1}])\mathbb{E}[l_{T-1}].
\end{aligned}
\tag{5.35}
$$

Since $w_t^i - \mathbb{E}[w_t^i]$, $\mathbb{E}[w_t^i]$, $v_t^i - \mathbb{E}[v_t^i]$, $\mathbb{E}[v_t^i]$, are $\mathcal{F}_t$-measurable,

$$
\begin{aligned}
&\mathbb{E}\Big[\big(n_{T-1}(w_{T-1} - \mathbb{E}[w_{T-1}])\big)^2\Big|\mathcal{F}_{T-1}\Big] = (n_{T-1})^2\big(w_{T-1} - \mathbb{E}[w_{T-1}]\big)^2,\\
&\mathbb{E}\Big[\Big(R'_{T-1}(v_{T-1} - \mathbb{E}[v_{T-1}])\Big)^2\Big|\mathcal{F}_{T-1}\Big]\\
&= \big(v_{T-1} - \mathbb{E}[v_{T-1}]\big)'\mathbb{E}[R_{T-1}R'_{T-1}]\big(v_{T-1} - \mathbb{E}[v_{T-1}]\big),\\
&\mathbb{E}\Big[\Big((R'_{T-1} - \mathbb{E}[R'_{T-1}])\mathbb{E}[v_{T-1}]\Big)^2\Big|\mathcal{F}_{T-1}\Big]\\
&= \mathbb{E}[(v_{T-1})']\Big(\mathbb{E}[R_{T-1}R'_{T-1}] - \mathbb{E}[R_{T-1}]\mathbb{E}[R'_{T-1}]\Big)\mathbb{E}[v_{T-1}],\\
&\mathbb{E}\Big[\big(c_{T-1} - \mathbb{E}[c_{T-1}]\big)^2\Big|\mathcal{F}_{T-1}\Big] = \mathbb{E}[c_{T-1}^2] - \mathbb{E}[c_{T-1}]^2,\\
&\mathbb{E}\Big[\big(q_{T-1}(l_{T-1} - \mathbb{E}[l_{T-1}])\big)^2\Big|\mathcal{F}_{T-1}\Big] = \mathbb{E}[q_{T-1}^2](l_{T-1} - \mathbb{E}[l_{T-1}])^2,\\
&\mathbb{E}\Big[\big((q_{T-1} - \mathbb{E}[q_{T-1}])\mathbb{E}[l_{T-1}])\big)^2\Big|\mathcal{F}_{T-1}\Big] = \Big(\mathbb{E}[q_{T-1}^2] - \mathbb{E}[q_{T-1}]^2\Big)\mathbb{E}[l_{T-1}]^2,\\
&\mathbb{E}\Big[2n_{T-1}(w_{T-1} - \mathbb{E}[w_{T-1}])R'_{T-1}(v_{T-1} - \mathbb{E}[v_{T-1}])\Big|\mathcal{F}_{T-1}\Big]\\
&= 2n_{T-1}\big(w_{T-1} - \mathbb{E}[w_{T-1}]\big)\mathbb{E}[R'_{T-1}]\big(v_{T-1} - \mathbb{E}[v_{T-1}]\big),\\
&\mathbb{E}\Big[2n_{T-1}\big(w_{T-1} - \mathbb{E}[w_{T-1}]\big)\big(R'_{T-1} - \mathbb{E}[R'_{T-1}]\big)\mathbb{E}[v_{T-1}]\Big|\mathcal{F}_{T-1}\Big] = 0,\\
&\mathbb{E}\Big[2n_{T-1}(w_{T-1} - \mathbb{E}[w_{T-1}])(c_{T-1} - \mathbb{E}[c_{T-1}])\Big|\mathcal{F}_{T-1}\Big] = 0,\\
&\mathbb{E}\Big[2n_{T-1}(w_{T-1} - \mathbb{E}[w_{T-1}])q_{T-1}(l_{T-1} - \mathbb{E}[l_{T-1}])\Big|\mathcal{F}_{T-1}\Big]\\
&= 2n_{T-1}\mathbb{E}[q_{T-1}](w_{T-1} - \mathbb{E}[w_{T-1}])(l_{T-1} - \mathbb{E}[l_{T-1}]),\\
&\mathbb{E}\Big[2n_{T-1}(w_{T-1} - \mathbb{E}[w_{T-1}])(q_{T-1} - \mathbb{E}[q_{T-1}])\mathbb{E}[l_{T-1}]\Big|\mathcal{F}_{T-1}\Big] = 0,\\
&\mathbb{E}\Big[2\big(v_{T-1} - \mathbb{E}[v_{T-1}]\big)'(R_{T-1})\Big(R'_{T-1} - \mathbb{E}[R'_{T-1}]\Big)\mathbb{E}[v_{T-1}]\Big|\mathcal{F}_{T-1}\Big]\\
&= 2\big(v_{T-1} - \mathbb{E}[v_{T-1}]\big)'\Big(\mathbb{E}[R_{T-1}R'_{T-1}] - \mathbb{E}[R_{T-1}]\mathbb{E}[R'_{T-1}]\Big)\mathbb{E}[v_{T-1}] = 0,\\
&\mathbb{E}\Big[2R'_{T-1}(v_{T-1} - \mathbb{E}[v_{T-1}])(c_{T-1} - \mathbb{E}[c_{T-1}])\Big|\mathcal{F}_{T-1}\Big]\\
&= 2\big(\mathbb{E}[c_{T-1}R'_{T-1}] - \mathbb{E}[c_{T-1}]\mathbb{E}[R'_{T-1}\big)(v_{T-1} - \mathbb{E}[v_{T-1}]) = 0,\\
&\mathbb{E}\Big[2R'_{T-1}(v_{T-1} - \mathbb{E}[v_{T-1}])q_{T-1}(l_{T-1} - \mathbb{E}[l_{T-1}])\Big|\mathcal{F}_{T-1}\Big]\\
&= 2\mathbb{E}[q_{T-1}R'_{T-1}](v_{T-1} - \mathbb{E}[v_{T-1}])(l_{T-1} - \mathbb{E}[l_{T-1}]),\\
&\mathbb{E}\Big[2R'_{T-1}(v_{T-1} - \mathbb{E}[v_{T-1}])(q_{T-1} - \mathbb{E}[q_{T-1}])\mathbb{E}[l_{T-1}]\Big|\mathcal{F}_{T-1}\Big]\\
&= 2\big(\mathbb{E}[q_{T-1}R'_{T-1}] - \mathbb{E}[q_{T-1}]\mathbb{E}[R'_{T-1}]\big)(v_{T-1} - \mathbb{E}[v_{T-1}])\mathbb{E}[l_{T-1}] = 0,
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbb{E}\Big[2\big(R'_{T-1}-\mathbb{E}[R'_{T-1}]\big)\mathbb{E}[v_{T-1}]\big(c_{T-1}-\mathbb{E}[c_{T-1}]\big)\Big|\mathcal{F}_{T-1}\Big]\\
&=2\Big(\mathbb{E}[c_{T-1}R'_{T-1}]-\mathbb{E}[c_{T-1}]\mathbb{E}[R'_{T-1}]\Big)\mathbb{E}[v_{T-1}],\\
&\mathbb{E}\Big[2(R'_{T-1}-\mathbb{E}[R'_{T-1}])\mathbb{E}[v_{T-1}]q_{T-1}(l_{T-1}-\mathbb{E}[l_{T-1}])\Big|\mathcal{F}_{T-1}\Big]=0,\\
&\mathbb{E}\Big[2(R'_{T-1}-\mathbb{E}[R'_{T-1}])\mathbb{E}[v_{T-1}](q_{T-1}-\mathbb{E}[q_{T-1}])\mathbb{E}[l_{T-1}]\Big|\mathcal{F}_{T-1}\Big]\\
&=2\big(\mathbb{E}[q_{T-1}R'_{T-1}]-\mathbb{E}[q_{T-1}]\mathbb{E}[R'_{T-1}]\big)\mathbb{E}[v_{T-1}]\mathbb{E}[l_{T-1}],\\
&\mathbb{E}\Big[2(c_{T-1}-\mathbb{E}[c_{T-1}])q_{T-1}(l_{T-1}-\mathbb{E}[l_{T-1}])\Big|\mathcal{F}_{T-1}\Big]\\
&=2\big(\mathbb{E}[q_{T-1}c_{T-1}]-\mathbb{E}[q_{T-1}]\mathbb{E}[c_{T-1}]\big)(l_{T-1}-\mathbb{E}[l_{T-1}])=0,\\
&\mathbb{E}\Big[2(c_{T-1}-\mathbb{E}[c_{T-1}])(q_{T-1}-\mathbb{E}[q_{T-1}])\mathbb{E}[l_{T-1}]\Big|\mathcal{F}_{T-1}\Big]\\
&=2\big(\mathbb{E}[q_{T-1}c_{T-1}]-\mathbb{E}[q_{T-1}]\mathbb{E}[c_{T-1}]\big)\mathbb{E}[l_{T-1}],\\
&\mathbb{E}\Big[2q_{T-1}(l_{T-1}-\mathbb{E}[l_{T-1}])(q_{T-1}-\mathbb{E}[q_{T-1}])\mathbb{E}[l_{T-1}]\Big|\mathcal{F}_{T-1}\Big]\\
&=2\big(\mathbb{E}[q_{T-1}^2]-\mathbb{E}[q_{T-1}]^2\big)(l_{T-1}-\mathbb{E}[l_{T-1}])\mathbb{E}[l_{T-1}]=0.
\end{aligned}
$$

Therefore, we can reduce (5.34) into

$$
\begin{aligned}
&\mathbb{E}\big[J_T(\mathbb{E}[w_T],w_T-\mathbb{E}[w_T],\mathbb{E}[l_T],l_T-\mathbb{E}[l_T])|\mathcal{F}_{T-1}\big]\\
=\;&\mathbb{E}\big[(w_T-l_T-\mathbb{E}[w_T-l_T])^2-\lambda\mathbb{E}[w_T-l_T]|\mathcal{F}_{T-1}\big]\\
=\;&\mathbb{E}\Big[\Big(n_{T-1}(w_{T-1}-\mathbb{E}[w_{T-1}])+R'_{T-1}(v_{T-1}-\mathbb{E}[v_{T-1}])\\
&+(R'_{T-1}-\mathbb{E}[R'_{T-1}])\mathbb{E}[v_{T-1}]+(c_{T-1}-\mathbb{E}[c_{T-1}])\\
&-q_{T-1}(l_{T-1}-\mathbb{E}[l_{T-1}])-(q_{T-1}-\mathbb{E}[q_{T-1}])\mathbb{E}[l_{T-1}]\Big)^2\\
&-\lambda\Big(n_{T-1}\mathbb{E}[w_{T-1}]+\mathbb{E}[R'_{T-1}]\mathbb{E}[v_{T-1}]+\mathbb{E}[c_{T-1}]-\mathbb{E}[q_{T-1}]\mathbb{E}[l_{T-1}]\Big)\Big|\mathcal{F}_{T-1}\Big].
\end{aligned}
$$

Thus,

$$
\begin{aligned}
& \mathbb{E}\big[J_T(\mathbb{E}[w_T], w_T-\mathbb{E}[w_T], \mathbb{E}[l_T], l_T-\mathbb{E}[l_T])|\mathcal{F}_{T-1}\big] \\
= \;& (n_{T-1})^2\big(w_{T-1}-\mathbb{E}[w_{T-1}]\big)^2+\big(v_{T-1}-\mathbb{E}[v_{T-1}]\big)'\mathbb{E}[R_{T-1}R'_{T-1}]\big(v_{T-1}-\mathbb{E}[v_{T-1}]\big) \\
& +\mathbb{E}[(v_{T-1})']\Big(\mathbb{E}[R_{T-1}R'_{T-1}]-\mathbb{E}[R_{T-1}]\mathbb{E}[R'_{T-1}]\Big)\mathbb{E}[v_{T-1}]+\mathbb{E}[(c_{T-1})^2] \\
& -\mathbb{E}[c_{T-1}]^2+\mathbb{E}[q^2_{T-1}](l_{T-1}-\mathbb{E}[l_{T-1}])^2+\Big(\mathbb{E}[q^2_{T-1}]-\mathbb{E}[q_{T-1}]^2\Big)\mathbb{E}[l_{T-1}]^2 \\
& +2n_{T-1}\big(w_{T-1}-\mathbb{E}[w_{T-1}]\big)\mathbb{E}[R'_{T-1}]\big(v_{T-1}-\mathbb{E}[v_{T-1}]\big) \\
& -2n_{T-1}\mathbb{E}[q_{T-1}](w_{T-1}-\mathbb{E}[w_{T-1}])(l_{T-1}-\mathbb{E}[l_{T-1}]) \\
& -2\mathbb{E}[q_{T-1}R'_{T-1}](v_{T-1}-\mathbb{E}[v_{T-1}])(l_{T-1}-\mathbb{E}[l_{T-1}]) \\
& +2\Big(\mathbb{E}[c_{T-1}R'_{T-1}]-\mathbb{E}[c_{T-1}]\mathbb{E}[R'_{T-1}]\Big)\mathbb{E}[v_{T-1}] \\
& -2\big(\mathbb{E}[q_{T-1}R'_{T-1}]-\mathbb{E}[q_{T-1}]\mathbb{E}[R'_{T-1}]\big)\mathbb{E}[v_{T-1}]\mathbb{E}[l_{T-1}] \\
& -2\big(\mathbb{E}[q_{T-1}c_{T-1}]-\mathbb{E}[q_{T-1}]\mathbb{E}[c_{T-1}]\big)\mathbb{E}[l_{T-1}] \\
& -\lambda n_{T-1}\mathbb{E}[w_{T-1}]-\lambda\mathbb{E}[R'_{T-1}]\mathbb{E}[v_{T-1}]-\lambda\mathbb{E}[c_{T-1}]+\lambda\mathbb{E}[q_{T-1}]\mathbb{E}[l_{T-1}].
\end{aligned}
\tag{5.36}
$$

The optimal strategies at period $T-1$ can be derived from the above equation corresponding to v_{T-1}

$$
\begin{aligned}
v_{T-1}-\mathbb{E}[v_{T-1}] &= -n_{T-1}(w_{T-1}-\mathbb{E}[w_{T-1}])\mathbb{E}^{-1}[R_{T-1}R'_{T-1}]\mathbb{E}[R_{T-1}] \\
&\quad +(l_{T-1}-\mathbb{E}[l_{T-1}])\mathbb{E}^{-1}[R_{T-1}R'_{T-1}]\mathbb{E}[q_{T-1}R_{T-1}], \\
\mathbb{E}[v_{T-1}] &= -\Big(\mathbb{E}[R_{T-1}R'_{T-1}]-\mathbb{E}[R_{T-1}]\mathbb{E}[R'_{T-1}]\Big)^{-1} \\
&\quad \times\Big(\mathbb{E}[c_{T-1}R_{T-1}]-\mathbb{E}[c_{T-1}]\mathbb{E}[R_{T-1}]-\frac{\lambda}{2}\mathbb{E}[R_{T-1}] \\
&\quad -\big(\mathbb{E}[q_{T-1}R_{T-1}]-\mathbb{E}[q_{T-1}]\mathbb{E}[R_{T-1}]\big)\mathbb{E}[l_{T-1}]\Big),
\end{aligned}
\tag{5.37}
$$

where

$$
\begin{aligned}
B_{T-1} = \;& \mathbb{E}[R'_{T-1}]\mathbb{E}^{-1}[R_{T-1}R'_{T-1}]\mathbb{E}[R_{T-1}], \\
\widehat{B}_{T-1} = \;& \mathbb{E}[c_{T-1}R'_{T-1}]\mathbb{E}^{-1}[R_{T-1}R'_{T-1}]\mathbb{E}[R_{T-1}], \\
\widetilde{B}_{T-1} = \;& \mathbb{E}[c_{T-1}R'_{T-1}]\mathbb{E}^{-1}[R_{T-1}R'_{T-1}]\mathbb{E}[c_{T-1}R_{T-1}], \\
\overline{B}_{T-1} = \;& \mathbb{E}[q_{T-1}R'_{T-1}]\mathbb{E}^{-1}[R_{T-1}R'_{T-1}]\mathbb{E}[R_{T-1}], \\
B'_{T-1} = \;& \mathbb{E}[q_{T-1}R'_{T-1}]\mathbb{E}^{-1}[R_{T-1}R'_{T-1}]\mathbb{E}[R_{T-1}q_{T-1}], \\
\overrightarrow{B}_{T-1} = \;& \mathbb{E}[c_{T-1}R'_{T-1}]\mathbb{E}^{-1}[R_{T-1}R'_{T-1}]\mathbb{E}[R_{T-1}q_{T-1}].
\end{aligned}
\tag{5.38}
$$

Substituting the optimal strategies back to (5.36), we obtain

$$
\begin{aligned}
& J_{T-1}(\mathbb{E}[w_{T-1}], w_{T-1}-\mathbb{E}[w_{T-1}], \mathbb{E}[l_{T-1}], l_{T-1}-\mathbb{E}[l_{T-1}]) \\
= \ & \mathbb{E}\big[J_T(\mathbb{E}[w_T], w_T-\mathbb{E}[w_T], \mathbb{E}[l_T], l_T-\mathbb{E}[l_T])|\mathcal{F}_{T-1}\big] \\
= \ & (n_{T-1})^2\big(w_{T-1}-\mathbb{E}[w_{T-1}]\big)^2 + (\mathbb{E}[(c_{T-1})^2]-\mathbb{E}[c_{T-1}]^2) \\
& +\mathbb{E}[q_{T-1}^2](l_{T-1}-\mathbb{E}[l_{T-1}])^2 + \Big(\mathbb{E}[q_{T-1}^2]-\mathbb{E}[q_{T-1}]^2\Big)\mathbb{E}[l_{T-1}]^2 \\
& -2n_{T-1}\mathbb{E}[q_{T-1}](w_{T-1}-\mathbb{E}[w_{T-1}])(l_{T-1}-\mathbb{E}[l_{T-1}]) - \lambda n_{T-1}\mathbb{E}[w_{T-1}] \\
& -\lambda\mathbb{E}[c_{T-1}] + \lambda\mathbb{E}[q_{T-1}]\mathbb{E}[l_{T-1}] - (n_{T-1})^2 B_{T-1}(w_{T-1}-\mathbb{E}[w_{T-1}])^2 \\
& -B'_{T-1}(l_{T-1}-\mathbb{E}[l_{T-1}])^2 + 2n_{T-1}\overline{B}_{T-1}(w_{T-1}-\mathbb{E}[w_{T-1}])(l_{T-1}-\mathbb{E}[l_{T-1}]) \\
& -2\big(\mathbb{E}[q_{T-1}c_{T-1}]-\mathbb{E}[q_{T-1}]\mathbb{E}[c_{T-1}]\big)\mathbb{E}[l_{T-1}] \\
& -\Big(\mathbb{E}[c_{T-1}R'_{T-1}]-\mathbb{E}[c_{T-1}]\mathbb{E}[R'_{T-1}]-\frac{\lambda}{2}\mathbb{E}[R'_{T-1}] \\
& -\big(\mathbb{E}[q_{T-1}R'_{T-1}]-\mathbb{E}[q_{T-1}]\mathbb{E}[R'_{T-1}]\big)\mathbb{E}[l_{T-1}]\Big) \\
& \times\Big(\mathbb{E}[R_{T-1}R'_{T-1}]-\mathbb{E}[R_{T-1}]\mathbb{E}[R'_{T-1}]\Big)^{-1} \\
& \times\Big(\mathbb{E}[c_{T-1}R_{T-1}]-\mathbb{E}[c_{T-1}]\mathbb{E}[R_{T-1}]-\frac{\lambda}{2}\mathbb{E}[R_{T-1}] \\
& -\big(\mathbb{E}[q_{T-1}R_{T-1}]-\mathbb{E}[q_{T-1}]\mathbb{E}[R_{T-1}]\big)\mathbb{E}[l_{T-1}]\Big) \\
= \ & (n_{T-1})^2(1-B_{T-1})\big(w_{T-1}-\mathbb{E}[w_{T-1}]\big)^2 + (\mathbb{E}[(c_{T-1})^2]-\mathbb{E}[c_{T-1}]^2) \\
& +\big(\mathbb{E}[q_{T-1}^2]-B'_{T-1}\big)(l_{T-1}-\mathbb{E}[l_{T-1}])^2 \\
& -2\big(n_{T-1}\mathbb{E}[q_{T-1}]-n_{T-1}\overline{B}_{T-1}\big)(w_{T-1}-\mathbb{E}[w_{T-1}])(l_{T-1}-\mathbb{E}[l_{T-1}]) \\
& +\Big(\mathbb{E}[q_{T-1}^2]-\mathbb{E}[q_{T-1}]^2\Big)\mathbb{E}[l_{T-1}]^2 - \lambda n_{T-1}\mathbb{E}[w_{T-1}] - \lambda\mathbb{E}[c_{T-1}] \\
& +\lambda\mathbb{E}[q_{T-1}]\mathbb{E}[l_{T-1}] - 2\big(\mathbb{E}[q_{T-1}c_{T-1}]-\mathbb{E}[q_{T-1}]\mathbb{E}[c_{T-1}]\big)\mathbb{E}[l_{T-1}] \\
& -\Big(\big(\mathbb{E}[c_{T-1}]+\frac{\lambda}{2}-\widehat{B}_{T-1}\big)^2\frac{B_{T-1}}{1-B_{T-1}} - 2\big(\mathbb{E}[c_{T-1}]+\frac{\lambda}{2}-\widehat{B}_{T-1}\big)\widehat{B}_{T-1} \\
& +\widetilde{B}_{T-1}-\widehat{B}_{T-1}^2\Big) \\
& +2\Big(\frac{\big(\mathbb{E}[c_{T-1}]+\frac{\lambda}{2}\big)\big(\mathbb{E}[q_{T-1}]B_{T-1}-\overline{B}_{T-1}\big)+\widehat{B}_{T-1}\overline{B}_{T-1}-\widehat{B}_{T-1}\mathbb{E}[q_{T-1}]}{1-B_{T-1}} \\
& +\overrightarrow{B}_{T-1}\Big)\mathbb{E}[l_{T-1}] - \Big(B'_{T-1}-\mathbb{E}[q_{T-1}]^2+\frac{\big(\overline{B}_{T-1}-\mathbb{E}[q_{T-1}]\big)^2}{1-B_{T-1}}\Big)\mathbb{E}[l_{T-1}]^2 \\
= \ & \xi_{T-1}\big(w_{T-1}-\mathbb{E}[w_{T-1}]\big)^2 - 2\eta_{T-1}(w_{T-1}-\mathbb{E}[w_{T-1}])(l_{T-1}-\mathbb{E}[l_{T-1}]) \\
& +\epsilon_{T-1}(l_{T-1}-\mathbb{E}[l_{T-1}])^2 - \lambda\zeta_{T-1}\mathbb{E}[w_{T-1}] + \lambda\theta_{T-1}\mathbb{E}[l_{T-1}] + \delta_{T-1}\mathbb{E}[l_{T-1}]^2 \\
& +\Delta_{T-1},
\end{aligned}
\tag{5.39}
$$

where

$$
\begin{aligned}
\xi_{T-1} = &\quad (n_{T-1})^2(1-B_{T-1}),\\
\eta_{T-1} = &\quad n_{T-1}\mathbb{E}[q_{T-1}] - n_{T-1}\overline{B}_{T-1},\\
\epsilon_{T-1} = &\quad \mathbb{E}[q_{T-1}^2] - B'_{T-1},\\
\zeta_{T-1} = &\quad n_{T-1},\\
\theta_{T-1} = &\quad \mathbb{E}[q_{T-1}] - \frac{2}{\lambda}\big(\mathbb{E}[q_{T-1}c_{T-1}] - \mathbb{E}[q_{T-1}]\mathbb{E}[c_{T-1}]\big)\\
&\quad + \frac{2}{\lambda}\Big(\frac{\big(\mathbb{E}[c_{T-1}] + \frac{\lambda}{2}\big)\big(\mathbb{E}[q_{T-1}]B_{T-1} - \overline{B}_{T-1}\big) + \widehat{B}_{T-1}\overline{B}_{T-1} - \widehat{B}_{T-1}\mathbb{E}[q_{T-1}]}{1-B_{T-1}}\\
&\quad + \overrightarrow{B}_{T-1}\Big),\\
\delta_{T-1} = &\quad \mathbb{E}[q_{T-1}^2] - \mathbb{E}[q_{T-1}]^2 - \Big(B'_{T-1} - \mathbb{E}[q_{T-1}]^2 + \frac{\big(\overline{B}_{T-1} - \mathbb{E}[q_{T-1}]\big)^2}{1-B_{T-1}}\Big),\\
\Delta_{T-1} = &\quad \mathbb{E}[(c_{T-1})^2] - \mathbb{E}[c_{T-1}]^2 - \lambda\mathbb{E}[c_{T-1}] - \Big(\big(\mathbb{E}[c_{T-1}] + \frac{\lambda}{2} - \widehat{B}_{T-1}\big)^2\frac{B_{T-1}}{1-B_{T-1}}\\
&\quad -2\big(\mathbb{E}[c_{T-1}] + \frac{\lambda}{2} - \widehat{B}_{T-1}\big)\widehat{B}_{T-1} + \widetilde{B}_{T-1} - \widehat{B}_{T-1}^2\Big).
\end{aligned}
\tag{5.40}
$$

Repeating the process at time $T-2$, we have

$$
\begin{aligned}
&\quad J_{T-2}(\mathbb{E}[w_{T-2}], w_{T-2} - \mathbb{E}[w_{T-2}], \mathbb{E}[l_{T-2}], l_{T-2} - \mathbb{E}[l_{T-2}])\\
= &\quad \mathbb{E}\big[J_{T-1}(\mathbb{E}[w_{T-1}], w_{T-1} - \mathbb{E}[w_{T-1}], \mathbb{E}[l_{T-1}], l_{T-1} - \mathbb{E}[l_{T-1}])|\mathcal{F}_{T-2}\big]\\
= &\quad \mathbb{E}\big[\xi_{T-1}\big(w_{T-1} - \mathbb{E}[w_{T-1}]\big)^2 - 2\eta_{T-1}(w_{T-1} - \mathbb{E}[w_{T-1}])(l_{T-1} - \mathbb{E}[l_{T-1}])\\
&\quad + \epsilon_{T-1}(l_{T-1} - \mathbb{E}[l_{T-1}])^2 - \lambda\zeta_{T-1}\mathbb{E}[w_{T-1}] + \lambda\theta_{T-1}\mathbb{E}[l_{T-1}]\\
&\quad + \delta_{T-1}\mathbb{E}[l_{T-1}]^2 + \Delta_{T-1}|\mathcal{F}_{T-2}\big].
\end{aligned}
$$

Thus,

$$
\begin{aligned}
& J_{T-2}(\mathbb{E}[w_{T-2}], w_{T-2}-\mathbb{E}[w_{T-2}], \mathbb{E}[l_{T-2}], l_{T-2}-\mathbb{E}[l_{T-2}]) \\
= \;& \mathbb{E}\Bigg[\xi_{T-1}\Big[n_{T-2}(w_{T-2}-\mathbb{E}[w_{T-2}]) + R'_{T-2}(v_{T-2}-\mathbb{E}[v_{T-2}]) \\
& +(R'_{T-2}-\mathbb{E}[R'_{T-2}])\mathbb{E}[v_{T-2}] + (c_{T-2}-\mathbb{E}[c_{T-2}])\Big]^2 \\
& -2\eta_{T-1}\Big[q_{T-2}(l_{T-2}-\mathbb{E}[l_{T-2}]) + (q_{T-2}-\mathbb{E}[q_{T-2}])\mathbb{E}[l_{T-2}]\Big] \\
& \times\Big[n_{T-2}(w_{T-2}-\mathbb{E}[w_{T-2}]) + R'_{T-2}(v_{T-2}-\mathbb{E}[v_{T-2}]) \\
& +(R'_{T-2}-\mathbb{E}[R'_{T-2}])\mathbb{E}[v_{T-2}] + (c_{T-2}-\mathbb{E}[c_{T-2}])\Big] \\
& +\epsilon_{T-1}\big[q_{T-2}(l_{T-2}-\mathbb{E}[l_{T-2}]) + (q_{T-2}-\mathbb{E}[q_{T-2}])\mathbb{E}[l_{T-2}]\big]^2 \\
& -\lambda\zeta_{T-1}\big(n_{T-2}\mathbb{E}[w_{T-2}] + \mathbb{E}[R'_{T-2}]\mathbb{E}[v_{T-2}] + \mathbb{E}[c_{T-2}]\big) \\
& +\lambda\theta_{T-1}\mathbb{E}[q_{T-2}]\mathbb{E}[l_{T-2}] + \delta_{T-1}(\mathbb{E}[q_{T-2}]\mathbb{E}[l_{T-2}])^2 + \Delta_{T-1}\Bigg|\mathcal{F}_{T-2}\Bigg] \\
= \;& \xi_{T-1}\Big[n^2_{T-2}\big(w_{T-2}-\mathbb{E}[w_{T-2}]\big)^2 + \big(v_{T-2}-\mathbb{E}[v_{T-2}]\big)'\mathbb{E}[R_{T-2}R'_{T-2}]\big(v_{T-2}-\mathbb{E}[v_{T-2}]\big) \\
& +2n_{T-2}\big(w_{T-2}-\mathbb{E}[w_{T-2}]\big)\mathbb{E}[R'_{T-2}]\big(v_{T-2}-\mathbb{E}[v_{T-2}]\big) \\
& +\mathbb{E}[v'_{T-2}]\big(\mathbb{E}[R_{T-2}R'_{T-2}] - \mathbb{E}[R_{T-2}]\mathbb{E}[R'_{T-2}]\big)\mathbb{E}[v_{T-2}] \\
& +(\mathbb{E}[(c_{T-2})^2] - \mathbb{E}[c_{T-2}]^2) + 2\Big(\mathbb{E}[c_{T-2}R'_{T-2}] - \mathbb{E}[c_{T-2}]\mathbb{E}[R'_{T-2}]\Big)\mathbb{E}[v_{T-2}]\Big] \\
& -2\eta_{T-1}\Big[n_{T-2}\mathbb{E}[q_{T-2}]\big(l_{T-2}-\mathbb{E}[l_{T-2}]\big)\big(w_{T-2}-\mathbb{E}[w_{T-2}]\big) \\
& +\mathbb{E}[q_{T-2}R'_{T-2}]\big(l_{T-2}-\mathbb{E}[l_{T-2}]\big)\big(v_{T-2}-\mathbb{E}[v_{T-2}]\big) \\
& +\big(\mathbb{E}[q_{T-2}R'_{T-2}] - \mathbb{E}[q_{T-2}]\mathbb{E}[R'_{T-2}]\big)\mathbb{E}[v_{T-2}]\mathbb{E}[l_{T-2}] \\
& +\big(\mathbb{E}[q_{T-2}c_{T-2}] - \mathbb{E}[q_{T-2}]\mathbb{E}[c_{T-2}]\big)\mathbb{E}[l_{T-2}]\Big] \\
& +\epsilon_{T-1}\Big[\mathbb{E}[q^2_{T-2}](l_{T-2}-\mathbb{E}[l_{T-2}])^2 + (\mathbb{E}[q^2_{T-2}]-(\mathbb{E}[q_{T-2}])^2)(\mathbb{E}[l_{T-2}])^2\Big] \\
& -\lambda\zeta_{T-1}\big(n_{T-2}\mathbb{E}[w_{T-2}] + \mathbb{E}[R'_{T-2}]\mathbb{E}[v_{T-2}] + \mathbb{E}[c_{T-2}]\big) \\
& +\lambda\theta_{T-1}\mathbb{E}[q_{T-2}]\mathbb{E}[l_{T-2}] + \delta_{T-1}(\mathbb{E}[q_{T-2}]\mathbb{E}[l_{T-2}])^2 + \Delta_{T-1}.
\end{aligned}
\tag{5.41}
$$

The optimal strategies at period $T-2$ can be derived from the above equation

corresponding to v_{T-2}

$$\begin{aligned}
v_{T-2} - \mathbb{E}[v_{T-2}] =& -n_{T-2}(w_{T-2} - \mathbb{E}[w_{T-2}])\mathbb{E}^{-1}[R_{T-2}R'_{T-2}]\mathbb{E}[(R_{T-2})] \\
& +\eta_{T-1}\xi_{T-1}^{-1}\big(l_{T-2} - \mathbb{E}[l_{T-2}]\big)\mathbb{E}^{-1}[R_{T-2}R'_{T-2}]\mathbb{E}[q_{T-2}R_{T-2}], \\
\mathbb{E}[v_{T-2}] =& -\Big(\mathbb{E}[R_{T-2}R'_{T-2}] - \mathbb{E}[R_{T-2}]\mathbb{E}[R'_{T-2}]\Big)^{-1} \\
& \times\Big(\mathbb{E}[c_{T-2}R_{T-2}] - \mathbb{E}[c_{T-2}]\mathbb{E}[R_{T-2}] - \frac{\lambda\zeta_{T-1}}{2\xi_{T-1}}\mathbb{E}[R_{T-2}] \\
& -\frac{\eta_{T-1}}{\xi_{T-1}}\big(\mathbb{E}[q_{T-2}R_{T-2}] - \mathbb{E}[q_{T-2}]\mathbb{E}[R_{T-2}]\big)\mathbb{E}[l_{T-2}]\Big).
\end{aligned} \tag{5.42}$$

Substituting the optimal strategies back to (5.41), we obtain

$$\begin{aligned}
& J_{T-2}(\mathbb{E}[w_{T-2}], w_{T-2} - \mathbb{E}[w_{T-2}], \mathbb{E}[l_{T-2}], l_{T-2} - \mathbb{E}[l_{T-2}]) \\
= \;& \xi_{T-1}(n_{T-2})^2\big(w_{T-2} - \mathbb{E}[w_{T-2}]\big)^2 + \xi_{T-1}(\mathbb{E}[(c_{T-2})^2] - \mathbb{E}[c_{T-2}]^2) \\
& +\epsilon_{T-1}\mathbb{E}[q_{T-2}^2](l_{T-2} - \mathbb{E}[l_{T-2}])^2 + \epsilon_{T-1}\Big(\mathbb{E}[q_{T-2}^2] - \mathbb{E}[q_{T-2}]^2\Big)\mathbb{E}[l_{T-2}]^2 \\
& -2\eta_{T-1}n_{T-2}\mathbb{E}[q_{T-2}](w_{T-2} - \mathbb{E}[w_{T-2}])(l_{T-2} - \mathbb{E}[l_{T-2}]) \\
& -\lambda\zeta_{T-1}n_{T-2}\mathbb{E}[w_{T-2}] - \lambda\zeta_{T-1}\mathbb{E}[c_{T-2}] + \lambda\theta_{T-1}\mathbb{E}[q_{T-2}]\mathbb{E}[l_{T-2}] \\
& +\delta_{T-1}(\mathbb{E}[q_{T-2}]\mathbb{E}[l_{T-2}])^2 + \Delta_{T-1} - \xi_{T-1}(n_{T-2})^2B_{T-2}(w_{T-2} - \mathbb{E}[w_{T-2}])^2 \\
& -\eta_{T-1}^2\xi_{T-1}^{-1}B'_{T-2}(l_{T-2} - \mathbb{E}[l_{T-2}])^2 \\
& +2\eta_{T-1}n_{T-2}\overline{B}_{T-2}(w_{T-2} - \mathbb{E}[w_{T-2}])(l_{T-2} - \mathbb{E}[l_{T-2}]) \\
& -2\eta_{T-1}\big(\mathbb{E}[q_{T-2}c_{T-2}] - \mathbb{E}[q_{T-2}]\mathbb{E}[c_{T-2}]\big)\mathbb{E}[l_{T-2}] \\
& -\xi_{T-1}\Big(\mathbb{E}[c_{T-2}R'_{T-2}] - \mathbb{E}[c_{T-2}]\mathbb{E}[R'_{T-2}] - \frac{\lambda\zeta_{T-1}}{2\xi_{T-1}}\mathbb{E}[R'_{T-2}] - \frac{\eta_{T-1}}{\xi_{T-1}}(\mathbb{E}[q_{T-2}R'_{T-2}] \\
& -\mathbb{E}[q_{T-2}]\mathbb{E}[R'_{T-2}])\mathbb{E}[l_{T-2}]\Big)\Big(\mathbb{E}[R_{T-2}R'_{T-2}] - \mathbb{E}[R_{T-2}]\mathbb{E}[R'_{T-2}]\Big)^{-1} \\
& \times\Big(\mathbb{E}[c_{T-2}R_{T-2}] - \mathbb{E}[c_{T-2}]\mathbb{E}[R_{T-2}] - \frac{\lambda\zeta_{T-1}}{2\xi_{T-1}}\mathbb{E}[R_{T-2}] \\
& -\frac{\eta_{T-1}}{\xi_{T-1}}\big(\mathbb{E}[q_{T-2}R_{T-2}] - \mathbb{E}[q_{T-2}]\mathbb{E}[R_{T-2}]\big)\mathbb{E}[l_{T-2}]\Big).
\end{aligned}$$

Thus,

$$
\begin{aligned}
& J_{T-2}(\mathbb{E}[w_{T-2}], w_{T-2}-\mathbb{E}[w_{T-2}], \mathbb{E}[l_{T-2}], l_{T-2}-\mathbb{E}[l_{T-2}]) \\
= \;& \xi_{T-1}(n_{T-2})^2(1-B_{T-2})\big(w_{T-2}-\mathbb{E}[w_{T-2}]\big)^2 + \xi_{T-1}(\mathbb{E}[(c_{T-2})^2]-\mathbb{E}[c_{T-2}]^2) \\
& +\big(\epsilon_{T-1}\mathbb{E}[q_{T-2}^2] - \eta_{T-1}^2\xi_{T-1}^{-1}B'_{T-2}\big)(l_{T-2}-\mathbb{E}[l_{T-2}])^2 \\
& -2\eta_{T-1}\big(n_{T-2}\mathbb{E}[q_{T-2}] - n_{T-2}\overline{B}_{T-2}\big)(w_{T-2}-\mathbb{E}[w_{T-2}])(l_{T-2}-\mathbb{E}[l_{T-2}]) \\
& +\epsilon_{T-1}\Big(\mathbb{E}[q_{T-2}^2]-\mathbb{E}[q_{T-2}]^2\Big)\mathbb{E}[l_{T-2}]^2 - \lambda\zeta_{T-1}n_{T-2}\mathbb{E}[w_{T-2}] \\
& -\lambda\zeta_{T-1}\mathbb{E}[c_{T-2}] + \lambda\theta_{T-1}\mathbb{E}[q_{T-2}]\mathbb{E}[l_{T-2}] + \delta_{T-1}(\mathbb{E}[q_{T-2}]\mathbb{E}[l_{T-2}])^2 \\
& -2\eta_{T-1}\big(\mathbb{E}[q_{T-2}c_{T-2}]-\mathbb{E}[q_{T-2}]\mathbb{E}[c_{T-2}]\big)\mathbb{E}[l_{T-2}] + \Delta_{T-1} \\
& -\xi_{T-1}\Big(\big(\mathbb{E}[c_{T-2}]+\frac{\lambda\zeta_{T-1}}{2\xi_{T-1}}-\widehat{B}_{T-2}\big)^2\frac{B_{T-2}}{1-B_{T-2}} \\
& -2\big(\mathbb{E}[c_{T-2}]+\frac{\lambda\zeta_{T-1}}{2\xi_{T-1}}-\widehat{B}_{T-2}\big)\widehat{B}_{T-2}+\widetilde{B}_{T-2}-\widehat{B}_{T-2}^2\Big) \\
& +2\eta_{T-1}\Big(\frac{\big(\mathbb{E}[c_{T-2}]+\frac{\lambda\zeta_{T-1}}{2\xi_{T-1}}\big)\big(\mathbb{E}[q_{T-2}]B_{T-2}-\overline{B}_{T-2}\big)+\widehat{B}_{T-2}\overline{B}_{T-2}-\widehat{B}_{T-2}\mathbb{E}[q_{T-2}]}{1-B_{T-2}} \\
& +\overrightarrow{B}_{T-2}\Big)\mathbb{E}[l_{T-2}] - \frac{\eta_{T-1}^2}{\xi_{T-1}}\Big(B'_{T-2}-\mathbb{E}[q_{T-2}]^2+\frac{\big(\overline{B}_{T-2}-\mathbb{E}[q_{T-2}]\big)^2}{1-B_{T-2}}\Big)\mathbb{E}[l_{T-2}]^2 \\
= \;& \xi_{T-2}\big(w_{T-2}-\mathbb{E}[w_{T-2}]\big)^2 - 2\eta_{T-2}(w_{T-2}-\mathbb{E}[w_{T-2}])(l_{T-2}-\mathbb{E}[l_{T-2}]) \\
& +\epsilon_{T-2}(l_{T-2}-\mathbb{E}[l_{T-2}])^2 - \lambda\zeta_{T-2}\mathbb{E}[w_{T-2}] + \lambda\theta_{T-2}\mathbb{E}[l_{T-2}] + \delta_{T-2}\mathbb{E}[l_{T-2}]^2 + \Delta_{T-2},
\end{aligned}
\tag{5.43}
$$

where

$$
\begin{aligned}
\xi_{T-2} = \quad & \xi_{T-1}(n_{T-2})^2(1-B_{T-2}),\\
\eta_{T-2} = \quad & \eta_{T-1}\big(n_{T-2}\mathbb{E}[q_{T-2}] - n_{T-2}\overline{B}_{T-2}\big),\\
\epsilon_{T-2} = \quad & \epsilon_{T-1}\mathbb{E}[q_{T-2}^2] - \eta_{T-1}^2\xi_{T-1}^{-1}B'_{T-2},\\
\zeta_{T-2} = \quad & \zeta_{T-1}n_{T-2},\\
\theta_{T-2} = \quad & \theta_{T-1}\mathbb{E}[q_{T-2}] - \frac{2}{\lambda}\eta_{T-1}\big(\mathbb{E}[q_{T-2}c_{T-2}] - \mathbb{E}[q_{T-2}]\mathbb{E}[c_{T-2}]\big) + \frac{2\eta_{T-1}}{\lambda}\\
& \times\Big(\frac{\big(\mathbb{E}[c_{T-2}] + \frac{\lambda\zeta_{T-1}}{2\xi_{T-1}}\big)\big(\mathbb{E}[q_{T-2}]B_{T-2} - \overline{B}_{T-2}\big) + \widehat{B}_{T-2}\overline{B}_{T-2} - \widehat{B}_{T-2}\mathbb{E}[q_{T-2}]}{1-B_{T-2}}\\
& + \overrightarrow{B}_{T-2}\Big),\\
\delta_{T-2} = \quad & \epsilon_{T-1}\Big(\mathbb{E}[q_{T-2}^2] - \mathbb{E}[q_{T-2}]^2\Big) + \delta_{T-1}(\mathbb{E}[q_{T-2}])^2\\
& - \frac{\eta_{T-1}^2}{\xi_{T-1}}\Big({B'}_{T-2} - \mathbb{E}[q_{T-2}]^2 + \frac{\big(\overline{B}_{T-2} - \mathbb{E}[q_{T-2}]\big)^2}{1-B_{T-2}}\Big),\\
\Delta_{T-2} = \quad & \Delta_{T-1} + \xi_{T-1}\mathbb{E}[(c_{T-2})^2] - \xi_{T-1}\mathbb{E}[c_{T-2}]^2 - \lambda\zeta_{T-1}\mathbb{E}[c_{T-2}]\\
& - \xi_{T-1}\Big(\big(\mathbb{E}[c_{T-2}] + \frac{\lambda\zeta_{T-1}}{2\xi_{T-1}} - \widehat{B}_{T-2}\big)^2\frac{B_{T-2}}{1-B_{T-2}}\\
& - 2\big(\mathbb{E}[c_{T-2}] + \frac{\lambda\zeta_{T-1}}{2\xi_{T-1}} - \widehat{B}_{T-2}\big)\widehat{B}_{T-2} + \widetilde{B}_{T-2} - \widehat{B}_{T-2}^2\Big).
\end{aligned}
\tag{5.44}
$$

Assume that the following equation (5.45) holds at time $t+1$, we prove it according to the mathematical induction.

$$
\begin{aligned}
& J_{t+1}(\mathbb{E}[w_{t+1}], w_{t+1} - \mathbb{E}[w_{t+1}], \mathbb{E}[l_{t+1}], l_{t+1} - \mathbb{E}[l_{t+1}])\\
= \quad & \xi_{t+1}\big(w_{t+1} - \mathbb{E}[w_{t+1}]\big)^2 - 2\eta_{t+1}(w_{t+1} - \mathbb{E}[w_{t+1}])(l_{t+1} - \mathbb{E}[l_{t+1}])\\
& + \epsilon_{t+1}(l_{t+1} - \mathbb{E}[l_{t+1}])^2 - \lambda\zeta_{t+1}\mathbb{E}[w_{t+1}] + \lambda\theta_{t+1}\mathbb{E}[l_{t+1}] + \delta_{t+1}\mathbb{E}[l_{t+1}]^2 + \Delta_{t+1},
\end{aligned}
\tag{5.45}
$$

where

$$
\begin{aligned}
\xi_{t+1} = &\ \ \xi_{t+2}(n_{t+1})^2(1-B_{t+1}),\\
\eta_{t+1} = &\ \ \eta_{t+2}\big(n_{t+1}\mathbb{E}[q_{t+1}] - n_{t+1}\overline{B}_{t+1}\big),\\
\epsilon_{t+1} = &\ \ \epsilon_{t+2}\mathbb{E}[q_{t+1}^2] - \eta_{t+2}^2\xi_{t+2}^{-1}B'_{t+1},\\
\zeta_{t+1} = &\ \ \zeta_{t+2}n_{t+1},\\
\theta_{t+1} = &\ \ \theta_{t+2}\mathbb{E}[q_{t+1}] - \frac{2}{\lambda}\eta_{t+2}\big(\mathbb{E}[q_{t+1}c_{t+1}] - \mathbb{E}[q_{t+1}]\mathbb{E}[c_{t+1}]\big) + \frac{2\eta_{t+2}}{\lambda}\\
&\times\Big(\frac{\big(\mathbb{E}[c_{t+1}] + \frac{\lambda\zeta_{t+2}}{2\xi_{t+2}}\big)\big(\mathbb{E}[q_{t+1}]B_{t+1} - \overline{B}_{t+1}\big) + \widehat{B}_{t+1}\overline{B}_{t+1} - \widehat{B}_{t+1}\mathbb{E}[q_{t+1}]}{1-B_{t+1}}\\
&+\overrightarrow{B}_{t+1}\Big),\\
\delta_{t+1} = &\ \ \epsilon_{t+2}\Big(\mathbb{E}[q_{t+1}^2] - \mathbb{E}[q_{t+1}]^2\Big) + \delta_{t+2}(\mathbb{E}[q_{t+1}])^2\\
&-\frac{\eta_{t+2}^2}{\xi_{t+2}}\Big({B'}_{t+1} - \mathbb{E}[q_{t+1}]^2 + \frac{\big(\overline{B}_{t+1} - \mathbb{E}[q_{t+1}]\big)^2}{1-B_{t+1}}\Big),\\
\Delta_{t+1} = &\ \ \Delta_{t+2} + \xi_{t+2}\mathbb{E}[(c_{t+1})^2] - \xi_{t+2}\mathbb{E}[c_{t+1}]^2 - \lambda\zeta_{t+2}\mathbb{E}[c_{t+1}]\\
&-\xi_{t+2}\Big(\big(\mathbb{E}[c_{t+1}] + \frac{\lambda\zeta_{t+2}}{2\xi_{t+2}} - \widehat{B}_{t+1}\big)^2\frac{B_{t+1}}{1-B_{t+1}}\\
&-2\big(\mathbb{E}[c_{t+1}] + \frac{\lambda\zeta_{t+2}}{2\xi_{t+2}} - \widehat{B}_{t+1}\big)\widehat{B}_{t+1} + \widetilde{B}_{t+1} - \widehat{B}_{t+1}^2\Big),
\end{aligned}
\tag{5.46}
$$

and

$$\xi_T = 1,\ \eta_T = 1,\ \epsilon_T = 1,\ \zeta_T = 1,\ \theta_T = 1,\ \delta_T = 0,\ \Delta_T = 0.$$

According to equations (5.45) and (5.46), we derive the result at time t.

$$
\begin{aligned}
& J_t(\mathbb{E}[w_t], w_t - \mathbb{E}[w_t], \mathbb{E}[l_t], l_t - \mathbb{E}[l_t]) \\
= \;& \mathbb{E}\big[J_{t+1}(\mathbb{E}[w_{t+1}], w_{t+1} - \mathbb{E}[w_{t+1}], \mathbb{E}[l_{t+1}], l_{t+1} - \mathbb{E}[l_{t+1}])|\mathcal{F}_t\big] \\
= \;& \mathbb{E}\big[\xi_{t+1}\big(w_{t+1} - \mathbb{E}[w_{t+1}]\big)^2 - 2\eta_{t+1}(w_{t+1} - \mathbb{E}[w_{t+1}])(l_{t+1} - \mathbb{E}[l_{t+1}]) \\
& +\epsilon_{t+1}(l_{t+1} - \mathbb{E}[l_{t+1}])^2 - \lambda\zeta_{t+1}\mathbb{E}[w_{t+1}] + \lambda\theta_{t+1}\mathbb{E}[l_{t+1}] + \delta_{t+1}\mathbb{E}[l_{t+1}]^2 \\
& +\Delta_{t+1}|\mathcal{F}_t\big] \\
= \;& \mathbb{E}\Bigg[\xi_{t+1}\Big[n_t(w_t - \mathbb{E}[w_t]) + R_t'(v_t - \mathbb{E}[v_t]) + (R_t' - \mathbb{E}[R_t'])\mathbb{E}[v_t] + (c_t - \mathbb{E}[c_t])\Big]^2 \\
& -2\eta_{t+1}\Big[q_t(l_t - \mathbb{E}[l_t]) + (q_t - \mathbb{E}[q_t])\mathbb{E}[l_t]\Big]\Big[n_t(w_t - \mathbb{E}[w_t]) + R_t'(v_t - \mathbb{E}[v_t]) \\
& +(R_t' - \mathbb{E}[R_t'])\mathbb{E}[v_t] + (c_t - \mathbb{E}[c_t])\Big] + \epsilon_{t+1}\Big[q_t(l_t - \mathbb{E}[l_t]) + (q_t - \mathbb{E}[q_t])\mathbb{E}[l_t]\Big]^2 \\
& -\lambda\zeta_{t+1}\big(n_t\mathbb{E}[w_t] + \mathbb{E}[R_t']\mathbb{E}[v_t] + \mathbb{E}[c_t]\big) + \lambda\theta_{t+1}\mathbb{E}[q_t]\mathbb{E}[l_t] + \delta_{t+1}(\mathbb{E}[q_t]\mathbb{E}[l_t])^2 \\
& +\Delta_{t+1}\Bigg|\mathcal{F}_t\Bigg] \\
= \;& \xi_{t+1}\Big[n_t^2\big(w_t - \mathbb{E}[w_t]\big)^2 + \big(v_t - \mathbb{E}[v_t]\big)'\mathbb{E}[R_tR_t']\big(v_t - \mathbb{E}[v_t]\big) \\
& +2n_t\big(w_t - \mathbb{E}[w_t]\big)\mathbb{E}[R_t']\big(v_t - \mathbb{E}[v_t]\big) + \mathbb{E}[v_t']\big(\mathbb{E}[R_tR_t'] - \mathbb{E}[R_t]\mathbb{E}[R_t']\big)\mathbb{E}[v_t] \\
& +(\mathbb{E}[(c_t)^2] - \mathbb{E}[c_t]^2) + 2\big(\mathbb{E}[c_tR_t'] - \mathbb{E}[c_t]\mathbb{E}[R_t']\big)\mathbb{E}[v_t]\Big] \\
& -2\eta_{t+1}\Big[n_t\mathbb{E}[q_t]\big(l_t - \mathbb{E}[l_t]\big)\big(w_t - \mathbb{E}[w_t]\big) + \mathbb{E}[q_tR_t']\big(l_t - \mathbb{E}[l_t]\big)\big(v_t - \mathbb{E}[v_t]\big) \\
& +\big(\mathbb{E}[q_tR_t'] - \mathbb{E}[q_t]\mathbb{E}[R_t']\big)\mathbb{E}[v_t]\mathbb{E}[l_t] + \big(\mathbb{E}[q_tc_t] - \mathbb{E}[q_t]\mathbb{E}[c_t]\big)\mathbb{E}[l_t]\Big] \\
& +\epsilon_{t+1}\Big[\mathbb{E}[q_t^2](l_t - \mathbb{E}[l_t])^2 + (\mathbb{E}[q_t^2] - (\mathbb{E}[q_t])^2)(\mathbb{E}[l_t])^2\Big] \\
& -\lambda\zeta_{t+1}\big(n_t\mathbb{E}[w_t] + \mathbb{E}[R_t']\mathbb{E}[v_t] + \mathbb{E}[c_t]\big) + \lambda\theta_{t+1}\mathbb{E}[q_t]\mathbb{E}[l_t] + \delta_{t+1}(\mathbb{E}[q_t]\mathbb{E}[l_t])^2 \\
& +\Delta_{t+1}.
\end{aligned}
\tag{5.47}
$$

The optimal strategies at period t can be derived from the above equation corresponding to v_t

$$\begin{aligned}
\pi_t^* - \mathbb{E}[\pi_t^*] =& -n_t(w_t - \mathbb{E}[w_t])\mathbb{E}^{-1}[R_tR_t']\mathbb{E}[(R_t)] \\
&+\eta_{t+1}\xi_{t+1}^{-1}\big(l_t - \mathbb{E}[l_t]\big)\mathbb{E}^{-1}[R_tR_t']\mathbb{E}[q_tR_t], \\
\mathbb{E}[\pi_t^*] =& -\Big(\mathbb{E}[R_tR_t'] - \mathbb{E}[R_t]\mathbb{E}[R_t']\Big)^{-1}\Big(\mathbb{E}[c_tR_t] - \mathbb{E}[c_t]\mathbb{E}[R_t] \\
&-\frac{\lambda\zeta_{t+1}}{2\xi_{t+1}}\mathbb{E}[R_t] - \frac{\eta_{t+1}}{\xi_{t+1}}\big(\mathbb{E}[q_tR_t] - \mathbb{E}[q_t]\mathbb{E}[R_t]\big)\mathbb{E}[l_t]\Big) \\
=& \Big(\mathbb{E}[c_t] + \frac{\lambda\zeta_{t+1}}{2\xi_{t+1}} - \widehat{B}_t + \frac{\eta_{t+1}\big(\overline{B}_t - \mathbb{E}[q_t]\big)\mathbb{E}[l_t]}{\xi_{t+1}}\Big)\frac{\mathbb{E}^{-1}[R_tR_t']\mathbb{E}[R_t]}{1-B_t} \\
&-\mathbb{E}^{-1}[R_tR_t']\mathbb{E}[c_tR_t] + \frac{\eta_{t+1}}{\xi_{t+1}}\mathbb{E}^{-1}[R_tR_t']\mathbb{E}[R_tq_t]\mathbb{E}[l_t].
\end{aligned} \tag{5.48}$$

Substituting the optimal strategies back to (5.47), we obtain

$$\begin{aligned}
& J_t(\mathbb{E}[w_t], w_t - \mathbb{E}[w_t], \mathbb{E}[l_t], l_t - \mathbb{E}[l_t]) \\
= \; & \xi_{t+1}(n_t)^2\big(w_t - \mathbb{E}[w_t]\big)^2 + \xi_{t+1}(\mathbb{E}[(c_t)^2] - \mathbb{E}[c_t]^2) \\
& +\epsilon_{t+1}\mathbb{E}[q_t^2](l_t - \mathbb{E}[l_t])^2 + \epsilon_{t+1}\Big(\mathbb{E}[q_t^2] - \mathbb{E}[q_t]^2\Big)\mathbb{E}[l_t]^2 \\
& -2\eta_{t+1}n_t\mathbb{E}[q_t](w_t - \mathbb{E}[w_t])(l_t - \mathbb{E}[l_t]) \\
& -\lambda\zeta_{t+1}n_t\mathbb{E}[w_t] - \lambda\zeta_{t+1}\mathbb{E}[c_t] + \lambda\theta_{t+1}\mathbb{E}[q_t]\mathbb{E}[l_t] + \delta_{t+1}(\mathbb{E}[q_t]\mathbb{E}[l_t])^2 + \Delta_{t+1} \\
& -\xi_{t+1}(n_t)^2B_t(w_t - \mathbb{E}[w_t])^2 - \eta_{t+1}^2\xi_{t+1}^{-1}B_t'(l_t - \mathbb{E}[l_t])^2 \\
& +2\eta_{t+1}n_t\overline{B}_t(w_t - \mathbb{E}[w_t])(l_t - \mathbb{E}[l_t]) \\
& -2\eta_{t+1}\big(\mathbb{E}[q_tc_t] - \mathbb{E}[q_t]\mathbb{E}[c_t]\big)\mathbb{E}[l_t] \\
& -\xi_{t+1}\Big(\mathbb{E}[c_tR_t'] - \mathbb{E}[c_t]\mathbb{E}[R_t'] - \frac{\lambda\zeta_{t+1}}{2\xi_{t+1}}\mathbb{E}[R_t'] - \frac{\eta_{t+1}}{\xi_{t+1}}(\mathbb{E}[q_tR_t'] - \mathbb{E}[q_t]\mathbb{E}[R_t'])\mathbb{E}[l_t]\Big) \\
& \times\Big(\mathbb{E}[R_tR_t'] - \mathbb{E}[R_t]\mathbb{E}[R_t']\Big)^{-1} \\
& \times\Big(\mathbb{E}[c_tR_t] - \mathbb{E}[c_t]\mathbb{E}[R_t] - \frac{\lambda\zeta_{t+1}}{2\xi_{t+1}}\mathbb{E}[R_t] - \frac{\eta_{t+1}}{\xi_{t+1}}\big(\mathbb{E}[q_tR_t] - \mathbb{E}[q_t]\mathbb{E}[R_t]\big)\mathbb{E}[l_t]\Big).
\end{aligned}$$

Thus,

$$
\begin{aligned}
& J_t(\mathbb{E}[w_t], w_t - \mathbb{E}[w_t], \mathbb{E}[l_t], l_t - \mathbb{E}[l_t]) \\
= \;& \xi_{t+1}(n_t)^2(1-B_t)\big(w_t - \mathbb{E}[w_t]\big)^2 + \xi_{t+1}(\mathbb{E}[(c_t)^2] - \mathbb{E}[c_t]^2) \\
& + \big(\epsilon_{t+1}\mathbb{E}[q_t^2] - \eta_{t+1}^2 \xi_{t+1}^{-1} B_t'\big)(l_t - \mathbb{E}[l_t])^2 \\
& -2\eta_{t+1}\big(n_t\mathbb{E}[q_t] - n_t\overline{B}_t\big)(w_t - \mathbb{E}[w_t])(l_t - \mathbb{E}[l_t]) \\
& + \epsilon_{t+1}\Big(\mathbb{E}[q_t^2] - \mathbb{E}[q_t]^2\Big)\mathbb{E}[l_t]^2 - \lambda\zeta_{t+1}n_t\mathbb{E}[w_t] - \lambda\zeta_{t+1}\mathbb{E}[c_t] + \lambda\theta_{t+1}\mathbb{E}[q_t]\mathbb{E}[l_t] \\
& + \delta_{t+1}(\mathbb{E}[q_t]\mathbb{E}[l_t])^2 - 2\eta_{t+1}\big(\mathbb{E}[q_t c_t] - \mathbb{E}[q_t]\mathbb{E}[c_t]\big)\mathbb{E}[l_t] + \Delta_{t+1} \\
& - \xi_{t+1}\Big(\big(\mathbb{E}[c_t] + \frac{\lambda\zeta_{t+1}}{2\xi_{t+1}} - \widehat{B}_t\big)^2 \frac{B_t}{1-B_t} - 2\big(\mathbb{E}[c_t] + \frac{\lambda\zeta_{t+1}}{2\xi_{t+1}} - \widehat{B}_t\big)\widehat{B}_t + \widetilde{B}_t - \widehat{B}_t^2\Big) \\
& + 2\eta_{t+1}\Big(\frac{\big(\mathbb{E}[c_t] + \frac{\lambda\zeta_{t+1}}{2\xi_{t+1}}\big)\big(\mathbb{E}[q_t]B_t - \overline{B}_t\big) + \widehat{B}_t\overline{B}_t - \widehat{B}_t\mathbb{E}[q_t]}{1-B_t} + \overrightarrow{B}_t\Big)\mathbb{E}[l_t] \\
& - \frac{\eta_{t+1}^2}{\xi_{t+1}}\Big(B'_t - \mathbb{E}[q_t]^2 + \frac{\big(\overline{B}_t - \mathbb{E}[q_t]\big)^2}{1-B_t}\Big)\mathbb{E}[l_t]^2 \\
= \;& \xi_t\big(w_t - \mathbb{E}[w_t]\big)^2 - 2\eta_t(w_t - \mathbb{E}[w_t])(l_t - \mathbb{E}[l_t]) + \epsilon_t(l_t - \mathbb{E}[l_t])^2 \\
& - \lambda\zeta_t\mathbb{E}[w_t] + \lambda\theta_t\mathbb{E}[l_t] + \delta_t\mathbb{E}[l_t]^2 + \Delta_t,
\end{aligned}
\tag{5.49}
$$

where

$$
\begin{aligned}
\xi_t = &\ \ \xi_{t+1}(n_t)^2(1-B_t),\\
\eta_t = &\ \ \eta_{t+1}\big(n_t\mathbb{E}[q_t]-n_t\overline{B}_t\big),\\
\epsilon_t = &\ \ \epsilon_{t+1}\mathbb{E}[q_t^2]-\eta_{t+1}^2\xi_{t+1}^{-1}B_t',\\
\zeta_t = &\ \ \zeta_{t+1}n_t,\\
\theta_t = &\ \ \theta_{t+1}\mathbb{E}[q_t]-\frac{2}{\lambda}\eta_{t+1}\big(\mathbb{E}[q_tc_t]-\mathbb{E}[q_t]\mathbb{E}[c_t]\big)\\
&+\frac{2\eta_{t+1}}{\lambda}\Big(\frac{\big(\mathbb{E}[c_t]+\frac{\lambda\zeta_{t+1}}{2\xi_{t+1}}\big)\big(\mathbb{E}[q_t]B_t-\overline{B}_t\big)+\widehat{B}_t\overline{B}_t-\widehat{B}_t\mathbb{E}[q_t]}{1-B_t}+\overrightarrow{B}_t\Big),\\
\delta_t = &\ \ \epsilon_{t+1}\Big(\mathbb{E}[q_t^2]-\mathbb{E}[q_t]^2\Big)+\delta_{t+1}(\mathbb{E}[q_t])^2-\frac{\eta_{t+1}^2}{\xi_{t+1}}\Big({B'}_t-\mathbb{E}[q_t]^2+\frac{\big(\overline{B}_t-\mathbb{E}[q_t]\big)^2}{1-B_t}\Big),\\
\Delta_t = &\ \ \Delta_{t+1}+\xi_{t+1}\mathbb{E}[(c_t)^2]-\xi_{t+1}\mathbb{E}[c_t]^2-\lambda\zeta_{t+1}\mathbb{E}[c_t]\\
&-\xi_{t+1}\Big(\big(\mathbb{E}[c_t]+\frac{\lambda\zeta_{t+1}}{2\xi_{t+1}}-\widehat{B}_t\big)^2\frac{B_t}{1-B_t}-2\big(\mathbb{E}[c_t]+\frac{\lambda\zeta_{t+1}}{2\xi_{t+1}}-\widehat{B}_t\big)\widehat{B}_t+\widetilde{B}_t-\widehat{B}_t^2\Big).
\end{aligned}
\tag{5.50}
$$

Substituting $\mathbb{E}[v_t^*]$ to the dynamics eqation in (5.30) yields

$$
\mathbb{E}[w_{t+1}]=n_t\mathbb{E}[w_t]+\Big(\mathbb{E}[c_t]+\frac{\lambda\zeta_{t+1}}{2\xi_{t+1}}-\widehat{B}_t-\mathbb{E}[q_t]+\overline{B}_t\Big)\frac{B_t}{1-B_t}-\widehat{B}_t+\overline{B}_t+\mathbb{E}[c_t],
\tag{5.51}
$$

which implies

$$
\begin{aligned}
\mathbb{E}[w_t]=w_0\prod_{k=0}^{t-1}n_k+\sum_{j=0}^{t-1}\Big(&\big(\mathbb{E}[c_j]+\frac{\lambda\zeta_{j+1}}{2\xi_{j+1}}-\widehat{B}_j-\mathbb{E}[q_j]+\overline{B}_j\big)\frac{B_j}{1-B_j}\\
&-\widehat{B}_j+\overline{B}_j+\mathbb{E}[c_j]\Big)\prod_{l=j+1}^{t-1}n_l.
\end{aligned}
\tag{5.52}
$$

Typically, we know the optimal value of (5.28) is equal to J_0. Thus,

$$
\begin{aligned}
& \mathrm{Var}(w_T) \\
= \; & J_0(\mathbb{E}[w_0], w_0 - \mathbb{E}[w_0], \mathbb{E}[l_0], l_0 - \mathbb{E}[l_0]) + \lambda \mathbb{E}[w_T] \\
= \; & -\lambda\zeta_0 w_0 + \lambda\theta_0 l_0 + \delta_0 l_0^2 + \Delta_0 + \lambda\Bigg(w_0 \prod_{k=0}^{T-1} n_k \\
& + \sum_{j=0}^{T-1} \Bigg((\mathbb{E}[c_j] + \frac{\lambda\zeta_{j+1}}{2\xi_{j+1}} - \widehat{B}_j - \mathbb{E}[q_j] + \overline{B}_j) \frac{B_j}{1 - B_j} \\
& - \widehat{B}_j + \overline{B}_j + \mathbb{E}[c_j] \Bigg) \prod_{l=j+1}^{T-1} n_l \Bigg).
\end{aligned}
\tag{5.53}
$$

5.3.2 Numerical Examples

Similar to the uncorrelation case, Table 5.2 gives the expected return, standard deviation and the correlation coefficients among three assets, cash flow and liability. We assume the risk-free return rate is defined as 5% ($n_t = 1.05$).

Table 5.2: Data for assets, cashflow and liability with correlation

	SP	EM	MS	cashflow	liability
Expected return	14%	16%	17%	1	10%
Standard deviation	18.5%	30%	24%	20%	20%
Correlation coefficient					
SP	1	0.64	0.79	ρ_1	$\widehat{\rho}_1$
EM	0.64	1	0.75	ρ_2	$\widehat{\rho}_2$
MS	0.79	0.75	1	ρ_3	$\widehat{\rho}_3$
cashflow	ρ_1	ρ_2	ρ_3	1	$\widehat{\rho}_4$
liability	$\widehat{\rho}_1$	$\widehat{\rho}_2$	$\widehat{\rho}_3$	$\widehat{\rho}_4$	1

Thus,

$$\mathbb{E}[R_t] = \begin{pmatrix} 0.09 \\ 0.11 \\ 0.12 \end{pmatrix}, \ \mathrm{Cov}(R_t) = \begin{pmatrix} 0.0342 & 0.0355 & 0.0351 \\ 0.0355 & 0.0900 & 0.0540 \\ 0.0351 & 0.0540 & 0.0576 \end{pmatrix},$$

$$\mathbb{E}[R_t R_t'] = \begin{pmatrix} 0.0423 & 0.0454 & 0.0459 \\ 0.0454 & 0.1021 & 0.0672 \\ 0.0459 & 0.0672 & 0.0720 \end{pmatrix}.$$

The correlation coefficient between cash flow and i-th asset are defined as $\rho = (\rho_1, \rho_2, \rho_3)$,

$$\rho_i = \frac{\mathrm{Cov}(c_t, R_t^i)}{\sqrt{\mathrm{Var}(c_t)}\sqrt{\mathrm{Var}(R_t^i)}}.$$

Therefore, we have the following eqation

$$\mathbb{E}[c_t R_t^i] = \mathbb{E}[c_t]\mathbb{E}[R_t^i] + \rho_i \sqrt{\mathrm{Var}(c_t)}\sqrt{\mathrm{Var}(R_t^i)}.$$

The correlation coefficient between liability and i-th asset are defined as $\widehat{\rho} = (\widehat{\rho_1}, \widehat{\rho_2}, \widehat{\rho_3})$,

$$\widehat{\rho_i} = \frac{\mathrm{Cov}(q_t, R_t^i)}{\sqrt{\mathrm{Var}(q_t)}\sqrt{\mathrm{Var}(R_t^i)}}.$$

Therefore, we have

$$\mathbb{E}[q_t R_t^i] = \mathbb{E}[q_t]\mathbb{E}[R_t^i] + \widehat{\rho_i} \sqrt{\mathrm{Var}(q_t)}\sqrt{\mathrm{Var}(R_t^i)}.$$

Typically, $\widehat{\rho_4}$ is defined as

$$\widehat{\rho_4} = \frac{\mathrm{Cov}(c_t, q_t)}{\sqrt{\mathrm{Var}(c_t)}\sqrt{\mathrm{Var}(q_t)}}.$$

Thus,

$$\mathbb{E}[c_t q_t] = \mathbb{E}[c_t]\mathbb{E}[q_t] + \widehat{\rho_4} \sqrt{\mathrm{Var}(c_t)}\sqrt{\mathrm{Var}(q_t)},$$

$$\mathbb{E}[c_t^2] = \mathbb{E}[c_t]^2 + \mathrm{Var}(c_t),$$

$$\mathbb{E}[q_t^2] = \mathbb{E}[q_t]^2 + \mathrm{Var}(q_t).$$

Assume that $\rho = (\rho_1, \rho_2, \rho_3) = (-0.25, 0.5, 0.25)$, $\widehat{\rho} = (\widehat{\rho_1}, \widehat{\rho_2}, \widehat{\rho_3}) = (-0.2, 0.4, 0.3)$ and $\widehat{\rho_4} = 0.1$. Thus,

$$\begin{aligned}
\mathrm{Cov}\left(\begin{pmatrix} R_t \\ c_t \end{pmatrix}\right) &= \begin{pmatrix} \mathrm{Cov}(R_t) & \mathrm{Cov}(c_t, R_t) \\ \mathrm{Cov}(c_t, R_t') & \mathrm{Var}(c_t) \end{pmatrix} \\
&= \begin{pmatrix} 0.0342 & 0.0355 & 0.0351 & -0.0092 \\ 0.0355 & 0.0900 & 0.0540 & 0.0300 \\ 0.0351 & 0.0540 & 0.0576 & 0.0120 \\ -0.0092 & 0.0300 & 0.0120 & 0.0400 \end{pmatrix} \\
&\succ 0.
\end{aligned}$$

In order to make the equations more clearly, we define some new notation as

$$Y_1 = \mathbb{E}^{-1}[R_t R_t']\mathbb{E}[R_t] = \begin{pmatrix} 1.0589 \\ -0.1196 \\ 1.1033 \end{pmatrix},$$

$$Y_2 = \mathbb{E}^{-1}[R_t R_t']\mathbb{E}[c_t R_t] = \begin{pmatrix} -0.3490 \\ 0.4493 \\ 1.6365 \end{pmatrix},$$

$$Y_3 = \mathbb{E}^{-1}[R_t R_t']\mathbb{E}[q_t R_t] = \begin{pmatrix} -1.0411 \\ 0.1754 \\ 0.8209 \end{pmatrix}.$$

$$\begin{aligned}
B_t = 0.2145, \quad \widehat{B}_t = 0.2144, \quad \widetilde{B}_t = 0.2507, \\
\overline{B}_t = 0.0241, \quad B'_t = 0.0218, \quad \overrightarrow{B}_t = 0.0488.
\end{aligned} \tag{5.54}$$

We assume the investor will exit the market at period $t = 5$. Therefore we define the optimal expected value as

$$\mathbb{E}[w] = (\mathbb{E}[w_1], \mathbb{E}[w_2], \mathbb{E}[w_3], \mathbb{E}[w_4], \mathbb{E}[w_5])$$

Substituting the data

$$\mathbb{E}[w] = (4.4487, 5.9181, 7.4184, 8.9584, 10.5464).$$

The initial wealth are defined as $w_0 = 3$ while initial liability $l_0 = 1$ and trade-off parameter $\lambda = 1$. Thus, we substitute the data to the optimal equation

$$
\begin{aligned}
v_0^* &= -1.05(w_0 - 5.2629)Y_1 - Y_2 + 0.000075Y_3l_0, \\
v_1^* &= -1.05(w_1 - 6.4820)Y_1 - Y_2 + 0.0008Y_3l_1, \\
v_2^* &= -1.05(w_2 - 7.7620)Y_1 - Y_2 + 0.0085Y_3l_2, \\
v_3^* &= -1.05(w_3 - 9.1060)Y_1 - Y_2 + 0.0920Y_3l_3, \\
v_4^* &= -1.05(w_4 - 10.5172)Y_1 - Y_2 + 1.0000Y_3l_4.
\end{aligned}
$$

The optimal variance at terminal time period is $\mathrm{Var}(w_5) = 0.6182$.

Chapter 6

Multi-Period Mean-Variance Asset-Liability Management with Cash Flow under an Uncertain Exit Time

Based on the framework of mean-variance model with mean-field formulation in Chapter 4, we directly assume that the return rates between assets and cash flow are correlated. Then we investigate the case when all the additional conditions mentioned above are taken into account. We derive the analytical form of optimal strategies and optimal value in expression for the new model. Finally, the numerical examples in both cases are given to illustrate the results established in this work.

6.1 The Model

We define the similar probability mass function under the construction of Chapter 4. Thus, we introduce the following multi-period mean-variance asset-liability model with cash flow and uncertain exit time.

$$\begin{cases} \min & \mathrm{Var}^{(\tau)}(w_{T\wedge\tau} - l_{T\wedge\tau}) - \lambda \mathbb{E}^{(\tau)}[w_{T\wedge\tau} - l_{T\wedge\tau}], \\ \text{s.t.} & w_{t+1} = \displaystyle\sum_{i=1}^{n} m_t^i v_t^i + \left(w_t - \sum_{i=1}^{n} v_t^i\right) n_t + c_t \\ & \quad\;\; = n_t w_t + R_t' v_t + c_t, \\ & l_{t+1} = q_t l_t, \quad \text{for } t = 0, 1, \cdots, T-1, \end{cases} \tag{6.1}$$

where $\lambda > 0$ represents the risk aversion. We define $\mathbb{E}^{(\tau)}[w_{T\wedge\tau}]$ and $\mathrm{Var}^{(\tau)}(w_{T\wedge\tau})$ as follows,

$$\begin{aligned} \mathbb{E}^{(\tau)}[w_{T\wedge\tau} - l_{T\wedge\tau}] &\triangleq \sum_{t=1}^{T} \mathbb{E}\big[w_{T\wedge\tau} - l_{T\wedge\tau} \big| T\wedge\tau = t\big] \Pr\{T\wedge\tau = t\} \\ &= \sum_{t=1}^{T} \mathbb{E}[w_t - l_t] p_t, \\ \mathrm{Var}^{(\tau)}(w_{T\wedge\tau} - l_{T\wedge\tau}) &\triangleq \sum_{t=1}^{T} \mathrm{Var}\big(w_{T\wedge\tau} - l_{T\wedge\tau} \big| T\wedge\tau = t\big) \Pr\{T\wedge\tau = t\} \\ &= \sum_{t=1}^{T} \mathrm{Var}(w_t - l_t) p_t. \end{aligned}$$

Then we can rewrite the model as ,

$$\begin{cases} \min & \displaystyle\sum_{t=1}^{T} p_t \Big\{ \mathrm{Var}(w_t - l_t) - \lambda \mathbb{E}[w_t - l_t] \Big\}, \\ \text{s.t.} & w_{t+1} = n_t w_t + R_t' v_t + c_t, \\ & l_{t+1} = q_t l_t, \quad \text{for } t = 0, 1, \cdots, T-1. \end{cases} \tag{6.2}$$

6.2 The Mean-Field Formulation

Similar to the construction we mentioned in Section 5.1, we have the following dynamic equation

$$\begin{cases} \mathbb{E}[w_{t+1}] = & n_t\mathbb{E}[w_t] + \mathbb{E}[R_t']\mathbb{E}[v_t] + \mathbb{E}[c_t], \\ \mathbb{E}[w_0] = & w_0, \end{cases}$$

$$\begin{cases} \mathbb{E}[l_{t+1}] = & \mathbb{E}[q_t]\mathbb{E}[l_t], \\ \mathbb{E}[l_0] = & l_0, \end{cases} \tag{6.3}$$

Combining the dynamic equations in (6.2) and (6.3), we have

$$\begin{cases} w_{t+1} - \mathbb{E}[w_{t+1}] = & n_t\big(w_t - \mathbb{E}[w_t]\big) + R_t'v_t - \mathbb{E}[R_t']\mathbb{E}[v_t] + (c_t - \mathbb{E}[c_t]) \\ = & n_t\big(w_t - \mathbb{E}[w_t]\big) + R_t'\big(v_t - \mathbb{E}[v_t]\big) + \big(R_t' - \mathbb{E}[R_t']\big)\mathbb{E}[v_t] \\ & +(c_t - \mathbb{E}[c_t]), \\ w_0 - \mathbb{E}[w_0] = & 0, \end{cases} \tag{6.4}$$

$$\begin{cases} l_{t+1} - \mathbb{E}[l_{t+1}] = & q_t(l_t - \mathbb{E}[l_t]) + (q_t - \mathbb{E}[q_t])\mathbb{E}[l_t], \\ l_0 - \mathbb{E}[l_0] = & 0, \end{cases} \tag{6.5}$$

where $t = 0, 1, \cdots, T-1$. Therefore, we can equivalently reformulate problem (6.2) to a linear quadratic optimal stochastic control problem in mean-field type

$$\begin{cases} \min & \sum_{t=1}^{T} p_t\Big\{\mathbb{E}\big[(w_T - l_T - \mathbb{E}[w_T - l_T])^2\big] - \lambda\mathbb{E}[w_T - l_T]\Big\}, \\ \text{s.t.} & \mathbb{E}[w_t] \text{ satisfies dynamic equation (6.3)}, \\ & \mathbb{E}[l_t] \text{ satisfies dynamic equation (6.3)}, \\ & w_t - \mathbb{E}[w_t] \text{ satisfies (6.4)}, \\ & l_t - \mathbb{E}[l_t] \text{ satisfies (6.5)}, \\ & \mathbb{E}(v_t - \mathbb{E}[v_t]) = \mathbf{0}, \\ & \mathbb{E}(l_t - \mathbb{E}[l_t]) = \mathbf{0}, \quad t = 0, 1, \cdots, T-1. \end{cases} \tag{6.6}$$

Thus, we are able to solve the multi-period mean-variance asset-liability model with cash flow and uncertain exit time in this mean-field formulation by dynamic programming.

6.3 The Optimal Strategy with Correlation under Uncertain Exit Time

Theorem 6.1. *We derive the optimal portfolio selection of problem (6.6) as*

$$
\begin{aligned}
v_t - \mathbb{E}[v_t] =& -n_t(w_t - \mathbb{E}[w_t])\mathbb{E}^{-1}[R_tR_t']\mathbb{E}[(R_t)] \\
&+\eta_{t+1}\xi_{t+1}^{-1}\big(l_t - \mathbb{E}[l_t]\big)\mathbb{E}^{-1}[R_tR_t']\mathbb{E}[q_tR_t], \\
\mathbb{E}[v_t] =& -\Big(\mathbb{E}[R_tR_t'] - \mathbb{E}[R_t]\mathbb{E}[R_t']\Big)^{-1}\Big(\mathbb{E}[c_tR_t] - \mathbb{E}[c_t]\mathbb{E}[R_t] \\
&-\frac{\lambda\zeta_{t+1}}{2\xi_{t+1}}\mathbb{E}[R_t] - \frac{\eta_{t+1}}{\xi_{t+1}}\big(\mathbb{E}[q_tR_t] - \mathbb{E}[q_t]\mathbb{E}[R_t]\big)\mathbb{E}[l_t]\Big)
\end{aligned}
\tag{6.7}
$$

the expect value of optimal wealth can be presented as

$$
\begin{aligned}
\mathbb{E}[w_t] = \quad & w_0\prod_{k=0}^{t-1} n_k + \sum_{j=0}^{t-1}\Big(\big(\mathbb{E}[c_j] + \frac{\lambda\zeta_{j+1}}{2\xi_{j+1}} - \widehat{B}_j - \mathbb{E}[q_j] + \overline{B}_j\big)\frac{B_j}{1-B_j} \\
&-\widehat{B}_j + \overline{B}_j + \mathbb{E}[c_j]\Big)\prod_{l=j+1}^{t-1} n_l,
\end{aligned}
\tag{6.8}
$$

for $t = 1, 2, \cdots, T$ and $\prod_{\emptyset}(\cdot) = 1$, $\sum_{\emptyset}(\cdot) = 0$ in this section.

Proof. Given an information set $\mathcal{F}_t$, we prove it by employing the backward recursion

$$
\begin{aligned}
J_t(\mathbb{E}[w_t], w_t - \mathbb{E}[w_t], \mathbb{E}[l_t], l_t - \mathbb{E}[l_t]) =& \min_{v_t}\mathbb{E}\big[J_{t+1}\big|\mathcal{F}_t\big] + p_t(w_t - l_t - \mathbb{E}[w_t - l_t])^2 \\
&- p_t\lambda\mathbb{E}[w_t - l_t],
\end{aligned}
\tag{6.9}
$$

with the boundary condition

$$
J_T(\mathbb{E}[w_T], w_T - \mathbb{E}[w_T], \mathbb{E}[l_T], l_T - \mathbb{E}[l_T]) = p_T\Big\{(w_T - l_T - \mathbb{E}[w_T - l_T])^2 - \lambda\mathbb{E}[w_T - l_T]\Big\}.
$$

Thus,

$$\begin{aligned}
& \mathbb{E}\Big[J_T(\mathbb{E}[w_T], w_T - \mathbb{E}[w_T], \mathbb{E}[l_T], l_T - \mathbb{E}[l_T])|\mathcal{F}_{T-1}\Big] \\
= \; & \mathbb{E}\Big[(w_T - l_T - \mathbb{E}[w_T - l_T])^2 - \lambda\mathbb{E}[w_T - l_T]|\mathcal{F}_{T-1}\Big] \\
= \; & \mathbb{E}\Big[\Big(n_{T-1}(w_{T-1} - \mathbb{E}[w_{T-1}]) + R'_{T-1}(v_{T-1} - \mathbb{E}[v_{T-1}]) + (R'_{T-1} - \mathbb{E}[R'_{T-1}])\mathbb{E}[v_{T-1}] \\
& + (c_{T-1} - \mathbb{E}[c_{T-1}]) - q_{T-1}(l_{T-1} - \mathbb{E}[l_{T-1}]) - (q_{T-1} - \mathbb{E}[q_{T-1}])\mathbb{E}[l_{T-1}]\Big)^2 \\
& -\lambda\Big(n_{T-1}\mathbb{E}[w_{T-1}] + \mathbb{E}[R'_{T-1}]\mathbb{E}[v_{T-1}] + \mathbb{E}[c_{T-1}] - \mathbb{E}[q_{T-1}]\mathbb{E}[l_{T-1}]\Big)\Big|\mathcal{F}_{T-1}\Big],
\end{aligned} \tag{6.10}$$

where

$$\begin{aligned}
& \Big(n_{T-1}(w_{T-1} - \mathbb{E}[w_{T-1}]) + R'_{T-1}(v_{T-1} - \mathbb{E}[v_{T-1}]) + (R'_{T-1} - \mathbb{E}[R'_{T-1}])\mathbb{E}[v_{T-1}] \\
& + (c_{T-1} - \mathbb{E}[c_{T-1}]) - q_{T-1}(l_{T-1} - \mathbb{E}[l_{T-1}]) - (q_{T-1} - \mathbb{E}[q_{T-1}])\mathbb{E}[l_{T-1}]\Big)^2 \\
= \; & \big(n_{T-1}(w_{T-1} - \mathbb{E}[w_{T-1}])\big)^2 + \big(R'_{T-1}(v_{T-1} - \mathbb{E}[v_{T-1}])\big)^2 \\
& + \big((R'_{T-1} - \mathbb{E}[R'_{T-1}])\mathbb{E}[v_{T-1}]\big)^2 + \big(c_{T-1} - \mathbb{E}[c_{T-1}]\big)^2 \\
& + \big(q_{T-1}(l_{T-1} - \mathbb{E}[l_{T-1}])\big)^2 + \big((q_{T-1} - \mathbb{E}[q_{T-1}])\mathbb{E}[l_{T-1}]\big)^2 \\
& + 2n_{T-1}(w_{T-1} - \mathbb{E}[w_{T-1}])R'_{T-1}(v_{T-1} - \mathbb{E}[v_{T-1}]) \\
& + 2n_{T-1}(w_{T-1} - \mathbb{E}[w_{T-1}])(R'_{T-1} - \mathbb{E}[R'_{T-1}])\mathbb{E}[v_{T-1}] \\
& + 2n_{T-1}(w_{T-1} - \mathbb{E}[w_{T-1}])(c_{T-1} - \mathbb{E}[c_{T-1}]) \\
& - 2n_{T-1}(w_{T-1} - \mathbb{E}[w_{T-1}])q_{T-1}(l_{T-1} - \mathbb{E}[l_{T-1}]) \\
& - 2n_{T-1}(w_{T-1} - \mathbb{E}[w_{T-1}])(q_{T-1} - \mathbb{E}[q_{T-1}])\mathbb{E}[l_{T-1}] \\
& + 2(v_{T-1} - \mathbb{E}[v_{T-1}])'R_{T-1}(R'_{T-1} - \mathbb{E}[R'_{T-1}])\mathbb{E}[v_{T-1}] \\
& + 2R'_{T-1}(v_{T-1} - \mathbb{E}[v_{T-1}])(c_{T-1} - \mathbb{E}[c_{T-1}]) \\
& - 2R'_{T-1}(v_{T-1} - \mathbb{E}[v_{T-1}])q_{T-1}(l_{T-1} - \mathbb{E}[l_{T-1}]) \\
& - 2R'_{T-1}(v_{T-1} - \mathbb{E}[v_{T-1}])(q_{T-1} - \mathbb{E}[q_{T-1}])\mathbb{E}[l_{T-1}] \\
& + 2(R'_{T-1} - \mathbb{E}[R'_{T-1}])\mathbb{E}[v_{T-1}](c_{T-1} - \mathbb{E}[c_{T-1}]) \\
& - 2(R'_{T-1} - \mathbb{E}[R'_{T-1}])\mathbb{E}[v_{T-1}]q_{T-1}(l_{T-1} - \mathbb{E}[l_{T-1}]) \\
& - 2(R'_{T-1} - \mathbb{E}[R'_{T-1}])\mathbb{E}[v_{T-1}](q_{T-1} - \mathbb{E}[q_{T-1}])\mathbb{E}[l_{T-1}] \\
& - 2(c_{T-1} - \mathbb{E}[c_{T-1}])q_{T-1}(l_{T-1} - \mathbb{E}[l_{T-1}]) \\
& - 2(c_{T-1} - \mathbb{E}[c_{T-1}])(q_{T-1} - \mathbb{E}[q_{T-1}])\mathbb{E}[l_{T-1}] \\
& + 2q_{T-1}(l_{T-1} - \mathbb{E}[l_{T-1}])(q_{T-1} - \mathbb{E}[q_{T-1}])\mathbb{E}[l_{T-1}].
\end{aligned} \tag{6.11}$$

Since $w_t^i - \mathbb{E}[w_t^i]$, $\mathbb{E}[w_t^i]$, $v_t^i - \mathbb{E}[v_t^i]$, $\mathbb{E}[v_t^i]$, are $\mathcal{F}_t$-measurable,

$$\begin{aligned}
&\mathbb{E}\Big[\big(n_{T-1}(w_{T-1} - \mathbb{E}[w_{T-1}])\big)^2\Big|\mathcal{F}_{T-1}\Big] = (n_{T-1})^2\big(w_{T-1} - \mathbb{E}[w_{T-1}]\big)^2,\\
&\mathbb{E}\Big[\Big(R'_{T-1}(v_{T-1} - \mathbb{E}[v_{T-1}])\Big)^2\Big|\mathcal{F}_{T-1}\Big]\\
&= \big(v_{T-1} - \mathbb{E}[v_{T-1}]\big)'\mathbb{E}[R_{T-1}R'_{T-1}]\big(v_{T-1} - \mathbb{E}[v_{T-1}]\big),\\
&\mathbb{E}\Big[\Big((R'_{T-1} - \mathbb{E}[R'_{T-1}])\mathbb{E}[v_{T-1}]\Big)^2\Big|\mathcal{F}_{T-1}\Big]\\
&= \mathbb{E}[(v_{T-1})']\Big(\mathbb{E}[R_{T-1}R'_{T-1}] - \mathbb{E}[R_{T-1}]\mathbb{E}[R'_{T-1}]\Big)\mathbb{E}[v_{T-1}],\\
&\mathbb{E}\Big[\big(c_{T-1} - \mathbb{E}[c_{T-1}]\big)^2\Big|\mathcal{F}_{T-1}\Big] = \mathbb{E}[c_{T-1}^2] - \mathbb{E}[c_{T-1}]^2,\\
&\mathbb{E}\Big[\big(q_{T-1}(l_{T-1} - \mathbb{E}[l_{T-1}])\big)^2\Big|\mathcal{F}_{T-1}\Big] = \mathbb{E}[q_{T-1}^2](l_{T-1} - \mathbb{E}[l_{T-1}])^2,\\
&\mathbb{E}\Big[\big((q_{T-1} - \mathbb{E}[q_{T-1}])\mathbb{E}[l_{T-1}]\big)\big)^2\Big|\mathcal{F}_{T-1}\Big] = \Big(\mathbb{E}[q_{T-1}^2] - \mathbb{E}[q_{T-1}]^2\Big)\mathbb{E}[l_{T-1}]^2,\\
&\mathbb{E}\Big[2n_{T-1}(w_{T-1} - \mathbb{E}[w_{T-1}])R'_{T-1}(v_{T-1} - \mathbb{E}[v_{T-1}])\Big|\mathcal{F}_{T-1}\Big]\\
&= 2n_{T-1}\big(w_{T-1} - \mathbb{E}[w_{T-1}]\big)\mathbb{E}[R'_{T-1}]\big(v_{T-1} - \mathbb{E}[v_{T-1}]\big),\\
&\mathbb{E}\Big[2n_{T-1}\big(w_{T-1} - \mathbb{E}[w_{T-1}]\big)\big(R'_{T-1} - \mathbb{E}[R'_{T-1}]\big)\mathbb{E}[v_{T-1}]\Big|\mathcal{F}_{T-1}\Big] = 0,\\
&\mathbb{E}\Big[2n_{T-1}(w_{T-1} - \mathbb{E}[w_{T-1}])(c_{T-1} - \mathbb{E}[c_{T-1}])\Big|\mathcal{F}_{T-1}\Big] = 0,\\
&\mathbb{E}\Big[2n_{T-1}(w_{T-1} - \mathbb{E}[w_{T-1}])q_{T-1}(l_{T-1} - \mathbb{E}[l_{T-1}])\Big|\mathcal{F}_{T-1}\Big]\\
&= 2n_{T-1}\mathbb{E}[q_{T-1}](w_{T-1} - \mathbb{E}[w_{T-1}])(l_{T-1} - \mathbb{E}[l_{T-1}]),\\
&\mathbb{E}\Big[2n_{T-1}(w_{T-1} - \mathbb{E}[w_{T-1}])(q_{T-1} - \mathbb{E}[q_{T-1}])\mathbb{E}[l_{T-1}]\Big|\mathcal{F}_{T-1}\Big] = 0,\\
&\mathbb{E}\Big[2\big(v_{T-1} - \mathbb{E}[v_{T-1}]\big)'(R_{T-1})\Big(R'_{T-1} - \mathbb{E}[R'_{T-1}]\Big)\mathbb{E}[v_{T-1}]\Big|\mathcal{F}_{T-1}\Big]\\
&= 2\big(v_{T-1} - \mathbb{E}[v_{T-1}]\big)'\Big(\mathbb{E}[R_{T-1}R'_{T-1}] - \mathbb{E}[R_{T-1}]\mathbb{E}[R'_{T-1}]\Big)\mathbb{E}[v_{T-1}] = 0,\\
&\mathbb{E}\Big[2R'_{T-1}(v_{T-1} - \mathbb{E}[v_{T-1}])(c_{T-1} - \mathbb{E}[c_{T-1}])\Big|\mathcal{F}_{T-1}\Big]\\
&= 2\big(\mathbb{E}[c_{T-1}R'_{T-1}] - \mathbb{E}[c_{T-1}]\mathbb{E}[R'_{T-1}\big)(v_{T-1} - \mathbb{E}[v_{T-1}]) = 0,\\
&\mathbb{E}\Big[2R'_{T-1}(v_{T-1} - \mathbb{E}[v_{T-1}])q_{T-1}(l_{T-1} - \mathbb{E}[l_{T-1}])\Big|\mathcal{F}_{T-1}\Big]\\
&= 2\mathbb{E}[q_{T-1}R'_{T-1}](v_{T-1} - \mathbb{E}[v_{T-1}])(l_{T-1} - \mathbb{E}[l_{T-1}]),\\
&\mathbb{E}\Big[2R'_{T-1}(v_{T-1} - \mathbb{E}[v_{T-1}])(q_{T-1} - \mathbb{E}[q_{T-1}])\mathbb{E}[l_{T-1}]\Big|\mathcal{F}_{T-1}\Big]\\
&= 2\big(\mathbb{E}[q_{T-1}R'_{T-1}] - \mathbb{E}[q_{T-1}]\mathbb{E}[R'_{T-1}]\big)(v_{T-1} - \mathbb{E}[v_{T-1}])\mathbb{E}[l_{T-1}] = 0,\\
&\mathbb{E}\Big[2\big(R'_{T-1} - \mathbb{E}[R'_{T-1}]\big)\mathbb{E}[v_{T-1}]\big(c_{T-1} - \mathbb{E}[c_{T-1}]\big)\Big|\mathcal{F}_{T-1}\Big]\\
&= 2\Big(\mathbb{E}[c_{T-1}R'_{T-1}] - \mathbb{E}[c_{T-1}]\mathbb{E}[R'_{T-1}]\Big)\mathbb{E}[v_{T-1}],\\
&\mathbb{E}\Big[2(R'_{T-1} - \mathbb{E}[R'_{T-1}])\mathbb{E}[v_{T-1}]q_{T-1}(l_{T-1} - \mathbb{E}[l_{T-1}])\Big|\mathcal{F}_{T-1}\Big] = 0,
\end{aligned}$$

and

$$
\begin{aligned}
&\mathbb{E}\Big[2(R'_{T-1}-\mathbb{E}[R'_{T-1}])\mathbb{E}[v_{T-1}](q_{T-1}-\mathbb{E}[q_{T-1}])\mathbb{E}[l_{T-1}]\Big|\mathcal{F}_{T-1}\Big]\\
&=2\big(\mathbb{E}[q_{T-1}R'_{T-1}]-\mathbb{E}[q_{T-1}]\mathbb{E}[R'_{T-1}]\big)\mathbb{E}[v_{T-1}]\mathbb{E}[l_{T-1}],\\
&\mathbb{E}\Big[2(c_{T-1}-\mathbb{E}[c_{T-1}])q_{T-1}(l_{T-1}-\mathbb{E}[l_{T-1}])\Big|\mathcal{F}_{T-1}\Big]\\
&=2\big(\mathbb{E}[q_{T-1}c_{T-1}]-\mathbb{E}[q_{T-1}]\mathbb{E}[c_{T-1}]\big)(l_{T-1}-\mathbb{E}[l_{T-1}])=0,\\
&\mathbb{E}\Big[2(c_{T-1}-\mathbb{E}[c_{T-1}])(q_{T-1}-\mathbb{E}[q_{T-1}])\mathbb{E}[l_{T-1}]\Big|\mathcal{F}_{T-1}\Big]\\
&=2\big(\mathbb{E}[q_{T-1}c_{T-1}]-\mathbb{E}[q_{T-1}]\mathbb{E}[c_{T-1}]\big)\mathbb{E}[l_{T-1}],\\
&\mathbb{E}\Big[2q_{T-1}(l_{T-1}-\mathbb{E}[l_{T-1}])(q_{T-1}-\mathbb{E}[q_{T-1}])\mathbb{E}[l_{T-1}]\Big|\mathcal{F}_{T-1}\Big]\\
&=2\big(\mathbb{E}[q_{T-1}^2]-\mathbb{E}[q_{T-1}]^2\big)(l_{T-1}-\mathbb{E}[l_{T-1}])\mathbb{E}[l_{T-1}]=0.
\end{aligned}
$$

Therefore, we can reduce (6.10) into

$$
\begin{aligned}
&\mathbb{E}\big[(w_T-l_T-\mathbb{E}[w_T-l_T])^2-\lambda\mathbb{E}[w_T-l_T]|\mathcal{F}_{T-1}\big]\\
=\;&\mathbb{E}\Big[\Big(n_{T-1}(w_{T-1}-\mathbb{E}[w_{T-1}])+R'_{T-1}(v_{T-1}-\mathbb{E}[v_{T-1}])\\
&+(R'_{T-1}-\mathbb{E}[R'_{T-1}])\mathbb{E}[v_{T-1}]+(c_{T-1}-\mathbb{E}[c_{T-1}])\\
&-q_{T-1}(l_{T-1}-\mathbb{E}[l_{T-1}])-(q_{T-1}-\mathbb{E}[q_{T-1}])\mathbb{E}[l_{T-1}]\Big)^2\\
&-\lambda\Big(n_{T-1}\mathbb{E}[w_{T-1}]+\mathbb{E}[R'_{T-1}]\mathbb{E}[v_{T-1}]+\mathbb{E}[c_{T-1}]-\mathbb{E}[q_{T-1}]\mathbb{E}[l_{T-1}]\Big)\Big|\mathcal{F}_{T-1}\Big]\\
=\;&(n_{T-1})^2\big(w_{T-1}-\mathbb{E}[w_{T-1}]\big)^2+\big(v_{T-1}-\mathbb{E}[v_{T-1}]\big)'\mathbb{E}[R_{T-1}R'_{T-1}]\big(v_{T-1}-\mathbb{E}[v_{T-1}]\big)\\
&+\mathbb{E}[(v_{T-1})']\Big(\mathbb{E}[R_{T-1}R'_{T-1}]-\mathbb{E}[R_{T-1}]\mathbb{E}[R'_{T-1}]\Big)\mathbb{E}[v_{T-1}]+\mathbb{E}[(c_{T-1})^2]\\
&-\mathbb{E}[c_{T-1}]^2+\mathbb{E}[q_{T-1}^2](l_{T-1}-\mathbb{E}[l_{T-1}])^2+\Big(\mathbb{E}[q_{T-1}^2]-\mathbb{E}[q_{T-1}]^2\Big)\mathbb{E}[l_{T-1}]^2\\
&+2n_{T-1}\big(w_{T-1}-\mathbb{E}[w_{T-1}]\big)\mathbb{E}[R'_{T-1}]\big(v_{T-1}-\mathbb{E}[v_{T-1}]\big)\\
&-2n_{T-1}\mathbb{E}[q_{T-1}](w_{T-1}-\mathbb{E}[w_{T-1}])(l_{T-1}-\mathbb{E}[l_{T-1}])\\
&-2\mathbb{E}[q_{T-1}R'_{T-1}](v_{T-1}-\mathbb{E}[v_{T-1}])(l_{T-1}-\mathbb{E}[l_{T-1}])\\
&+2\Big(\mathbb{E}[c_{T-1}R'_{T-1}]-\mathbb{E}[c_{T-1}]\mathbb{E}[R'_{T-1}]\Big)\mathbb{E}[v_{T-1}]\\
&-2\big(\mathbb{E}[q_{T-1}R'_{T-1}]-\mathbb{E}[q_{T-1}]\mathbb{E}[R'_{T-1}]\big)\mathbb{E}[v_{T-1}]\mathbb{E}[l_{T-1}]\\
&-2\big(\mathbb{E}[q_{T-1}c_{T-1}]-\mathbb{E}[q_{T-1}]\mathbb{E}[c_{T-1}]\big)\mathbb{E}[l_{T-1}]\\
&-\lambda n_{T-1}\mathbb{E}[w_{T-1}]-\lambda\mathbb{E}[R'_{T-1}]\mathbb{E}[v_{T-1}]-\lambda\mathbb{E}[c_{T-1}]+\lambda\mathbb{E}[q_{T-1}]\mathbb{E}[l_{T-1}].
\end{aligned}
\tag{6.12}
$$

The optimal strategies at period $T-1$ can be derived from the above equation

corresponding to v_{T-1}

$$\begin{aligned}
v_{T-1} - \mathbb{E}[v_{T-1}] &= -n_{T-1}(w_{T-1} - \mathbb{E}[w_{T-1}])\mathbb{E}^{-1}[R_{T-1}R'_{T-1}]\mathbb{E}[R_{T-1}] \\
&\quad + (l_{T-1} - \mathbb{E}[l_{T-1}])\mathbb{E}^{-1}[R_{T-1}R'_{T-1}]\mathbb{E}[q_{T-1}R_{T-1}], \\
\mathbb{E}[v_{T-1}] &= -\Big(\mathbb{E}[R_{T-1}R'_{T-1}] - \mathbb{E}[R_{T-1}]\mathbb{E}[R'_{T-1}]\Big)^{-1} \\
&\quad \times \Big(\mathbb{E}[c_{T-1}R_{T-1}] - \mathbb{E}[c_{T-1}]\mathbb{E}[R_{T-1}] - \frac{\lambda}{2}\mathbb{E}[R_{T-1}] \\
&\quad - \big(\mathbb{E}[q_{T-1}R_{T-1}] - \mathbb{E}[q_{T-1}]\mathbb{E}[R_{T-1}]\big)\mathbb{E}[l_{T-1}]\Big),
\end{aligned} \tag{6.13}$$

where

$$\begin{aligned}
B_{T-1} &= \mathbb{E}[R'_{T-1}]\mathbb{E}^{-1}[R_{T-1}R'_{T-1}]\mathbb{E}[R_{T-1}], \\
\widehat{B}_{T-1} &= \mathbb{E}[c_{T-1}R'_{T-1}]\mathbb{E}^{-1}[R_{T-1}R'_{T-1}]\mathbb{E}[R_{T-1}], \\
\widetilde{B}_{T-1} &= \mathbb{E}[c_{T-1}R'_{T-1}]\mathbb{E}^{-1}[R_{T-1}R'_{T-1}]\mathbb{E}[c_{T-1}R_{T-1}], \\
\overline{B}_{T-1} &= \mathbb{E}[q_{T-1}R'_{T-1}]\mathbb{E}^{-1}[R_{T-1}R'_{T-1}]\mathbb{E}[R_{T-1}], \\
B'_{T-1} &= \mathbb{E}[q_{T-1}R'_{T-1}]\mathbb{E}^{-1}[R_{T-1}R'_{T-1}]\mathbb{E}[R_{T-1}q_{T-1}], \\
\overrightarrow{B}_{T-1} &= \mathbb{E}[c_{T-1}R'_{T-1}]\mathbb{E}^{-1}[R_{T-1}R'_{T-1}]\mathbb{E}[R_{T-1}q_{T-1}].
\end{aligned} \tag{6.14}$$

Substituting the optimal strategies back to (6.12), we obtain

$$
\begin{aligned}
& J_{T-1}(\mathbb{E}[w_{T-1}], w_{T-1}-\mathbb{E}[w_{T-1}], \mathbb{E}[l_{T-1}], l_{T-1}-\mathbb{E}[l_{T-1}]) \\
= \;& \mathbb{E}\big[J_T(\mathbb{E}[w_T], w_T-\mathbb{E}[w_T], \mathbb{E}[l_T], l_T-\mathbb{E}[l_T])|\mathcal{F}_{T-1}\big] \\
& +p_{T-1}(w_{T-1}-l_{T-1}-\mathbb{E}[w_{T-1}-l_{T-1}])^2 - p_{T-1}\lambda\mathbb{E}[w_{T-1}-l_{T-1}] \\
= \;& p_T\Bigg\{(n_{T-1})^2\big(w_{T-1}-\mathbb{E}[w_{T-1}]\big)^2 + (\mathbb{E}[(c_{T-1})^2]-\mathbb{E}[c_{T-1}]^2) \\
& +\mathbb{E}[q_{T-1}^2](l_{T-1}-\mathbb{E}[l_{T-1}])^2 + \Big(\mathbb{E}[q_{T-1}^2]-\mathbb{E}[q_{T-1}]^2\Big)\mathbb{E}[l_{T-1}]^2 \\
& -2n_{T-1}\mathbb{E}[q_{T-1}](w_{T-1}-\mathbb{E}[w_{T-1}])(l_{T-1}-\mathbb{E}[l_{T-1}]) \\
& -\lambda n_{T-1}\mathbb{E}[w_{T-1}] - \lambda\mathbb{E}[c_{T-1}] + \lambda\mathbb{E}[q_{T-1}]\mathbb{E}[l_{T-1}] \\
& -(n_{T-1})^2 B_{T-1}(w_{T-1}-\mathbb{E}[w_{T-1}])^2 - B'_{T-1}(l_{T-1}-\mathbb{E}[l_{T-1}])^2 \\
& +2n_{T-1}\overline{B}_{T-1}(w_{T-1}-\mathbb{E}[w_{T-1}])(l_{T-1}-\mathbb{E}[l_{T-1}]) \\
& -2\big(\mathbb{E}[q_{T-1}c_{T-1}]-\mathbb{E}[q_{T-1}]\mathbb{E}[c_{T-1}]\big)\mathbb{E}[l_{T-1}] \\
& -\Big(\mathbb{E}[c_{T-1}R'_{T-1}]-\mathbb{E}[c_{T-1}]\mathbb{E}[R'_{T-1}]-\frac{\lambda}{2}\mathbb{E}[R'_{T-1}] \\
& -\big(\mathbb{E}[q_{T-1}R'_{T-1}]-\mathbb{E}[q_{T-1}]\mathbb{E}[R'_{T-1}]\big)\mathbb{E}[l_{T-1}]\Big) \\
& \times\Big(\mathbb{E}[R_{T-1}R'_{T-1}]-\mathbb{E}[R_{T-1}]\mathbb{E}[R'_{T-1}]\Big)^{-1} \\
& \times\Big(\mathbb{E}[c_{T-1}R_{T-1}]-\mathbb{E}[c_{T-1}]\mathbb{E}[R_{T-1}]-\frac{\lambda}{2}\mathbb{E}[R_{T-1}] \\
& -\big(\mathbb{E}[q_{T-1}R_{T-1}]-\mathbb{E}[q_{T-1}]\mathbb{E}[R_{T-1}]\big)\mathbb{E}[l_{T-1}]\Big)\Bigg\} \\
& +p_{T-1}(w_{T-1}-l_{T-1}-\mathbb{E}[w_{T-1}-l_{T-1}])^2 - p_{T-1}\lambda\mathbb{E}[w_{T-1}-l_{T-1}].
\end{aligned}
$$

Thus,

$$
\begin{aligned}
& J_{T-1}(\mathbb{E}[w_{T-1}], w_{T-1}-\mathbb{E}[w_{T-1}], \mathbb{E}[l_{T-1}], l_{T-1}-\mathbb{E}[l_{T-1}]) \\
= \; & p_T\Big\{(n_{T-1})^2(1-B_{T-1})\big(w_{T-1}-\mathbb{E}[w_{T-1}]\big)^2+(\mathbb{E}[(c_{T-1})^2]-\mathbb{E}[c_{T-1}]^2) \\
& +\big(\mathbb{E}[q_{T-1}^2]-B'_{T-1}\big)(l_{T-1}-\mathbb{E}[l_{T-1}])^2 \\
& -2\big(n_{T-1}\mathbb{E}[q_{T-1}]-n_{T-1}\overline{B}_{T-1}\big)(w_{T-1}-\mathbb{E}[w_{T-1}])(l_{T-1}-\mathbb{E}[l_{T-1}]) \\
& +\Big(\mathbb{E}[q_{T-1}^2]-\mathbb{E}[q_{T-1}]^2\Big)\mathbb{E}[l_{T-1}]^2-\lambda n_{T-1}\mathbb{E}[w_{T-1}]-\lambda\mathbb{E}[c_{T-1}]+\lambda\mathbb{E}[q_{T-1}]\mathbb{E}[l_{T-1}] \\
& -2\big(\mathbb{E}[q_{T-1}c_{T-1}]-\mathbb{E}[q_{T-1}]\mathbb{E}[c_{T-1}]\big)\mathbb{E}[l_{T-1}]-\Big(\big(\mathbb{E}[c_{T-1}]+\frac{\lambda}{2}-\widehat{B}_{T-1}\big)^2\frac{B_{T-1}}{1-B_{T-1}} \\
& -2\big(\mathbb{E}[c_{T-1}]+\frac{\lambda}{2}-\widehat{B}_{T-1}\big)\widehat{B}_{T-1}+\widetilde{B}_{T-1}-\widehat{B}_{T-1}^2\Big) \\
& +2\Big(\frac{\big(\mathbb{E}[c_{T-1}]+\frac{\lambda}{2}\big)\big(\mathbb{E}[q_{T-1}]B_{T-1}-\overline{B}_{T-1}\big)+\widehat{B}_{T-1}\overline{B}_{T-1}-\widehat{B}_{T-1}\mathbb{E}[q_{T-1}]}{1-B_{T-1}} \\
& +\overrightarrow{B}_{T-1}\Big)\mathbb{E}[l_{T-1}]-\Big(B'_{T-1}-\mathbb{E}[q_{T-1}]^2+\frac{\big(\overline{B}_{T-1}-\mathbb{E}[q_{T-1}]\big)^2}{1-B_{T-1}}\Big)\mathbb{E}[l_{T-1}]^2\Big\} \\
& +p_{T-1}(w_{T-1}-l_{T-1}-\mathbb{E}[w_{T-1}-l_{T-1}])^2-p_{T-1}\lambda\mathbb{E}[w_{T-1}-l_{T-1}] \\
= \; & \xi_{T-1}\big(w_{T-1}-\mathbb{E}[w_{T-1}]\big)^2-2\eta_{T-1}(w_{T-1}-\mathbb{E}[w_{T-1}])(l_{T-1}-\mathbb{E}[l_{T-1}]) \\
& +\epsilon_{T-1}(l_{T-1}-\mathbb{E}[l_{T-1}])^2-\lambda\zeta_{T-1}\mathbb{E}[w_{T-1}]+\lambda\theta_{T-1}\mathbb{E}[l_{T-1}]+\delta_{T-1}\mathbb{E}[l_{T-1}]^2+\Delta_{T-1},
\end{aligned}
\tag{6.15}
$$

where

$$\begin{aligned}
\xi_{T-1} = \quad & p_T(n_{T-1})^2(1-B_{T-1}) + p_{T-1}, \\
\eta_{T-1} = \quad & p_T(n_{T-1}\mathbb{E}[q_{T-1}] - n_{T-1}\overline{B}_{T-1}) + p_{T-1}, \\
\epsilon_{T-1} = \quad & p_T(\mathbb{E}[q_{T-1}^2] - B'_{T-1}) + p_{T-1}, \\
\zeta_{T-1} = \quad & p_T n_{T-1} + p_{T-1}, \\
\theta_{T-1} = \quad & p_T\Bigg(\mathbb{E}[q_{T-1}] - \frac{2}{\lambda}\big(\mathbb{E}[q_{T-1}c_{T-1}] - \mathbb{E}[q_{T-1}]\mathbb{E}[c_{T-1}]\big) \\
& + \frac{2}{\lambda}\Big(\frac{\big(\mathbb{E}[c_{T-1}] + \frac{\lambda}{2}\big)\big(\mathbb{E}[q_{T-1}]B_{T-1} - \overline{B}_{T-1}\big) + \widehat{B}_{T-1}\overline{B}_{T-1} - \widehat{B}_{T-1}\mathbb{E}[q_{T-1}]}{1-B_{T-1}} \\
& + \overrightarrow{B}_{T-1}\Big)\Bigg) + p_{T-1}, \\
\delta_{T-1} = \quad & p_T\Bigg(\mathbb{E}[q_{T-1}^2] - \mathbb{E}[q_{T-1}]^2 - \Big(B'_{T-1} - \mathbb{E}[q_{T-1}]^2 + \frac{\big(\overline{B}_{T-1} - \mathbb{E}[q_{T-1}]\big)^2}{1-B_{T-1}}\Big)\Bigg), \\
\Delta_{T-1} = \quad & p_T\Bigg(\mathbb{E}[(c_{T-1})^2] - \mathbb{E}[c_{T-1}]^2 - \lambda\mathbb{E}[c_{T-1}] \\
& - \Big(\big(\mathbb{E}[c_{T-1}] + \frac{\lambda}{2} - \widehat{B}_{T-1}\big)^2\frac{B_{T-1}}{1-B_{T-1}} - 2\big(\mathbb{E}[c_{T-1}] + \frac{\lambda}{2} - \widehat{B}_{T-1}\big)\widehat{B}_{T-1} \\
& + \widetilde{B}_{T-1} - \widehat{B}_{T-1}^2\Big)\Bigg).
\end{aligned} \tag{6.16}$$

Repeating the process at time $T-2$, we have

$$
\begin{aligned}
& J_{T-2}(\mathbb{E}[w_{T-2}], w_{T-2}-\mathbb{E}[w_{T-2}], \mathbb{E}[l_{T-2}], l_{T-2}-\mathbb{E}[l_{T-2}]) \\
= \; & \mathbb{E}\big[J_{T-1}(\mathbb{E}[w_{T-1}], w_{T-1}-\mathbb{E}[w_{T-1}], \mathbb{E}[l_{T-1}], l_{T-1}-\mathbb{E}[l_{T-1}])|\mathcal{F}_{T-2}\big] \\
& + p_{T-2}(w_{T-2}-l_{T-2}-\mathbb{E}[w_{T-2}-l_{T-2}])^2 - p_{T-2}\lambda\mathbb{E}[w_{T-2}-l_{T-2}] \\
= \; & \mathbb{E}\big[\xi_{T-1}\big(w_{T-1}-\mathbb{E}[w_{T-1}]\big)^2 - 2\eta_{T-1}(w_{T-1}-\mathbb{E}[w_{T-1}])(l_{T-1}-\mathbb{E}[l_{T-1}]) \\
& + \epsilon_{T-1}(l_{T-1}-\mathbb{E}[l_{T-1}])^2 - \lambda\zeta_{T-1}\mathbb{E}[w_{T-1}] + \lambda\theta_{T-1}\mathbb{E}[l_{T-1}] + \delta_{T-1}\mathbb{E}[l_{T-1}]^2 \\
& + \Delta_{T-1}|\mathcal{F}_{T-2}\big] + p_{T-2}(w_{T-2}-l_{T-2}-\mathbb{E}[w_{T-2}-l_{T-2}])^2 - p_{T-2}\lambda\mathbb{E}[w_{T-2}-l_{T-2}] \\
= \; & \mathbb{E}\Big[\xi_{T-1}\Big[n_{T-2}(w_{T-2}-\mathbb{E}[w_{T-2}]) + R'_{T-2}(v_{T-2}-\mathbb{E}[v_{T-2}]) + (R'_{T-2}-\mathbb{E}[R'_{T-2}])\mathbb{E}[v_{T-2}] \\
& + (c_{T-2}-\mathbb{E}[c_{T-2}])\Big]^2 - 2\eta_{T-1}\Big[q_{T-2}(l_{T-2}-\mathbb{E}[l_{T-2}]) + (q_{T-2}-\mathbb{E}[q_{T-2}])\mathbb{E}[l_{T-2}]\Big] \\
& \Big[n_{T-2}(w_{T-2}-\mathbb{E}[w_{T-2}]) + R'_{T-2}(v_{T-2}-\mathbb{E}[v_{T-2}]) + (R'_{T-2}-\mathbb{E}[R'_{T-2}])\mathbb{E}[v_{T-2}] \\
& + (c_{T-2}-\mathbb{E}[c_{T-2}])\Big] + \epsilon_{T-1}\big[q_{T-2}(l_{T-2}-\mathbb{E}[l_{T-2}]) + (q_{T-2}-\mathbb{E}[q_{T-2}])\mathbb{E}[l_{T-2}]\big]^2 \\
& - \lambda\zeta_{T-1}\big(n_{T-2}\mathbb{E}[w_{T-2}] + \mathbb{E}[R'_{T-2}]\mathbb{E}[v_{T-2}] + \mathbb{E}[c_{T-2}]\big) \\
& + \lambda\theta_{T-1}\mathbb{E}[q_{T-2}]\mathbb{E}[l_{T-2}] + \delta_{T-1}(\mathbb{E}[q_{T-2}]\mathbb{E}[l_{T-2}])^2 + \Delta_{T-1}\Big|\mathcal{F}_{T-2}\Big] \\
& + p_{T-2}(w_{T-2}-l_{T-2}-\mathbb{E}[w_{T-2}-l_{T-2}])^2 - p_{T-2}\lambda\mathbb{E}[w_{T-2}-l_{T-2}].
\end{aligned}
$$

Thus,

$$
\begin{aligned}
& J_{T-2}(\mathbb{E}[w_{T-2}], w_{T-2}-\mathbb{E}[w_{T-2}], \mathbb{E}[l_{T-2}], l_{T-2}-\mathbb{E}[l_{T-2}]) \\
= \;& \xi_{T-1}\Big[n_{T-2}^2\big(w_{T-2}-\mathbb{E}[w_{T-2}]\big)^2+\big(v_{T-2}-\mathbb{E}[v_{T-2}]\big)'\mathbb{E}[R_{T-2}R_{T-2}']\big(v_{T-2}-\mathbb{E}[v_{T-2}]\big) \\
& +2n_{T-2}\big(w_{T-2}-\mathbb{E}[w_{T-2}]\big)\mathbb{E}[R_{T-2}']\big(v_{T-2}-\mathbb{E}[v_{T-2}]\big) \\
& +\mathbb{E}[v_{T-2}']\big(\mathbb{E}[R_{T-2}R_{T-2}']-\mathbb{E}[R_{T-2}]\mathbb{E}[R_{T-2}']\big)\mathbb{E}[v_{T-2}] \\
& +(\mathbb{E}[(c_{T-2})^2]-\mathbb{E}[c_{T-2}]^2)+2\Big(\mathbb{E}[c_{T-2}R_{T-2}']-\mathbb{E}[c_{T-2}]\mathbb{E}[R_{T-2}']\Big)\mathbb{E}[v_{T-2}]\Big] \\
& -2\eta_{T-1}\Big[n_{T-2}\mathbb{E}[q_{T-2}]\big(l_{T-2}-\mathbb{E}[l_{T-2}]\big)\big(w_{T-2}-\mathbb{E}[w_{T-2}]\big) \\
& +\mathbb{E}[q_{T-2}R_{T-2}']\big(l_{T-2}-\mathbb{E}[l_{T-2}]\big)\big(v_{T-2}-\mathbb{E}[v_{T-2}]\big) \\
& +\big(\mathbb{E}[q_{T-2}R_{T-2}']-\mathbb{E}[q_{T-2}]\mathbb{E}[R_{T-2}']\big)\mathbb{E}[v_{T-2}]\mathbb{E}[l_{T-2}] \\
& +\big(\mathbb{E}[q_{T-2}c_{T-2}]-\mathbb{E}[q_{T-2}]\mathbb{E}[c_{T-2}]\big)\mathbb{E}[l_{T-2}]\Big] \\
& +\epsilon_{T-1}\Big[\mathbb{E}[q_{T-2}^2](l_{T-2}-\mathbb{E}[l_{T-2}])^2+(\mathbb{E}[q_{T-2}^2]-(\mathbb{E}[q_{T-2}])^2)(\mathbb{E}[l_{T-2}])^2\Big] \\
& -\lambda\zeta_{T-1}\big(n_{T-2}\mathbb{E}[w_{T-2}]+\mathbb{E}[R_{T-2}']\mathbb{E}[v_{T-2}]+\mathbb{E}[c_{T-2}]\big) \\
& +\lambda\theta_{T-1}\mathbb{E}[q_{T-2}]\mathbb{E}[l_{T-2}]+\delta_{T-1}(\mathbb{E}[q_{T-2}]\mathbb{E}[l_{T-2}])^2+\Delta_{T-1} \\
& +p_{T-2}(w_{T-2}-l_{T-2}-\mathbb{E}[w_{T-2}-l_{T-2}])^2-p_{T-2}\lambda\mathbb{E}[w_{T-2}-l_{T-2}].
\end{aligned}
\tag{6.17}
$$

The optimal strategies at period $T-2$ can be derived by from above equation corresponding to v_{T-2}

$$
\begin{aligned}
v_{T-2}-\mathbb{E}[v_{T-2}] =& -n_{T-2}(w_{T-2}-\mathbb{E}[w_{T-2}])\mathbb{E}^{-1}[R_{T-2}R_{T-2}']\mathbb{E}[(R_{T-2})] \\
& +\eta_{T-1}\xi_{T-1}^{-1}\big(l_{T-2}-\mathbb{E}[l_{T-2}]\big)\mathbb{E}^{-1}[R_{T-2}R_{T-2}']\mathbb{E}[q_{T-2}R_{T-2}], \\
\mathbb{E}[v_{T-2}] =& -\Big(\mathbb{E}[R_{T-2}R_{T-2}']-\mathbb{E}[R_{T-2}]\mathbb{E}[R_{T-2}']\Big)^{-1} \\
& \times\Big(\mathbb{E}[c_{T-2}R_{T-2}]-\mathbb{E}[c_{T-2}]\mathbb{E}[R_{T-2}]-\frac{\lambda\zeta_{T-1}}{2\xi_{T-1}}\mathbb{E}[R_{T-2}] \\
& -\frac{\eta_{T-1}}{\xi_{T-1}}\big(\mathbb{E}[q_{T-2}R_{T-2}]-\mathbb{E}[q_{T-2}]\mathbb{E}[R_{T-2}]\big)\mathbb{E}[l_{T-2}]\Big).
\end{aligned}
\tag{6.18}
$$

Substituting the optimal strategies back to (6.17), we obtain

$$
\begin{aligned}
& J_{T-2}(\mathbb{E}[w_{T-2}], w_{T-2}-\mathbb{E}[w_{T-2}], \mathbb{E}[l_{T-2}], l_{T-2}-\mathbb{E}[l_{T-2}]) \\
= \ & \xi_{T-1}(n_{T-2})^2\big(w_{T-2}-\mathbb{E}[w_{T-2}]\big)^2 + \xi_{T-1}(\mathbb{E}[(c_{T-2})^2]-\mathbb{E}[c_{T-2}]^2) \\
& +\epsilon_{T-1}\mathbb{E}[q_{T-2}^2](l_{T-2}-\mathbb{E}[l_{T-2}])^2 + \epsilon_{T-1}\Big(\mathbb{E}[q_{T-2}^2]-\mathbb{E}[q_{T-2}]^2\Big)\mathbb{E}[l_{T-2}]^2 \\
& -2\eta_{T-1}n_{T-2}\mathbb{E}[q_{T-2}](w_{T-2}-\mathbb{E}[w_{T-2}])(l_{T-2}-\mathbb{E}[l_{T-2}]) \\
& -\lambda\zeta_{T-1}n_{T-2}\mathbb{E}[w_{T-2}] - \lambda\zeta_{T-1}\mathbb{E}[c_{T-2}] + \lambda\theta_{T-1}\mathbb{E}[q_{T-2}]\mathbb{E}[l_{T-2}] \\
& +\delta_{T-1}(\mathbb{E}[q_{T-2}]\mathbb{E}[l_{T-2}])^2 + \Delta_{T-1} - \xi_{T-1}(n_{T-2})^2 B_{T-2}(w_{T-2}-\mathbb{E}[w_{T-2}])^2 \\
& -\eta_{T-1}^2\xi_{T-1}^{-1}B'_{T-2}(l_{T-2}-\mathbb{E}[l_{T-2}])^2 \\
& +2\eta_{T-1}n_{T-2}\overline{B}_{T-2}(w_{T-2}-\mathbb{E}[w_{T-2}])(l_{T-2}-\mathbb{E}[l_{T-2}]) \\
& -2\eta_{T-1}\big(\mathbb{E}[q_{T-2}c_{T-2}]-\mathbb{E}[q_{T-2}]\mathbb{E}[c_{T-2}]\big)\mathbb{E}[l_{T-2}] \\
& -\xi_{T-1}\Big(\mathbb{E}[c_{T-2}R'_{T-2}]-\mathbb{E}[c_{T-2}]\mathbb{E}[R'_{T-2}] - \frac{\lambda\zeta_{T-1}}{2\xi_{T-1}}\mathbb{E}[R'_{T-2}] \\
& -\frac{\eta_{T-1}}{\xi_{T-1}}(\mathbb{E}[q_{T-2}R'_{T-2}]-\mathbb{E}[q_{T-2}]\mathbb{E}[R'_{T-2}])\mathbb{E}[l_{T-2}]\Big) \\
& \times\Big(\mathbb{E}[R_{T-2}R'_{T-2}]-\mathbb{E}[R_{T-2}]\mathbb{E}[R'_{T-2}]\Big)^{-1} \\
& \times\Big(\mathbb{E}[c_{T-2}R_{T-2}]-\mathbb{E}[c_{T-2}]\mathbb{E}[R_{T-2}] - \frac{\lambda\zeta_{T-1}}{2\xi_{T-1}}\mathbb{E}[R_{T-2}] \\
& -\frac{\eta_{T-1}}{\xi_{T-1}}\big(\mathbb{E}[q_{T-2}R_{T-2}]-\mathbb{E}[q_{T-2}]\mathbb{E}[R_{T-2}]\big)\mathbb{E}[l_{T-2}]\Big) \\
& +p_{T-2}(w_{T-2}-l_{T-2}-\mathbb{E}[w_{T-2}-l_{T-2}])^2 - p_{T-2}\lambda\mathbb{E}[w_{T-2}-l_{T-2}].
\end{aligned}
$$

Thus,

$$
\begin{aligned}
& J_{T-2}(\mathbb{E}[w_{T-2}], w_{T-2}-\mathbb{E}[w_{T-2}], \mathbb{E}[l_{T-2}], l_{T-2}-\mathbb{E}[l_{T-2}]) \\
= \;& \xi_{T-1}(n_{T-2})^2(1-B_{T-2})\big(w_{T-2}-\mathbb{E}[w_{T-2}]\big)^2 + \xi_{T-1}(\mathbb{E}[(c_{T-2})^2]-\mathbb{E}[c_{T-2}]^2) \\
& + \big(\epsilon_{T-1}\mathbb{E}[q_{T-2}^2] - \eta_{T-1}^2\xi_{T-1}^{-1}B'_{T-2}\big)(l_{T-2}-\mathbb{E}[l_{T-2}])^2 \\
& -2\eta_{T-1}\big(n_{T-2}\mathbb{E}[q_{T-2}] - n_{T-2}\overline{B}_{T-2}\big)(w_{T-2}-\mathbb{E}[w_{T-2}])(l_{T-2}-\mathbb{E}[l_{T-2}]) \\
& +\epsilon_{T-1}\Big(\mathbb{E}[q_{T-2}^2]-\mathbb{E}[q_{T-2}]^2\Big)\mathbb{E}[l_{T-2}]^2 \\
& -\lambda\zeta_{T-1}n_{T-2}\mathbb{E}[w_{T-2}] - \lambda\zeta_{T-1}\mathbb{E}[c_{T-2}] + \lambda\theta_{T-1}\mathbb{E}[q_{T-2}]\mathbb{E}[l_{T-2}] \\
& +\delta_{T-1}(\mathbb{E}[q_{T-2}]\mathbb{E}[l_{T-2}])^2 - 2\eta_{T-1}\big(\mathbb{E}[q_{T-2}c_{T-2}]-\mathbb{E}[q_{T-2}]\mathbb{E}[c_{T-2}]\big)\mathbb{E}[l_{T-2}] + \Delta_{T-1} \\
& -\xi_{T-1}\Big(\big(\mathbb{E}[c_{T-2}] + \frac{\lambda\zeta_{T-1}}{2\xi_{T-1}} - \widehat{B}_{T-2}\big)^2\frac{B_{T-2}}{1-B_{T-2}} - 2\big(\mathbb{E}[c_{T-2}] + \frac{\lambda\zeta_{T-1}}{2\xi_{T-1}} - \widehat{B}_{T-2}\big)\widehat{B}_{T-2} \\
& +\widetilde{B}_{T-2} - \widehat{B}_{T-2}^2\Big) \\
& +2\eta_{T-1}\Big(\frac{\big(\mathbb{E}[c_{T-2}] + \frac{\lambda\zeta_{T-1}}{2\xi_{T-1}}\big)\big(\mathbb{E}[q_{T-2}]B_{T-2} - \overline{B}_{T-2}\big) + \widehat{B}_{T-2}\overline{B}_{T-2} - \widehat{B}_{T-2}\mathbb{E}[q_{T-2}]}{1-B_{T-2}} \\
& +\overrightarrow{B}_{T-2}\Big)\mathbb{E}[l_{T-2}] - \frac{\eta_{T-1}^2}{\xi_{T-1}}\Big(B'_{T-2} - \mathbb{E}[q_{T-2}]^2 + \frac{\big(\overline{B}_{T-2}-\mathbb{E}[q_{T-2}]\big)^2}{1-B_{T-2}}\Big)\mathbb{E}[l_{T-2}]^2 \\
& +p_{T-2}(w_{T-2}-l_{T-2}-\mathbb{E}[w_{T-2}-l_{T-2}])^2 - p_{T-2}\lambda\mathbb{E}[w_{T-2}-l_{T-2}] \\
= \;& \xi_{T-2}\big(w_{T-2}-\mathbb{E}[w_{T-2}]\big)^2 - 2\eta_{T-2}(w_{T-2}-\mathbb{E}[w_{T-2}])(l_{T-2}-\mathbb{E}[l_{T-2}]) \\
& +\epsilon_{T-2}(l_{T-2}-\mathbb{E}[l_{T-2}])^2 - \lambda\zeta_{T-2}\mathbb{E}[w_{T-2}] + \lambda\theta_{T-2}\mathbb{E}[l_{T-2}] + \delta_{T-2}\mathbb{E}[l_{T-2}]^2 + \Delta_{T-2},
\end{aligned}
\tag{6.19}
$$

where

$$\begin{aligned}
\xi_{T-2} = &\quad \xi_{T-1}(n_{T-2})^2(1-B_{T-2})+p_{T-2},\\
\eta_{T-2} = &\quad \eta_{T-1}\big(n_{T-2}\mathbb{E}[q_{T-2}]-n_{T-2}\overline{B}_{T-2}\big)+p_{T-2},\\
\epsilon_{T-2} = &\quad \epsilon_{T-1}\mathbb{E}[q_{T-2}^2]-\eta_{T-1}^2\xi_{T-1}^{-1}B'_{T-2}+p_{T-2},\\
\zeta_{T-2} = &\quad \zeta_{T-1}n_{T-2}+p_{T-2},\\
\theta_{T-2} = &\quad \theta_{T-1}\mathbb{E}[q_{T-2}]-\frac{2}{\lambda}\eta_{T-1}\big(\mathbb{E}[q_{T-2}c_{T-2}]-\mathbb{E}[q_{T-2}]\mathbb{E}[c_{T-2}]\big)+\frac{2\eta_{T-1}}{\lambda}\\
&\times\Bigg(\frac{\big(\mathbb{E}[c_{T-2}]+\frac{\lambda\zeta_{T-1}}{2\xi_{T-1}}\big)\big(\mathbb{E}[q_{T-2}]B_{T-2}-\overline{B}_{T-2}\big)+\widehat{B}_{T-2}\overline{B}_{T-2}-\widehat{B}_{T-2}\mathbb{E}[q_{T-2}]}{1-B_{T-2}}\\
&+\overrightarrow{B}_{T-2}\Bigg)+p_{T-2},\\
\delta_{T-2} = &\quad \epsilon_{T-1}\Big(\mathbb{E}[q_{T-2}^2]-\mathbb{E}[q_{T-2}]^2\Big)+\delta_{T-1}(\mathbb{E}[q_{T-2}])^2\\
&-\frac{\eta_{T-1}^2}{\xi_{T-1}}\Big(B'_{T-2}-\mathbb{E}[q_{T-2}]^2+\frac{\big(\overline{B}_{T-2}-\mathbb{E}[q_{T-2}]\big)^2}{1-B_{T-2}}\Big),\\
\Delta_{T-2} = &\quad \Delta_{T-1}+\xi_{T-1}\mathbb{E}[(c_{T-2})^2]-\xi_{T-1}\mathbb{E}[c_{T-2}]^2-\lambda\zeta_{T-1}\mathbb{E}[c_{T-2}]\\
&-\xi_{T-1}\Big(\big(\mathbb{E}[c_{T-2}]+\frac{\lambda\zeta_{T-1}}{2\xi_{T-1}}-\widehat{B}_{T-2}\big)^2\frac{B_{T-2}}{1-B_{T-2}}\\
&-2\big(\mathbb{E}[c_{T-2}]+\frac{\lambda\zeta_{T-1}}{2\xi_{T-1}}-\widehat{B}_{T-2}\big)\widehat{B}_{T-2}+\widetilde{B}_{T-2}-\widehat{B}_{T-2}^2\Big).
\end{aligned} \tag{6.20}$$

Assume that the following equation (6.21) holds at time $t+1$, we prove it according to the mathematical induction.

$$\begin{aligned}
&J_{t+1}(\mathbb{E}[w_{t+1}],w_{t+1}-\mathbb{E}[w_{t+1}],\mathbb{E}[l_{t+1}],l_{t+1}-\mathbb{E}[l_{t+1}])\\
= &\quad \xi_{t+1}\big(w_{t+1}-\mathbb{E}[w_{t+1}]\big)^2-2\eta_{t+1}(w_{t+1}-\mathbb{E}[w_{t+1}])(l_{t+1}-\mathbb{E}[l_{t+1}])\\
&+\epsilon_{t+1}(l_{t+1}-\mathbb{E}[l_{t+1}])^2-\lambda\zeta_{t+1}\mathbb{E}[w_{t+1}]+\lambda\theta_{t+1}\mathbb{E}[l_{t+1}]+\delta_{t+1}\mathbb{E}[l_{t+1}]^2+\Delta_{t+1},
\end{aligned} \tag{6.21}$$

where

$$
\begin{aligned}
\xi_{t+1} = \;& \xi_{t+2}(n_{t+1})^2(1-B_{t+1})+p_{t+1},\\
\eta_{t+1} = \;& \eta_{t+2}\big(n_{t+1}\mathbb{E}[q_{t+1}]-n_{t+1}\overline{B}_{t+1}\big)+p_{t+1},\\
\epsilon_{t+1} = \;& \epsilon_{t+2}\mathbb{E}[q_{t+1}^2]-\eta_{t+2}^2\xi_{t+2}^{-1}B'_{t+1}+p_{t+1},\\
\zeta_{t+1} = \;& \zeta_{t+2}n_{t+1}+p_{t+1},\\
\theta_{t+1} = \;& \theta_{t+2}\mathbb{E}[q_{t+1}]-\frac{2}{\lambda}\eta_{t+2}\big(\mathbb{E}[q_{t+1}c_{t+1}]-\mathbb{E}[q_{t+1}]\mathbb{E}[c_{t+1}]\big)+\frac{2\eta_{t+2}}{\lambda}\\
& \times\Big(\frac{\big(\mathbb{E}[c_{t+1}]+\frac{\lambda\zeta_{t+2}}{2\xi_{t+2}}\big)\big(\mathbb{E}[q_{t+1}]B_{t+1}-\overline{B}_{t+1}\big)+\widehat{B}_{t+1}\overline{B}_{t+1}-\widehat{B}_{t+1}\mathbb{E}[q_{t+1}]}{1-B_{t+1}}\\
& +\overrightarrow{B}_{t+1}\Big)+p_{t+1},\\
\delta_{t+1} = \;& \epsilon_{t+2}\Big(\mathbb{E}[q_{t+1}^2]-\mathbb{E}[q_{t+1}]^2\Big)+\delta_{t+2}(\mathbb{E}[q_{t+1}])^2\\
& -\frac{\eta_{t+2}^2}{\xi_{t+2}}\Big({B'}_{t+1}-\mathbb{E}[q_{t+1}]^2+\frac{\big(\overline{B}_{t+1}-\mathbb{E}[q_{t+1}]\big)^2}{1-B_{t+1}}\Big),\\
\Delta_{t+1} = \;& \Delta_{t+2}+\xi_{t+2}\mathbb{E}[(c_{t+1})^2]-\xi_{t+2}\mathbb{E}[c_{t+1}]^2-\lambda\zeta_{t+2}\mathbb{E}[c_{t+1}]\\
& -\xi_{t+2}\Big(\big(\mathbb{E}[c_{t+1}]+\frac{\lambda\zeta_{t+2}}{2\xi_{t+2}}-\widehat{B}_{t+1}\big)^2\frac{B_{t+1}}{1-B_{t+1}}\\
& -2\big(\mathbb{E}[c_{t+1}]+\frac{\lambda\zeta_{t+2}}{2\xi_{t+2}}-\widehat{B}_{t+1}\big)\widehat{B}_{t+1}+\widetilde{B}_{t+1}-\widehat{B}_{t+1}^2\Big),
\end{aligned}
\tag{6.22}
$$

and

$$\xi_T = p_T,\ \eta_T = p_T,\ \epsilon_T = p_T,\ \zeta_T = p_T,\ \theta_T = p_T,\ \delta_T = 0,\ \Delta_T = 0.$$

According to equation (6.21) and (6.22), we derive the result at time t.

$$\begin{aligned}
& J_t(\mathbb{E}[w_t], w_t-\mathbb{E}[w_t], \mathbb{E}[l_t], l_t-\mathbb{E}[l_t]) \\
= \;& \mathbb{E}\big[J_{t+1}(\mathbb{E}[w_{t+1}], w_{t+1}-\mathbb{E}[w_{t+1}], \mathbb{E}[l_{t+1}], l_{t+1}-\mathbb{E}[l_{t+1}])|\mathcal{F}_t\big] \\
& +p_t(w_t-l_t-\mathbb{E}[w_t-l_t])^2 - p_t\lambda\mathbb{E}[w_t-l_t] \\
= \;& \mathbb{E}\big[\xi_{t+1}\big(w_{t+1}-\mathbb{E}[w_{t+1}]\big)^2 - 2\eta_{t+1}(w_{t+1}-\mathbb{E}[w_{t+1}])(l_{t+1}-\mathbb{E}[l_{t+1}]) + \epsilon_{t+1}(l_{t+1}-\mathbb{E}[l_{t+1}])^2 \\
& \quad -\lambda\zeta_{t+1}\mathbb{E}[w_{t+1}] + \lambda\theta_{t+1}\mathbb{E}[l_{t+1}] + \delta_{t+1}\mathbb{E}[l_{t+1}]^2 + \Delta_{t+1}|\mathcal{F}_t\big] \\
& +p_t(w_t-l_t-\mathbb{E}[w_t-l_t])^2 - p_t\lambda\mathbb{E}[w_t-l_t] \\
= \;& \mathbb{E}\Big[\xi_{t+1}\Big[n_t(w_t-\mathbb{E}[w_t]) + R_t'(v_t-\mathbb{E}[v_t]) + (R_t'-\mathbb{E}[R_t'])\mathbb{E}[v_t] + (c_t-\mathbb{E}[c_t])\Big]^2 \\
& -2\eta_{t+1}\Big[q_t(l_t-\mathbb{E}[l_t]) + (q_t-\mathbb{E}[q_t])\mathbb{E}[l_t]\Big]\Big[n_t(w_t-\mathbb{E}[w_t]) + R_t'(v_t-\mathbb{E}[v_t]) \\
& +(R_t'-\mathbb{E}[R_t'])\mathbb{E}[v_t] + (c_t-\mathbb{E}[c_t])\Big] + \epsilon_{t+1}\big[q_t(l_t-\mathbb{E}[l_t]) + (q_t-\mathbb{E}[q_t])\mathbb{E}[l_t]\big]^2 \\
& -\lambda\zeta_{t+1}\big(n_t\mathbb{E}[w_t] + \mathbb{E}[R_t']\mathbb{E}[v_t] + \mathbb{E}[c_t]\big) + \lambda\theta_{t+1}\mathbb{E}[q_t]\mathbb{E}[l_t] + \delta_{t+1}(\mathbb{E}[q_t]\mathbb{E}[l_t])^2 + \Delta_{t+1}\Big|\mathcal{F}_t\Big] \\
& +p_t(w_t-l_t-\mathbb{E}[w_t-l_t])^2 - p_t\lambda\mathbb{E}[w_t-l_t] \\
= \;& \xi_{t+1}\Big[n_t^2\big(w_t-\mathbb{E}[w_t]\big)^2 + \big(v_t-\mathbb{E}[v_t]\big)'\mathbb{E}[R_tR_t']\big(v_t-\mathbb{E}[v_t]\big) \\
& +2n_t\big(w_t-\mathbb{E}[w_t]\big)\mathbb{E}[R_t']\big(v_t-\mathbb{E}[v_t]\big) + \mathbb{E}[v_t']\big(\mathbb{E}[R_tR_t'] - \mathbb{E}[R_t]\mathbb{E}[R_t']\big)\mathbb{E}[v_t] \\
& +(\mathbb{E}[(c_t)^2] - \mathbb{E}[c_t]^2) + 2\Big(\mathbb{E}[c_tR_t'] - \mathbb{E}[c_t]\mathbb{E}[R_t']\Big)\mathbb{E}[v_t]\Big] \\
& -2\eta_{t+1}\Big[n_t\mathbb{E}[q_t]\big(l_t-\mathbb{E}[l_t]\big)\big(w_t-\mathbb{E}[w_t]\big) + \mathbb{E}[q_tR_t']\big(l_t-\mathbb{E}[l_t]\big)\big(v_t-\mathbb{E}[v_t]\big) \\
& +\big(\mathbb{E}[q_tR_t'] - \mathbb{E}[q_t]\mathbb{E}[R_t']\big)\mathbb{E}[v_t]\mathbb{E}[l_t] + \big(\mathbb{E}[q_tc_t] - \mathbb{E}[q_t]\mathbb{E}[c_t]\big)\mathbb{E}[l_t]\Big]\Big] \\
& +\epsilon_{t+1}\Big[\mathbb{E}[q_t^2](l_t-\mathbb{E}[l_t])^2 + (\mathbb{E}[q_t^2]-(\mathbb{E}[q_t])^2)(\mathbb{E}[l_t])^2\Big] \\
& -\lambda\zeta_{t+1}\big(n_t\mathbb{E}[w_t] + \mathbb{E}[R_t']\mathbb{E}[v_t] + \mathbb{E}[c_t]\big) + \lambda\theta_{t+1}\mathbb{E}[q_t]\mathbb{E}[l_t] + \delta_{t+1}(\mathbb{E}[q_t]\mathbb{E}[l_t])^2 + \Delta_{t+1} \\
& +p_t(w_t-l_t-\mathbb{E}[w_t-l_t])^2 - p_t\lambda\mathbb{E}[w_t-l_t].
\end{aligned} \tag{6.23}$$

The optimal strategies at period t can be derived from the above equation corresponding to v_t

$$\begin{aligned}
v_t-\mathbb{E}[v_t] &= -n_t(w_t-\mathbb{E}[w_t])\mathbb{E}^{-1}[R_tR_t']\mathbb{E}[(R_t)] + \eta_{t+1}\xi_{t+1}^{-1}\big(l_t-\mathbb{E}[l_t]\big)\mathbb{E}^{-1}[R_tR_t']\mathbb{E}[q_tR_t], \\
\mathbb{E}[v_t] &= -\Big(\mathbb{E}[R_tR_t'] - \mathbb{E}[R_t]\mathbb{E}[R_t']\Big)^{-1} \\
& \quad \times\Big(\mathbb{E}[c_tR_t] - \mathbb{E}[c_t]\mathbb{E}[R_t] - \frac{\lambda\zeta_{t+1}}{2\xi_{t+1}}\mathbb{E}[R_t] - \frac{\eta_{t+1}}{\xi_{t+1}}\big(\mathbb{E}[q_tR_t] - \mathbb{E}[q_t]\mathbb{E}[R_t]\big)\mathbb{E}[l_t]\Big).
\end{aligned} \tag{6.24}$$

Substituting the optimal strategies back to (6.23), we obtain

$$
\begin{aligned}
& J_t(\mathbb{E}[w_t], w_t - \mathbb{E}[w_t], \mathbb{E}[l_t], l_t - \mathbb{E}[l_t]) \\
= \quad & \xi_{t+1}(n_t)^2\big(w_t - \mathbb{E}[w_t]\big)^2 + \xi_{t+1}(\mathbb{E}[(c_t)^2] - \mathbb{E}[c_t]^2) \\
& +\epsilon_{t+1}\mathbb{E}[q_t^2](l_t - \mathbb{E}[l_t])^2 + \epsilon_{t+1}\Big(\mathbb{E}[q_t^2] - \mathbb{E}[q_t]^2\Big)\mathbb{E}[l_t]^2 \\
& -2\eta_{t+1}n_t\mathbb{E}[q_t](w_t - \mathbb{E}[w_t])(l_t - \mathbb{E}[l_t]) \\
& -\lambda\zeta_{t+1}n_t\mathbb{E}[w_t] - \lambda\zeta_{t+1}\mathbb{E}[c_t] + \lambda\theta_{t+1}\mathbb{E}[q_t]\mathbb{E}[l_t] + \delta_{t+1}(\mathbb{E}[q_t]\mathbb{E}[l_t])^2 + \Delta_{t+1} \\
& -\xi_{t+1}(n_t)^2B_t(w_t - \mathbb{E}[w_t])^2 - \eta_{t+1}^2\xi_{t+1}^{-1}B_t'(l_t - \mathbb{E}[l_t])^2 \\
& +2\eta_{t+1}n_t\overline{B}_t(w_t - \mathbb{E}[w_t])(l_t - \mathbb{E}[l_t]) \\
& -2\eta_{t+1}\big(\mathbb{E}[q_tc_t] - \mathbb{E}[q_t]\mathbb{E}[c_t]\big)\mathbb{E}[l_t] \\
& -\xi_{t+1}\Big(\mathbb{E}[c_tR_t'] - \mathbb{E}[c_t]\mathbb{E}[R_t'] - \frac{\lambda\zeta_{t+1}}{2\xi_{t+1}}\mathbb{E}[R_t'] - \frac{\eta_{t+1}}{\xi_{t+1}}(\mathbb{E}[q_tR_t'] - \mathbb{E}[q_t]\mathbb{E}[R_t'])\mathbb{E}[l_t]\Big) \\
& \times\Big(\mathbb{E}[R_tR_t'] - \mathbb{E}[R_t]\mathbb{E}[R_t']\Big)^{-1} \\
& \times\Big(\mathbb{E}[c_tR_t] - \mathbb{E}[c_t]\mathbb{E}[R_t] - \frac{\lambda\zeta_{t+1}}{2\xi_{t+1}}\mathbb{E}[R_t] - \frac{\eta_{t+1}}{\xi_{t+1}}\big(\mathbb{E}[q_tR_t] - \mathbb{E}[q_t]\mathbb{E}[R_t]\big)\mathbb{E}[l_t]\Big) \\
& +p_t(w_t - l_t - \mathbb{E}[w_t - l_t])^2 - p_t\lambda\mathbb{E}[w_t - l_t] \\
= \quad & \xi_{t+1}(n_t)^2(1 - B_t)\big(w_t - \mathbb{E}[w_t]\big)^2 + \xi_{t+1}(\mathbb{E}[(c_t)^2] - \mathbb{E}[c_t]^2) \\
& +\big(\epsilon_{t+1}\mathbb{E}[q_t^2] - \eta_{t+1}^2\xi_{t+1}^{-1}B_t'\big)(l_t - \mathbb{E}[l_t])^2 \\
& -2\eta_{t+1}\big(n_t\mathbb{E}[q_t] - n_t\overline{B}_t\big)(w_t - \mathbb{E}[w_t])(l_t - \mathbb{E}[l_t]) \\
& +\epsilon_{t+1}\Big(\mathbb{E}[q_t^2] - \mathbb{E}[q_t]^2\Big)\mathbb{E}[l_t]^2 - \lambda\zeta_{t+1}n_t\mathbb{E}[w_t] - \lambda\zeta_{t+1}\mathbb{E}[c_t] + \lambda\theta_{t+1}\mathbb{E}[q_t]\mathbb{E}[l_t] \\
& +\delta_{t+1}(\mathbb{E}[q_t]\mathbb{E}[l_t])^2 - 2\eta_{t+1}\big(\mathbb{E}[q_tc_t] - \mathbb{E}[q_t]\mathbb{E}[c_t]\big)\mathbb{E}[l_t] + \Delta_{t+1} \\
& -\xi_{t+1}\Big(\big(\mathbb{E}[c_t] + \frac{\lambda\zeta_{t+1}}{2\xi_{t+1}} - \widehat{B}_t\big)^2\frac{B_t}{1 - B_t} - 2\big(\mathbb{E}[c_t] + \frac{\lambda\zeta_{t+1}}{2\xi_{t+1}} - \widehat{B}_t\big)\widehat{B}_t + \widetilde{B}_t - \widehat{B}_t^2\Big) \\
& +2\eta_{t+1}\Big(\frac{\big(\mathbb{E}[c_t] + \frac{\lambda\zeta_{t+1}}{2\xi_{t+1}}\big)\big(\mathbb{E}[q_t]B_t - \overline{B}_t\big) + \widehat{B}_t\overline{B}_t - \widehat{B}_t\mathbb{E}[q_t]}{1 - B_t} + \overrightarrow{B}_t\Big)\mathbb{E}[l_t] \\
& -\frac{\eta_{t+1}^2}{\xi_{t+1}}\Big(B'_t - \mathbb{E}[q_t]^2 + \frac{\big(\overline{B}_t - \mathbb{E}[q_t]\big)^2}{1 - B_t}\Big)\mathbb{E}[l_t]^2 \\
& +p_t(w_t - l_t - \mathbb{E}[w_t - l_t])^2 - p_t\lambda\mathbb{E}[w_t - l_t].
\end{aligned}
$$

Thus,

$$
\begin{aligned}
& J_t(\mathbb{E}[w_t], w_t - \mathbb{E}[w_t], \mathbb{E}[l_t], l_t - \mathbb{E}[l_t]) \\
= \; & \xi_t\big(w_t - \mathbb{E}[w_t]\big)^2 - 2\eta_t(w_t - \mathbb{E}[w_t])(l_t - \mathbb{E}[l_t]) + \epsilon_t(l_t - \mathbb{E}[l_t])^2 \\
& -\lambda\zeta_t\mathbb{E}[w_t] + \lambda\theta_t\mathbb{E}[l_t] + \delta_t\mathbb{E}[l_t]^2 + \Delta_t,
\end{aligned}
\tag{6.25}
$$

where

$$
\begin{aligned}
\xi_t = \; & \xi_{t+1}(n_t)^2(1 - B_t) + p_t, \\
\eta_t = \; & \eta_{t+1}\big(n_t\mathbb{E}[q_t] - n_t\overline{B}_t\big) + p_t, \\
\epsilon_t = \; & \epsilon_{t+1}\mathbb{E}[q_t^2] - \eta_{t+1}^2\xi_{t+1}^{-1}B_t' + p_t, \\
\zeta_t = \; & \zeta_{t+1}n_t + p_t, \\
\theta_t = \; & \theta_{t+1}\mathbb{E}[q_t] - \frac{2}{\lambda}\eta_{t+1}\big(\mathbb{E}[q_tc_t] - \mathbb{E}[q_t]\mathbb{E}[c_t]\big) \\
& + \frac{2\eta_{t+1}}{\lambda}\Bigg(\frac{\big(\mathbb{E}[c_t] + \frac{\lambda\zeta_{t+1}}{2\xi_{t+1}}\big)\big(\mathbb{E}[q_t]B_t - \overline{B}_t\big) + \widehat{B}_t\overline{B}_t - \widehat{B}_t\mathbb{E}[q_t]}{1 - B_t} + \overrightarrow{B}_t\Bigg) + p_t, \\
\delta_t = \; & \epsilon_{t+1}\Big(\mathbb{E}[q_t^2] - \mathbb{E}[q_t]^2\Big) + \delta_{t+1}(\mathbb{E}[q_t])^2 - \frac{\eta_{t+1}^2}{\xi_{t+1}}\Big({B'}_t - \mathbb{E}[q_t]^2 + \frac{\big(\overline{B}_t - \mathbb{E}[q_t]\big)^2}{1 - B_t}\Big), \\
\Delta_t = \; & \Delta_{t+1} + \xi_{t+1}\mathbb{E}[(c_t)^2] - \xi_{t+1}\mathbb{E}[c_t]^2 - \lambda\zeta_{t+1}\mathbb{E}[c_t] \\
& -\xi_{t+1}\Big(\big(\mathbb{E}[c_t] + \frac{\lambda\zeta_{t+1}}{2\xi_{t+1}} - \widehat{B}_t\big)^2\frac{B_t}{1 - B_t} - 2\big(\mathbb{E}[c_t] + \frac{\lambda\zeta_{t+1}}{2\xi_{t+1}} - \widehat{B}_t\big)\widehat{B}_t + \widetilde{B}_t - \widehat{B}_t^2\Big).
\end{aligned}
\tag{6.26}
$$

Substituting $\mathbb{E}[v_t^*]$ to the dynamic equation in (6.3) yields

$$
\mathbb{E}[w_{t+1}] = n_t\mathbb{E}[w_t] + \Big(\mathbb{E}[c_t] + \frac{\lambda\zeta_{t+1}}{2\xi_{t+1}} - \widehat{B}_t - \mathbb{E}[q_t] + \overline{B}_t\Big)\frac{B_t}{1 - B_t} - \widehat{B}_t + \overline{B}_t + \mathbb{E}[c_t].
\tag{6.27}
$$

Therefore,

$$
\begin{aligned}
\mathbb{E}[w_t] = \; & w_0\prod_{k=0}^{t-1}n_k + \sum_{j=0}^{t-1}\Big(\big(\mathbb{E}[c_j] + \frac{\lambda\zeta_{j+1}}{2\xi_{j+1}} - \widehat{B}_j - \mathbb{E}[q_j] + \overline{B}_j\big)\frac{B_j}{1 - B_j} \\
& -\widehat{B}_j + \overline{B}_j + \mathbb{E}[c_j]\Big)\prod_{l=j+1}^{t-1}n_l.
\end{aligned}
\tag{6.28}
$$

Typically, we know the optimal value of (6.6) is equal to J_0. Thus,

$$
\begin{aligned}
& \sum_{t=1}^{T} p_t \mathrm{Var}(w_t) \\
= \;& J_0(\mathbb{E}[w_0], w_0 - \mathbb{E}[w_0], \mathbb{E}[l_0], l_0 - \mathbb{E}[l_0]) + \sum_{t=1}^{T} p_t \lambda \mathbb{E}[w_t] \\
= \;& -\lambda\zeta_0 w_0 + \lambda\theta_0 l_0 + \delta_0 l_0^2 + \Delta_0 + \sum_{t=1}^{T} p_t \lambda \Bigg(w_0 \prod_{k=0}^{t-1} n_k + \sum_{j=0}^{t-1} \Big((\mathbb{E}[c_j] + \frac{\lambda\zeta_{j+1}}{2\xi_{j+1}} \\
& -\widehat{B}_j - \mathbb{E}[q_j] + \overline{B}_j) \tfrac{B_j}{1-B_j} - \widehat{B}_j + \overline{B}_j + \mathbb{E}[c_j] \Big) \prod_{l=j+1}^{t-1} n_l \Bigg).
\end{aligned}
\tag{6.29}
$$

□

The optimal strategy obtained in Theorem 6.1 will keep the same as the result established in section 5.2 if the exit time is determined by T.

6.4 Numerical Example

Example 6.1. Consider the example in Section 5. Here we ignore the case of uncorrelation between R_t and c_t, i.e., the return rates and cash flow are correlated.

Table 6.1: Data for assets and cashflow example

	SP	EM	MS	cashflow	liability
Expected return	14%	16%	17%	1	10%
Standard deviation	18.5%	30%	24%	20%	20%
Correlation coefficient					
SP	1	0.64	0.79	ρ_1	$\widehat{\rho}_1$
EM	0.64	1	0.75	ρ_2	$\widehat{\rho}_2$
MS	0.79	0.75	1	ρ_3	$\widehat{\rho}_3$
cashflow	ρ_1	ρ_2	ρ_3	1	$\widehat{\rho}_4$
liability	$\widehat{\rho}_1$	$\widehat{\rho}_2$	$\widehat{\rho}_3$	$\widehat{\rho}_4$	1

Thus,

$$\mathbb{E}[R_t] = \begin{pmatrix} 0.09 \\ 0.11 \\ 0.12 \end{pmatrix}, \ \mathrm{Cov}(R_t) = \begin{pmatrix} 0.0342 & 0.0355 & 0.0351 \\ 0.0355 & 0.0900 & 0.0540 \\ 0.0351 & 0.0540 & 0.0576 \end{pmatrix},$$

$$\mathbb{E}[R_t R_t'] = \begin{pmatrix} 0.0423 & 0.0454 & 0.0459 \\ 0.0454 & 0.1021 & 0.0672 \\ 0.0459 & 0.0672 & 0.0720 \end{pmatrix}.$$

Using the formula mentioned in section 5.3, we have $\mathbb{E}[c_t R_t] = (0.0898, 0.1510, 0.1440)'$. Moreover,

$$Y_1 = \mathbb{E}^{-1}[R_t R_t']\mathbb{E}[R_t] = \begin{pmatrix} 1.0589 \\ -0.1196 \\ 1.1033 \end{pmatrix},$$

$$Y_2 = \mathbb{E}^{-1}[R_t R_t']\mathbb{E}[c_t R_t] = \begin{pmatrix} -0.3490 \\ 0.4493 \\ 1.6365 \end{pmatrix},$$

$$Y_3 = \mathbb{E}^{-1}[R_t R_t']\mathbb{E}[q_t R_t] = \begin{pmatrix} -1.0411 \\ 0.1754 \\ 0.8209 \end{pmatrix}.$$

$$\begin{aligned} &B_t = 0.2145, \quad \widehat{B}_t = 0.2144, \quad \widetilde{B}_t = 0.2507, \\ &\overline{B}_t = 0.0241, \quad {B'}_t = 0.0218, \quad \overrightarrow{B}_t = 0.0488. \end{aligned} \tag{6.30}$$

The probability mass function of an exit time τ is

$$(p_1, p_2, p_3, p_4, p_5) = (0.10, 0.15, 0.2, 0.25, 0.3),$$

respectively, for $t = 1, 2, 3, 4, 5$. Then we define the optimal expected value as follow

$$\mathbb{E}[w] = (\mathbb{E}[w_1], \mathbb{E}[w_2], \mathbb{E}[w_3], \mathbb{E}[w_4], \mathbb{E}[w_5])$$

substituting the data we have

$$\mathbb{E}[w] = (4.3710, 5.7834, 7.2450, 8.7621, 10.3403).$$

The initial wealth, initial liability and trade-off parameter are defined as $w_0 = 3$, $l_0 = 1$ and $\lambda = 1$ respectively. Thus, we substitute the data to the optimal equation

$$v_0^* = -1.05(w_0 - 4.9032)Y_1 - Y_2 + 0.1595Y_3l_0,$$

$$v_1^* = -1.05(w_1 - 6.1660)Y_1 - Y_2 + 0.2377Y_3l_1,$$

$$v_2^* = -1.05(w_2 - 7.4853)Y_1 - Y_2 + 0.3458Y_3l_2,$$

$$v_3^* = -1.05(w_3 - 8.8694)Y_1 - Y_2 + 0.5373Y_3l_3,$$

$$v_4^* = -1.05(w_4 - 10.3209)Y_1 - Y_2 + 1.0000Y_3l_4.$$

The mean and variance of the final optimal value are $\mathbb{E}^{(\tau)}(w_{5\wedge\tau}) = 8.0462$ and $\mathrm{Var}^{(\tau)}(w_{5\wedge\tau}) = 0.3901$, respectively.

Chapter 7

Conclusions and Future Work

Conclusions of the thesis are given in this chapter. We also indicate some new directions and future work related to the Mean-Variance model.

7.1 Conclusions

The thesis investigates the multi-period asset-liability mean-variance portfolio selection with cash flow. It is a nonseparable dynamic programming problem since it cannot be solved under the backward recursion. In this thesis, we first formulate the problem in deterministic terminal expectation and solve it by mean-field method. By transforming the original model to mean-field formulation, we turn it into a solvable stochastic control problem. Second, we employ dynamic programming and mathematical induction to deal with the model. By these two methods, we derive the analytical optimal strategies and optimal value of multi-period mean-variance portfolio selection problems with various kinds of constraints, such as with cash flow, uncertain exit time, liability. The relations of them are given and the effects of different constraints are illustrated by numerical examples. Our methods are showed to be much more efficient and accuracy compared with other methods in the literature.

7.2 Future Work

We list the future work of the related topics as follows.

1. This thesis supposes that there is only one deterministic market state. However, the underlying market environment is random and there are various market states in the real world. In recent years, regime-switching models have become popular for reflecting the various states of the financial market. In the future, using mean-field formulation to tackle mean-variance model with regime-switching is worthwhile and challenging.

2. Although asset-liability mean-variance portfolio selection is an important issue in modern finance theory, the time-consistent problem has not attracted much attention. In the future work, seeking for time-inconsistent optimal strategy and efficient frontier for asset-liability management is indeed meaningful.

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