DIRECTION FINDING USING TWO DIFFERENTIAL SENSORS OF POSSIBLY UNEQUAL ORDERS, WITH PERPENDICULAR ORIENTATION AND POSSIBLE DISPLACEMENT

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Direction Finding Using Two Differential Sensors of Possibly Unequal Orders, with Perpendicular Orientation and Possible Displacement

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CERTIFICATE OF ORIGINALITY

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Abstract

A differential sensor has a directional response which approximates a measurement of the kth-order spatial derivative of the pressure field. A sensor array composed of a pair perpendicularly-oriented higher-order (possibly unequal orders) differential sensors can be used to find the direction of acoustic signal sources. The objective of my research is to devise a signal-processing algorithm to estimate the direction-of-arrival (DOA) of a signal incident upon this type of sensor array. A viable approach is herein proposed. The new direction finding method relies on well-known eigen-based DOA estimation algorithms as a preceding stage. The closed-form formulas can be then directly applied when certain prior knowledge is available. This dissertation presents the derivation of closed-form formulas for elevation and azimuth angle estimation with nine possible configurations. Necessary assumptions and prior information needed for the algorithm are also discussed. Monte Carlo simulations result shows that the proposed algorithm can successfully estimate the elevation and azimuth angle of incident signal. The corresponding Cramér-Rao Bounds (CRB) are derived so as to evaluate the performance of proposed algorithm.

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Contents

List of Figures

List of Tables

Chapter 1

Introduction

1.1 Microphone Array

A microphone array is composed of a group of sensors placed in a certain spatial pattern such that the array response carries the spatial information of the incoming signals. The main objective of microphone array signal processing is to estimate parameters or the signal of interest by using the information available at the outputs of the microphone array [1]. Typical assumptions made for microphone array processing include isotropic and non-dispersive medium, plane wave propagation, far field and zero mean white noise. Applications of microphone arrays are commonly applied to following area: noise reduction, echo reduction, source separation, estimation of the number of sources and localization of signal sources.

The acoustic signal in microphone reception are often interfered by noise and reverberation. Microphone array can be effective in handling these problems by beamforming technique. Beamforming, which is widely used in sensor array signal processing, is achieved by controlling or analyzing the phase and amplitude of the received signal in each sensor. Besides the capability of noise mitigating, another important application of microphone array is the direction of arrival estimation which utilizes its advantage of directivity. Typical application of microphone array includes noise reduction, dereverberation, echo reduction, direction of arrival estimation and estimation of the number of sources.

1.2 Direction of Arrival

Direction of arrival (DOA) estimation (a.k.a direction finding), as a popular topic of array signal processing, has been widely investigated over decades. In general, the objective of DOA estimation is to find the elevation angle and azimuth angle of an incident signal, as shown in Figure 1.2. Namely, it is to locate the emitter which generates the incident signal. For acoustic signal, the emitter of

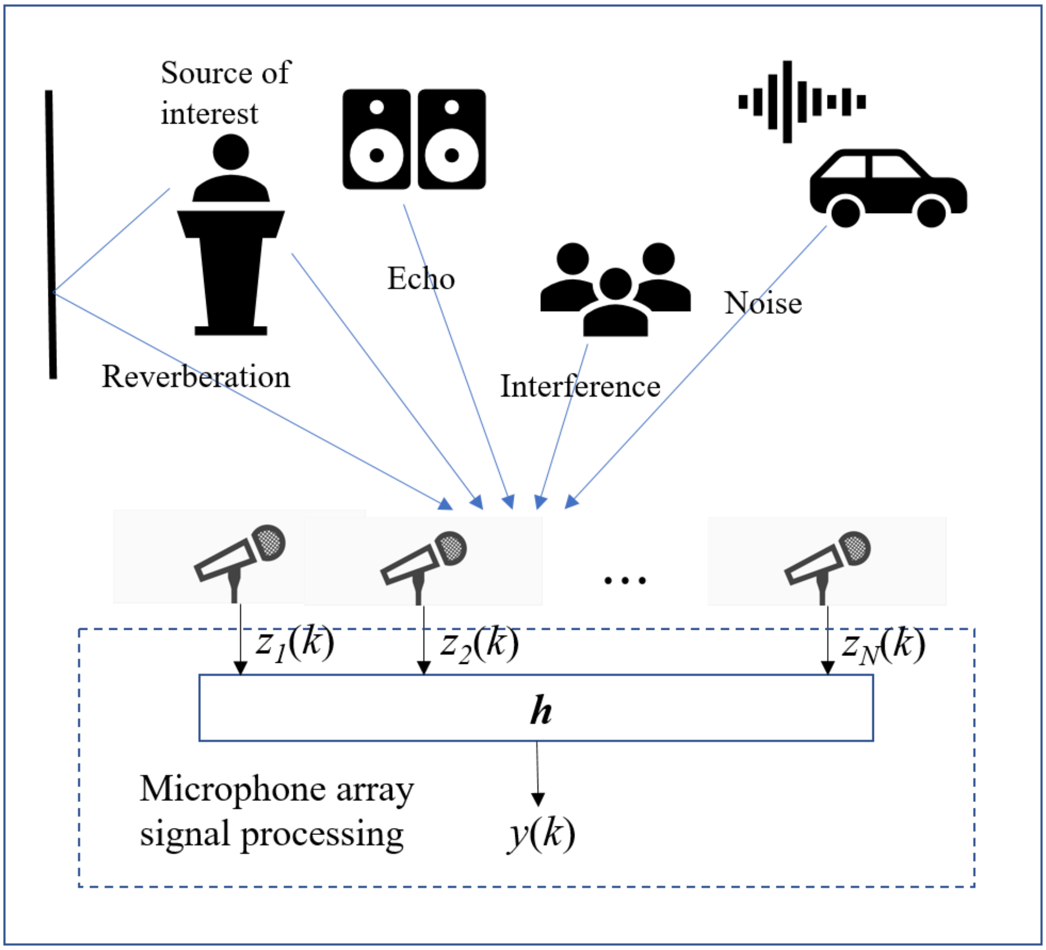

Figure 1.1: Microphone array signal processing and applications

interests usually includes

- a talker in video conferencing,
- a submarine,
- a gun shot or an artillery,
- a whale, etc.

There are already a great number of practical applications for microphone array direction finding. For example, the gun shot localization system developed by Microflown Technologies can be adopted to detect the location of sniper in battle field. Another example is the "Voice Tracker" designed by Acoustic Magic which can automatically estimate the location of a speaker.

Direction of arrival estimation algorithm is one of the popular topics in microphone (sensor) array signal processing. One important theoretical fundamental is the vector space signal model popularized by Schmidt and others. They brought

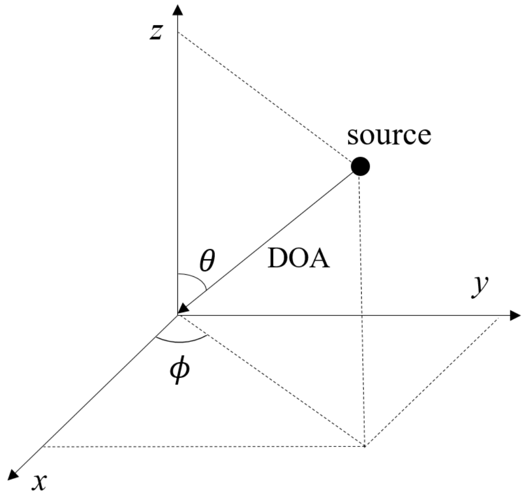

Figure 1.2: Direction of arrival

up the concept of a parametric array manifold which connects the array signal processing and powerful estimation tools including Minimum Mean-Square Estimation (MMSE), Maximum Likelihood (ML) and the Cramér-Rao Bound (CRB). These theoretic tools provide a benchmark for the sensor array direction finding performance. However, conventional DOA estimation algorithm, such as the multiple signal classification (MUSIC) [2] and ESPRIT [3], are only applicable to the pressure sensors array. These algorithms estimate the steering vector $\hat{\mathbf{a}}$ of incident signal with ambiguity of an unknown complex-value (i.e. $\hat{\mathbf{a}} = c\mathbf{a}$) by eigen-decomposing the data correlation matrix. The measurement model of acoustic vector sensor (AVS) array was first introduced by Nehorai and Paldi [4]. Ever since then, lots of research has been conducted for extending conventional DOA estimation algorithm to differential sensors. Nonetheless, little has been done for direction finding with a pair of higher order differential sensors. This M.Phil. research investigates the direction finding approach for a pair of orthogonally oriented differential sensors (with possibly unequal order) based on eigen-based parameter estimation algorithms.

1.3 Differential Sensors

Highly directive microphones are useful, especially for enhanced "random efficiency" (i.e., for improved suppression of background noises/interference off-axis) and for a farther "distance factor" (i.e., the spatial reach of the microphone on-axis). One type of directional microphones/hydrophones is the differential sensor.

A *first*-order differential sensor (a.k.a. a "pressure gradient" sensor) is often implemented by measuring the pressure difference on the two sides of a diaphragm. As this first-order spatial derivative is proportional to the particle velocity, the first-order differential sensor is also called a uni-axial "velocity sensor" or a "velocity hydrophone". More mathematically, such a first-order differential sensor would have a dipole-like directional response of $\cos(\phi)$, where $\phi \in [0, 2\pi)$ denotes the incident source's angle measured with respect to the sensor axis. This represents a figure-8, because their $\cos(\phi)$ gain response resembles the digit "8". This response is bidirectional in nature, sensitive equally to incident energy from the front as well as from the back, but little sideway pickup.

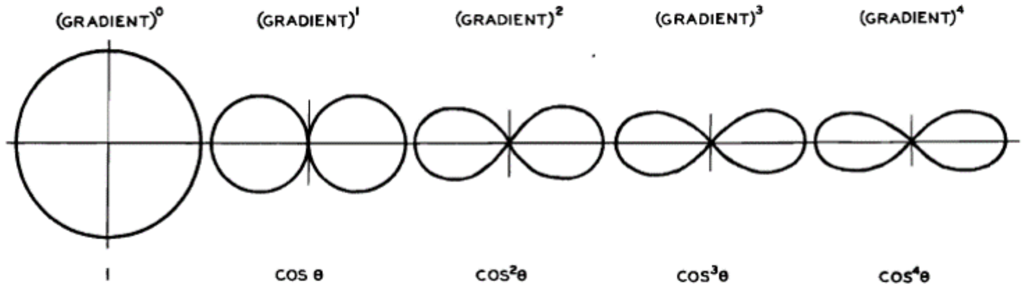

Figure 1.3: Gain pattern of higher-order differential sensors

As generalization to the above-mentioned *first*-order differential sensor – a kth-order differential sensor (chapter 8.5 of [5], chapter 2.2 of [11]) measures a pressure field at $k+1$ closely spaced points along a straight line, then computes the kth-order finite difference among them, to approximate a measurement of the kth-order partial derivative of the pressure field [18, 19]). A kth-order differential sensor has a directional gain response equal to the kth-order spatial derivative of the pressure field. More mathematically, a k-order differential sensor has a directional response of $\cos^k(\phi)$.

1.4 Principle of Differential Sensors

The response of a pressure gradient microphone (first-order differential sensor) directly corresponds to the pressure difference between two points in space. If the ratio of the displacement between two points to the wavelength is low, the pressure gradient will be proportional to the particle velocity. While a velocity microphone's response directly corresponds to the particle velocity field.

Similarly, a n-th order differential sensor can be approximated as taking n-th finite difference on pressure sensors. A general differential equation on the pressure field may give us an insight of the underlying principle of the differential sensors. Assume a plane-wave signal impinging on the sensors, the acoustic pressure field of signal can be written as [11]

$$p(q, \mathbf{r}, t) = P_0 e^{j(\omega t - qr\cos\theta)} \tag{1.1}$$

where P_0 is the power of the signal source, ω is the angular frequency, $q = \frac{\omega}{c} = \frac{2\pi}{\lambda}$, and $r = \|\mathbf{r}\|$ where $\mathbf{r}$ is the direction vector with respect to (0, 0). Omit the time dependence and take n-th order spatial derivative with respect to $\mathbf{r}$ on (1.1) derives

$$\frac{d^n}{dr^n} p(q, r, \theta) = P_0(-jq\cos(\theta))^n e^{-jqr\cos(\theta)} \tag{1.2}$$

The spatial derivative above can be approximated by computing the nth-order finite difference multiplied by a bias error term. For example, the first-order spatial derivative of pressure is

$$\frac{d}{dr} p(q, r, \theta) = -jP_0 q\cos(\theta) e^{-jqr\cos(\theta)} \tag{1.3}$$

The first-order finite difference can be written as [11]

$$\begin{aligned} \frac{\Delta p(q, r, \theta)}{\Delta r} &= \frac{p(q, r + d/2, \theta) - p(q, r - d/2, \theta)}{d} \\ &= \frac{-j2P_0 \sin(qd/2\cos\theta) e^{-jqr\cos(\theta)}}{d} \end{aligned} \tag{1.4}$$

where d is the distance between pressure sensors .

Then the bias error term can be defined as dividing (1.4) by (1.3):

$$\delta = \frac{\Delta p/\Delta r}{dp/dr} \tag{1.5}$$

If the distance between sensor elements is small to a certain level (relative to the wavelength), i.e. usually $\frac{d}{\lambda} < 1/4$ in which case the error term is less than 1dB, the following approximation will be acceptable:

$$\frac{dp(q, r, \theta)}{dr} \approxeq \frac{\Delta p(q, r, \theta)}{\Delta r} \tag{1.6}$$

Similar derivations and conclusions could be obtained for higher-order case.

A typical hardware realization of the first-order differential sensor can be achieved by measuring the pressure-difference across the two sides of a diaphragm. A relationship between the pressure field and acoustic particle velocity can be expressed by Euler's equation as [11]

$$-\nabla p = \rho \frac{\partial \mathbf{v}}{\partial t}, \tag{1.7}$$

where $\mathbf{v}$ is the acoustic particle velocity and ρ is the static density of medium.

The equation above shows that the derivative of particle velocity with respect to time is directly related to the pressure gradient. Therefore, the response of a first-order differential sensor contains not only the scalar pressure but also one component of the particle velocity at that point. Based on the first-order differential sensor, a higher-order differential microphone can be designed as a combination of first-order microphones, where the order of new sensor is the sum of all component orders.

1.5 A Bi-Axial Pair of Differential Sensors in Spatial Collocation and Perpendicular Orientation

First-order differential sensors have been used for decades as a collocated *pair* in perpendicular orientation, giving an array manifold of

$$\mathbf{a}_1(\phi) = \begin{bmatrix} \cos(\phi) \\ \sin(\phi) \end{bmatrix}. \tag{1.8}$$

The above array manifold has a key advantage of independence from the frequency, spectrum and bandwidth of the incident signal, thereby decoupling the frequency coordinate from the direction-of-arrival coordinate. Such a pair is sometimes called a "u-u probe". It has been implemented in hardware [7–9]. Its beam patterns and directivity have been studied in [12, 14]. Direction-finding formulas have been advanced for it in [16]. Please refer to [20] for a literature review. [1]

Similarly, for two higher-order differential sensors of order k_1 and k_2, arranged bi-axially in perpendicularity and in spatial collocation, the pair's array manifold is equal to

$$\mathbf{a}_{k_1,k_2}(\phi) = \begin{bmatrix} \cos^{k_1}(\phi) \\ \sin^{k_2}(\phi) \end{bmatrix}. \tag{1.9}$$

This array manifold retains the frequency-decoupling advantage of (1.8). [2]

Such dipole-like bidirectional higher-order sensors have been used as a triad, with three component-sensors being collocated and perpendicularly oriented, for direction finding. The azimuth-elevation direction-of-arrival estimation formulas have been advanced in [16]. [3]

For a *pair* of perpendicularly oriented uniaxial *first*-order velocity-sensors, direction-finding formulas have been advanced in [20].

This research will propose direction-finding formulas for a pair of *higher*-order dipole-like differential sensors perpendicularly oriented, whether collocated or not, and regardless if the two sensors are of the same higher order.

[1] Incidentally, first-order differential sensors have been used in a collocated and perpendicular *triad*, called a "tri-axial velocity-sensor", or a "velocity-sensor triad", or a "vector sensor", or a "vector hydrophone". For comprehensive reviews of the "tri-axial velocity-sensor" literature, please consult [13, 15, 17].

[2] Incidentally, for a *triad* of *higher*-order differential sensors that are collocated in space and perpendicular in orientation, direction-finding formulas have been advanced in [16].

[3] If each such higher-order sensor is realized by a linear array of isotropic sensors, the noise statistics is analyzed in [18, 19].

1.6 The Advantages of using the differential sensors as an array (pair or triad)

A differential sensor triad/pair is formed by three/two perpendicularly-oriented differential sensors with a possible inner displacement. Figure 1.4 shows a triad consists of three identical differential sensors oriented towards x, y, z axis respectively and collocated (zero inner displacement) as a point-like array.

Each component sensor of the triad/pair reflects the particle-velocity (or n-th power of particle-velocity) in its according orientation with a gain pattern given in Figure 1.3. Such an array will deem the acoustic signal in a particle-velocity-field manner instead of pressure field. The response of a differential sensor triad/pair carries more information about the direction of incident signal than that of a pressure sensor array with similar geometry. Thus, it has a characteristic of high directivity. A higher-order differential microphone possesses higher directivity (narrower beam pattern), but as a trade-off, the hardware design could be more complicated and the price is often higher.

The accuracy of direction finding (DF) using a sensor array is, to a large extent, related to the array aperture. A larger array aperture will bring a better resolution for DF. Although an assumption of collocated sensor components gives a relatively simple expression of manifold, the hardware realization of such design could be very difficult. In fact, the existence of displacement between sensors is advantageous in extending the spatial aperture while keeping the number of sensors unchanged. For a pair of differential sensors, the displacement is necessary for unambiguous direction-of-arrival estimation.

For a first order differential sensor triad, its special array manifold will introduce a benefit of self-normalization to the eigen-based direction finding algorithm [22]. The eigen-based (subspace-based) algorithm could be adopted to estimate the steering vector ($\hat{\mathbf{a}}$) of incident signal by applying eigen-decomposition on the data-correlation matrix, but with an ambiguity of a unknown complex coefficient ($\hat{\mathbf{a}} = c\mathbf{a}$). Applying self-normalization method on $\hat{\mathbf{a}}$ would give a *un*ambiguous estimation without the unknown complex c.

Such eigen-based algorithm has been widely applied in first-order differential sensor array (vector sensor) including "Estimation of Signal Parameters via a Rotation Invariance Technique (ESPRIT) [22–26], "Multiple SIgnal Classification (MUSIC) and Root-MUSIC [27–29].

1.7 Organization of This Dissertation

This dissertation consists of seven sections. Chapter 2 will introduce the measurement model adopted for this project as well as the core idea of the proposed

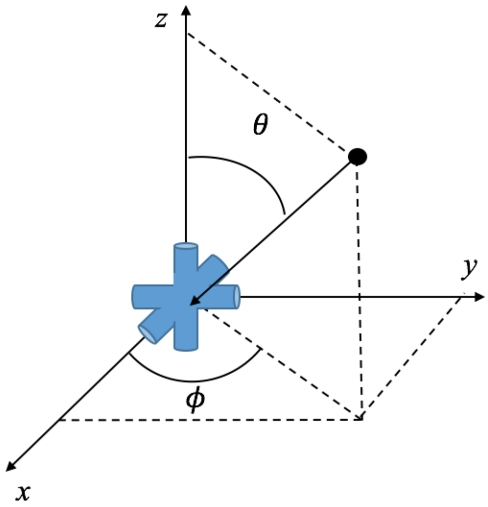

Figure 1.4: A schematic of differential sensor triad

algorithm. Chapter 3 and 4 will present a qualitative discussion on how to derive direction finding formulas. Detailed steps will be given, including deduction of equations and the investigation of prior knowledge requirements. Section 4.10 will briefly summarize the possibility of extending the current approach to the collocated case. Chapter 5 will show MATLAB simulation results. Monte Carlo simulations are employed to test the proposed direction finding method. Chapter 6 will present the Cramér-Rao bounds for elevation and azimuth angle direction-of-arrival estimation. Finally, Chapter 7 will conclude the dissertation.

Chapter 2

The Measurement Model And Direction Finding

In this project, we are considering a sensor array which consists of two *uni*-axial differential sensors (with possibly different order) oriented orthogonally and displaced along the axis. Without changing the coordinate system, there are totally nine configurations (three possible combinations of sensors' orientation $\times$ three types of displacement) for this type of sensor array, as shown in Figure 2.1. Similar model has been adopted by Song [20].

Array manifold is often used to describe the responses of sensor array. For three *higher*-order differential sensors shown in Figure 2.2, the array manifold is written as:

$$\mathbf{a} = \begin{bmatrix} (\sin(\theta)\cos(\phi))^{k_x}\, e^{j(2\pi\Delta_x/\lambda)\sin(\theta)\cos(\phi)} \\ (\sin(\theta)\sin(\phi))^{k_y}\, e^{j(2\pi\Delta_y/\lambda)\sin(\theta)\sin(\phi)} \\ \cos^{k_z}(\theta) e^{j(2\pi\Delta_z/\lambda)\cos(\theta)} \end{bmatrix} = \begin{bmatrix} u^{k_x} e^{j(2\pi\Delta_x/\lambda)u} \\ v^{k_y} e^{j(2\pi\Delta_y/\lambda)v} \\ w^{k_z} e^{j(2\pi\Delta_z/\lambda)w} \end{bmatrix}$$

This general array manifold consists of three components corresponding to k_x, k_y and k_z-th order spatial derivative of the pressure field, which can be measured by the differential sensor aligned in parallel to x, y and z axis respectively.

The ultimate objective is to solve the following problem: How to estimate the elevation and azimuth angle of an acoustic incident signal, as far-field narrow-band plane wave, impinging on the sensor array shown in Figure 2.1.

The data model is given as

$$\mathbf{z}(t_n) = \mathbf{a}s(t_n) + \mathbf{n}(t_n)$$

where $\mathbf{n}(t_n)$ is zero-mean white noise vector, $s(t_n)$ is a pure tone signal, $\mathbf{z}(t_n)$ is the received array response.

Since the parameter of interest only exists in the array manifold, the first step is to estimate $\hat{\mathbf{a}}$. The proposed algorithm employs eigen-based (subspace-based) parameter estimation algorithm as a preceding step, which may derive an estima-

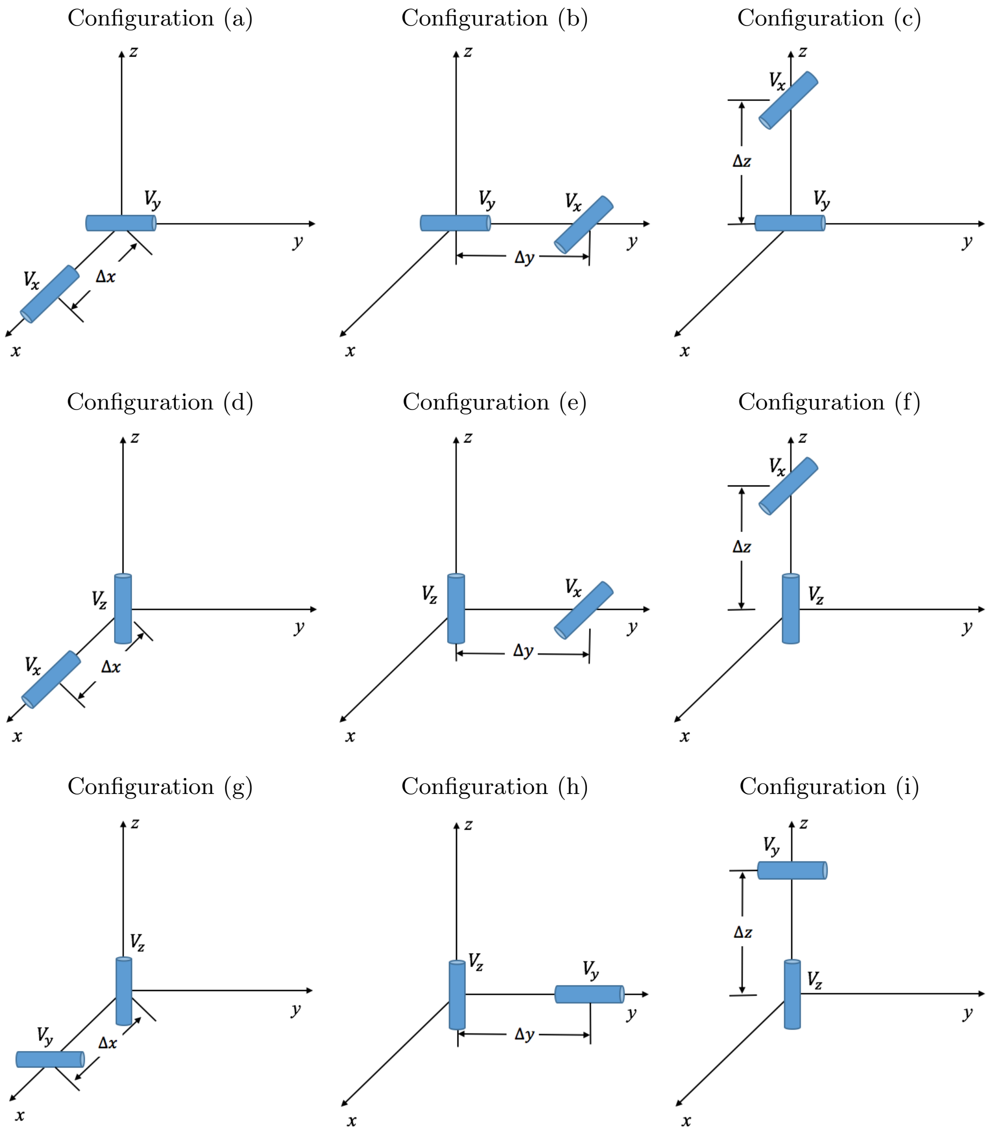

Figure 2.1: The nine configurations for a pair of differential sensors in orthogonality along Cartesian axes

tion of $\hat{\mathbf{a}}$ but correct to only within an unknown complex scalar. For N received snapshots, an eigen-based parameter estimation algorithm will eigen-decompose the data covariance matrix

$$\begin{aligned}\hat{\mathbf{C}} &= \frac{1}{N}\sum_{n=1}^{N}\mathbf{z}(t_n)\mathbf{z}(t_n)^H \\ &\approx E\left[\mathbf{z}(t_n)\mathbf{z}(t_n)^H\right] \\ &= P_s\mathbf{a}\mathbf{a}^H + P_n\mathbf{I}\end{aligned}$$

where P_s and P_n are the signal power and noise power respectively.

$\hat{\mathbf{C}}$ has a principle eigenvector $c\mathbf{a}$ where c can be any complex number has a

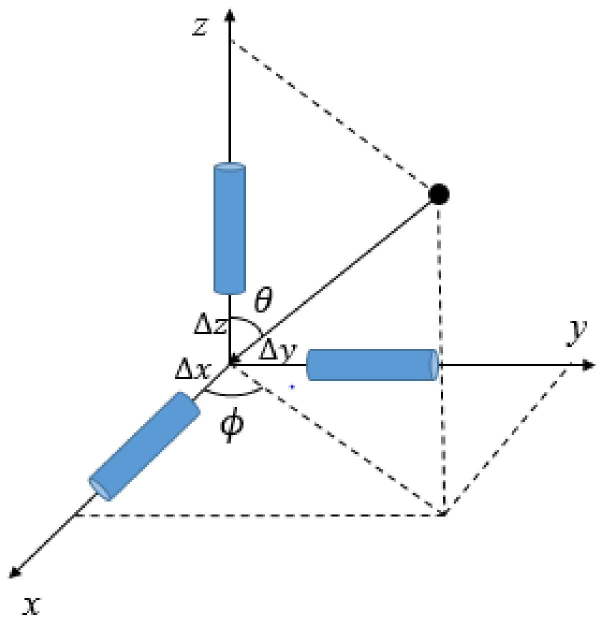

Figure 2.2: Three orthogonally oriented differential sensors

magnitude of $1/\|\mathbf{a}\|$.

Comparing to other conventional parameter estimation approaches such as Maximum Likelihood Estimation (MLE), eigen-based parameter algorithm is advantaged in following aspects

1) No prior knowledge of the noise statistics is needed.

2) No iteration needed, hence computationally faster and no initial coarse estimation needed.

3) The eigen-decomposition separates the noise subspace from the signal+noise subspace. This separation raises the signal-to-noise ratio in the sinal+noise subspace,from which the unknown parameters are to be estimated.

For a sensor array which consists of two *higher*-order differential sensors, the estimate of array manifold can be written in a general form as

$$\hat{\mathbf{a}}_{\zeta_1,\zeta_2}^{(\epsilon-\text{axis})} \approx c\mathbf{a}_{\zeta_1,\zeta_2}^{(\epsilon-\text{axis})} = c \begin{bmatrix} \eta_1^{k_1} \mathrm{e}^{j(2\pi\lambda\Delta_\epsilon)/\mu} \\ \eta_2^{k_2} \end{bmatrix} \tag{2.1}$$

where

$$\eta_1 = \begin{cases} u, & \text{if } \zeta_1 = x, \\ v, & \text{if } \zeta_1 = y, \\ w, & \text{if } \zeta_1 = z; \end{cases}$$

$$\eta_2 = \begin{cases} u, & \text{if } \zeta_2 = x, \\ v, & \text{if } \zeta_2 = y, \\ w, & \text{if } \zeta_2 = z; \end{cases}$$

$$\mu = \begin{cases} u, & \text{if } \epsilon = x, \\ v, & \text{if } \epsilon = y, \\ w, & \text{if } \epsilon = z; \end{cases}$$

the superscript ϵ specifies the displacement-axis between two uniaxial sensors; $\zeta_i \in \{x, y, z\}$ denotes the orientation of ith uniaxial sensor, with i=1,2, and $\zeta_1 \neq \zeta_2$; k_1 and k_2 are the order of first and second differential sensor respectively. $\mathrm{e}^{j(2\pi\Delta_\epsilon)/\lambda\mu}$ is the phase factor introduced by displacement of the two sensors. Δ_ϵ is the distance between two sensors as shown in Figure 2.1. λ refers to the smallest wavelength of the incident signal.

Since there are only two uniaxial sensors, the complex-value scalar c can not be estimated as in tri-axial sensor array [16]. Unknown c can be eliminated by

$$\frac{\left[\hat{\mathbf{a}}^{(\epsilon-\text{axis})}_{\zeta_1,\zeta_2}\right]_1}{\left[\hat{\mathbf{a}}^{(\epsilon-\text{axis})}_{\zeta_1,\zeta_2}\right]_2} = \frac{\eta_1^{k_1}}{\eta_2^{k_2}} \mathrm{e}^{j(2\pi\Delta_\epsilon)/\lambda\mu} \tag{2.2}$$

$$\Rightarrow \begin{cases} \widehat{\left(\frac{\eta_1^{k_1}}{\eta_2^{k_2}}\right)} = \left|\frac{\left[\hat{\mathbf{a}}^{(\epsilon-\text{axis})}_{\zeta_1,\zeta_2}\right]_1}{\left[\hat{\mathbf{a}}^{(\epsilon-\text{axis})}_{\zeta_1,\zeta_2}\right]_2}\right| \operatorname{sgn}\left(\frac{\eta_1^{k_1}}{\eta_2^{k_2}}\right), & (2.3) \\ \hat{\mu} = \frac{\lambda}{2\pi\Delta_\epsilon} \angle \left(\operatorname{sgn}\left(\frac{\eta_1^{k_1}}{\eta_2^{k_2}}\right) \frac{\left[\hat{\mathbf{a}}^{(\epsilon-\text{axis})}_{\zeta_1,\zeta_2}\right]_1}{\left[\hat{\mathbf{a}}^{(\epsilon-\text{axis})}_{\zeta_1,\zeta_2}\right]_2} \right). & (2.4) \end{cases}$$

where $[\cdot]_i$ symbolizes the ith element of the vector inside the square brackets. Equation (2.3) and (2.4) require prior knowledge of the sign of $\eta_1^{k_1}/\eta_2^{k_2}$.

For above derivation, equation (2.2) is obtained by dividing the first element of $\hat{\mathbf{a}}^{(\epsilon-\text{axis})}_{\zeta_1,\zeta_2}$ by its second element. Equation (2.3) and (2.4) are obtained by taking the magnitude and phase from the left-hand-side of (2.2) respectively. In this way, we derived two non-linear equations which only contain two unknown parameters (θ and ϕ). Thus, the closed-form formulas for these two parameters could be derived from (2.3) and (2.4).

Chapter 3

Derivation of Closed-Form Estimators of elevation and azimuth angle for the General Case

This chapter will elaborate the derivation of closed-form solution to the direction of arrival estimator ($\hat{\theta}$, $\hat{\phi}$) based on Equation (2.2), (2.3) and (2.4) in Chapter 2.

For configuration (a), (b), (d), (f), (h), and (i) (refer to Figure 2.1), where either $\eta_1 = \mu$ or $\eta_2 = \mu$, the closed-form solutions are able to be derived. For configuration (c), (e) and (g) where neither $\eta_1 = \mu$ nor $\eta_2 = \mu$, the closed-form formula is derivable only if the order of two sensors are equivalent. Section 3.7 will explain this in detail. The closed-form formulas for these three special configurations will be given in Chapter 4 under the assumption that the two sensors are equal-order. For the definition of the nine configurations please refer to Figure 2.1.

To simplify the expressions in later derivation, Equations (2.3) and (2.4) can be written as

$$\begin{cases} \widehat{\left(\dfrac{\eta_1^{k_1}}{\eta_2^{k_2}}\right)} = \left| \dfrac{\left[\hat{\mathbf{a}}^{(\epsilon-\text{axis})}_{\zeta_1,\zeta_2}\right]_1}{\left[\hat{\mathbf{a}}^{(\epsilon-\text{axis})}_{\zeta_1,\zeta_2}\right]_2} \right| \operatorname{sgn}\left(\dfrac{\eta_1^{k_1}}{\eta_2^{k_2}}\right) \triangleq \alpha\left(\operatorname{sgn}\left(\dfrac{\eta_1^{k_1}}{\eta_2^{k_2}}\right)\right), & (3.1) \\[2ex] \hat{\mu} = \dfrac{\lambda}{2\pi\Delta_\epsilon}\angle\left(\operatorname{sgn}\left(\dfrac{\eta_1^{k_1}}{\eta_2^{k_2}}\right)\dfrac{\left[\hat{\mathbf{a}}^{(\epsilon-\text{axis})}_{\zeta_1,\zeta_2}\right]_1}{\left[\hat{\mathbf{a}}^{(\epsilon-\text{axis})}_{\zeta_1,\zeta_2}\right]_2}\right) \triangleq \beta\left(\operatorname{sgn}\left(\dfrac{\eta_1^{k_1}}{\eta_2^{k_2}}\right)\right). & (3.2) \end{cases}$$

where $\alpha(\cdot)$ and $\beta(\cdot)$ refer to two functions.

Some assumptions are needed (as given below) in order to apply the proposed algorithm. Assumption [AS-1] is required such that $\widehat{\left(\frac{\eta_1^{k_1}}{\eta_2^{k_2}}\right)}$ and $\hat{\eta}_3$ can be determined through equations (3.1) and (3.2). Besides [AS-1], none/some of the other assumptions may be also required in order to unambiguously determine $\hat{\phi}$ and $\hat{\theta}$

with a specific configuration. For convenience of references, these assumptions are listed and labeled below:

[AS-1] $\operatorname{sgn}\left(\frac{\eta_1^{k_1}}{\eta_2^{k_2}}\right)$ is known for that specific configuration,

[AS-2] $\operatorname{sgn}\left(\frac{v}{u}\right)$ is known,

[AS-3] $\operatorname{sgn}(u)$, i.e., $\phi \in \left[0, \frac{\pi}{2}\right) \cup \left[\frac{3\pi}{2}, 2\pi\right)$ or $\phi \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right)$, is known,

[AS-4] $\operatorname{sgn}(v)$, i.e., $\phi \in [0, \pi)$ or $\phi \in [\pi, 2\pi)$, is known,

[AS-5] $\operatorname{sgn}(w)$, i.e., $\theta \in \left[0, \frac{\pi}{2}\right)$ or $\theta \in \left[\frac{\pi}{2}, \pi\right]$, is known.

3.1 y-axis and x-axis oriented sensors with displacement along x-axis

Figure 2.1 ‘configuration (a)’ illustrates this configuration.
From equations (3.1) and (3.2).

$$\begin{cases} \widehat{\left(\dfrac{u^{k_1}}{v^{k_2}}\right)} = \alpha\left(\operatorname{sgn}\left(\dfrac{u^{k_1}}{v^{k_2}}\right)\right) & (3.3) \\ \hat{u} = \beta\left(\operatorname{sgn}\left(\dfrac{u^{k_1}}{v^{k_2}}\right)\right) & (3.4) \end{cases}$$

$\left((3.4)^{k_1}/(3.3)\right)^{1/k_2}/(3.4)$ derives

$$\left|\tan(\hat{\phi})\right| = \left|\widehat{\left(\frac{v}{u}\right)}\right| = \left|\frac{\left(\frac{\beta^{k_1}}{\alpha}\right)^{\frac{1}{k_2}}}{\beta}\right|$$

which results

$$\tan(\hat{\phi}) = \left|\frac{\beta^{\frac{k_1}{k_2}-1}}{\alpha^{\frac{1}{k_2}}}\right| \operatorname{sgn}\left(\frac{v}{u}\right). \tag{3.5}$$

With assumption [AS-2], $\tan(\hat{\phi})$ can be determined via equation (3.5). Additionally, with assumption [AS-3], $\hat{\phi}$ in (3.5) can be unambiguously determined as

$$\hat{\phi} = \begin{cases} \dfrac{\pi}{2}[1 - \operatorname{sgn}(u)] + \tan^{-1}\left(\operatorname{sgn}(u)\left|\dfrac{\beta^{k_1/k_2-1}}{\alpha^{1/k_2}}\right|\right), \\ \qquad\qquad \text{if } \operatorname{sgn}(v) > 0; \\ \dfrac{\pi}{2}[3 + \operatorname{sgn}(u)] - \tan^{-1}\left(\operatorname{sgn}(u)\left|\dfrac{\beta^{k_1/k_2-1}}{\alpha^{1/k_2}}\right|\right), \\ \qquad\qquad \text{if } \operatorname{sgn}(v) < 0; \end{cases}. \tag{3.6}$$

From (3.4) and (3.6),

$$\sin(\hat{\theta}) = \frac{\hat{u}}{\cos(\hat{\phi})} = \beta \sec(\hat{\phi}). \tag{3.7}$$

With assumption [AS-5], $\hat{\theta}$ in (3.7) can be unambiguously determined as

$$\hat{\theta} = \begin{cases} \sin^{-1}\left(\beta \sec(\hat{\phi})\right), & \text{if } \theta \in \left[0, \dfrac{\pi}{2}\right); \\ \pi - \sin^{-1}\left(\beta \sec(\hat{\phi})\right), & \text{if } \theta \in \left[\dfrac{\pi}{2}, \pi\right]. \end{cases} \tag{3.8}$$

From the above derivation, (3.6) and (3.8) are obtained when assumptions [AS-1], [AS-2], [AS-3], and [AS-5] hold, which is equivalent to [AS-3], [AS-4] and [AS-5] hold. That is, the prior information required to unambiguously determine $\hat{\phi}$ and $\hat{\theta}$ for configuration (a) is $\phi \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right)$ or $\left[\pi, \frac{3\pi}{2}\right)$ or $\left[\frac{3\pi}{2}, 2\pi\right)$, and $\theta \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right]$.

3.2 y-axis and x-axis oriented sensors with displacement along y-axis

Figure 2.1 'configuration (b)' illustrates this configuration.

From (3.1) and (3.2)

$$\begin{cases} \widehat{\left(\dfrac{u^{k_1}}{v^{k_2}}\right)} = \alpha\left(\operatorname{sgn}\left(\dfrac{u^{k_1}}{v^{k_2}}\right)\right) & (3.9) \\ \hat{v} = \beta\left(\operatorname{sgn}\left(\dfrac{u^{k_1}}{v^{k_2}}\right)\right) & (3.10) \end{cases}$$

$(3.10)/\left((3.9)(3.10)^{k_2}\right)^{1/k_1}$ derives

$$\left|\tan(\hat{\phi})\right| = \left|\widehat{\left(\frac{v}{u}\right)}\right| = \left|\frac{\beta}{(\alpha\ \beta^{k_2})^{1/k_1}}\right|$$

which results

$$\tan(\hat{\phi}) = \left|\frac{\beta^{1-k_2/k_1}}{\alpha^{1/k_1}}\right| \operatorname{sgn}\left(\frac{v}{u}\right). \tag{3.11}$$

With assumption [AS-2], $\tan(\hat{\phi})$ can be determined via equation (3.11). Additionally, with assumption [AS-3], $\hat{\phi}$ in (3.11) can be unambiguously determined as

$$\hat{\phi} = \begin{cases} \dfrac{\pi}{2}[1-\operatorname{sgn}(u)] + \tan^{-1}\left(\operatorname{sgn}(u)\left|\dfrac{\beta^{1-k_2/k_1}}{\alpha^{1/k_1}}\right|\right), \\ \hfill \text{if } \operatorname{sgn}(v) > 0; \\ \dfrac{\pi}{2}[3+\operatorname{sgn}(u)] - \tan^{-1}\left(\operatorname{sgn}(u)\left|\dfrac{\beta^{1-k_2/k_1}}{\alpha^{1/k_1}}\right|\right), \\ \hfill \text{if } \operatorname{sgn}(v) < 0; \end{cases} \tag{3.12}$$

From (3.10) and (3.12),

$$\sin(\hat{\theta}) = \frac{\hat{v}}{\sin(\hat{\phi})} = \beta \csc(\hat{\phi}). \tag{3.13}$$

With assumption [AS-5], $\hat{\theta}$ in (3.13) can be unambiguously determined as

$$\hat{\theta} = \begin{cases} \sin^{-1}\left(\beta \csc(\hat{\phi})\right), & \text{if } \theta \in \left[0, \dfrac{\pi}{2}\right); \\ \pi - \sin^{-1}\left(\beta \csc(\hat{\phi})\right), & \text{if } \theta \in \left[\dfrac{\pi}{2}, \pi\right]. \end{cases} \tag{3.14}$$

From the above derivation, (3.12) and (3.14) are obtained when assumptions [AS-1], [AS-2], [AS-3], and [AS-5] hold, which is equivalent to [AS-3], [AS-4] and [AS-5] hold. That is, the prior information required to unambiguously determine $\hat{\phi}$ and $\hat{\theta}$ for configuration (b) is $\phi \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right)$ or $\left[\pi, \frac{3\pi}{2}\right)$ or $\left[\frac{3\pi}{2}, 2\pi\right)$, and $\theta \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right]$.

3.3 z-axis and x-axis oriented sensors with displacement along x-axis

Figure 2.1 'configuration (d)' illustrates this configuration.

From (3.1) and (3.2),

$$\begin{cases} \widehat{\left(\dfrac{u^{k_1}}{w^{k_2}}\right)} = \alpha\left(\operatorname{sgn}\left(\dfrac{u^{k_1}}{w^{k_2}}\right)\right) & (3.15) \\ \hat{u} = \beta\left(\operatorname{sgn}\left(\dfrac{u^{k_1}}{w^{k_2}}\right)\right). & (3.16) \end{cases}$$

Straightforward calculation from (3.16) shows that

$$\left|\cos(\hat{\theta})\right| = |\hat{w}| = \left|\left(\frac{\beta^{k_1}}{\alpha}\right)^{1/k_2}\right|$$

which results

$$\cos(\hat{\theta}) = \left|\frac{\beta^{k_1/k_2}}{\alpha^{1/k_2}}\right| \operatorname{sgn}(w). \tag{3.17}$$

With assumption [AS-5], $\cos(\hat{\theta})$ can be determined via equation (3.17). And $\hat{\theta}$ in (3.17) can be unambiguously determined as

$$\hat{\theta} = \cos^{-1}\left(\operatorname{sgn}(w)\left|\frac{\beta^{k_1/k_2}}{\alpha^{1/k_2}}\right|\right). \tag{3.18}$$

From (3.16) and (3.18),

$$\cos(\hat{\phi}) = \frac{\hat{u}}{\sin(\hat{\theta})} = \beta\csc(\hat{\theta}). \tag{3.19}$$

With assumption [AS-4], $\hat{\phi}$ in (3.19) can be unambiguously determined as

$$\hat{\phi} = \begin{cases} \cos^{-1}\left(\beta\csc(\hat{\theta})\right), & \text{if } \phi \in [0,\pi)\,; \\ 2\pi - \cos^{-1}\left(\beta\csc(\hat{\theta})\right), & \text{if } \phi \in [\pi, 2\pi)\,. \end{cases} \tag{3.20}$$

From the above derivation, (4.12) and (3.18) are obtained when assumptions [AS-1], [AS-4], and [AS-5] hold, which is equivalent to

i) [AS-4] and [AS-5] hold if k_1 is even. That is, the prior information required to unambiguously determine $\hat{\phi}$ and $\hat{\theta}$ for configuration (d) is $\phi \in [0,\pi)$ or $[\pi, 2\pi)$, and $\theta \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right]$.

ii) [AS-3], [AS-4] and [AS-5] hold if k_1 is odd. That is, the prior information required to unambiguously determine $\hat{\phi}$ and $\hat{\theta}$ for configuration (d) is $\phi \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right)$ or $\left[\pi, \frac{3\pi}{2}\right)$ or $\left[\frac{3\pi}{2}, 2\pi\right)$, and $\theta \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right]$.

3.4 z-axis and x-axis oriented sensors with displacement along z-axis

Figure 2.1 'configuration (f)' illustrates this configuration.
From (3.1) and (3.2)

$$\begin{cases} \widehat{\left(\frac{u^{k_1}}{w^{k_2}}\right)} = \alpha\left(\operatorname{sgn}\left(\frac{u^{k_1}}{w^{k_2}}\right)\right) & (3.21) \\ \hat{w} = \beta\left(\operatorname{sgn}\left(\frac{u^{k_1}}{w^{k_2}}\right)\right) & (3.22) \end{cases}$$

Straightforward calculation shows that

$$\cos(\hat{\theta}) = \beta. \tag{3.23}$$

Thus, $\hat{\theta}$ can be unambiguously determined through (3.23)

$$\hat{\theta} = \cos^{-1}(\beta)\,. \tag{3.24}$$

From (3.21) and (3.22),

$$\left|\sin(\hat{\theta})\cos(\hat{\phi})\right| = |\hat{u}| = \left|\left(\alpha\beta^{k_2}\right)^{1/k_1}\right| \tag{3.25}$$

which results

$$\cos(\hat{\phi}) = \csc(\hat{\theta})\left|\alpha^{1/k_1}\beta^{k_2/k_1}\right|\mathrm{sgn}(u) \tag{3.26}$$

With assumption [AS-3], $\cos(\hat{\phi})$ can be determined via equation (3.26). Additionally, with assumption [AS-4], $\hat{\phi}$ in (3.26) can be unambiguously determined as

$$\hat{\phi} = \begin{cases} \cos^{-1}\left(\mathrm{sgn}(u)\csc(\hat{\theta})\left|\alpha^{1/k_1}\beta^{k_2/k_1}\right|\right), & \text{if } \phi \in [0,\pi)\,; \\ 2\pi - \cos^{-1}\left(\mathrm{sgn}(u)\csc(\hat{\theta})\left|\alpha^{1/k_1}\beta^{k_2/k_1}\right|\right), & \text{if } \phi \in [\pi, 2\pi)\,. \end{cases} \tag{3.27}$$

From the above derivation, (3.24) and (3.27) are obtained when assumptions [AS-1], [AS-3] and [AS-4] hold, which is equivalent to

i) [AS-3] and [AS-4] hold if k_2 is even. That is, the prior information required to unambiguously determine $\hat{\phi}$ and $\hat{\theta}$ for configuration (f) is $\phi \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right)$ or $\left[\pi, \frac{3\pi}{2}\right)$ or $\left[\frac{3\pi}{2}, 2\pi\right)$.

ii) [AS-1], [AS-3] and [AS-4] hold if k_2 is odd. That is, the prior information required to unambiguously determine $\hat{\phi}$ and $\hat{\theta}$ for configuration (f) is $\phi \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right)$ or $\left[\pi, \frac{3\pi}{2}\right)$ or $\left[\frac{3\pi}{2}, 2\pi\right)$, and $\theta \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right]$.

3.5 z-axis and y-axis oriented sensors with displacement along y-axis

Figure 2.1 ‘configuration (h)’ illustrates this configuration.

From (3.1) and (3.2)

$$\begin{cases} \widehat{\left(\dfrac{v^{k_1}}{w^{k_2}}\right)} = \alpha\left(\mathrm{sgn}\left(\dfrac{v^{k_1}}{w^{k_2}}\right)\right) & (3.28) \\ \hat{v} = \beta\left(\mathrm{sgn}\left(\dfrac{v^{k_1}}{w^{k_2}}\right)\right). & (3.29) \end{cases}$$

Straightforward calculation shows that

$$\left|\cos(\hat{\theta})\right| = |\hat{w}| = \left|\left(\frac{\beta^{k_1}}{\alpha}\right)^{1/k_2}\right| = \left|\frac{\beta^{k_1/k_2}}{\alpha^{1/k_2}}\right|$$

which results

$$\cos(\hat{\theta}) = \left|\frac{\beta^{k_1/k_2}}{\alpha^{1/k_2}}\right|\mathrm{sgn}(w). \tag{3.30}$$

With assumption [AS-5], $\cos(\hat{\theta})$ can be determined via equation (3.30). And $\hat{\theta}$ in (3.30) can be unambiguously determined as

$$\hat{\theta} = \cos^{-1}\left(\operatorname{sgn}(w)\left|\frac{\beta^{k_1/k_2}}{\alpha^{1/k_2}}\right|\right). \tag{3.31}$$

From (3.29) and (3.31),

$$\sin(\hat{\phi}) = \frac{\hat{v}}{\sin(\hat{\theta})} = \beta\csc(\hat{\theta}). \tag{3.32}$$

With assumption [AS-3], $\hat{\phi}$ in (3.32) can be unambiguously determined as

$$\hat{\phi} = \begin{cases} \pi[1-\operatorname{sgn}(\psi)-\delta(\psi)] + \sin^{-1}\left(\beta\csc(\hat{\theta})\right), \\ \qquad\qquad \text{if } \phi \in \left[0, \frac{\pi}{2}\right) \cup \left[\frac{3\pi}{2}, 2\pi\right); \\ \pi - \sin^{-1}\left(\beta\csc(\hat{\theta})\right), \quad \text{if } \phi \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right); \end{cases} \tag{3.33}$$

where $\psi = \sin^{-1}\left(\frac{\lambda}{2\pi\Delta_y}\csc(\hat{\theta})\angle\left(\operatorname{sgn}\left(\frac{v^{k_1}}{w^{k_2}}\right)\frac{\left[\hat{\mathbf{a}}_{y,z}^{(y-\text{axis})}\right]_1}{\left[\hat{\mathbf{a}}_{y,z}^{(y-\text{axis})}\right]_2}\right)\right)$, and $\delta(\psi) = \begin{cases} 1, & \text{if } \psi = 0 \\ 0, & \text{if } \psi \neq 0 \end{cases}$.

From the above derivation, (3.31) and (3.33) are obtained when assumptions [AS-1], [AS-3] and [AS-5] hold, which is equivalent to

i) [AS-3] and [AS-5] hold if k_1 is even. That is, the prior information required to unambiguously determine $\hat{\phi}$ and $\hat{\theta}$ for configuration (h) is $\phi \in \left[0, \frac{\pi}{2}\right) \cup \left[\frac{3\pi}{2}, 2\pi\right)$ or $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right)$, and $\theta \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right]$.

ii) [AS-1], [AS-3] and [AS-5] hold if k_1 is odd. That is, the prior information required to unambiguously determine $\hat{\phi}$ and $\hat{\theta}$ for configuration (h) is $\phi \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right)$ or $\left[\pi, \frac{3\pi}{2}\right)$ or $\left[\frac{3\pi}{2}, 2\pi\right)$, and $\theta \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right]$.

3.6 z-axis and y-axis oriented sensors with displacement along z-axis

Figure 2.1 'configuration (i)' illustrates this configuration.

From (3.1) and (3.2)

$$\begin{cases} \widehat{\left(\frac{v^{k_1}}{w^{k_2}}\right)} = \alpha\left(\operatorname{sgn}\left(\frac{v^{k_1}}{w^{k_2}}\right)\right) & (3.34) \\ \hat{w} = \beta\left(\operatorname{sgn}\left(\frac{v^{k_1}}{w^{k_2}}\right)\right). & (3.35) \end{cases}$$

Straightforward calculation shows that

$$\cos(\hat{\theta}) = \beta. \tag{3.36}$$

Thus, $\hat{\theta}$ can be unambiguously determined via (3.36),

$$\hat{\theta} = \cos^{-1}(\beta). \tag{3.37}$$

From (3.34) and (3.35),

$$\begin{aligned} &\left|\sin(\hat{\theta})\sin(\hat{\phi})\right| = |\hat{v}| = \left|\left(\alpha\beta^{k_2}\right)^{1/k_1}\right| \\ &\quad \Rightarrow |\sin(\phi)| = \csc(\hat{\theta})\left|\alpha^{1/k_1}\beta^{k_2/k_1}\right|. \end{aligned} \tag{3.38}$$

which results

$$\sin(\hat{\phi}) = \csc(\hat{\theta})\left|\alpha^{1/k_1}\beta^{k_2/k_1}\right|\operatorname{sgn}(v) \tag{3.39}$$

With assumption [AS-4], $\sin(\hat{\phi})$ can be determined via equation (3.39). Additionally, with assumption [AS-3], $\hat{\phi}$ in (3.39) can be unambiguously determined as

$$\hat{\phi} = \begin{cases} \pi[1-\operatorname{sgn}(v)-\delta(v)] + \sin^{-1}\left(\operatorname{sgn}(v)\csc(\hat{\theta})\left|\alpha^{1/k_1}\beta^{k_2/k_1}\right|\right), \\ \hfill \text{if } \phi \in \left[0, \frac{\pi}{2}\right) \cup \left[\frac{3\pi}{2}, 2\pi\right); \\ \pi - \sin^{-1}\left(\operatorname{sgn}(v)\csc(\hat{\theta})\left|\alpha^{1/k_1}\beta^{k_2/k_1}\right|\right), \quad \text{if } \phi \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right); \end{cases} \tag{3.40}$$

$$\delta(v) = \begin{cases} 1, & \text{if } v = 0 \\ 0, & \text{if } v \neq 0 \end{cases}$$

From the above derivation, (3.37) and (3.40) are obtained when assumptions [AS-1], [AS-3] and [AS-4] hold, which is equivalent to

i) [AS-3] and [AS-4] hold if k_2 is even. That is, the prior information required to unambiguously determine $\hat{\phi}$ and $\hat{\theta}$ for configuration (i) is $\phi \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right)$ or $\left[\pi, \frac{3\pi}{2}\right)$ or $\left[\frac{3\pi}{2}, 2\pi\right)$.

ii) [AS-1], [AS-3] and [AS-4] hold if k_2 is odd. That is, the prior information required to unambiguously determine $\hat{\phi}$ and $\hat{\theta}$ for configuration (i) is $\phi \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right)$ or $\left[\pi, \frac{3\pi}{2}\right)$ or $\left[\frac{3\pi}{2}, 2\pi\right)$, and $\theta \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right]$.

3.7 Three Special Configurations

For configuration (c), (e) and (g) (refer to Figure 2.1), it's not able to derive the close-form solution. The common point of these three configurations is that neither of η_1 and η_2 equals to μ; that is $\eta_1 \neq \eta_2 \neq \mu$. In this case, it is not possible to eliminate η_1 or η_2 in (3.1) by μ. Therefore, there is no way to get rid of k_1 and k_2 on the left-hand-side of (3.1) and (3.2). To find the unknown θ and ϕ, we have to solve a high order equation set which may not has closed-form solutions.

For example, in configuration (e) $\eta_1 = u$, $\eta_2 = w$, $\mu = v$, thus we have

$$\begin{cases} \dfrac{u^{k_1}}{w^{k_2}} = \alpha\left(\operatorname{sgn}\left(\dfrac{\eta_1^{k_1}}{\eta_2^{k_2}}\right)\right) & (3.41) \\ v = \beta\left(\operatorname{sgn}\left(\dfrac{\eta_1^{k_1}}{\eta_2^{k_2}}\right)\right). & (3.42) \end{cases}$$

Take square on both sides of (3.41) get

$$\begin{aligned} \frac{\left(\sin^2(\theta)\cos^2(\phi)\right)^{k_1}}{\left(\cos^2(\theta)\right)^{k_2}} &= \alpha^2 \\ \Rightarrow \frac{\left(\sin^2(\theta)\left(1-\sin^2(\phi)\right)\right)^{k_1}}{\left(1-\sin^2(\theta)\right)^{k_2}} &= \alpha^2. \end{aligned}$$

According to (3.42) $\sin(\phi) = \beta/\sin(\theta)$, substitute $\beta/\sin(\theta)$ for $\sin(\phi)$ in the equation above, we derive

$$\begin{aligned} \Rightarrow \frac{\left(\sin^2(\theta)\left(1-\frac{\beta^2}{\sin^2(\theta)}\right)\right)^{k_1}}{\left(1-\sin^2(\theta)\right)^{k_2}} &= \alpha^2 \\ \Rightarrow \frac{\left(\sin^2(\theta)-\beta^2\right)^{k_1}}{\left(1-\sin^2(\theta)\right)^{k_2}} &= \alpha^2. \end{aligned}$$

Let $x = \sin^2(\theta)$, and expand this equation by binomial expansion,

$$\frac{\sum_{r=0}^{k_1}\binom{k_1}{r}x^{k_1-r}\left(-\beta^2\right)^r}{\sum_{l=0}^{k_2}\binom{k_2}{l}\left(-x\right)^l} = \alpha^2. \tag{3.43}$$

This is a high order equation about x. There is no closed-form solution for such case. So we may not be able to derive the closed-form solutions for configuration (e), and neither (c) and (g) for the same reason. So in the rest part of this section I will show the derivation of closed-form solutions for the other six configurations.

3.8 Summary

For the general case (k_1 and k_2 can possibly be unequal),

1. the closed-form estimators of ϕ and θ for configurations (a), (b), (d), (f), (h) and (i) have been derived;

2. neither the estimation of ϕ nor the estimation of θ is possible for configuration (c), (e), and (g).

Table 3.1 summarizes all the closed-form formulas. The general prior information required is to know in which octant the source resides.

Table 3.1: Formulas Applicable For Any k_1 and k_2

Configuration	Formulas	Prior Information Required
(a)	$\hat{\phi} = \begin{cases} \frac{\pi}{2}[1-\operatorname{sgn}(u)] + \tan^{-1}\left(\operatorname{sgn}(u)\left\lvert \frac{\beta\left(\operatorname{sgn}\left(\frac{u^{k_1}}{v^{k_2}}\right)\right)^{k_1/k_2-1}}{\alpha\left(\operatorname{sgn}\left(\frac{u^{k_1}}{v^{k_2}}\right)\right)^{1/k_2}} \right\rvert\right), & \text{if } \operatorname{sgn}(v) > 0; \\ \frac{\pi}{2}[3+\operatorname{sgn}(u)] - \tan^{-1}\left(\operatorname{sgn}(u)\left\lvert \frac{\beta\left(\operatorname{sgn}\left(\frac{u^{k_1}}{v^{k_2}}\right)\right)^{k_1/k_2-1}}{\alpha\left(\operatorname{sgn}\left(\frac{u^{k_1}}{v^{k_2}}\right)\right)^{1/k_2}} \right\rvert\right), & \text{if } \operatorname{sgn}(v) < 0; \end{cases}$ $\hat{\theta} = \begin{cases} \sin^{-1}\left(\beta\left(\operatorname{sgn}\left(\frac{u^{k_1}}{v^{k_2}}\right)\right)\sec(\hat{\phi})\right), & \text{if } \theta \in \left[0, \frac{\pi}{2}\right); \\ \pi - \sin^{-1}\left(\beta\left(\operatorname{sgn}\left(\frac{u^{k_1}}{v^{k_2}}\right)\right)\sec(\hat{\phi})\right), & \text{if } \theta \in \left[\frac{\pi}{2}, \pi\right]; \end{cases}$	$\phi \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right)$ or $\left[\pi, \frac{3\pi}{2}\right)$ or $\left[\frac{3\pi}{2}, 2\pi\right)$, $\theta \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right]$
(b)	$\hat{\phi} = \begin{cases} \frac{\pi}{2}[1-\operatorname{sgn}(u)] + \tan^{-1}\left(\operatorname{sgn}(u)\left\lvert \frac{\beta\left(\operatorname{sgn}\left(\frac{u^{k_1}}{v^{k_2}}\right)\right)^{1-k_2/k_1}}{\alpha\left(\operatorname{sgn}\left(\frac{u^{k_1}}{v^{k_2}}\right)\right)^{1/k_1}} \right\rvert\right), & \text{if } \operatorname{sgn}(v) > 0; \\ \frac{\pi}{2}[3+\operatorname{sgn}(u)] - \tan^{-1}\left(\operatorname{sgn}(u)\left\lvert \frac{\beta\left(\operatorname{sgn}\left(\frac{u^{k_1}}{v^{k_2}}\right)\right)^{1-k_2/k_1}}{\alpha\left(\operatorname{sgn}\left(\frac{u^{k_1}}{v^{k_2}}\right)\right)^{1/k_1}} \right\rvert\right), & \text{if } \operatorname{sgn}(v) < 0; \end{cases}$ $\hat{\theta} = \begin{cases} \sin^{-1}\left(\beta\left(\operatorname{sgn}\left(\frac{u^{k_1}}{v^{k_2}}\right)\right)\csc(\hat{\phi})\right), & \text{if } \theta \in \left[0, \frac{\pi}{2}\right); \\ \pi - \sin^{-1}\left(\beta\left(\operatorname{sgn}\left(\frac{u^{k_1}}{v^{k_2}}\right)\right)\csc(\hat{\phi})\right), & \text{if } \theta \in \left[\frac{\pi}{2}, \pi\right]; \end{cases}$	$\phi \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right)$ or $\left[\pi, \frac{3\pi}{2}\right)$ or $\left[\frac{3\pi}{2}, 2\pi\right)$, $\theta \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right]$
(d)	$\hat{\phi} = \begin{cases} \cos^{-1}\left(\beta\left(\operatorname{sgn}\left(\frac{u^{k_1}}{w^{k_2}}\right)\right)\csc(\hat{\theta})\right), & \text{if } \phi \in [0, \pi); \\ 2\pi - \cos^{-1}\left(\beta\left(\operatorname{sgn}\left(\frac{u^{k_1}}{w^{k_2}}\right)\right)\csc(\hat{\theta})\right), & \text{if } \phi \in [\pi, 2\pi); \end{cases}$ $\hat{\theta} = \cos^{-1}\left(\operatorname{sgn}(w)\left\lvert \frac{\beta\left(\operatorname{sgn}\left(\frac{u^{k_1}}{w^{k_2}}\right)\right)^{k_1/k_2}}{\alpha\left(\operatorname{sgn}\left(\frac{u^{k_1}}{w^{k_2}}\right)\right)^{1/k_2}} \right\rvert\right)$	If k_1 is odd: $\phi \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right)$ or $\left[\pi, \frac{3\pi}{2}\right)$ or $\left[\frac{3\pi}{2}, 2\pi\right)$, $\theta \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right]$ If k_1 is even: $\phi \in [0, \pi)$ or $[\pi, 2\pi)$, $\theta \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right]$

Configuration	Formulas	Prior Information Required
(f)	$\hat{\phi} = \begin{cases} \cos^{-1}\left(\operatorname{sgn}(u)\csc(\hat{\theta})\left\lvert \alpha\left(\operatorname{sgn}\left(\frac{u^{k_1}}{w^{k_2}}\right)\right)^{1/k_1} \beta\left(\operatorname{sgn}\left(\frac{u^{k_1}}{w^{k_2}}\right)\right)^{k_2/k_1}\right\rvert\right), & \text{if } \phi \in [0, \pi)\,; \\ 2\pi - \cos^{-1}\left(\operatorname{sgn}(u)\csc(\hat{\theta})\left\lvert \alpha\left(\operatorname{sgn}\left(\frac{u^{k_1}}{w^{k_2}}\right)\right)^{1/k_1} \beta\left(\operatorname{sgn}\left(\frac{u^{k_1}}{w^{k_2}}\right)\right)^{k_2/k_1}\right\rvert\right), & \text{if } \phi \in [\pi, 2\pi)\,; \end{cases}$ $\hat{\theta} = \cos^{-1}\left(\beta\left(\operatorname{sgn}\left(\frac{u^{k_1}}{w^{k_2}}\right)\right)\right)$	If k_2 is odd: $\phi \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right)$ or $\left[\pi, \frac{3\pi}{2}\right)$ or $\left[\frac{3\pi}{2}, 2\pi\right)$, $\theta \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right]$ If k_2 is even: $\phi \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right)$ or $\left[\pi, \frac{3\pi}{2}\right)$ or $\left[\frac{3\pi}{2}, 2\pi\right)$
(h)	$\hat{\phi} = \begin{cases} \pi[1 - \operatorname{sgn}(\psi) - \delta(\psi)] + \sin^{-1}\left(\beta\left(\operatorname{sgn}\left(\frac{v^{k_1}}{w^{k_2}}\right)\right)\csc(\hat{\theta})\right), & \text{if } \phi \in \left[0, \frac{\pi}{2}\right) \cup \left[\frac{3\pi}{2}, 2\pi\right)\,; \\ \pi - \sin^{-1}\left(\beta\left(\operatorname{sgn}\left(\frac{v^{k_1}}{w^{k_2}}\right)\right)\csc(\hat{\theta})\right), & \text{if } \phi \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right)\,; \end{cases}$ $\hat{\theta} = \cos^{-1}\left(\operatorname{sgn}(w)\left\lvert\frac{\beta\left(\operatorname{sgn}\left(\frac{v^{k_1}}{w^{k_2}}\right)\right)^{k_1/k_2}}{\alpha\left(\operatorname{sgn}\left(\frac{v^{k_1}}{w^{k_2}}\right)\right)^{1/k_2}}\right\rvert\right)$	If k_1 is odd: $\phi \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right)$ or $\left[\pi, \frac{3\pi}{2}\right)$ or $\left[\frac{3\pi}{2}, 2\pi\right)$, $\theta \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right]$ If k_1 is even: $\phi \in \left[0, \frac{\pi}{2}\right) \cup \left[\frac{3\pi}{2}, 2\pi\right)$ or $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right)$, $\theta \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right]$
(i)	$\hat{\phi} = \begin{cases} \pi[1 - \operatorname{sgn}(v) - \delta(v)] + \sin^{-1}\left(\operatorname{sgn}(v)\csc(\hat{\theta})\left\lvert \alpha\left(\operatorname{sgn}\left(\frac{v^{k_1}}{w^{k_2}}\right)\right)^{1/k_1} \beta\left(\operatorname{sgn}\left(\frac{v^{k_1}}{w^{k_2}}\right)\right)^{k_2/k_1}\right\rvert\right), & \text{if } \phi \in \left[0, \frac{\pi}{2}\right) \cup \left[\frac{3\pi}{2}, 2\pi\right)\,; \\ \pi - \sin^{-1}\left(\operatorname{sgn}(v)\csc(\hat{\theta})\left\lvert \alpha\left(\operatorname{sgn}\left(\frac{v^{k_1}}{w^{k_2}}\right)\right)^{1/k_1} \beta\left(\operatorname{sgn}\left(\frac{v^{k_1}}{w^{k_2}}\right)\right)^{k_2/k_1}\right\rvert\right), & \text{if } \phi \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right)\,; \end{cases}$ $\hat{\theta} = \cos^{-1}\left(\operatorname{sgn}(w)\left\lvert\frac{\beta\left(\operatorname{sgn}\left(\frac{v^{k_1}}{w^{k_2}}\right)\right)^{k_1/k_2}}{\alpha\left(\operatorname{sgn}\left(\frac{v^{k_1}}{w^{k_2}}\right)\right)^{1/k_2}}\right\rvert\right)$	If k_2 is odd: $\phi \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right)$ or $\left[\pi, \frac{3\pi}{2}\right)$ or $\left[\frac{3\pi}{2}, 2\pi\right)$, $\theta \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right]$ If k_2 is even: $\phi \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right)$ or $\left[\pi, \frac{3\pi}{2}\right)$ or $\left[\frac{3\pi}{2}, 2\pi\right)$

where $\psi = \sin^{-1}\left(\frac{\lambda}{2\pi\Delta_y}\csc(\hat{\theta})\angle\left(\operatorname{sgn}\left(\frac{v^{k_1}}{w^{k_2}}\right)\frac{\left[\hat{\mathbf{a}}_{y,z}^{(y-\text{axis})}\right]_1}{\left[\hat{\mathbf{a}}_{y,z}^{(y-\text{axis})}\right]_2}\right)\right)$, and $\delta(\psi) = \begin{cases} 1, & \text{if } \psi = 0 \\ 0, & \text{if } \psi \neq 0 \end{cases}$.

Chapter 4

Degenerated Cases for Equal-Order sensors

This chapter discusses the closed-form solutions to the direction-of-arrival estimator ($\hat{\theta}$, $\hat{\phi}$) under the assumption that the two sensors are equal-order, i.e. $k_1 = k_2$. In this case, the closed-form formulas for most configurations can be derived based on Table 3.1 except configuration (c), (e) and (g). In addition, section 4.10 also explores the possibility of deriving close-form formulas for collocated differential sensor pair. For the definition of assumptions used ([AS-1] to [AS-5]) in this chapter please refer to Chapter 3.

4.1 y-axis and x-axis oriented sensors with displacement along x-axis

For the degenerated case where the uniaxial velocity-sensors have the identical order, the estimators $\hat{\phi}$ and $\hat{\theta}$ can be directly obtained from equations (3.6) and (3.8) in Table 3.1 by setting $k_1 = k_2 = k$,

$$\hat{\phi} = \begin{cases} \frac{\pi}{2}[1 - \text{sgn}(u)] + \tan^{-1}\left(\text{sgn}(u)\left|\alpha^{-1/k}\right|\right), \text{if } \text{sgn}(v) > 0; \\ \frac{\pi}{2}[3 + \text{sgn}(u)] - \tan^{-1}\left(\text{sgn}(u)\left|\alpha^{-1/k}\right|\right), \text{if } \text{sgn}(v) < 0; \end{cases} \tag{4.1}$$

and

$$\hat{\theta} = \begin{cases} \sin^{-1}\left(\beta \sec(\hat{\phi})\right), & \text{if } \theta \in \left[0, \frac{\pi}{2}\right); \\ \pi - \sin^{-1}\left(\beta \sec(\hat{\phi})\right), & \text{if } \theta \in \left[\frac{\pi}{2}, \pi\right]. \end{cases} \tag{4.2}$$

where the prior knowledge required is $\phi \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right)$ or $\left[\pi, \frac{3\pi}{2}\right)$ or $\left[\frac{3\pi}{2}, 2\pi\right)$, and $\theta \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right]$.

4.2 y-axis and x-axis oriented sensors with displacement along y-axis

For the degenerated case where the uniaxial velocity-sensors have the identical order, the estimators $\hat{\phi}$ and $\hat{\theta}$ can be directly obtained from equations (3.12) and (3.14) in Table 3.1 by setting $k_1 = k_2 = k$,

$$\hat{\phi} = \begin{cases} \frac{\pi}{2}[1 - \text{sgn}(u)] + \tan^{-1}\left(\text{sgn}(u)\left|\alpha^{-1/k}\right|\right), \text{if sgn}(v) > 0; \\ \frac{\pi}{2}[3 + \text{sgn}(u)] - \tan^{-1}\left(\text{sgn}(u)\left|\alpha^{-1/k}\right|\right), \text{if sgn}(v) < 0; \end{cases} \tag{4.3}$$

and

$$\hat{\theta} = \begin{cases} \sin^{-1}\left(\beta \csc(\hat{\phi})\right), & \text{if } \theta \in \left[0, \frac{\pi}{2}\right); \\ \pi - \sin^{-1}\left(\beta \csc(\hat{\phi})\right), & \text{if } \theta \in \left[\frac{\pi}{2}, \pi\right]. \end{cases} \tag{4.4}$$

where prior knowledge required is $\phi \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right)$ or $\left[\pi, \frac{3\pi}{2}\right)$ or $\left[\frac{3\pi}{2}, 2\pi\right)$, and $\theta \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right]$.

4.3 y-axis and x-axis oriented sensors with displacement along z-axis

For this configuration, the estimation of ϕ and θ is not possible when $k_1 \neq k_2$, as shown in Table 3.1. However, when $k_1 = k_2 = k$, from equation (3.1) and (3.2)

$$\begin{cases} \widehat{\left(\frac{u^k}{v^k}\right)} = \alpha\left(\text{sgn}\left(\frac{u^k}{v^k}\right)\right) & (4.5) \\ \hat{w} = \beta\left(\text{sgn}\left(\frac{u^k}{v^k}\right)\right) & (4.6) \end{cases}$$

Straightforward calculation shows that

$$\left|\tan(\hat{\phi})\right| = \left|\widehat{\left(\frac{v}{u}\right)}\right| = \left|\alpha^{-1/k}\right|$$

which results

$$\tan(\hat{\phi}) = \left|\alpha^{-1/k}\right| \text{sgn}\left(\frac{v}{u}\right). \tag{4.7}$$

With assumption [AS-2], $\tan(\hat{\phi})$ can be determined via equation (4.7). Additionally, with assumption [AS-3], $\hat{\phi}$ in (4.7) can be unambiguously determined as

$$\hat{\phi} = \begin{cases} \frac{\pi}{2}[1 - \text{sgn}(u)] + \tan^{-1}\left(\text{sgn}(u)\left|\alpha^{-1/k}\right|\right), & \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } \text{sgn}(v) > 0; & \\ \frac{\pi}{2}[3 + \text{sgn}(u)] - \tan^{-1}\left(\text{sgn}(u)\left|\alpha^{-1/k}\right|\right), & \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } \text{sgn}(v) < 0; & \end{cases} \tag{4.8}$$

From (4.6),

$$\cos(\hat{\theta}) = \hat{w} = \beta. \tag{4.9}$$

Thus, $\hat{\theta}$ in (4.9) can be unambiguously determined as

$$\hat{\theta} = \cos^{-1}(\beta). \tag{4.10}$$

From the above derivation, (4.8) and (4.10) are obtained when assumptions [AS-1], [AS-2] and [AS-3] hold, which is equivalent to [AS-3], [AS-4] hold. That is, the prior information required to unambiguously determine $\hat{\phi}$ and $\hat{\theta}$ for configuration (c) is $\phi \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right)$ or $\left[\pi, \frac{3\pi}{2}\right)$ or $\left[\frac{3\pi}{2}, 2\pi\right)$.

4.4 z-axis and x-axis oriented sensors with displacement along x-axis

For the degenerated case where the uniaxial velocity-sensors have the identical order, the estimators $\hat{\phi}$ and $\hat{\theta}$ can be directly obtained from equations (3.18) and (4.12) in Table 3.1 by setting $k_1 = k_2 = k$,

$$\hat{\theta} = \cos^{-1}\left(\text{sgn}(w)\left|\frac{\beta}{\alpha^{1/k}}\right|\right). \tag{4.11}$$

and

$$\hat{\phi} = \begin{cases} \cos^{-1}\left(\beta\csc(\hat{\theta})\right), & \text{if } \phi \in [0, \pi); \\ 2\pi - \cos^{-1}\left(\beta\csc(\hat{\theta})\right), & \text{if } \phi \in [\pi, 2\pi). \end{cases} \tag{4.12}$$

where the prior knowledge required is $\phi \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right)$ or $\left[\pi, \frac{3\pi}{2}\right)$ or $\left[\frac{3\pi}{2}, 2\pi\right)$, and $\theta \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right]$. Specially, if k is a even number, less prior knowledge is required, i.e. $\phi \in [0, \pi)$ or $[\pi, 2\pi)$, and $\theta \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right]$.

4.5 z-axis and x-axis oriented sensors with displacement along y-axis

For this configuration, the estimation of ϕ and θ is not possible when $k_1 \neq k_2$, as shown in Table 3.1. However, when $k_1 = k_2 = k$, from equation (3.1) and (3.2)

$$\begin{cases} \widehat{\left(\dfrac{u^k}{w^k}\right)} = \left(\dfrac{\sin(\hat{\theta})\cos(\hat{\phi})}{\cos(\hat{\theta})}\right)^k = \alpha\left(\operatorname{sgn}\left(\dfrac{u^k}{w^k}\right)\right), & (4.13) \\ \hat{v} = \sin(\hat{\theta})\sin(\hat{\phi}) = \beta\left(\operatorname{sgn}\left(\dfrac{u^k}{w^k}\right)\right). & (4.14) \end{cases}$$

Straightforward deduction from (4.13) and (4.14):

$$\begin{aligned} & \frac{\sin^2(\hat{\theta})\cos^2(\hat{\phi})}{\cos^2(\hat{\theta})} = \alpha^{2/k} \\ \Rightarrow \quad & \frac{\sin^2(\hat{\theta})\left(1-\sin^2(\hat{\phi})\right)}{1-\sin^2(\hat{\theta})} = \alpha^{2/k} \\ \Rightarrow \quad & \frac{\sin^2(\hat{\theta})\left(1-\beta^2/\sin^2(\hat{\theta})\right)}{1-\sin^2(\hat{\theta})} = \alpha^{2/k} \\ \Rightarrow \quad & \frac{\sin^2(\hat{\theta})-\beta^2}{1-\sin^2(\hat{\theta})} = \alpha^{2/k} \\ \Rightarrow \quad & \sin^2(\hat{\theta}) = \frac{\alpha^{2/k}+\beta^2}{1+\alpha^{2/k}}. \end{aligned} \tag{4.15}$$

Since $\theta \in [0, \pi]$, $\sin(\theta) \geq 0$. Thus, we may obtain $\sin(\hat{\theta})$ from (4.15) as

$$\sin(\hat{\theta}) = \sqrt{\frac{\alpha^{2/k}+\beta^2}{1+\alpha^{2/k}}}. \tag{4.16}$$

With assumption [AS-5], $\hat{\theta}$ in (4.16) can be unambiguously determined as

$$\hat{\theta} = \begin{cases} \sin^{-1}\left(\sqrt{\dfrac{\alpha^{2/k}+\beta^2}{1+\alpha^{2/k}}}\right), & \text{if } \operatorname{sgn}(w) > 0; \\ \pi - \sin^{-1}\left(\sqrt{\dfrac{\alpha^{2/k}+\beta^2}{1+\alpha^{2/k}}}\right), & \text{if } \operatorname{sgn}(w) < 0; \end{cases} \tag{4.17}$$

From (4.14) and (4.17),

$$\sin(\hat{\phi}) = \frac{\hat{v}}{\sin(\hat{\theta})} = \beta\csc(\hat{\theta}) \tag{4.18}$$

With assumption [AS-3], $\hat{\phi}$ in (4.18) can be unambiguously determined as

$$\hat{\phi} = \begin{cases} \pi[1 - \mathrm{sgn}(\chi) - \delta(\chi)] + \sin^{-1}\left(\csc(\hat{\theta})\beta\right), & \text{if } \mathrm{sgn}(u) > 0; \\ \pi - \sin^{-1}\left(\csc(\hat{\theta})\beta\right), & \text{if } \mathrm{sgn}(u) < 0; \end{cases} \tag{4.19}$$

where $\chi = \sin^{-1}\left(\frac{\lambda}{2\pi\Delta_y}\csc(\hat{\theta})\angle\left(\mathrm{sgn}\left(\frac{u^k}{w^k}\right)\frac{\left[\hat{\mathbf{a}}_{x,z}^{(y-\text{axis})}\right]_1}{\left[\hat{\mathbf{a}}_{x,z}^{(y-\text{axis})}\right]_2}\right)\right)$, and $\delta(\chi) = \begin{cases} 1, & \text{if } \chi = 0 \\ 0, & \text{if } \chi \neq 0 \end{cases}$.

From the above derivation, (4.17) and (4.19) are obtained when assumption [AS-1], [AS-3] and [AS-5] hold, which is equivalent to [AS-3] and [AS-5] hold. That is, the prior information required to unambiguously determine $\hat{\phi}$ and $\hat{\theta}$ for configuration (e) is $\phi \in \left[0, \frac{\pi}{2}\right) \cup \left[\frac{3\pi}{2}, 2\pi\right)$ or $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right)$ and $\theta \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right]$.

4.6 z-axis and x-axis oriented sensors with displacement along z-axis

For the degenerated case where the uniaxial velocity-sensors have the identical order, the estimators $\hat{\phi}$ and $\hat{\theta}$ can be directly obtained from equations (3.27) and (3.24) in Table 3.1 by setting $k_1 = k_2 = k$,

$$\hat{\theta} = \cos^{-1}(\beta). \tag{4.20}$$

and

$$\hat{\phi} = \begin{cases} \cos^{-1}\left(\mathrm{sgn}(u)\csc(\hat{\theta})\left|\alpha^{1/k}\beta\right|\right), & \text{if } \phi \in [0, \pi); \\ 2\pi - \cos^{-1}\left(\mathrm{sgn}(u)\csc(\hat{\theta})\left|\alpha^{1/k}\beta\right|\right), & \text{if } \phi \in [\pi, 2\pi). \end{cases} \tag{4.21}$$

where the prior knowledge required is $\phi \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right)$ or $\left[\pi, \frac{3\pi}{2}\right)$ or $\left[\frac{3\pi}{2}, 2\pi\right)$, and $\theta \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right]$. Specially, if k is a even number, less prior knowledge is required, i.e. $\phi \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right)$ or $\left[\pi, \frac{3\pi}{2}\right)$ or $\left[\frac{3\pi}{2}, 2\pi\right)$.

4.7 z-axis and y-axis oriented sensors with displacement along x-axis

For this configuration, the estimation of ϕ and θ is not possible when $k_1 \neq k_2$, as shown in Table 3.1. However, when $k_1 = k_2 = k$, from equation (3.1) and (3.2)

$$\begin{cases} \widehat{\left(\frac{v^k}{w^k}\right)} = \left(\frac{\sin(\hat{\theta})\sin(\hat{\phi})}{\cos(\hat{\theta})}\right)^k = \alpha\left(\mathrm{sgn}\left(\frac{v^k}{w^k}\right)\right), & (4.22) \\ \hat{v} = \sin(\hat{\theta})\cos(\hat{\phi}) = \beta\left(\mathrm{sgn}\left(\frac{v^k}{w^k}\right)\right). & (4.23) \end{cases}$$

Straightforward deduction from (4.22) and (4.23):

$$
\begin{aligned}
& \frac{\sin^2(\hat{\theta})\sin^2(\hat{\phi})}{\cos^2(\hat{\theta})} = \alpha^{2/k} \\
\Rightarrow\quad & \frac{\sin^2(\hat{\theta})\left(1-\cos^2(\phi)\right)}{1-\sin^2(\hat{\theta})} = \alpha^{2/k} \\
\Rightarrow\quad & \frac{\sin^2(\hat{\theta})-\beta^2}{1-\sin^2(\hat{\theta})} = \alpha^{2/k} \\
\Rightarrow\quad & \sin^2(\hat{\theta}) = \frac{\alpha^{2/k}+\beta^2}{1+\alpha^{2/k}}. \qquad (4.24)
\end{aligned}
$$

Since $\theta \in [0, \pi]$, $\sin(\theta) \geq 0$. Thus, we may obtain $\sin(\hat{\theta})$ from (4.24) as

$$
\sin(\hat{\theta}) = \sqrt{\frac{\alpha^{2/k}+\beta^2}{1+\alpha^{2/k}}}. \qquad (4.25)
$$

With assumption [AS-5], $\hat{\theta}$ in (4.25) can be unambiguously determined as

$$
\hat{\theta} = \begin{cases} \sin^{-1}\left(\sqrt{\dfrac{\alpha^{2/k}+\beta^2}{1+\alpha^{2/k}}}\right), & \text{if } \operatorname{sgn}(w) > 0; \\ \pi - \sin^{-1}\left(\sqrt{\dfrac{\alpha^{2/k}+\beta^2}{1+\alpha^{2/k}}}\right), & \text{if } \operatorname{sgn}(w) < 0; \end{cases} \qquad (4.26)
$$

From (4.23) and (4.26),

$$
\cos(\hat{\phi}) = \frac{\hat{v}}{\sin(\hat{\theta})} = \beta\csc(\hat{\theta}). \qquad (4.27)
$$

With assumption [AS-4], $\hat{\phi}$ in (4.27) can be unambiguously determined as

$$
\hat{\phi} = \begin{cases} \cos^{-1}\left(\csc(\hat{\theta})\beta\right), & \text{if } \operatorname{sgn}(v) > 0; \\ 2\pi - \cos^{-1}\left(\csc(\hat{\theta})\beta\right), & \text{if } \operatorname{sgn}(v) < 0. \end{cases} \qquad (4.28)
$$

From the above derivation, (4.26) and (4.28) are obtained when assumption [AS-1], [AS-4] and [AS-5] hold, which is equivalent to [AS-4] and [AS-5] hold. That is, the prior information required to unambiguously determine $\hat{\phi}$ and $\hat{\theta}$ for configuration (e) is $\phi \in [0, \pi)$ or $[\pi, 2\pi)$ and $\theta \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right]$.

4.8 z-axis and y-axis oriented sensors with displacement along y-axis

For the degenerated case where the uniaxial velocity-sensors have the identical order, the estimators $\hat{\phi}$ and $\hat{\theta}$ can be directly obtained from equations (3.31) and (3.33) in Table 3.1 by setting $k_1 = k_2 = k$,

$$\hat{\theta} = \cos^{-1}\left(\mathrm{sgn}(w)\left|\frac{\beta}{\alpha^{1/k}}\right|\right). \tag{4.29}$$

and

$$\hat{\phi} = \begin{cases} \pi[1 - \mathrm{sgn}(\psi) - \delta(\psi)] + \sin^{-1}\left(\beta\csc(\hat{\theta})\right), \\ \qquad\qquad \text{if } \phi \in \left[0, \frac{\pi}{2}\right) \cup \left[\frac{3\pi}{2}, 2\pi\right); \\ \pi - \sin^{-1}\left(\beta\csc(\hat{\theta})\right), \quad \text{if } \phi \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right); \end{cases} \tag{4.30}$$

where $\psi = \sin^{-1}\left(\frac{\lambda}{2\pi\Delta_y}\csc(\hat{\theta})\angle\left(\mathrm{sgn}\left(\frac{v^k}{w^k}\right)\frac{\left[\hat{\mathbf{a}}_{y,z}^{(y-\text{axis})}\right]_1}{\left[\hat{\mathbf{a}}_{y,z}^{(y-\text{axis})}\right]_2}\right)\right)$, and $\delta(\psi) = \begin{cases} 1, & \text{if } \psi = 0 \\ 0, & \text{if } \psi \neq 0 \end{cases}$

The prior knowledge required is $\phi \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right)$ or $\left[\pi, \frac{3\pi}{2}\right)$ or $\left[\frac{3\pi}{2}, 2\pi\right)$, and $\theta \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right]$. Specially, if k is a even number, less prior information is required, i.e. $\phi \in \left[0, \frac{\pi}{2}\right) \cup \left[\frac{3\pi}{2}, 2\pi\right)$ or $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right)$, and $\theta \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right]$.

4.9 z-axis and y-axis oriented sensors with displacement along z-axis

For the degenerated case where the uniaxial velocity-sensors have the identical order, the estimators $\hat{\phi}$ and $\hat{\theta}$ can be directly obtained from equations (3.37) and (3.40) in Table 3.1 by setting $k_1 = k_2 = k$,

$$\hat{\theta} = \cos^{-1}(\beta). \tag{4.31}$$

and

$$\hat{\phi} = \begin{cases} \pi[1 - \mathrm{sgn}(v) - \delta(v)] + \sin^{-1}\left(\mathrm{sgn}(v)\csc(\hat{\theta})\left|\alpha^{1/k}\beta\right|\right), \\ \qquad\qquad \text{if } \phi \in \left[0, \frac{\pi}{2}\right) \cup \left[\frac{3\pi}{2}, 2\pi\right); \\ \pi - \sin^{-1}\left(\mathrm{sgn}(v)\csc(\hat{\theta})\left|\alpha^{1/k}\beta\right|\right), \quad \text{if } \phi \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right); \end{cases} \tag{4.32}$$

$$\delta(v) = \begin{cases} 1, & \text{if } v = 0 \\ 0, & \text{if } v \neq 0 \end{cases}.$$

The prior knowledge required is $\phi \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right)$ or $\left[\pi, \frac{3\pi}{2}\right)$ or $\left[\frac{3\pi}{2}, 2\pi\right)$, and $\theta \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right]$. Specially, if k is a even number, less prior knowledge is required, i.e. $\phi \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right)$ or $\left[\pi, \frac{3\pi}{2}\right)$ or $\left[\frac{3\pi}{2}, 2\pi\right)$.

4.10 Degenerated Cases for Collocated Velocity-sensors

For collocated sensors, i.e. $\Delta_\epsilon = 0$, the number of configuration reduces to three:

(1) $\eta_1 = u$, $\eta_2 = v$;

(2) $\eta_1 = u$, $\eta_2 = w$;

(3) $\eta_1 = v$, $\eta_2 = w$.

The estimate of array manifold can be written by simplifying (2.1) as

$$\hat{\mathbf{a}}_{\zeta_1,\zeta_2} \approx c\mathbf{a}_{\zeta_1,\zeta_2} = c \begin{bmatrix} \eta_1^{k_1} \\ \eta_2^{k_2} \end{bmatrix}. \tag{4.33}$$

In this case, we can find only one equation about $\hat{\theta}$ and $\hat{\phi}$ by

$$\widehat{\left(\frac{\eta_1^{k_1}}{\eta_2^{k_2}}\right)} = \frac{[\hat{\mathbf{a}}_{\zeta_1,\zeta_2}]_1}{[\hat{\mathbf{a}}_{\zeta_1,\zeta_2}]_2}. \tag{4.34}$$

It is not enough to solve both unknown variables, as concluded below:

For configuration (1), the closed-form formula for azimuth angle $\hat{\phi}$ is derivable only when $k_1 = k_2$. Closed-form formula for elevation angle $\hat{\theta}$ is underivable.

For configuration (2), neither formula can be deduced.

For configuration (3), neither formula can be deduced.

4.11 Summary

For the case of $k_1 = k_2 = k$, Table 4.1 shows all closed-form formulas of azimuth and elevation angle estimator for nine configurations. The estimators of ϕ and θ for configuration (a), (b), (d), (f), (h) and (i) can be deduced from Table 3.1. The closed-form solutions of $\hat{\phi}$ and $\hat{\theta}$ for configuration (c), (e) and (g) are also available in this case.

The prior information required for configuration (c), (e), (g) is less than other configurations. Prior information about DOA required for these tree configurations: know in which quadrant the source resides. Prior information required for other configurations: know in which octant the source resides.

For a pair of collocated differential sensor, i.e. $\Delta = 0$, the closed-form formula is not derivable.

Table 4.1: Formulas Applicable For Any k_1 and k_2

Configuration	Formulas	Prior Information Required
(a)	$\hat{\phi} = \begin{cases} \frac{\pi}{2}[1-\mathrm{sgn}(u)] + \tan^{-1}\left(\mathrm{sgn}(u)\left\lvert\alpha\left(\mathrm{sgn}\left(\frac{u^k}{v^k}\right)\right)^{-1/k}\right\rvert\right), & \text{if } \mathrm{sgn}(v) > 0; \\ \frac{\pi}{2}[3+\mathrm{sgn}(u)] - \tan^{-1}\left(\mathrm{sgn}(u)\left\lvert\alpha\left(\mathrm{sgn}\left(\frac{u^k}{v^k}\right)\right)^{-1/k}\right\rvert\right), & \text{if } \mathrm{sgn}(v) < 0; \end{cases}$ $\hat{\theta} = \begin{cases} \sin^{-1}\left(\beta\left(\mathrm{sgn}\left(\frac{u^k}{v^k}\right)\right)\sec(\hat{\phi})\right), & \text{if } \theta \in \left[0, \frac{\pi}{2}\right); \\ \pi - \sin^{-1}\left(\beta\left(\mathrm{sgn}\left(\frac{u^k}{v^k}\right)\right)\sec(\hat{\phi})\right), & \text{if } \theta \in \left[\frac{\pi}{2}, \pi\right]; \end{cases}$	$\phi \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right)$ or $\left[\pi, \frac{3\pi}{2}\right)$ or $\left[\frac{3\pi}{2}, 2\pi\right)$ $\theta \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right]$
(b)	$\hat{\phi} = \begin{cases} \frac{\pi}{2}[1-\mathrm{sgn}(u)] + \tan^{-1}\left(\mathrm{sgn}(u)\left\lvert\alpha\left(\mathrm{sgn}\left(\frac{u^k}{v^k}\right)\right)^{-1/k}\right\rvert\right), & \text{if } \mathrm{sgn}(v) > 0; \\ \frac{\pi}{2}[3+\mathrm{sgn}(u)] - \tan^{-1}\left(\mathrm{sgn}(u)\left\lvert\alpha\left(\mathrm{sgn}\left(\frac{u^k}{v^k}\right)\right)^{-1/k}\right\rvert\right), & \text{if } \mathrm{sgn}(v) < 0; \end{cases}$ $\hat{\theta} = \begin{cases} \sin^{-1}\left(\beta\left(\mathrm{sgn}\left(\frac{u^k}{v^k}\right)\right)\csc(\hat{\phi})\right), & \text{if } \theta \in \left[0, \frac{\pi}{2}\right); \\ \pi - \sin^{-1}\left(\beta\left(\mathrm{sgn}\left(\frac{u^k}{v^k}\right)\right)\csc(\hat{\phi})\right), & \text{if } \theta \in \left[\frac{\pi}{2}, \pi\right]; \end{cases}$	$\phi \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right)$ or $\left[\pi, \frac{3\pi}{2}\right)$ or $\left[\frac{3\pi}{2}, 2\pi\right)$ $\theta \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right]$
(c)	$\hat{\phi} = \begin{cases} \frac{\pi}{2}[1-\mathrm{sgn}(u)] + \tan^{-1}\left(\mathrm{sgn}(u)\left\lvert\alpha\left(\mathrm{sgn}\left(\frac{u^k}{v^k}\right)\right)^{-1/k}\right\rvert\right), & \text{if } \mathrm{sgn}(v) > 0; \\ \frac{\pi}{2}[3+\mathrm{sgn}(u)] - \tan^{-1}\left(\mathrm{sgn}(u)\left\lvert\alpha\left(\mathrm{sgn}\left(\frac{u^k}{v^k}\right)\right)^{-1/k}\right\rvert\right), & \text{if } \mathrm{sgn}(v) < 0; \end{cases}$ $\hat{\theta} = \cos^{-1}\left(\beta\left(\mathrm{sgn}\left(\tfrac{u^k}{v^k}\right)\right)\right)$	$\phi \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right)$ or $\left[\pi, \frac{3\pi}{2}\right)$ or $\left[\frac{3\pi}{2}, 2\pi\right)$

Configuration	Formulas	Prior Information Required
(d)	$\hat{\phi} = \begin{cases} \cos^{-1}\left(\beta\left(\operatorname{sgn}\left(\frac{u^k}{w^k}\right)\right)\csc(\hat{\theta})\right), & \text{if } \phi \in [0,\pi)\,; \\ 2\pi - \cos^{-1}\left(\beta\left(\operatorname{sgn}\left(\frac{u^k}{w^k}\right)\right)\csc(\hat{\theta})\right), & \text{if } \phi \in [\pi, 2\pi)\,; \end{cases}$ $\hat{\theta} = \cos^{-1}\left(\operatorname{sgn}(w)\left\lvert\frac{\beta\left(\operatorname{sgn}\left(\frac{u^k}{w^k}\right)\right)}{\alpha\left(\operatorname{sgn}\left(\frac{u^k}{w^k}\right)\right)^{1/k}}\right\rvert\right)$	If k is odd: $\phi \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right)$ or $\left[\pi, \frac{3\pi}{2}\right)$ or $\left[\frac{3\pi}{2}, 2\pi\right)$ $\theta \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right]$ If k is even: $\phi \in [0,\pi)$ or $[\pi, 2\pi)$ $\theta \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right]$
(e)	$\hat{\phi} = \begin{cases} \pi[1 - \operatorname{sgn}(\chi) - \delta(\chi)] + \sin^{-1}\left(\csc(\hat{\theta})\beta\left(\operatorname{sgn}\left(\frac{u^k}{w^k}\right)\right)\right), & \text{if } \operatorname{sgn}(u) > 0; \\ \pi - \sin^{-1}\left(\csc(\hat{\theta})\beta\left(\operatorname{sgn}\left(\frac{u^k}{w^k}\right)\right)\right), & \text{if } \operatorname{sgn}(u) < 0; \end{cases}$ $\hat{\theta} = \begin{cases} \sin^{-1}\left(\sqrt{\frac{\alpha\left(\operatorname{sgn}\left(\frac{u^k}{w^k}\right)\right)^{2/k} + \beta\left(\operatorname{sgn}\left(\frac{u^k}{w^k}\right)\right)^2}{1 + \alpha\left(\operatorname{sgn}\left(\frac{u^k}{w^k}\right)\right)^{2/k}}}\right), & \text{if } \operatorname{sgn}(w) > 0; \\ \pi - \sin^{-1}\left(\sqrt{\frac{\alpha\left(\operatorname{sgn}\left(\frac{u^k}{w^k}\right)\right)^{2/k} + \beta\left(\operatorname{sgn}\left(\frac{u^k}{w^k}\right)\right)^2}{1 + \alpha\left(\operatorname{sgn}\left(\frac{u^k}{w^k}\right)\right)^{2/k}}}\right), & \text{if } \operatorname{sgn}(w) < 0; \end{cases}$	$\phi \in \left[0, \frac{\pi}{2}\right) \cup \left[\frac{3\pi}{2}, 2\pi\right)$ or $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right)$ $\theta \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right]$.
(f)	$\hat{\phi} = \begin{cases} \cos^{-1}\left(\operatorname{sgn}(u)\csc(\hat{\theta})\left\lvert\alpha\left(\operatorname{sgn}\left(\frac{u^k}{w^k}\right)\right)^{1/k}\beta\left(\operatorname{sgn}\left(\frac{u^k}{w^k}\right)\right)\right\rvert\right), & \text{if } \phi \in [0,\pi)\,; \\ 2\pi - \cos^{-1}\left(\operatorname{sgn}(u)\csc(\hat{\theta})\left\lvert\alpha\left(\operatorname{sgn}\left(\frac{u^k}{w^k}\right)\right)^{1/k}\beta\left(\operatorname{sgn}\left(\frac{u^k}{w^k}\right)\right)\right\rvert\right), & \text{if } \phi \in [\pi, 2\pi)\,; \end{cases}$ $\hat{\theta} = \cos^{-1}\left(\beta\left(\operatorname{sgn}\left(\frac{u^k}{w^k}\right)\right)\right)$	If k is odd: $\phi \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right)$ or $\left[\pi, \frac{3\pi}{2}\right)$ or $\left[\frac{3\pi}{2}, 2\pi\right)$ $\theta \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right]$ If k is even: $\phi \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right)$ or $\left[\pi, \frac{3\pi}{2}\right)$ or $\left[\frac{3\pi}{2}, 2\pi\right)$

where $\chi = \sin^{-1}\left(\frac{\lambda}{2\pi\Delta_y}\csc(\hat{\theta})\angle\left(\operatorname{sgn}\left(\frac{u^k}{w^k}\right)\frac{\left[\hat{\mathbf{a}}_{x,z}^{(y-\text{axis})}\right]_1}{\left[\hat{\mathbf{a}}_{x,z}^{(y-\text{axis})}\right]_2}\right)\right)$, and $\delta(\chi) = \begin{cases} 1, & \text{if } \chi = 0 \\ 0, & \text{if } \chi \neq 0 \end{cases}$.

Configuration	Formulas	Prior Information Required
(g)	$\hat{\phi} = \begin{cases} \cos^{-1}\left(\csc(\hat{\theta})\beta\left(\mathrm{sgn}\left(\frac{v^k}{w^k}\right)\right)\right), & \text{if } \mathrm{sgn}(v) > 0; \\ 2\pi - \cos^{-1}\left(\csc(\hat{\theta})\beta\left(\mathrm{sgn}\left(\frac{v^k}{w^k}\right)\right)\right), & \text{if } \mathrm{sgn}(v) < 0; \end{cases}$ $\hat{\theta} = \begin{cases} \sin^{-1}\left(\sqrt{\frac{\alpha\left(\mathrm{sgn}\left(\frac{v^k}{w^k}\right)\right)^{2/k} + \beta\left(\mathrm{sgn}\left(\frac{v^k}{w^k}\right)\right)^2}{1+\alpha\left(\mathrm{sgn}\left(\frac{v^k}{w^k}\right)\right)^{2/k}}}\right), & \text{if } \mathrm{sgn}(w) > 0; \\ \pi - \sin^{-1}\left(\sqrt{\frac{\alpha\left(\mathrm{sgn}\left(\frac{v^k}{w^k}\right)\right)^{2/k} + \beta\left(\mathrm{sgn}\left(\frac{v^k}{w^k}\right)\right)^2}{1+\alpha\left(\mathrm{sgn}\left(\frac{v^k}{w^k}\right)\right)^{2/k}}}\right), & \text{if } \mathrm{sgn}(w) < 0; \end{cases}$	$\phi \in [0, \pi)$ or $[\pi, 2\pi)$ $\theta \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right]$
(h)	$\hat{\phi} = \begin{cases} \pi[1 - \mathrm{sgn}(\psi) - \delta(\psi)] + \sin^{-1}\left(\beta\left(\mathrm{sgn}\left(\frac{v^k}{w^k}\right)\right)\csc(\hat{\theta})\right), & \text{if } \phi \in \left[0, \frac{\pi}{2}\right) \cup \left[\frac{3\pi}{2}, 2\pi\right); \\ \pi - \sin^{-1}\left(\beta\left(\mathrm{sgn}\left(\frac{v^k}{w^k}\right)\right)\csc(\hat{\theta})\right), & \text{if } \phi \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right); \end{cases}$ $\hat{\theta} = \cos^{-1}\left(\mathrm{sgn}(w)\left\lvert\frac{\beta\left(\mathrm{sgn}\left(\frac{v^k}{w^k}\right)\right)}{\alpha\left(\mathrm{sgn}\left(\frac{v^k}{w^k}\right)\right)^{1/k}}\right\rvert\right)$	If k is odd: $\phi \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right)$ or $\left[\pi, \frac{3\pi}{2}\right)$ or $\left[\frac{3\pi}{2}, 2\pi\right)$ $\theta \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right]$ If k is even: $\phi \in \left[0, \frac{\pi}{2}\right) \cup \left[\frac{3\pi}{2}, 2\pi\right)$ or $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right)$, $\theta \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right]$
(i)	$\hat{\phi} = \begin{cases} \pi[1 - \mathrm{sgn}(v) - \delta(v)] + \sin^{-1}\left(\mathrm{sgn}(v)\csc(\hat{\theta})\left\lvert\alpha\left(\mathrm{sgn}\left(\frac{v^k}{w^k}\right)\right)^{1/k}\beta\left(\mathrm{sgn}\left(\frac{v^k}{w^k}\right)\right)\right\rvert\right), & \\ & \text{if } \phi \in \left[0, \frac{\pi}{2}\right) \cup \left[\frac{3\pi}{2}, 2\pi\right); \\ \pi - \sin^{-1}\left(\mathrm{sgn}(v)\csc(\hat{\theta})\left\lvert\alpha\left(\mathrm{sgn}\left(\frac{v^k}{w^k}\right)\right)^{1/k}\beta\left(\mathrm{sgn}\left(\frac{v^k}{w^k}\right)\right)\right\rvert\right), & \text{if } \phi \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right); \end{cases}$ $\hat{\theta} = \cos^{-1}\left(\beta\left(\mathrm{sgn}\left(\frac{v^k}{w^k}\right)\right)\right)$	If k is odd: $\phi \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right)$ or $\left[\pi, \frac{3\pi}{2}\right)$ or $\left[\frac{3\pi}{2}, 2\pi\right)$ $\theta \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right]$ If k is even: $\phi \in \left[0, \frac{\pi}{2}\right)$ or $\left[\frac{\pi}{2}, \pi\right)$ or $\left[\pi, \frac{3\pi}{2}\right)$ or $\left[\frac{3\pi}{2}, 2\pi\right)$

where $\psi = \sin^{-1}\left(\frac{\lambda}{2\pi\Delta_y}\csc(\hat{\theta})\angle\left(\mathrm{sgn}\left(\frac{v^{k_1}}{w^{k_2}}\right)\frac{\left[\hat{\mathbf{a}}_{y,z}^{(y-\text{axis})}\right]_1}{\left[\hat{\mathbf{a}}_{y,z}^{(y-\text{axis})}\right]_2}\right)\right)$, and $\delta(\psi) = \begin{cases} 1, & \text{if } \psi = 0 \\ 0, & \text{if } \psi \neq 0 \end{cases}$.

Chapter 5

Monte Carlo Simulation

For the simulation of afore-mentioned direction finding method, we consider a pair of spatially-*spread* higher-order differential sensors with perpendicular orientation. The coordinate system is defined in Figure 2.1 corresponding to each possible configuration. This approach requires the eigen-based DOA estimation algorithm as a preceding step to derive an estimate of steering vector $\hat{\mathbf{a}}$. Suppose a far-field narrow-band plane wave signal impinging on the sensors as $s(t_n)$, the following data model is adopted in the simulation:

$$\mathbf{z}(t_n) = \mathbf{a}s(t_n) + \mathbf{n}(t_n)$$

where $\mathbf{n}(t_n)$ is zero-mean white noise vector, $\mathbf{z}(t_n)$ is the received array response.

By eigen-decomposing the covariance matrix

$$\mathbf{R} = \frac{1}{N}\mathbf{Z}\mathbf{Z}^H$$

where $\mathbf{Z} = [\mathbf{z}(t_0), \mathbf{z}(t_1), ..., \mathbf{z}(t_N)]^T$, the estimate of $\hat{\mathbf{a}}$ for each incident signal can be then obtained within a unknown complex scalar.

The Monte Carlo simulations will verify the efficiency and accuracy of the proposed scheme. The following settings are applied: $\theta = 45°$, $\phi = 45°$, $\Delta/\lambda = 1/2$. Δ/λ is known as the inter-antenna spacing parameter, with Δ referring to the displacement of two sensors and λ referring to the wavelength. Figure 2 plots the RMSE of source's elevation and azimuth angle estimates versus SNR, ranging from 0dB to 50dB. The RMSE is defined as

$$\text{RMSE} = \sqrt{\frac{1}{100}\sum_{i=1}^{100}(\hat{\gamma}_i - \gamma)^2}$$

where $\gamma = \theta$ or ϕ, $\hat{\gamma}_i$ symbolizes the estimation result of ith Monte Carlo experiment among total 100.

Figure 5.1-5.2 and Figure 5.3 clearly shows that the proposed algorithm successfully obtained the estimates of elevation and azimuth angle for the incident signal. The RMSE is directly proportional to SNR which accords with our expectation. However, when SNR is lower than a certain level, about 10dB, this algorithm may fail. This is due to the inverse trigonometric functions in closed-form formulas which may return a complex number if input exceeds the general domain of trigonometric functions.

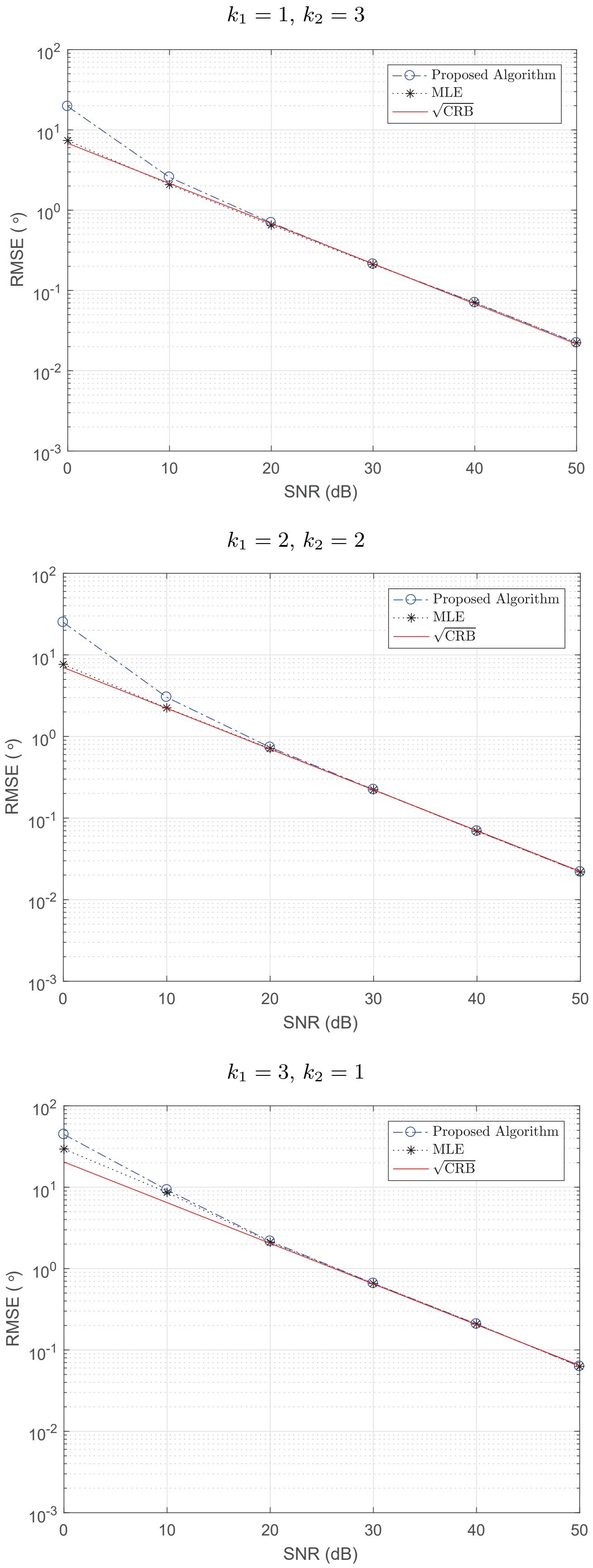

Figure 5.1: Monte Carlo Simulations of Proposed Algorithm vs. Maximum Likelihood Estimation vs. Cramér-Rao Bound for the estimator of elevation angle $\hat{\theta}$ in configuration (a) with following settings: $M = 100$, $\theta = 45°$, $\phi = 45°$, $\Delta_x/\lambda = 1/2$

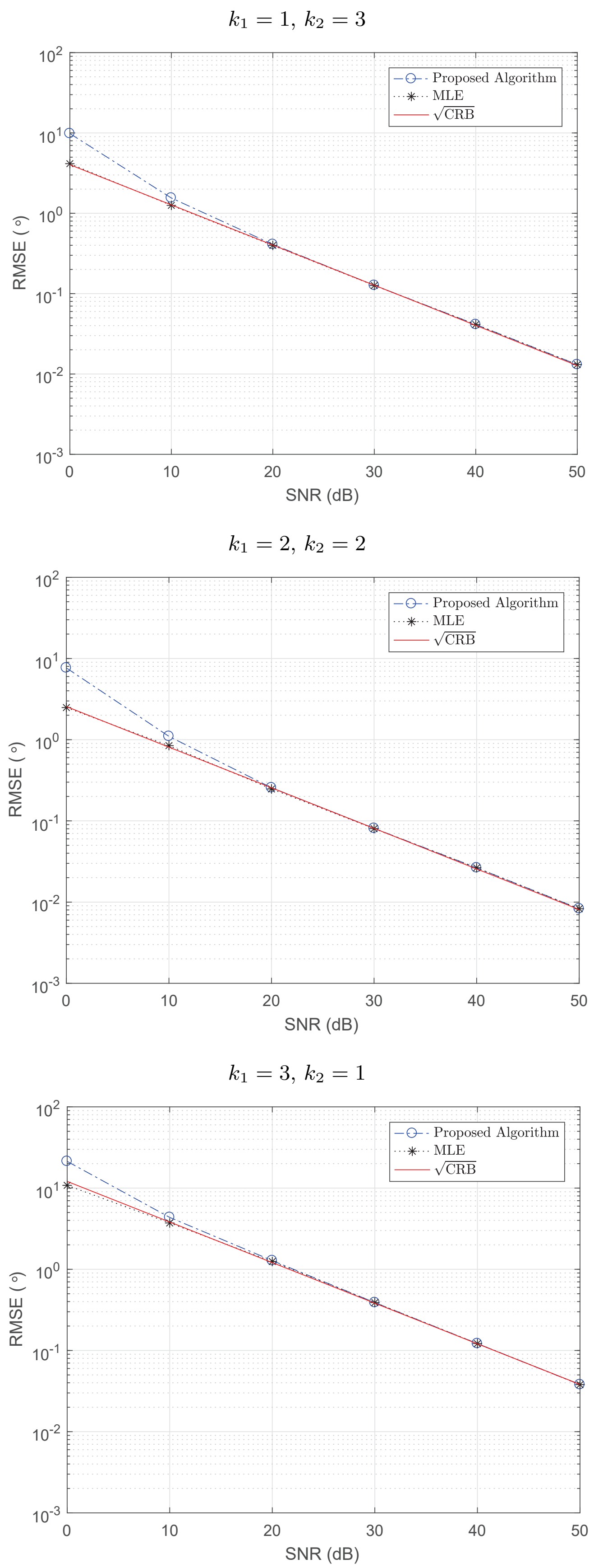

Figure 5.2: Monte Carlo Simulations of Proposed Algorithm vs. Maximum Likelihood Estimation vs. Cramér-Rao Bound for the estimator of azimuth angle $\hat{\phi}$ in configuration (a) with following settings: $M = 100$, $\theta = 45°$, $\phi = 45°$, $\Delta_x/\lambda = 1/2$

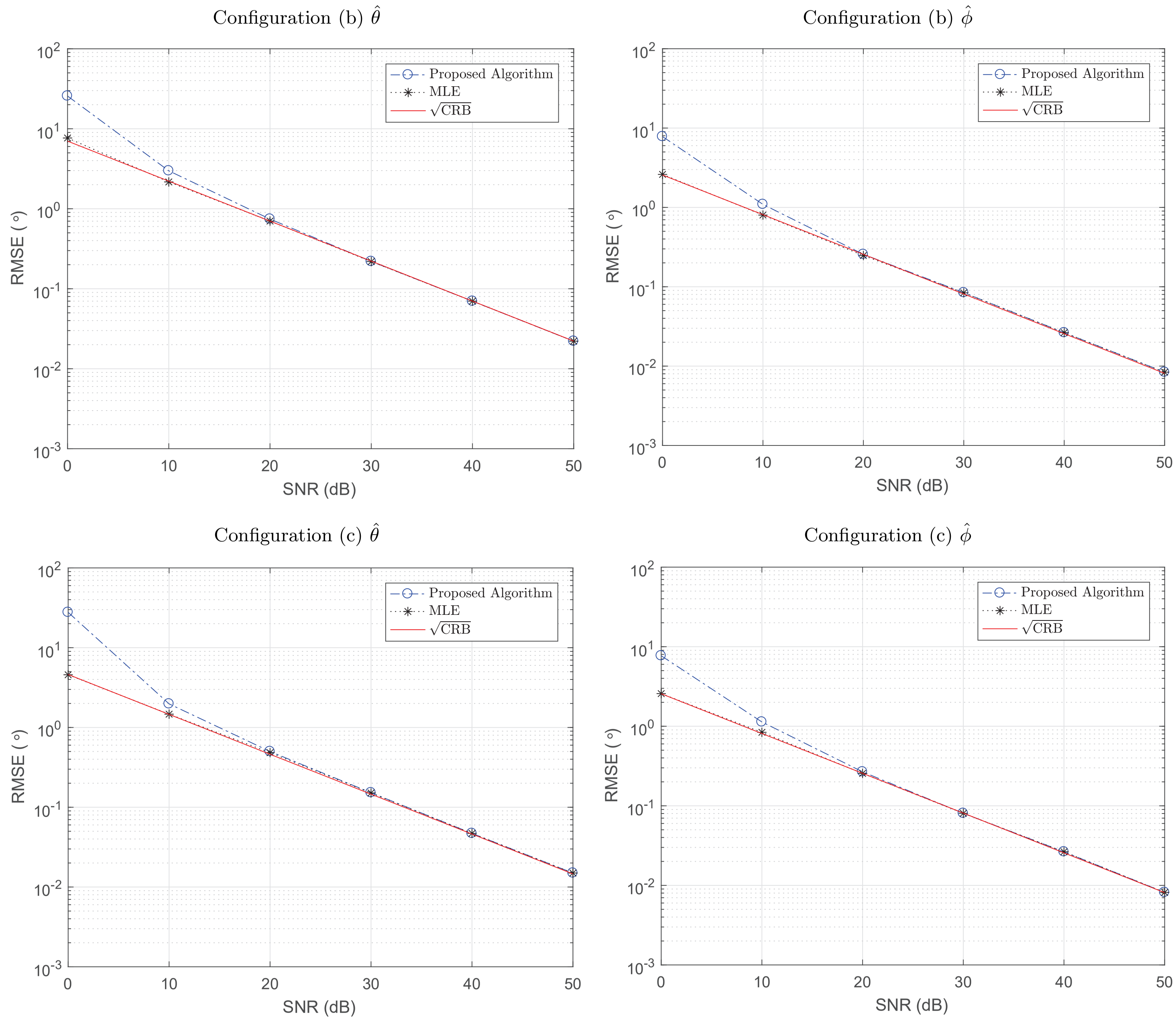

Figure 5.3: Monte Carlo Simulations of Proposed Algorithm vs. Maximum Likelihood Estimation vs. Cramér-Rao Bound with Following Settings: $k_1 = k_2 = 2$,$M = 100$, $\theta = 45°$, $\phi = 45°$, $\Delta_x/\lambda = 1/2$

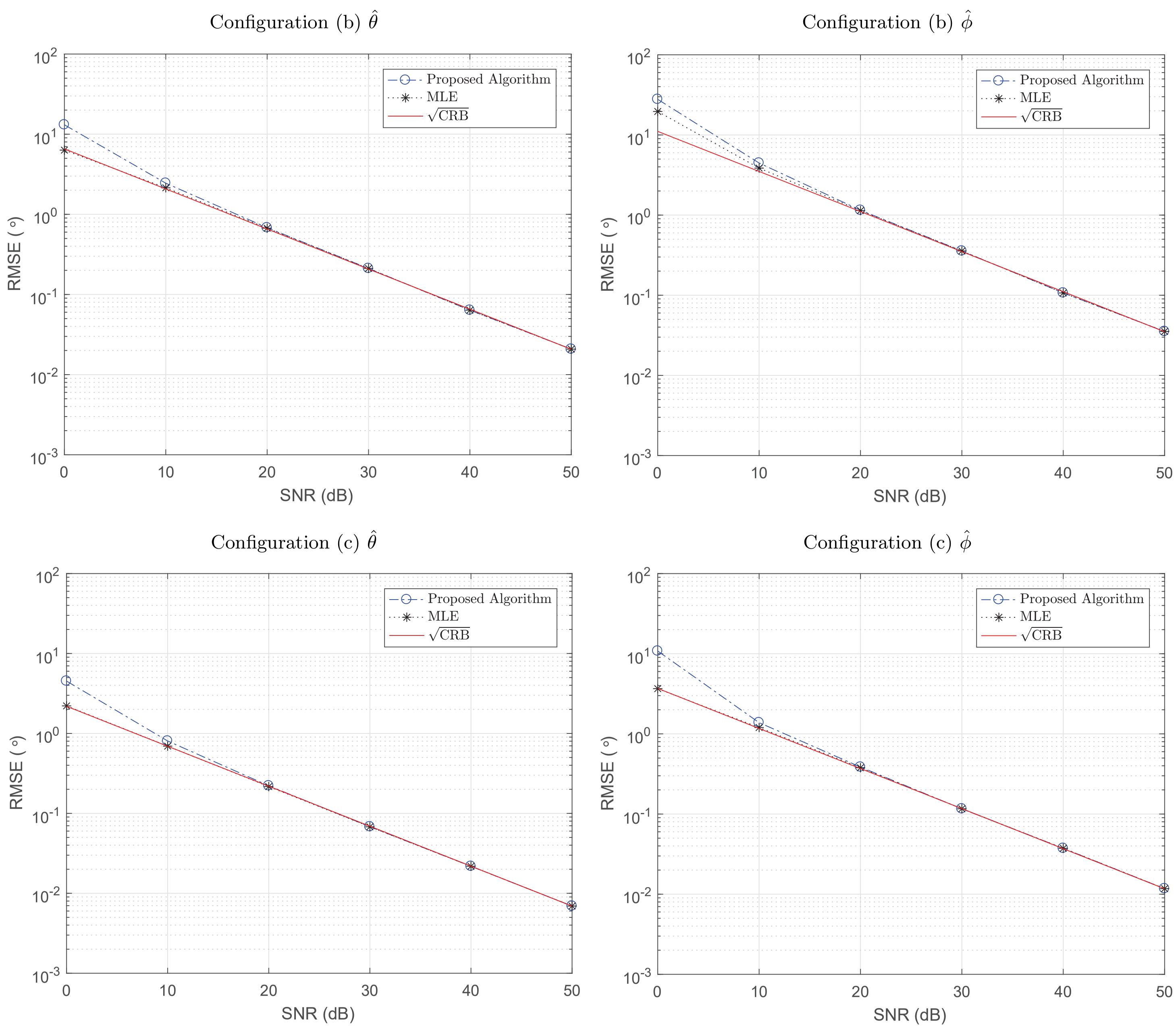

Figure 5.4: Monte Carlo Simulations of Proposed Algorithm vs. Maximum Likelihood Estimation vs. Cramér-Rao Bound with Following Settings: $k_1 = k_2 = 2$,$M = 100$, $\theta = 45^\circ$, $\phi = 45^\circ$, $\Delta_x/\lambda = 1/2$

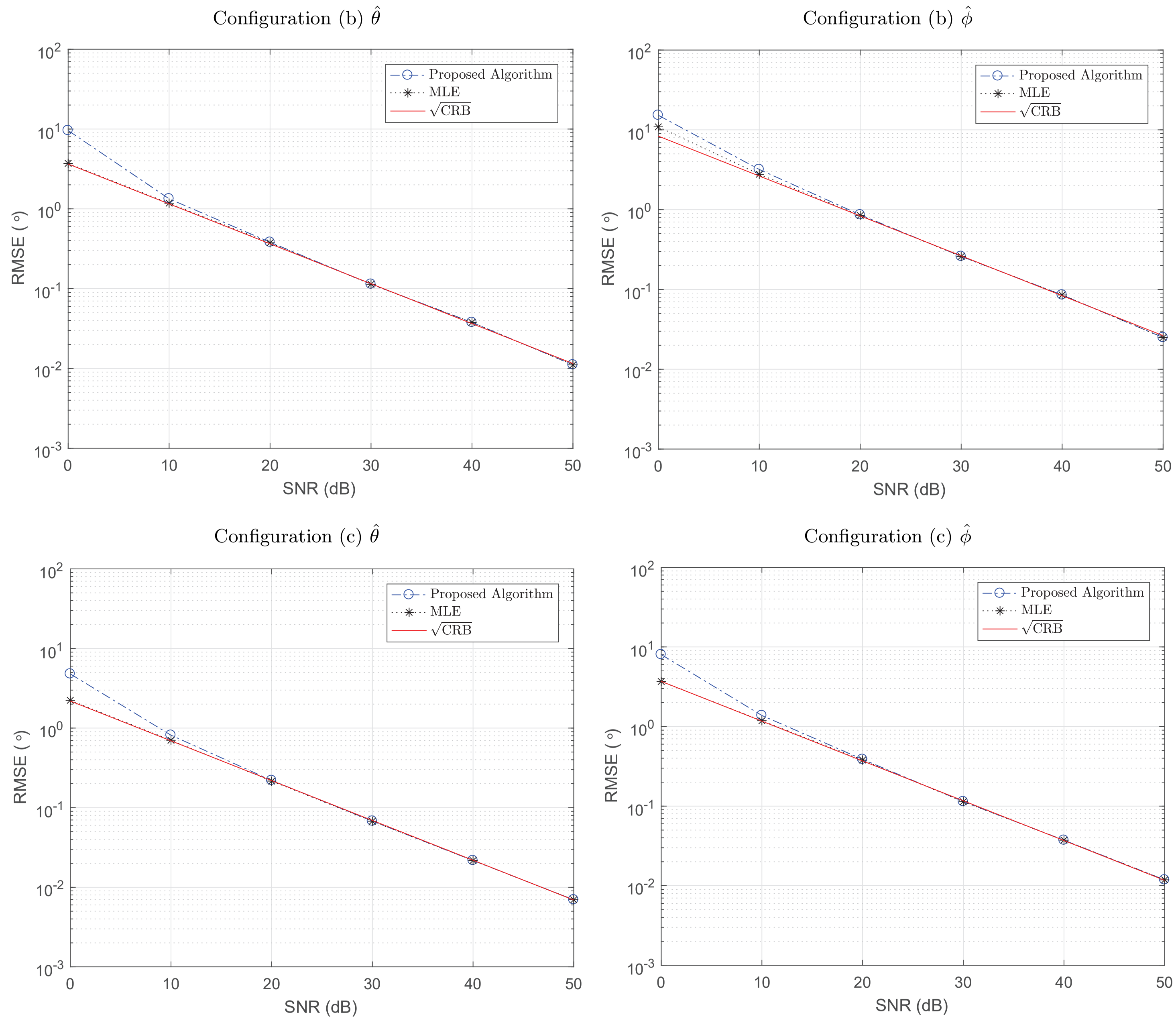

Figure 5.5: Monte Carlo Simulations of Proposed Algorithm vs. Maximum Likelihood Estimation vs. Cramér-Rao Bound with Following Settings: $k_1 = k_2 = 2$,$M = 100$, $\theta = 45°$, $\phi = 45°$, $\Delta_x/\lambda = 1/2$

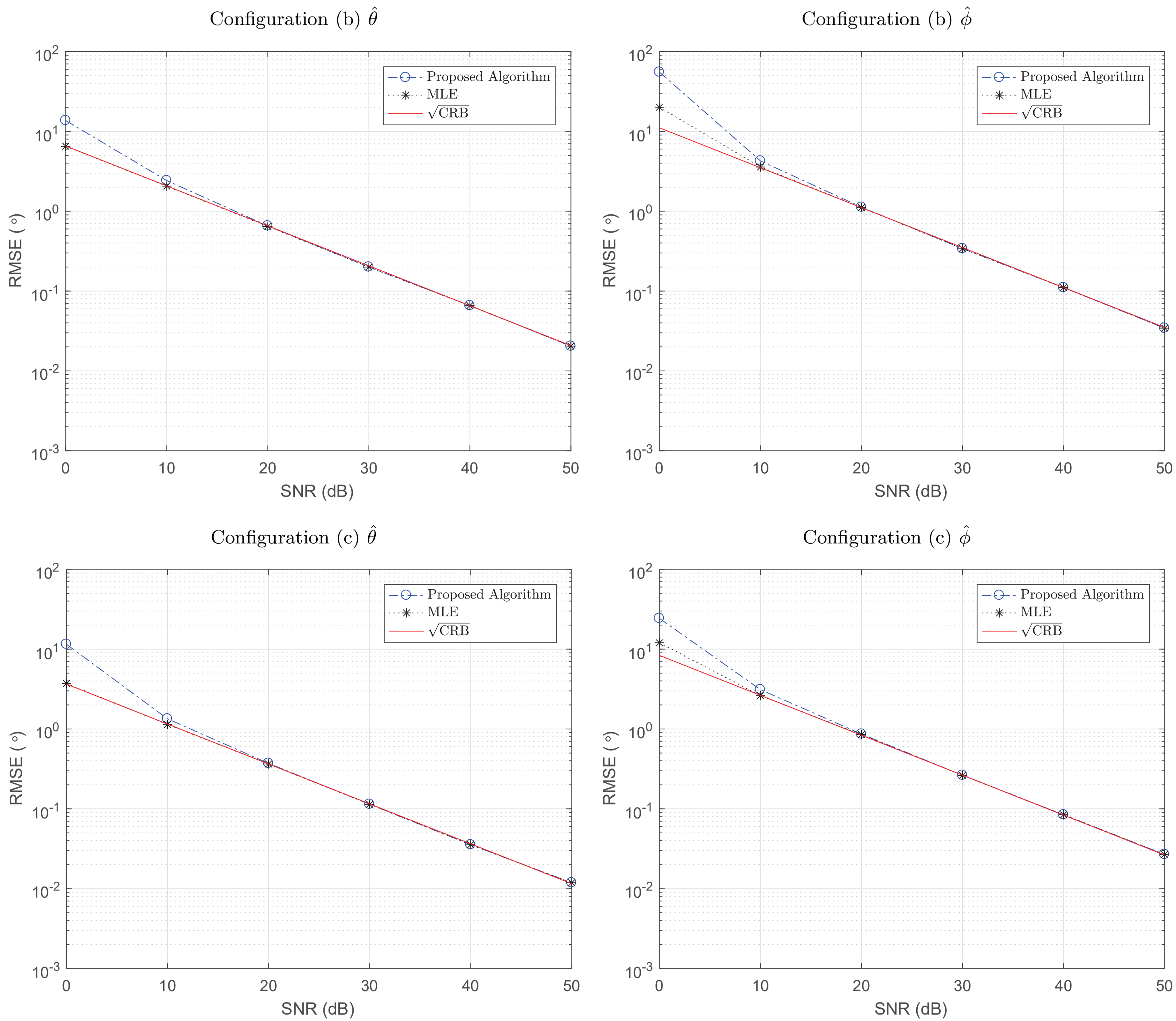

Figure 5.6: Monte Carlo Simulations of Proposed Algorithm vs. Maximum Likelihood Estimation vs. Cramér-Rao Bound with Following Settings: $k_1 = k_2 = 2$,$M = 100$, $\theta = 45°$, $\phi = 45°$, $\Delta_x/\lambda = 1/2$

Chapter 6

Cramér-Rao Bound (CRB) Derivation

In parameter estimation theory, the Cramér-Rao Bound gives the floor of the error variance for any unbiased estimator. Given a statistical data model, the Cramér-Rao Bound lower bounds show the best possible performance any unbiased estimator could achieve. An unbiased estimator is said to be (fully) efficient if its mean-square-error is equal to the Cramér-Rao Bound. A comparison with the CRB would indicate how much improvement is possible in terms of estimation error variance by using more complex estimation algorithms. Sometimes the possible improvement is not worth the increased complexity.

A brief summary of the steps for deriving CRB:

(i) The first step is to derive the Fisher information matrix $\mathbf{J}$. This step is important and straightforward, but the process could be tedious and time-consuming.

(ii) In many problems, the unknown parameter vector $\boldsymbol{\theta}$ contains two subvectors: parameter of interest $\boldsymbol{\theta}_{\mathrm{w}}$ and nuisance parameters $\boldsymbol{\theta}_{\mathrm{u}}$. The Fisher information matrix $\mathbf{J}$ is partitioned into blocks corresponding to $\boldsymbol{\theta}_{\mathrm{w}}$, and blocks corresponding to other (unwanted) parameters $\boldsymbol{\theta}_{\mathrm{u}}$, and blocks corresponding to the cross-terms. Then, partition CRB matrix $\mathbf{C}_{CR}$ similarly and derive the block of $\mathbf{C}_{CR}$ corresponding to $\boldsymbol{\theta}_{\mathrm{w}}$ by the formula for the inverse of block matrices. This step often involves complicated expressions.

For an N-element array, given the probability density for a single snapshot as $p_{\mathbf{x}|\boldsymbol{\theta}}(\mathbf{x})$, where $\mathbf{x}$ is an $N \times 1$ complex Gaussian random variable and $\boldsymbol{\theta}$ is a $D \times 1$ nonrandom unknown vector that we want to estimate, the log-likelihood function for a single snapshot is

$$
\begin{aligned}
L_{\mathbf{x}}(\boldsymbol{\theta}) &\triangleq \ln p_{\mathbf{x}|\boldsymbol{\theta}}(\mathbf{x}) \\
&= -\ln \det \left[\pi \mathbf{K}_{\mathbf{x}}(\boldsymbol{\theta})\right] - \left\{ \left(\mathbf{x}^H - \mathbf{m}^H(\boldsymbol{\theta})\right) \mathbf{K}_{\mathbf{x}}{}^{-1}(\boldsymbol{\theta}) \left(\mathbf{x} - \mathbf{m}(\boldsymbol{\theta})\right) \right\}. \quad (6.1)
\end{aligned}
$$

where $\mathbf{m}^H(\boldsymbol{\theta})$ is the mean and ${\mathbf{K_x}}^{-1}(\boldsymbol{\theta})$ is the covariance matrix.

The Cramér-Rao Bound provides a lower bound $\mathbf{C}_{CR}(\boldsymbol{\theta})$ on the covariance matrix of any unbiased estimate of $\boldsymbol{\theta}$. The covariance matrix of the estimation error is given as

$$\mathbf{C}(\boldsymbol{\theta}) \triangleq E\left[\left(\hat{\boldsymbol{\theta}}-\boldsymbol{\theta}\right)\left(\hat{\boldsymbol{\theta}}-\boldsymbol{\theta}\right)^T\right]. \tag{6.2}$$

The classic Cramér-Rao Bounds states that

$$\mathbf{C}(\boldsymbol{\theta}) \geq \mathbf{C}_{CR}(\boldsymbol{\theta}) \triangleq \mathbf{J}^{-1}, \tag{6.3}$$

Matrix $\mathbf{J}$ is commonly called the Fisher Information Matrix (FIM), whose (i,j)th entry is

$$\begin{aligned} J_{ij} &\triangleq E\left[\frac{\partial L_{\mathbf{x}}(\boldsymbol{\theta})}{\partial \theta_i}\cdot\frac{\partial L_{\mathbf{x}}(\boldsymbol{\theta})}{\partial \theta_j}\right] \\ &= -E\left[\frac{\partial^2 L_{\mathbf{x}}(\boldsymbol{\theta})}{\partial \theta_i \partial \theta_j}\right]. \end{aligned} \tag{6.4}$$

Using the same data model as in the aforementioned Monte Carlo simulations, with M number of time samples, then the $2M \times 1$ data set equals

$$\begin{aligned} \mathbf{z} &= \left[\tilde{\mathbf{z}}(T_s)^T, \cdots, \tilde{\mathbf{z}}(MT_s)^T\right]^T \\ &= \underbrace{\mathbf{s}\otimes\mathbf{a}}_{\stackrel{\text{def}}{=}\boldsymbol{\mu}} + \underbrace{\left[\tilde{\mathbf{n}}(T_s)^T, \cdots, \tilde{\mathbf{n}}(MT_s)^T\right]}_{\stackrel{\text{def}}{=}\mathbf{n}}, \end{aligned} \tag{6.5}$$

where $\mathbf{s} = e^{j\varphi}\left[e^{jT_s\omega}, \cdots, e^{jMT_s\omega}\right]^T$ represents a pure tone signal, $\otimes$ is the Kronecker product, $\mathbf{n}$ is a $2M \times 1$ noise vector with a covariance matrix $\boldsymbol{\Gamma} = \mathbf{I}_M \otimes \boldsymbol{\Gamma}_0$.

In this case we have a 2×2 Fisher information matrix containing a (i,j)th entry of

$$[\mathbf{J}]_{ij} = 2\mathrm{Re}\left[\left(\frac{\partial\boldsymbol{\mu}}{\partial[\psi]_i}\right)^H \boldsymbol{\Gamma}^{-1}\left(\frac{\partial\boldsymbol{\mu}}{\partial[\psi]_j}\right)\right] + \mathrm{Tr}\left[\boldsymbol{\Gamma}^{-\mathbf{1}}\frac{\partial\boldsymbol{\Gamma}}{\partial[\psi]_i}\boldsymbol{\Gamma}^{-1}\frac{\partial\boldsymbol{\Gamma}}{\partial[\psi]_j}\right], \tag{6.6}$$

where $\psi = [\theta, \phi]^T$, $\mathrm{Tr}[\cdot]$ denotes the trace operator.
Because only θ and ϕ are estimated, the CRBs for these estimators are the diagonal elements of $\mathbf{J}^{-1}$. For each configuration among the nine configurations, I need to derive the CRBs for elevation and azimuth angle estimates.

Figure 6.2 shows an example of Cramér-Rao bounds for configuration (a), i.e. a pair of higher-order differential sensors with orientation along x-axis (at ($\lambda/2$, 0, 0)) and y-axis (at (0, 0, 0)). The coefficients used for this configuration are: $\sigma = 0.1$, $M = 100$, $k_1 = 1, 2, 3$ and $k_2 = 1, 2, 3$.

The following relations among CRBs were observed:

- If k_1 and k_2 are fixed, the CRBs for the configurations with x,z-oriented differential sensors and y,z-oriented differential sensors can be obtained from each other by shifting along a certain direction. Specifically, configuration (d) and (h) are identical except a shift in ϕ; configuration (e) and (g) are identical except a shift in ϕ; configuration (f) and (i) are identical except a shift in ϕ.

- All Cramér-Rao bound figures are symmetric, either with respect to $\theta = \pi/2$ or $\phi = 0$.

- With the order of the sensors increasing, i.e. k_1 and k_2 increase, the Cramér-Rao bounds is higher at the edge but lower at other region (refer to the contour map in Figure 6.3), which reveals that a higher order sensor will result in a better performance for elevation-azimuth angle direction-of-arrival estimation but poorer effective range.

The third point mentioned above can be explained by the gain pattern of the differential sensors as shown in Figure 6.1. With the order of the sensor increasing, the slope of sensor's response is sharper in the middle region (near to 45°) which results in a better sensitivity, while flatter in the edge region (near to 0° and 90°) which results in a poorer sensitivity.

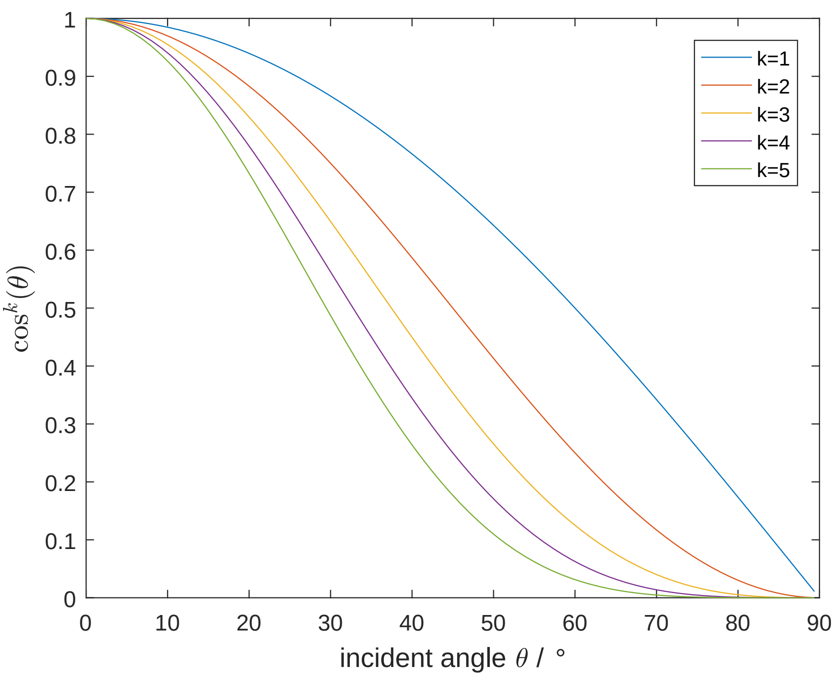

Figure 6.1: An illustration of the differential sensors response

The orders of the sensors will also affect the proposed algorithm in following aspects:

- The prior information required of the source's incident region may be reduced (from one octant to one quadrant of the sphere) if an even order is used.

- For three special configurations (orientation of both sensors are different from the orientation of the displacement), the two sensors's order must be equal.
- Generally the lower the order, the lower will be for the computational complexity as well as the hardware complexity.

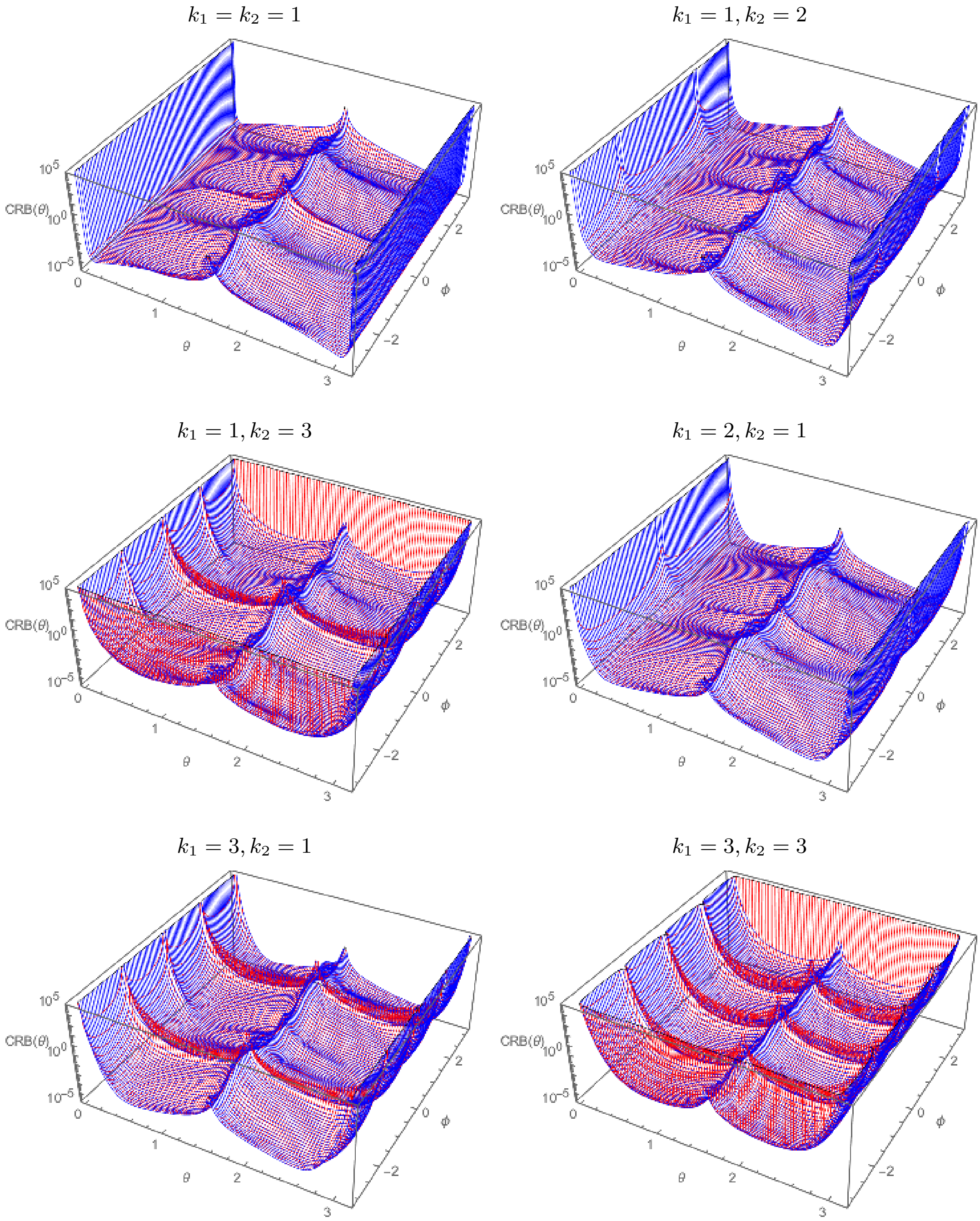

Figure 6.2: The Cramér-Rao bound for elevation-azimuth direction-of-arrival estimation using a pair of higher-order differential sensors: x-axis-oriented sensor at $(\lambda/2,\ 0,\ 0)$ and y-axis-oriented sensor at $(0,\ 0,\ 0)$.

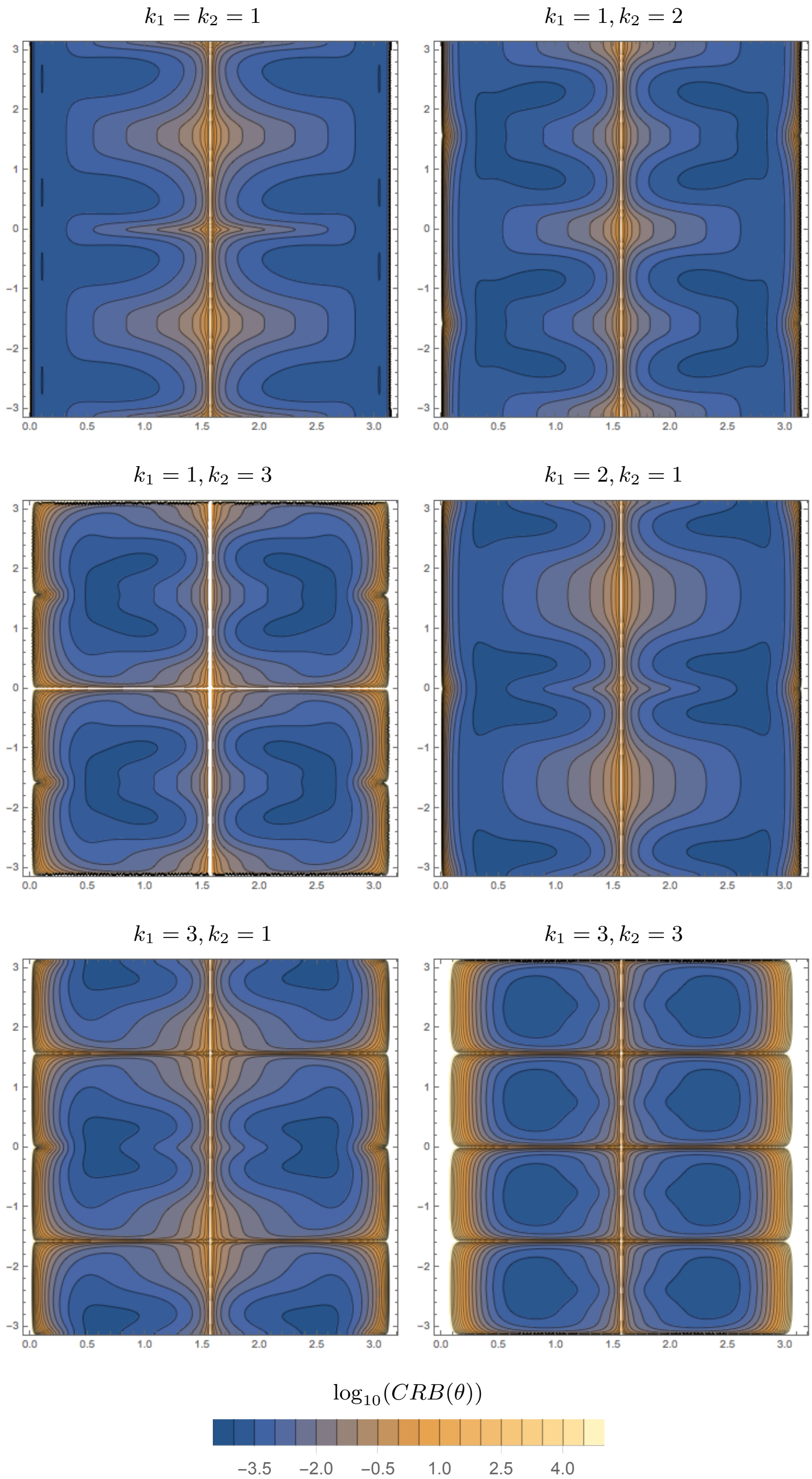

Figure 6.3: Contour plots of the Cramér-Rao bound for elevation-azimuth direction-of-arrival estimation using a pair of higher-order differential sensors: x-axis-oriented sensor at ($\lambda/2$, 0, 0) and y-axis-oriented sensor at (0, 0, 0).. The contour is in logarithmic scale.

Chapter 7

Conclusion

Differential sensor array is advantageous in acoustic signal direction finding because of its small size and high directivity. With certain prior information known, a pair of higher-order differential sensors is sufficient for the estimation of both elevation angle and azimuth angle. This dissertation devises a DOA estimation algorithm for a pair of orthogonal differential sensors. The proposed method can be summarized as follow:

1. Compute the data covariance matrix from the received snapshots,
2. Apply eigen-decomposition on the data covariance matrix to obtain an estimate of the steering vector $\hat{\mathbf{a}}$,
3. Use the closed-form formulas in Table 3.1 and Table 4.1 to calculate the estimation of elevation angle and azimuth angle.

Chapter 3 and Chapter 4 elaborates the derivation of closed-form formulas of DOA estimation for high order biaxial differential sensors. The problem is divided into two parts: i) sensors with equal order, i.e. $k_1 = k_2 = k$ and ii) sensors with unequal order, i.e. $k_1 \neq k_2$. For case i), the closed-form solutions for all nine configurations could be derived under proper assumptions. Prior knowledge about DOA (which octant the signal source resides in) and the ratio of the inter-sensor spacing to the incident signals wavelength are required. For similar situation, Song [16] has derived the formulas for first-order cases, i.e. $k_1 = k_2 = 1$. If $k_1 = k_2 = k$, the prior information required for higher order case is the same as first order case. For case ii), the closed-form formulas for configuration (c), (e) and (g) are not derivable. The common point of these three configurations is that the orientation of displacement is not the same as either of the two sensors' orientation. The equations derived in Chapter 3 (for case ii) are also valid for case i). Table 3.1 and Table 4.1 provide clear summary of all derived formulas.

Monte Carlo simulation examined the validity of proposed direction finding scheme. Figure 5.1, 5.2 and Figure 5.3 shows that the root mean square error curve approaches the Cramér-Rao bound as signal-to-noise ratio increasing.

Cramér-Rao bound 3-D graphs reveal the performance of the differential sensor pair for different incident angle. The sensor's order affects the performance in a way that it shifts the most sensitive DOA region of the senor pair. The estimation accuracy of the elevation and azimuth angle differs from one another, depending on which configuration is applied.

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