The Hong Kong Polytechnic University

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Coherent Feedback Control for Linear Quantum Systems and Two-strategy Evolutionary Game Theory

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Certificate of Originality

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__________________________(Signed)

CUI Lei (Name of student)

Dedicate to my parents.

Abstract

The thesis is mainly concerned with two fields: coherent feedback control for linear quantum systems and two-strategy evolutionary game theory. Three topics are considered:

1. Design mixed linear quadratic Gaussian (LQG) and H_∞ coherent feedback controllers for linear quantum systems.

2. Develop a new classical strategy model to solve the problem that defectors always dominate cooperators in a static two-strategy game.

3. Generalize the classical game theory into the quantum domain and state the advantages of quantum strategies over classical strategies.

For topic 1, a class of closed-loop linear quantum systems is formulated in terms of quantum stochastic differential equations (QSDEs) in quadrature form, where both the plant and the controller are quantum systems. Under this framework, the mixed feedback control problem is synthesized. After proving a general result for the lower bound of LQG index, two methods, rank constrained LMI method and genetic-algorithm-based method, are proposed for controller design. A passive system (cavity) and a non-passive one (degenerate parametric amplifier, DPA), used as numerical examples, demonstrate the effectiveness of these two proposed algorithms. Furthermore, the superiority of genetic algorithm (GA) is verified by the comparison between the numerical results of two proposed methods.

For topic 2, by adding tags to the game players, a class of classical two-strategy evolutionary model with finite population is proposed in Chapter 4. Tags represent different characteristics of individuals in the game, and each player can choose how many tags she/he expresses in the game. When an individual expresses more tags, she/he will obtain higher probability to have common tags, which will make probabilistically higher payoff. Nevertheless, the players should also pay for the expressed tags, the cost of expressed tags increases with its numbers arising. Upon this model, the evolutionary dynamics and stationary distributions of infinite time are synthesized. Finally, three kinds of feasible strategies for C-strategy individual invading into D-strategy population are obtained from numerical examples.

Topic 3 is presented in Chapter 5. Upon the standard classical memory-one Iterated Prisoner's Dilemma (IPD) game, the classical zero-determinant (ZD) strategies in Press and Dyson (2012) is introduced. Then a class of two-strategy evolutionary game theory, named quantum zero-determinant (ZD) strategies, is attained by generalizing the classical ZD strategies into the quantum domain. Three kinds of numerical examples are given, which illustrate that the quantum zero-determinant strategies have significant advantage over the classical zero-determinant strategies. Meanwhile, when both players choose quantum zero-determinant strategies, it is the same as the classical case, which are named quasi-classical zero-determinant (ZD) strategies.

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Contents

List of Figures

List of Tables

List of Notations

$\mathbb{R}$	set of real numbers
$\mathbb{C}$	set of complex numbers
$\mathbb{R}^{m\times n}$	set of $m \times n$ real matrices
$\mathbb{C}^{m\times n}$	set of $m \times n$ complex matrices
x^T	transpose of a matrix/vector x
$x^\#$	adjoint of each element of a matrix/vector x
$x^\dagger$	$:= (x^\#)^T$
$\breve{x}$	$:= [x^T \ x^\dagger]^T$
I_N	identity matrix of dimension N
J_N	$:= \begin{bmatrix} I_N & 0 \\ 0 & -I_N \end{bmatrix}$
$X^\flat$	$:= J_M X^\dagger J_N$
F	$:= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
Θ	$:= \mathrm{diag}(F, F, \ldots, F)$
$[A, B]$	$:= AB - BA$
$\Delta(U, V)$	$:= \begin{bmatrix} U & V \\ V^\# & U^\# \end{bmatrix}$

Chapter 1

Introduction

1.1 Background

1.1.1 Quantum optimal control theory

With the rapid development of quantum technology in recent years, more and more researchers are paying attention to quantum control systems, which are an important part in quantum information science. On the other hand, it is found that many methodologies in classical (namely non-quantum) control theory, can be extended into the quantum regime (Bouten et al., 2007; Doherty and Jacobs, 1999; Doherty et al., 2000; Hamerly and Mabuchi, 2013; James et al., 2008; Wang and James, 2015; Zhang et al., 2012). Meanwhile, quantum control has its special features absent in the classical case, see e.g. Wiseman and Milburn (2010), Wang et al. (2013), Zhang and James (2011) and Zhang and James (2012). For example, a controller in a quantum feedback control system may be classical, quantum or even mixed quantum/classical (James et al., 2008). Generally speaking, in the quantum control literature, the feedback control problem in which the designed controller is also a fully quantum system is often named as “coherent feedback control”.

Optimal control, as a vital concept in classical control theory (Zhou et al., 1996), has been widely studied. H_2 and H_∞ control are the two foremost control methods in classical control theory, which aim to minimize cost functions with specific forms from exogenous inputs (disturbances or noises) to pertinent performance outputs. When the disturbances and measurement noises are Gaussian stochastic processes with known power spectral densities, and the objective is a quadratic performance criterion, then the problem of minimizing this quadratic cost function of linear systems is named as LQG control problem, which has been proved to be equivalent to an H_2 optimal control problem (Zhou et al., 1996). On the other hand, H_∞ control problem mainly concerns the robustness of a system to parameter uncertainty or external disturbance signals, and a controller to be designed should make the closed-loop system stable, meanwhile minimizing the influence of disturbances or system uncertainties on the system performance in terms of the H_∞ norm of a certain transfer function. Furthermore, the mixed LQG (or H_2) and H_∞ control problem for classical systems has been studied intensively during the last three decades. When the control system is subject to both white noises and signals with bounded power, not only optimal performance (measured in H_2 norm) but also robustness specifications (in terms of an H_∞ constraint) should be taken into account, which is one of the main motivations for considering the mixed control problem (Zhou et al., 1994); see also Campos-Delgado and Zhou (2003), Doyle et al. (1994), Khargonekar and Rotea (1991), Neumann and Araujo (2004), Qiu et al. (2015), Zhou et al. (1996) and Zhou et al. (1994) and the references therein.

Very recently, researchers have turned to consider the optimal control problem of quantum systems. For instance, H_∞ control of linear quantum stochastic systems is investigated in James et al. (2008), three types of controllers are proposed. Nurdin et al. (2009) proposes a method for quantum LQG control, for which the designed controller is also a fully quantum system. In Zhang and James (2011), direct coupling and indirect coupling for quantum linear systems have been discussed. It is shown in Zhang et al. (2012) that phase shifters and squeezers can be used in feedback loop for better control performance. Nevertheless, all of above papers mainly focus on the vacuum inputs, while the authors in Hamerly and Mabuchi (2013) concern not only the vacuum input, but also the thermal input. They also discussed how to design both classical and non-classical controllers for LQG control problem. Besides, because of non-linear and non-convex constraints in the coherent quantum controller synthesis, Harno and Petersen (2015) uses a differential evolution algorithm to construct an optimal linear coherent quantum controller. Notwithstanding the above research, to our best knowledge, there is little research on the mixed LQG and H_∞ coherent control problem for linear quantum systems, except Bian et al. (2015).

Similar to the classical case, in mixed LQG and H_∞ quantum coherent control, LQG and H_∞ performances are not independent. Moreover, because the controller to be designed is another quantum-mechanical system, it has to satisfy additional constraints, which are called "physical realizability conditions" (James et al., 2008; Zhang and James, 2012). For more details, please refer to Section 3.2.

1.1.2 Classical game theory

Game theory is the study of mathematical models of multi-person decision problems in some conflict situation. It was developed in the 1930's and then firstly formalized in von Neumann and Morgenstern (1944). Now, game theory has been widely used to illuminate economic, logical, political, psychological and biological phenomena.

Games can be divided into two types, cooperative game and non-cooperative game (Myerson, 1997). A game is cooperative if the players are able to form binding commitments externally enforced, while a game is non-cooperative if players cannot form alliances or if all agreements need to be self-enforcing. Cooperative game theory provides a high-level approach, since it focuses on predicting which coalitions will form, the joint actions that groups take and the resulting collective payoffs, whereas the non-cooperative game theory focuses on predicting individual players' actions and payoffs. Thus, the non-cooperative game theory is more general and we mainly focus on non-cooperative games in this thesis.

When considering a non-cooperative game, Nash Equilibrium (NE) is one of the foundational concepts. The Nash Equilibrium is a solution concept of a non-cooperative game involving two or more players in which each player is assumed to know the equilibrium strategies of the other players, and no player has anything to gain by changing only his or her own strategy (Osborne and Rubinstein, 1994).

Prisoner's Dilemma (PD) is a standard example of a game analyzed in game theory that shows why two completely "rational" individuals might not cooperate, even if it appears that it is in their best interests to do so. However, it has been proved that the Nash Equilibrium of a traditional Prisoner's Dilemma game is mutual defection, which is not the globally best strategy. Therefore, more and more researchers turn to focus on repeated/iterated Prisoner's Dilemma (IPD), and further the evolution of cooperation, in order to design some kinds of strategies and succeed to achieve cooperation. Please see Axelrod and Hamilton (1981), Axelrod (1984), Kendall et al. (2007), Nowak and Sigmund (1993), Nowak et al. (2004), Nowak (2006), Press and Dyson (2012) and Rubinstein (1986). Moreover, in the standard evolutionary model of the repeated/iterated Prisoner's Dilemma, two important issues, evolutionary game dynamics and evolutionary stable strategies (ESS), are also been widely studied (Binmore and Samuelson, 1992; Fudenberg and Harris, 1992; Ficici and Pollack, 2000; Hofbauer and Sigmund, 2003; Nowak et al., 2004; Schaffer, 1988; Taylor and Jonker, 1978; Hilbe et al., 2013, 2014; Pan et al., 2015).

1.1.3 Quantum game theory

With the recent interest in quantum computing and quantum information theory, there is necessity to extend the classical game theory into the quantum domain, and hence study the effect of quantum superposition and entanglement on the agents' optimal strategies.

In the quantum game theory, there are three main differences from the classical game theory: 1. superposed initial states; 2. quantum entanglement of initial states; 3. superposition of strategies to be used on the initial states. Thus, the quantum game theory is very different to the classical game theory, and may be useful in studying quantum communication, which can be considered as a game where the objective is to maximize effective communication. Inspired by quantum game theory, some researchers have tried to apply this theory in some traditional areas, see e.g., Piotrowski and Sladkowski (2001), Pakula et al. (2006).

Several results on quantum game theory have been obtained. For example, a simple coin game is considered in Meyer (1999), while a player with quantum superposition could certainty win against a classical player. Eisert et al. (1999) generalized the classical prisoners' dilemma to a kind of general two-strategy quantum games between two game players, with the superior performance of the quantum strategies if entanglement is present. Furthermore, Benjamin and Hayden extend the result in Eisert et al. (1999) to the multi-player games in Benjamin and Hayden (2001). Marinatto and Weber (2000) studies a quantum approach for static games while Iqbal and Toor (2002b) considers the repeated quantum games. There are also results for quantum game theory, by combining the game theory definitions and methods with quantum properties and applications, see e.g. Du et al. (2002), Iqbal and Toor (2001), Iqbal and Toor (2002a) and Li et al. (2001) and the references therein.

Inspired by classical zero-determinant (ZD) strategies, developed in Press and Dyson (2012), and upon the above successful results with extension of classical strategies to the quantum domain, a class of quantum zero-determinant strategies could be considered. Moreover, we may expect that the quantum zero-determinant strategies will have advantages over the classical zero-determinant strategies.

1.2 Summary of contributions of the thesis

The original contributions of this thesis are as follows.

- A mixed LQG and H_∞ coherent feedback control problem has been studied, while most of the present literatures (except the conference paper Bian et al. (2015)) only focus on LQG or H_∞ control problem separately. For a typical quantum optical system, there exist quantum white noise as well as finite energy signals (like lasers). Quantum white noise can be dealt with by LQG control, while finite energy disturbance can be better handled by H_∞ control. As a result, it is important to study the mixed control problem. A general result for the lower bound of LQG index is proved, which is the extension of Theorem 4.1 in Zhang et al. (2012). The genetic-algorithm-based method is proposed to design a coherent controller for this mixed problem. In contrast to the numerical algorithm proposed in the earlier conference paper (Bian et al., 2015), the new algorithm is much simpler and is able to produce better results, as clearly demonstrated by numerical studies in Chapter 3.

- By adding tags to express the characteristics of the game players, a novel model of classical two-strategy repeated/iterated game in finite populations is proposed. The players' payoffs are affected by the numbers of common tags, as described: when an individual express more tags, it will obtain higher probability to have common tags, which will yield probabilistic higher payoff. Nevertheless, the players should also pay for the expressed tags, the cost of expressed tags increases with its numbers arising. The effects are subject to the interaction rate $r(x)$ and the penalty parameter δ. Since the transition matrix of each time step is a Markov matrix, the stationary distribution, as well as the evolutionary dynamics, are discussed. Three kinds of feasible strategies for

C-strategy individual invading into D-strategy population are obtained from the numerical examples, which also indicate the effectiveness of the proposed model.

- Upon two-player Iterated Prisoner's Dilemma (IPD), the classical zero-determinant (ZD) strategies, developed in Press and Dyson (2012) are extended into the quantum domain, yielding a class of quantum zero-determinant strategies. Three kinds of quantum zero-determinant strategies are chosen as the strategy for one player in the two-player IPD game. The effectiveness of the proposed quantum zero-determinant strategies are demonstrated by numerical results. Furthermore, the advantages and features of quantum ZD strategies over classical ZD strategies are indicated in the game, between one player (with quantum ZD strategies) and another player (with classical/quantum ZD strategies).

1.3 Organization of the thesis

The thesis is structured as follows.

- Chapter 2 reviews the preliminary knowledge of linear quantum systems firstly. Two representation forms of linear quantum systems, annihilation-creation operator representation and amplitude-phase quadrature representation, are presented. Then physical realizability conditions in terms of QSDEs are stated, followed by Direct coupling, which is used in numerical simulations. Secondly, some basic concepts and terminologies of game theory are reviewed. The definitions of static games and Nash Equilibrium (NE) are introduced. Upon the original Prisoner's Dilemma (PD) and its general form presented in Subsection 2.2.2, the Iterated Prisoner's Dilemma (IPD) is discussed at last.

- Chapter 3 focuses on a mixed LQG and H_∞ coherent feedback control problem for linear quantum systems. A general model of linear quantum composite plant-controller systems and corresponding physical realizability conditions are presented. Then the mixed LQG and H_∞ coherent feedback control problem is formulated in terms of the close-loop system. Two different algorithms, rank constrained LMI method (Bian et al., 2015) and genetic-algorithm-based method are proposed to solve the mixed problem. The numerical results (subject to a passive cavity and a non-passive DPA) illustrate the effectiveness of the proposed algorithms, and demonstrate the advantage of the GA method over the rank constrained LMI method.

- Chapter 4 states a kind of classical evolutionary game theory in terms of players with tags. The standard classical evolutionary game model in finite populations is presented firstly. By adding tags to game players to describe their characteristics, we synthesize the evolutionary dynamics and stationary distributions for the new model. Finally, three classes of numerical examples are chosen to do the simulations, indicating the effectiveness of the proposed model, while getting the feasible strategies. Moreover, the comparison between strategy factors is also presented.

- Chapter 5 generalizes the classical game theory into the quantum domain. Firstly, the standard classical Iterated Prisoner's Dilemma (IPD) is introduced. Upon two-player memory-one iterated game, a kind of classical strategies, named zero-determinant (ZD) strategies, is presented. Then the general protocol of a two-player quantum game is reviewed. Depending on the quantum game theory, we synthesize a class of quantum zero-determinant strategies. At last, the effectiveness of the quantum ZD strategies, as well as their advantages over the classical ZD strategies are illustrated by numerical simulations. Furthermore, it is interesting that when both two players use quantum strategies, the quantum ZD strategies degenerate to the classical ones (named quasi-classical ZD strategies).

- Chapter 6 concludes the whole thesis and plans for the future work.

Chapter 2

Preliminaries of linear quantum systems and iterated game theory in finite populations

This chapter has two sections. The first section mainly concerns basic concepts and results of linear quantum systems. Firstly, we introduce two kinds of representations for an open linear quantum system, namely annihilation-creation operator form and amplitude-phase quadrature form. Then physical realizability conditions are reviewed, which are necessary for a class of stochastic differential equations to represent the dynamics of a meaningful physical system. Direct coupling is also introduced for the use of Chapter 3. We give brief introductions of traditional game theory in the second section. Firstly, the definitions of static games and Nash Equilibrium (NE) are given. Then we introduce a famous kind of games, Prisoner's Dilemma (PD), and the canonical form of Prisoner's Dilemma, as well as some conditions. For the use of Chapter 4, we introduce the Iterated Prisoner's Dilemma (IPD) and state the reason for considering it.

2.1 Linear quantum systems

2.1.1 Operator representation of linear quantum systems

An open linear quantum system G consists of N quantum harmonic oscillators $a = [a_1 \ \cdots \ a_N]^T$ interacting with N_w-channel quantum fields. Here, a_j is the *annihilation operator* of the jth quantum harmonic oscillator and a_j^* is the *creation operator*. They satisfy canonical commutation relations (CCR): $[a_j, a_k^*] = \delta_{jk}$, and $[a_j, a_k] = [a_j^*, a_k^*] = 0$ $(j, k = 1, ..., N)$, where the asterisk $*$ indicates Hilbert space adjoint or complex conjugation. Such a linear quantum system can be specified by a triple of physical parameters (S, L, H) (Gough and James, 2009).

In this triple, S is a unitary scattering matrix of dimension N_w. L is a vector of coupling operators defined by

$$L = C_- a + C_+ a^{\#} \tag{2.1}$$

where C_- and $C_+ \in \mathbb{C}^{N_w \times N}$. H is the Hamiltonian describing the self-energy of the system, satisfying

$$H = \frac{1}{2}\breve{a}^{\dagger}\Delta(\Omega_-, \Omega_+)\breve{a} \tag{2.2}$$

where Ω_- and $\Omega_+ \in \mathbb{C}^{N \times N}$ with $\Omega_- = \Omega_-^{\dagger}$ and $\Omega_+ = \Omega_+^T$.

The *annihilation-creation operator representation* for linear quantum stochastic systems can be written as the following quantum stochastic differential equations (QSDEs)

$$\begin{aligned} d\breve{a}(t) &= \breve{A}\breve{a}(t)dt + \breve{B}d\breve{B}_{in}(t), \quad \breve{a}(0) = \breve{a}_0 \\ d\breve{y}(t) &= \breve{C}\breve{a}(t)dt + \breve{D}d\breve{B}_{in}(t), \end{aligned} \tag{2.3}$$

where $B_{in}(t) = \int_0^t b_{in}(\tau)d\tau$ and $b_{in}(t)$ describes the N_w-channel input field before interaction, $\breve{y}(t)$ is the system output $\breve{B}_{out}(t)$ and $B_{out}(t)$ describes the N_y-channel output field after interaction. The correspondences between system matrices

$(\breve{A}, \breve{B}, \breve{C}, \breve{D})$ and parameters (S, L, H) are as follows

$$\begin{aligned}
&\breve{A} = -\frac{1}{2}\breve{C}^\flat \breve{C} - iJ_N \Delta(\Omega_-, \Omega_+), \quad \breve{B} = -\breve{C}^\flat \Delta(S, 0), \\
&\breve{C} = \Delta(C_-, C_+), \quad \breve{D} = \Delta(S, 0),
\end{aligned} \tag{2.4}$$

where $i = \sqrt{-1}$ is the imaginary unit and $N_y = N_w$.

2.1.2 Quadrature representation of linear quantum systems

In addition to annihilation-creation operator representation, there is an alternative form, *amplitude-phase quadrature representation*, where all the operators are observables (self-adjoint operators) and all corresponding matrices are real, so this form is more convenient for standard matrix analysis software packages and programmes (Bian et al., 2015; Zhang et al., 2012).

Firstly, denote $q_j = (a_j + a_j^*)/\sqrt{2}$ as the *real* or *amplitude quadrature*, and $p_j = (-ia_j + ia_j^*)/\sqrt{2}$ as the *imaginary* or *phase quadrature*. It is easy to show these two quadratures satisfy the CCR $[q_j, p_k] = i\delta_{jk}$ and $[q_j, q_k] = [p_j, p_k] = 0$ $(j, k = 1, ..., N)$.

By defining a coordinate transform

$$x := \Lambda_n \breve{a}, \quad w := \Lambda_{n_w} \breve{b}_{in}, \quad y := \Lambda_{n_y} \breve{y}, \tag{2.5}$$

where

$$\Lambda_n = \frac{1}{\sqrt{2}} \begin{bmatrix} I & I \\ -iI & iI \end{bmatrix}_{n \times n} \tag{2.6}$$

is a unitary matrix, we could get

$$\begin{aligned}
&dx(t) = Ax(t)dt + Bdw(t), \quad x(0) = x_0 \\
&dy(t) = Cx(t)dt + Ddw(t),
\end{aligned} \tag{2.7}$$

where $n = 2N$, $n_w = 2N_w$, $n_y = 2N_y$ are positive even integers, and $x(t) = [q_1(t) \ \cdots \ q_N(t) \ \ p_1(t) \ \cdots \ p_N(t)]^T$ is the vector of system variables, $w(t) = [w_1(t) \ \cdots \ w_{n_w}(t)]^T$

is the vector of input signals, including control input signals, noises and disturbances, $y(t) = [y_1(t) \ \cdots \ y_{n_y}(t)]^T$ is the vector of outputs. A, B, C and D are matrices in $\mathbb{R}^{n\times n}$, $\mathbb{R}^{n\times n_w}$, $\mathbb{R}^{n_y\times n}$ and $\mathbb{R}^{n_y\times n_w}$, respectively. The correspondences between the coefficient matrices of two different representations are

$$\begin{aligned} A &= \Lambda_n \breve{A} \Lambda_n^\dagger, \quad B = \Lambda_n \breve{B} \Lambda_{n_w}^\dagger, \\ C &= \Lambda_{n_y} \breve{C} \Lambda_n^\dagger, \quad D = \Lambda_{n_y} \breve{D} \Lambda_{n_w}^\dagger. \end{aligned} \tag{2.8}$$

Remark 2.1. *For simplicity in calculation, we usually do a permutation to obtain* $x(t) = [x_1(t) \ \cdots \ x_n(t)]^T = [q_1(t) \ p_1(t) \ \cdots \ q_N(t) \ p_N(t)]^T$*, and similarly for* $w(t)$*,* $y(t)$ *and corresponding matrices (Zhang et al., 2012). In the rest of this Chapter and Chapter 3, we focus on the quadrature form after this permutation.*

Assumption 2.1. *We give some assumptions for the quantum system (2.7) (Bian et al., 2015; James et al., 2008; Nurdin et al., 2009).*

1. *The initial system variable* $x(0) = x_0$ *is Gaussian.*

2. *The vector of inputs* $w(t)$ *could be decomposed as* $dw(t) = \beta_w(t)dt + d\tilde{w}(t)$*, where* $\beta_w(t)$ *is a self-adjoint adapted process,* $\tilde{w}(t)$ *is the noise part of* $w(t)$*, and satisfies* $d\tilde{w}(t)d\tilde{w}^T(t) = F_{\tilde{w}}dt$*, where* $F_{\tilde{w}}$ *is a nonnegative Hermitian matrix. In quantum optics,* $\tilde{w}(t)$ *is quantum white noise, and* $\beta_w(t)$ *is the signal, which in many cases is* L_2 *integrable.*

3. *The components of* $\beta_w(t)$ *commute with those of* $d\tilde{w}(t)$ *and also those of* $x(t)$ *for all* $t \geqslant 0$.

2.1.3 Physical realizability conditions of linear QSDEs

The QSDEs (2.7) may not necessarily represent the dynamics of a meaningful physical system, because quantum mechanics demands physical systems to evolve in a unitary manner. This implies the preservation of canonical commutation relations

$x(t)x^T(t) - (x(t)x^T(t))^T = i\Theta$ for all $t \geqslant 0$, and also another constraint related to the output signal. These constraints are formulated as physically realizability conditions of quantum linear systems in James et al. (2008).

A linear noncommutative stochastic system of quadrature form (2.7) is physically realizable if and only if

$$iA\Theta + i\Theta A^T + BT_{\tilde{w}}B^T = 0, \tag{2.9a}$$

$$B = \Theta C^T \mathrm{diag}_{N_y}(F), \tag{2.9b}$$

$$D = I_{n_y}, \tag{2.9c}$$

where $T_{\tilde{w}} = \frac{1}{2}(F_{\tilde{w}} - F_{\tilde{w}}^T)$, the first equation determines the Hamiltonian and coupling operators, while the others relate to the required form of the output equation.

2.1.4 Direct coupling

There are also some additional components in quantum systems, such as direct coupling, phase shifter, ideal squeezer, etc. Interested readers could refer to e.g. Zhang and James (2011), Zhang and James (2012), Zhang et al. (2012). Depending on the need of this thesis, we just briefly introduce the *direct coupling*.

In quantum mechanics, two independent systems G_1 and G_2 may interact by exchanging energy directly. This energy exchange can be described by an interaction Hamiltonian H_{int}. In this case, it is said that these two systems are directly coupled. When they are expressed in the annihilation-creation operator form,

$$d\breve{a}_1(t) = \breve{A}_1\breve{a}_1(t)dt + \breve{B}_{12}\breve{a}_2(t)dt,$$

$$d\breve{a}_2(t) = \breve{A}_2\breve{a}_2(t)dt + \breve{B}_{21}\breve{a}_1(t)dt,$$

where the subscript 1 means that corresponding terms belong to the system G_1, and similar for subscript 2. B_{12} and B_{21} denote the direct coupling between two systems, and satisfy the following relations

$$B_{12} = -\Delta(K_-, K_+)^\flat,$$

$$B_{21} = -B_{12}^\flat = \Delta(K_-, K_+),$$

where K_- and K_+ are arbitrary constant matrices of appropriate dimensions.

Definition 2.1. *For a quantum linear system in the annihilation-creation operator form which is defined by parameters $(C_-, C_+, \Omega_-, \Omega_+, K_-, K_+)$, there will have the following classifications:*

1. *If all "plus" parameters (i.e. C_+, Ω_+ and K_+) are equal to 0, the system is called a passive system;*

2. *Otherwise, it is called a non-passive system.*

For example, consider a optical cavity system taken from Section VII of James et al. (2008). The dynamics of the cavity is described by Eq. (2.10).

$$\begin{aligned} dx\,(t) &= -\frac{\gamma}{2} I_2 x\,(t)\,dt - \sqrt{\kappa_1} I_2 dv(t) - \sqrt{\kappa_2} I_2 dw\,(t) - \sqrt{\kappa_3} I_2 du(t), \\ dy(t) &= \sqrt{\kappa_2} I_2 x(t) dt + I_2 dw(t) \end{aligned} \tag{2.10}$$

with parameters $\gamma = \kappa_1 + \kappa_2 + \kappa_3$, $\kappa_1 = 2.6$, $\kappa_2 = \kappa_3 = 0.2$.

For this open system, it can be seen that $K_- = K_+ = 0$. By Eqs. (2.7) and (2.8), it is clearly that $\breve{A} = A = -1.5I_2$, $\breve{B} = B = -\sqrt{0.2}I_2$, $\breve{C} = C = \sqrt{0.2}I_2$ and $\breve{D} = D = I_2$. Therefore, by Eq. (2.4), we could easily calculate that:

$$C_- = \sqrt{0.2},\ C_+ = 0,\ \Omega_- = 1.4,\ \Omega_+ = 0,$$

and the corresponding Hamiltonian $H = 0.7(aa + a^* a^*)$.

Consequently, as $C_+ = \Omega_+ = K_+ = 0$, by Definition 2.1, this example (optical cavity) is a passive system. Similarly, it is easy to verify that the example 2 (DPA) in Section 3.4 is a non-passive system.

2.2 Brief introduction of game theory

In this section, we introduce some basic terminologies and ideas of game theory.

2.2.1 Static games and Nash Equilibrium (NE)

Game theory attempts to mathematically model a situation where agents interact (Flitney and Abbott, 2002). The agents in the game are called *players*, their possible actions are called *moves* or *actions*, and a prescription that specifies the particular move to be made in all possible game situations is called a *strategy*. The *utility* to a player of a game's outcome is a numerical measure of the desirability of that outcome for the player, and a *payoff matrix* gives numerical values to the players' utility for all the game outcomes.

Static games are also called *simultaneous-move games*, which mean that the players choose actions simultaneously, after that the players receive payoffs that depend on the combination of actions just chosen. In this Subsection, we only discuss static games.

Generally speaking, classical game theory consists of two parts: *cooperative game* and *non-cooperative game*. In this thesis, we mainly focus on the non-cooperative game.

Definition 2.2. *(Cheng and Liu, 2015) A normal non-cooperative game is a triple* $G = (N, S, c)$, *where*

1. $N = \{1, 2, \ldots, n\}$ *is the set of players. That is, there are* n *players named as* $1, 2, \ldots, n$ *in the game.*

2. $S_i = \{1, 2, \ldots, k_i\}$ *is the strategy set of the player* i, $i = 1, \ldots, n$. *That is, the player* i *has* k_i *different strategies (or actions). The Cartesian product of* S_i, *denoted as* $S = \prod_{i=1}^{n} S_i$, *is called the profile of the game.*

3. $c_j(s) : S \to \mathbb{R}$ *is called the payoff function of the player* j*, which means how much the player* j *can get from the game,* $j = 1, \ldots, n$*. Putting them together, we have* $c = \{c_1, \ldots, c_n\}$*.*

For non-cooperative games, the *Nash Equilibrium (NE)* is the most important concept, which is considered as the "solution" to non-cooperative games.

Definition 2.3. *(Gibbons, 1992) In a normal non-cooperative game* G*, a profile* $s = (s_1^*, \ldots, s_n^*) \in S$ *is called a Nash equilibrium, if for each player* j*,* s_j^* *is the player* j*'s best response to the strategies specified for the other* $n-1$ *players* $(s_1^*, \ldots, s_{j-1}^*, s_{j+1}^*, \ldots, s_n^*)$*, i.e.*

$$c_j(s_1^*, \ldots, s_{j-1}^*, s_j^*, s_{j+1}^*, \ldots, s_n^*) \geqslant c_j(s_1^*, \ldots, s_{j-1}^*, s_j, s_{j+1}^*, \ldots, s_n^*), \quad \forall s_j \in S_j,\ j = 1, \ldots, n.$$

That is, s_j^* *solves*

$$\max_{s_j \in S_j} c_j(s_1^*, \ldots, s_{j-1}^*, s_j, s_{j+1}^*, \ldots, s_n^*).$$

	L	C	R
T	(0, 4)	(4, 0)	(5, 3)
M	(4, 0)	(0, 4)	(5, 3)
B	(3, 5)	(3, 5)	(6, 6)

$\Longrightarrow$

	L	C	R
T	(0, $\underline{4}$)	($\underline{4}$, 0)	(5, 3)
M	($\underline{4}$, 0)	(0, $\underline{4}$)	(5, 3)
B	(3, 5)	(3, 5)	($\underline{6}$, $\underline{6}$)

Figure 2.1: A example to show how to calculate the Nash Equilibrium (NE).

To be more concrete, we now look at a simple example. Consider a normal-form game shown in Figure 2.1, where T, M, B denote the row player's strategies, and L, C, R are strategies for the column player. A brute-force approach to finding a game's Nash equilibrium is simply to check whether each possible combination of strategies satisfies condition (NE) in Definition 2.3. In a two-player game, this approach begins as follows: for each player, and for each feasible strategy of that player, determine the other player's best response to that strategy. The right figure in Figure 2.1 does this for the game in the left figure in Figure 2.1, by underlining the payoff to player j's best response to each of player i's feasible strategies. For instance, if the column player were to play C, then the row player's best response would be T, since 4 exceeds 0 and 3, so the row player's payoff of 4 in the (T, C) cell is underlined.

A pair of strategies satisfies condition (NE) if each player's strategy is the best response to the other's, i.e., if both payoffs are underlined in the corresponding cell. Thus, the cell (B, R) in Figure 2.1 is the only strategy pair that satisfies (NE), indicating that this strategy pair is the unique Nash equilibrium of the game.

2.2.2 Prisoner's Dilemma (PD)

The Prisoner's Dilemma (PD), named by Albert W. Tucker (Tucker, 1950; Straffin, 1980), is a standard example of games analyzed in game theory that shows why two completely "rational" individuals might not cooperate, even if it appears that it is in their best interests to do so. It is presented as follows (Poundstone, 1992).

Two members of a criminal gang are arrested and imprisoned. Each prisoner is in solitary confinement with no means of communication with the other. The prosecutors lack sufficient evidence to convict the pair on the principal charge. They hope to get both sentenced to a year in prison on a lesser charge. Simultaneously, the prosecutors offer each prisoner a bargain. Each prisoner is given the opportunity either to: betray the other by testifying that the other committed the crime, or to cooperate with the other by remaining silent. The offer is:

1. If A and B each betrays the other, each of them serves 2 years in prison,

2. If A betrays B but B remains silent, A will be set free and B will serve 3 years in prison (and vice versa),

3. If A and B both remain silent, both of them will only serve 1 year in prison (on the lesser charge),

as shown in Figure 2.2, where the first entry in the parenthesis for prisoner A and the second number for prisoner B.

		Prisoner B	
		Silent	Betray
Prisoner A	Silent	(-1, -1)	(-3, 0)
	Betray	(0, -3)	(-2, -2)

Figure 2.2: The illustration of Prisoner's Dilemma (PD).

It is assumed that prisoners A and B only focus on their own prison sentences, and their decision will not affect their reputation in the future. Then it is clear to see that, prisoner A will serve less years in prison when prisoner A betrays prisoner B than cooperates with prisoner B (since: $0 < 1$, $2 < 3$), and similar to prisoner B. So the only possible outcome for two purely rational prisoners is betraying each other, corresponding to the cell $(-2, -2)$. Furthermore, we notice that the cell $(-2, -2)$ is the unique Nash equilibrium of PD, which can be calculated depending on Definition 2.3.

The structure of the traditional Prisoner's Dilemma can be generalized from its original prisoner setting. Suppose that the two players are represented by Alice and Bob, respectively. Each player can choose one of two strategies, "Cooperate" (corresponding to "Silent" in original PD) or "Defect" (corresponding to "Betray" in original PD), which can be expressed in Table 2.1.

Table 2.1 shows the payoff matrix for a general Prisoner's Dilemma, where the first entry in the parenthesis denotes the payoff of the player Alice and the second number for the payoff of the player Bob. When Alice and Bob cooperate (corresponds

Table 2.1: Payoff matrix of the Canonical Prisoner's Dilemma.

		Player: Bob	
		Strategy: C	Strategy: D
Player: Alice	Strategy: C	(R, R)	(S, T)
	Strategy: D	(T, S)	(P, P)

to strategy C), each player earns a payoff R (Reward). If one defects (corresponds to strategy D), the defector gets an larger payment T (Temptation), and the naive cooperator gets S ("Sucker's" payoff). However, if both of them defect, then both get a meager payment P (Punishment).

Moreover, to be a PD game in the strong sense, the following two inequality conditions must be held.

1. $T > R > P > S$ guarantees that the Nash equilibrium of the game is mutual defection, where $T > R$ and $P > S$ imply that defection is the dominant strategy for both of them, while $R > P$ implies the mutual cooperation is superior to mutual defection.

2. $2R > T + S$ makes mutual cooperation the globally best outcome.

2.2.3 Iterated Prisoner's Dilemma (IPD)

It is clearly that the Canonical Prisoner's Dilemma discussed in Subsection 2.2.2 is a type of static games, and its unique Nash equilibrium is the cell (P, P) (i.e. both two players choose defection), which is not the globally best strategy.

If two players play Prisoner's Dilemma game more than once in succession and they remember previous actions of their opponent and change their strategy accordingly, the game is called Iterated (or repeated) Prisoner's Dilemma (IPD).

Definition 2.4. *(Gibbons, 1992) Given a static game G,*

1. *$G(T)$, named as **finitely repeated game**, if G is played T times, with the outcomes of all preceding plays observed before the next play begins.*

2. *If the game G is played infinitely, we have an **infinitely repeated game**.*

When we consider the Nash equilibrium of a finitely repeated game, there is an important result shown in Lemma 2.1.

Lemma 2.1. *(Gibbons, 1992) If the static game G has a unique Nash equilibrium, then for any finite time T, the finitely repeated game $G(T)$ has a unique subgame-perfect outcome: the Nash equilibrium of G is played in every stage.*

By Lemma 2.1, it is easily obtained that, in every period of a finitely repeated Prisoner's Dilemma, the Nash equilibrium is each player chooses strategy D. When a game goes on indefinitely, if there is a future period in which one player's action is C, then this player can punish the other player by choosing strategy D instead of C in the next period (Osborne, 2004). This fact suggests that an infinitely repeated game may be a suitable model which captures the idea that cooperation may be sustained by "punishment" strategies when players interact repeatedly.

Consequently, for classical game theory, we consider an infinitely repeated/iterated Prisoner's Dilemma game consisting of multiple, successive plays by the same opponents in Chapter 4. Opponents may now condition their play on their opponent's strategy insofar as each can deduce it from the previous play. However, we give each player only a finite memory of previous play (Press and Dyson, 2012; Hauert and Schuster, 1997). It has been proved that, for any strategy of the longer-memory player, shorter-memory player's score is exactly the same as if the longer-memory player had played a certain shorter-memory strategy. For the detailed proof and recent results on memory-one and long-memory strategies, interested readers could refer to Appendix A in Press and Dyson (2012), as well as Hilbe et al. (2017).

Chapter 3

Mixed LQG and H_∞ coherent feedback control for linear quantum systems

In this chapter, we focus on a mixed LQG and H_∞ coherent feedback control problem. Firstly, for a class of open linear quantum systems, a quantum controller is designed to form the closed-loop system. We also derive the specific physical realizability conditions, which are necessary for coherent feedback control problems. Secondly, we discuss both the LQG and H_∞ control problems for the closed-loop system, as well as the synthesis of the mixed coherent feedback control problem. We also prove a general result for the lower bound of the LQG index, which is an extension of Theorem 4.1 in Zhang et al. (2012). Then two algorithms, rank constrained LMI method (Bian et al., 2015) and genetic-algorithm-based method, are proposed for the mixed problem. Finally, we test the two proposed algorithms by means of a passive system (cavity) and a non-passive one (degenerate parametric amplifier, DPA). The numerical results illustrate the effectiveness of the proposed algorithms. Furthermore, the genetic-algorithm-based method is much simpler and is able to produce better results, as clearly demonstrated by numerical studies.

3.1 Formulation of linear quantum systems

In this section, we firstly formulate the QSDEs for a closed-loop system, in which both the plant and controller are quantum systems. We also propose physical realizability conditions for the controller.

3.1.1 Composite plant-controller system

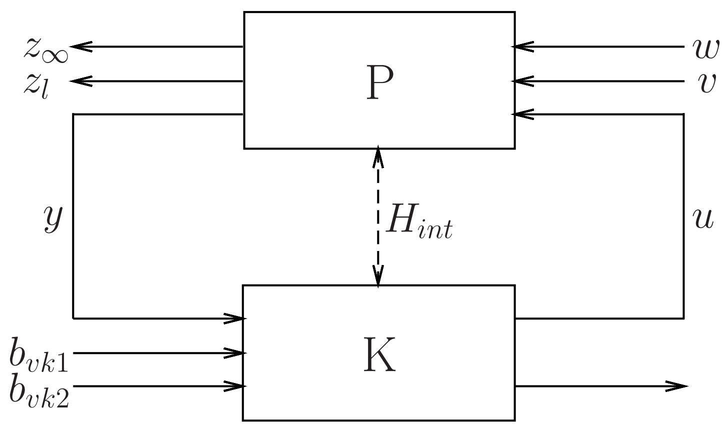


Figure 3.1: Schematic of the closed-loop plant-controller system.

Consider the closed-loop system as shown in Figure 3.1. The quantum plant P is described by QSDEs in the quadrature form (Bian et al., 2015),

$$\begin{aligned} dx(t) &= Ax(t)dt + B_0dv(t) + B_1dw(t) + B_2du(t), \\ dy(t) &= C_2x(t)dt + D_{20}dv(t) + D_{21}dw(t), \\ dz_\infty(t) &= C_1x(t)dt + D_{12}du(t), \\ z_l(t) &= C_zx(t) + D_z\beta_u(t), \end{aligned} \tag{3.1}$$

where A, B_0, B_1, B_2, C_2, D_{20}, D_{21}, C_1, D_{12}, C_z and D_z are real matrices in $\mathbb{R}^{n\times n}$, $\mathbb{R}^{n\times n_v}$, $\mathbb{R}^{n\times n_w}$, $\mathbb{R}^{n\times n_u}$, $\mathbb{R}^{n_y\times n}$, $\mathbb{R}^{n_y\times n_v}$, $\mathbb{R}^{n_y\times n_w}$, $\mathbb{R}^{n_\infty\times n}$, $\mathbb{R}^{n_\infty\times n_u}$, $\mathbb{R}^{n_l\times n}$, $\mathbb{R}^{n_l\times n_u}$ respectively, n, n_v, n_w, n_u, n_y are positive even numbers, and n_∞, n_l are positive integers. $x(t) = [x_1(t)\ \cdots\ x_n(t)]^T$ is the vector of self-adjoint possibly noncommutative system variables; $u(t) = [u_1(t) \ \cdots \ u_{n_u}(t)]^T$ is the controlled input; $v(t) = [v_1(t) \ \cdots \ v_{n_v}(t)]^T$ and $w(t) = [w_1(t) \ \cdots \ w_{n_w}(t)]^T$ contain quantum white noises and signal inputs. $z_\infty(t) = [z_{\infty_1}(t) \ \cdots \ z_{\infty_{n_\infty}}(t)]^T$ and $z_l(t) = [z_{l_1}(t) \ \cdots \ z_{l_{n_l}}(t)]^T$ are controlled outputs which are used for H_∞ and LQG performance, respectively.

The purpose is to design a coherent feedback controller K to minimize the LQG index and the H_∞ norm of the closed-loop system simultaneously, and K has the following form

$$\begin{aligned} d\xi(t) &= A_k\xi(t)dt + B_{k1}dB_{vk1}(t) + B_{k2}dB_{vk2}(t) + B_{k3}dy(t), \\ du(t) &= C_k\xi(t)dt + dB_{vk1}(t), \end{aligned} \tag{3.2}$$

where $\xi(t) = [\xi_1(t) \ \cdots \ \xi_{n_k}(t)]^T$ is a vector of self-adjoint variables, and matrices A_k, B_{k1}, B_{k2}, B_{k3}, C_k have appropriate dimensions.

Assumption 3.1. *Similarly with Assumption 2.1, we give additional assumptions for the plant and controller in Figure 3.1.*

1. *The inputs $w(t)$ and $u(t)$ have the decompositions: $dw(t) = \beta_w(t)dt + d\tilde{w}(t)$, $du(t) = \beta_u(t)dt + d\tilde{u}(t)$, where the meanings of β_w, β_u and $\tilde{w}$, $\tilde{u}$ are similar as those in Assumption 2.1;*

2. *The controller state variable $\xi(t)$ has the same order as the plant state variable $x(t)$;*

3. *$v(t)$, $\tilde{w}(t)$, $b_{vk1}(t)$ and $b_{vk2}(t)$ are independent quantum noises in vacuum state;*

4. *The initial plant state $x(0)$ and controller state $\xi(0)$ satisfy the following relations: $x(0)x^T(0) - (x(0)x^T(0))^T = i\Theta$, $\xi(0)\xi^T(0) - (\xi(0)\xi^T(0))^T = i\Theta_k$, $x(0)\xi^T(0) - (\xi(0)x^T(0))^T = 0$, where Θ_k means a matrix Θ with appropriate dimension for the controller K.*

By the identification $\beta_u(t) \equiv C_k\xi(t)$ and $\tilde{u}(t) \equiv b_{vk1}(t)$, the closed-loop system is obtained as

$$\begin{aligned} d\eta(t) &= M\eta(t)dt + Nd\tilde{w}_{cl}(t) + H\beta_w(t)dt, \\ dz_\infty(t) &= \Gamma\eta(t)dt + \Pi d\tilde{w}_{cl}(t), \\ z_l(t) &= \Psi\eta(t), \end{aligned} \tag{3.3}$$

where $\eta(t) = [x^T(t)\ \xi^T(t)]^T$ denotes the state of the closed-loop system, $\beta_w(t)$ is the disturbance, while $\tilde{w}_{cl}(t) = [v^T(t)\ \tilde{w}^T(t)\ b_{vk1}^T(t)\ b_{vk2}^T(t)]^T$ contains all quantum white noises. The coefficient matrices are

$$\begin{aligned} M &= \begin{bmatrix} A & B_2C_k \\ B_{k3}C_2 & A_k \end{bmatrix}, \\ N &= \begin{bmatrix} B_0 & B_1 & B_2 & 0 \\ B_{k3}D_{20} & B_{k3}D_{21} & B_{k1} & B_{k2} \end{bmatrix}, \\ H &= \begin{bmatrix} B_1 \\ B_{k3}D_{21} \end{bmatrix}, \quad \Gamma = \begin{bmatrix} C_1 & D_{12}C_k \end{bmatrix}, \\ \Pi &= \begin{bmatrix} 0 & 0 & D_{12} & 0 \end{bmatrix}, \quad \Psi = \begin{bmatrix} C_z & D_zC_k \end{bmatrix}. \end{aligned}$$

3.1.2 Physical realizability conditions

For the plant P introduced in the previous subsection, we want to design a controller K which is also a quantum-mechanical system. Hence from James et al. (2008) and Zhang et al. (2012), Eq. (3.2) should also satisfy the following physical realizability conditions

$$\begin{aligned} A_k\Theta_k + \Theta_kA_k^T &+ B_{k1}\mathrm{diag}_{n_{vk1}/2}(F)B_{k1}^T \\ &+ B_{k2}\mathrm{diag}_{n_{vk2}/2}(F)B_{k2}^T \\ &+ B_{k3}\mathrm{diag}_{n_{vk3}/2}(F)B_{k3}^T = 0, \end{aligned} \tag{3.4a}$$

$$B_{k1} = \Theta_kC_k^T\mathrm{diag}_{n_u/2}(F). \tag{3.4b}$$

3.2 Synthesis of mixed LQG and H_∞ coherent feedback control problem

For the system given in Section 3.1, we firstly present the standard LQG and H_∞ control problem separately. For the LQG index, we prove a general result for the lower bound of it. On the other hand, since it is general difficult to derive a general result for lower bounds of H_∞ index, we get a simple example to calculate it, by using algebraic Riccati equations (James et al., 2008). At last, the mixed LQG and H_∞ control problem is discussed.

3.2.1 LQG control problem

LQG control problem, i.e. Linear Quadratic Gaussian control problem, is an optimal control problem about designing a controller to minimize the quadratic cost function of the system which is disturbed by additive Gaussian white noise. With the closed-loop system (3.3), we associate a quadratic performance index

$$J(t_f) = \int_0^{t_f} \langle z_l^T(t) z_l(t) \rangle dt, \tag{3.5}$$

where the notation $\langle \cdot \rangle$ is standard and refers to as quantum expectation (Merzbacher, 1998).

Remark 3.1. *In classical control, $\int_0^\infty (x(t)^T P x(t) + u(t)^T Q u(t)) dt$ is the standard form for LQG performance index, where x is the system variable and u is the control input. However, things are more complicated in the quantum regime. By Eq. (3.2), we can see that $u(t)$ is a function of both $\xi(t)$ (the controller variable) and $b_{vk1}(t)$ (input quantum white noise). If we use $u(t)$ in Eq. (3.2) directly, then there will be quantum white noise in the LQG performance index, which yields an unbounded LQG control performance. On the other hand, by Eq. (3.5), the LQG performance* *index is a function of $x(t)$ (the system variable) and $\xi(t)$ (the controller variable). This is the appropriate counterpart of the classical case.*

Generally, we always focus on the infinite horizon case $t_f \to \infty$. Therefore, as in Nurdin et al. (2009), assume that M is asymptotically stable, by standard analysis methods, we have the infinite-horizon LQG performance index as

$$J_\infty = \lim_{t_f \to \infty} \frac{1}{t_f} \int_0^{t_f} \langle z_l^T(t) z_l(t) \rangle dt = \mathrm{Tr}(\Psi P \Psi^T), \tag{3.6}$$

where P is the unique symmetric positive definite solution of the Lyapunov equation

$$MP + PM^T + \frac{1}{2} NN^T = 0. \tag{3.7}$$

Problem 3.1. *The LQG coherent feedback control problem is to find a quantum controller K of Eq. (3.2) that minimizes the LQG performance index $J_\infty = \mathrm{Tr}(\Psi P \Psi^T)$. Here P is the unique solution of Eq. (3.7), and coefficient matrices of controller satisfy the physical realization conditions (3.4).*

When considering minimizing LQG performance index, firstly we want to know the minimum. But in general, it is too complicated to get theoretical results. We choose the orders of plant and controller to be 2. In this case, because $C_z^T C_z$ and $D_z^T D_z$ are both 2-by-2 positive semi-definite real matrices, we denote

$$C_z^T C_z = \begin{bmatrix} c_1 & c_2 \\ c_2 & c_3 \end{bmatrix}, D_z^T D_z = \begin{bmatrix} d_1 & d_2 \\ d_2 & d_3 \end{bmatrix}, C_k = \begin{bmatrix} c_{k1} & c_{k2} \\ c_{k3} & c_{k4} \end{bmatrix}, \tag{3.8}$$

where all parameters in these matrices are real scalars.

The following result is an extension of Theorem 4.1 in Zhang et al. (2012).

Theorem 3.1. *(The lower bound of LQG index) Assume that both the plant and the controller defined in Subsection 3.1.1 are in the ground state, then the LQG* *performance index*

$$J_\infty \geqslant \frac{c_1 + c_3}{2} + d_2(c_{k1}c_{k3} + c_{k2}c_{k4}),$$

where c_* and d_* come from the matrices in Eq. (3.8).

Proof. Since $z_l = C_z x + D_z \beta_u = C_z x + D_z C_k \xi$, we could easily get

$$\begin{aligned}\langle z_l^T z_l \rangle &= \langle (C_z x + D_z C_k \xi)^T (C_z x + D_z C_k \xi) \rangle \\ &= \langle x^T C_z^T C_z x \rangle + \langle \xi^T C_k^T D_z^T D_z C_k \xi \rangle \\ &\quad + \langle x^T C_z^T D_z C_k \xi \rangle + \langle \xi^T C_k^T D_z^T C_z x \rangle,\end{aligned} \tag{3.9}$$

where

$$x = \begin{bmatrix} q \\ p \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{bmatrix} a \\ a^* \end{bmatrix}, \xi = \begin{bmatrix} q_k \\ p_k \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{bmatrix} a_k \\ a_k^* \end{bmatrix}.$$

Then we have

$$\begin{aligned}\langle x^T C_z^T C_z x \rangle &= \frac{1}{2} \langle [a \ a^*] \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \begin{bmatrix} c_1 & c_2 \\ c_2 & c_3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{bmatrix} a \\ a^* \end{bmatrix} \rangle \\ &= \frac{1}{2} \langle [a \ a^*] \begin{bmatrix} c_1 - c_3 - 2ic_2 & c_1 + c_3 \\ c_1 + c_3 & c_1 - c_3 + 2ic_2 \end{bmatrix} \begin{bmatrix} a \\ a^* \end{bmatrix} \rangle \\ &= \frac{1}{2} \langle (c_1 + c_3) a^* a + (c_1 + c_3) a a^* \\ &\quad + (c_1 - c_3 - 2ic_2) aa + (c_1 - c_3 + 2ic_2) a^* a^*] \rangle \\ &= \langle (c_1 + c_3) a^* a + \frac{c_1 + c_3}{2} \rangle,\end{aligned} \tag{3.10}$$

where the last equality follows from our assumption that the plant are in the ground state, and $[a, a^*] = 1 \Rightarrow aa^* = 1 + a^* a$. The second term of equation (3.9) becomes

$$\begin{aligned}&\langle \xi^T C_k^T D_z^T D_z C_k \xi \rangle \\ &= \langle [q_k \ p_k] \begin{bmatrix} c_{k1} & c_{k3} \\ c_{k2} & c_{k4} \end{bmatrix} \begin{bmatrix} d_1 & d_2 \\ d_2 & d_3 \end{bmatrix} \begin{bmatrix} c_{k1} & c_{k2} \\ c_{k3} & c_{k4} \end{bmatrix} \begin{bmatrix} q_k \\ p_k \end{bmatrix} \rangle \\ &= \langle [q_k \ p_k] \begin{bmatrix} e_1 & e_2 \\ e_2 & e_3 \end{bmatrix} \begin{bmatrix} q_k \\ p_k \end{bmatrix} \rangle \\ &= \langle e_1 q_k^2 + e_3 p_k^2 + e_2 (q_k p_k + p_k q_k) \rangle,\end{aligned} \tag{3.11}$$

where $e_1 = d_1c_{k1}^2 + d_3c_{k3}^2 + 2d_2c_{k1}c_{k3}$, $e_3 = d_1c_{k2}^2 + d_3c_{k4}^2 + 2d_2c_{k2}c_{k4}$, $e_2 = d_1c_{k1}c_{k2} + d_3c_{k3}c_{k4} + d_2(c_{k1}c_{k4} + c_{k2}c_{k3})$.

While $q_k = \frac{a_k+a_k^*}{\sqrt{2}}$ and $p_k = \frac{-ia_k+ia_k^*}{\sqrt{2}}$, we get

$$q_k^2 = \frac{1}{2}[a_k^2 + (a_k^*)^2 + 2a_k^*a_k + 1],$$

$$p_k^2 = -\frac{1}{2}[a_k^2 + (a_k^*)^2 - 2a_k^*a_k - 1],$$

$$q_kp_k + p_kq_k = -i[a_k^2 - (a_k^*)^2],$$

and

$$\begin{aligned}&\langle\xi^TC_k^TD_z^TD_zC_k\xi\rangle\\ =&\langle\frac{e_1}{2}[a_k^2 + (a_k^*)^2 + 2a_k^*a_k + 1]\\ &-\frac{e_3}{2}[a_k^2 + (a_k^*)^2 - 2a_k^*a_k - 1] - e_2i[a_k^2 - (a_k^*)^2]\rangle.\end{aligned} \tag{3.12}$$

Since both the plant and the controller are in the ground state, all terms containing a, a^*, a_k and a_k^* are 0; and the plant state x commutes with the controller state ξ, so the third and fourth terms of equation (3.9) are also 0. By substituting (3.10) and (3.12) into (3.9), we obtain

$$\begin{aligned}\langle z_l^Tz_l\rangle &= \frac{c_1+c_3}{2} + \frac{e_1+e_3}{2}\\ &= \frac{d_1(c_{k1}^2 + c_{k2}^2) + d_3(c_{k3}^2 + c_{k4}^2) + 2d_2(c_{k1}c_{k3} + c_{k2}c_{k4})}{2}\\ &\quad+\frac{c_1+c_3}{2}.\end{aligned} \tag{3.13}$$

Consequently, all square terms are not less than 0, so $J_\infty \geqslant \frac{c_1+c_3}{2} + d_2(c_{k1}c_{k3} + c_{k2}c_{k4})$. The proof is completed. □

Remark 3.2. *Sometimes for simplicity, we could choose a coefficient matrix D_z satisfying $d_2 = 0$, then the lower bound of the LQG index becomes $J_\infty \geqslant \frac{c_1+c_3}{2}$, which* *is a constant, independent of the designed controller. This is consistent with the result in Zhang et al. (2012).*

Meanwhile, it is easy to see that physical realizability conditions (3.4) of the coherent controller K are polynomial equality constraints, so they are difficult to solve numerically using general existing optimization algorithms. Hence sometimes we reformulate Problem 3.1 into a rank constrained LMI feasibility problem, by letting the LQG performance index $J_\infty < \gamma_l$ for a prespecified constant $\gamma_l > 0$. This is given by the following result.

Lemma 3.1. *(Relaxed LQG problem (Nurdin et al., 2009)) Given Θ_k and $\gamma_l > 0$, if there exist symmetric matrix $P_L = P^{-1}$, Q and coefficient matrices of the controller such that physical realizability constraints (3.4) and following inequality constraints*

$$\begin{aligned} \begin{bmatrix} M^T P_L + P_L M & P_L N \\ N^T P_L & -I \end{bmatrix} &< 0, \\ \begin{bmatrix} P_L & \Psi^T \\ \Psi & Q \end{bmatrix} &> 0, \\ \mathrm{Tr}(Q) &< \gamma_l \end{aligned} \tag{3.14}$$

hold, then the LQG coherent feedback control problem admits a coherent feedback controller K of the form (3.2).

3.2.2 H_∞ control problem

H_∞ control problem mainly concerns the robustness of a system to parameter uncertainty or some external disturbance signals, a controller is designed to make the closed-loop system stable and minimizes the influence of disturbances on the system performance. For linear systems, the H_∞ norm can be expressed as follows

$$\|T\|_\infty = \sup_{\omega\in\mathbb{R}} \sigma_{\max}[T(j\omega)] = \sup_{\omega\in\mathbb{R}} \sqrt{\lambda_{\max}(T^*(j\omega)T(j\omega))} \tag{3.15}$$

where $\sigma_{\max}$ is the largest singular value of a matrix, and $\lambda_{\max}$ is the largest eigenvalue of an Hermitian matrix.

Since we consider the H_∞ control problem for the closed-loop system (3.3), and only β_w part contains exogenous signals while the others are all white noises, we interpret $\beta_w \to z_\infty$ as the robustness channel for measuring the H_∞ performance, and our objective to be minimized is

$$\begin{aligned} \|G_{\beta_w \to z_\infty}\|_\infty &= \|D_{cl} + C_{cl}(sI - A_{cl})^{-1}B_{cl}\|_\infty \\ &= \|\Gamma(sI - M)^{-1}H\|_\infty. \end{aligned} \tag{3.16}$$

Problem 3.2. *The H_∞ coherent feedback control problem is to find a quantum controller K of form (3.2) that minimizes the H_∞ performance index $\|G_{\beta_w \to z_\infty}\|_\infty$, while coefficient matrices A_k, B_{k1}, B_{k2}, B_{k3} and C_k of the controller satisfy constraints (3.4).*

Similarly to the LQG case, we proceed to relax Problem 3.2 into a rank constrained LMI feasibility problem, i.e. let $\|G_{\beta_w \to z_\infty}\|_\infty < \gamma_\infty$ for a prespecified constant $\gamma_\infty > 0$, then we get the following lemma.

Lemma 3.2. *(Relaxed H_∞ problem (Zhang and James, 2011)) Given Θ_k and $\gamma_\infty > 0$, if there exist A_k, B_{k1}, B_{k2}, B_{k3}, C_k and a symmetric matrix P_H such that physical realizability constraints (3.4) and following inequality constraints*

$$\begin{aligned} \begin{bmatrix} M^T P_H + P_H M & P_H H & \Gamma^T \\ H^T P_H & -\gamma_\infty I & 0 \\ \Gamma & 0 & -\gamma_\infty I \end{bmatrix} &< 0, \\ P_H &> 0 \end{aligned} \tag{3.17}$$

hold, then the H_∞ coherent feedback control problem admits a coherent feedback controller K of the form (3.2).

We want to know the lower bound of the H_∞ performance index. It is in general difficult to derive the minimum value of the H_∞ index analytically, here we just present a simple example. We begin with the following remark.

Remark 3.3. *By referring to James et al. (2008), there exists an H_∞ controller of form (3.2) for the quantum system (3.1), if and only if the following pair of algebraic Riccati equations*

$$\begin{aligned}&(A - B_2 E_1^{-1} D_{12}^T C_1)^T X + X(A - B_2 E_1^{-1} D_{12}^T C_1)\\ &\quad + X(B_1 B_1^T - \gamma_\infty^2 B_2 E_1^{-1} B_2^T)X\\ &\quad + \gamma_\infty^{-2} C_1^T (I - D_{12} E_1^{-1} D_{12}^T) C_1 = 0\end{aligned} \tag{3.18}$$

and

$$\begin{aligned}&(A - B_1 D_{21}^T E_2^{-1} C_2)Y + Y(A - B_1 D_{21}^T E_2^{-1} C_2)^T\\ &\quad + Y(\gamma_\infty^{-2} C_1^T C_1 - C_2^T E_2^{-1} C_2)Y\\ &\quad + B_1 (I - D_{21}^T E_2^{-1} D_{21}) B_1^T = 0\end{aligned} \tag{3.19}$$

have positive definite solutions X and Y, where $D_{12}^T D_{12} = E_1 > 0$, $D_{21} D_{21}^T = E_2 > 0$.

We consider a simple example. The system equations are

$$\begin{aligned}dx(t) &= -\frac{1}{2}\begin{bmatrix}0.89 & 0\\ 0 & 0.91\end{bmatrix} x(t)dt - \sqrt{0.5}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} dv(t)\\ &\quad - \sqrt{0.2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} dw(t) - \sqrt{0.2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} du(t),\\ dy(t) &= \sqrt{0.5}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} x(t)dt + \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} dv(t) + \delta\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} dw(t),\\ dz_\infty(t) &= \sqrt{0.2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} x(t)dt + \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} du(t),\end{aligned}$$

where δ is a very small positive real number.

There has no problem to calculate the first Riccati equation (3.18). For the second one (3.19), denote $Y = \begin{bmatrix}y_1 & y_2\\ y_2 & y_3\end{bmatrix}$, we get

$$\begin{bmatrix}(\frac{0.2}{\gamma_\infty^2} - \frac{0.5}{\delta^2})(y_1^2 + y_2^2) - (0.89 - \frac{2\sqrt{0.1}}{\delta})y_1 & [(\frac{0.2}{\gamma_\infty^2} - \frac{0.5}{\delta^2})(y_1 + y_3) - (0.9 - \frac{2\sqrt{0.1}}{\delta})]y_2\\ [(\frac{0.2}{\gamma_\infty^2} - \frac{0.5}{\delta^2})(y_1 + y_3) - (0.9 - \frac{2\sqrt{0.1}}{\delta})]y_2 & (\frac{0.2}{\gamma_\infty^2} - \frac{0.5}{\delta^2})(y_2^2 + y_3^2) - (0.91 - \frac{2\sqrt{0.1}}{\delta})y_3\end{bmatrix} = 0. \tag{3.20}$$

Notice that, since δ is very small, $0.89-\frac{2\sqrt{0.1}}{\delta}$, $0.9-\frac{2\sqrt{0.1}}{\delta}$ and $0.91-\frac{2\sqrt{0.1}}{\delta}$ are negative.

From the (1,2) term in Eq. (3.20), we make a classification: $y_2 = 0$ or $y_2 \neq 0$.

1. *$y_2 = 0$: Since (1,1) and (2,2) terms are 0, we get*

$$y_1 = 0 \quad or \quad y_1 = \frac{0.89 - \frac{2\sqrt{0.1}}{\delta}}{\frac{0.2}{\gamma_\infty^2} - \frac{0.5}{\delta^2}},$$

$$y_3 = 0 \quad or \quad y_3 = \frac{0.91 - \frac{2\sqrt{0.1}}{\delta}}{\frac{0.2}{\gamma_\infty^2} - \frac{0.5}{\delta^2}}.$$

2. *$y_2 \neq 0$: From the (1,2) term we get*

$$y_1 + y_3 = \frac{0.9 - \frac{2\sqrt{0.1}}{\delta}}{\frac{0.2}{\gamma_\infty^2} - \frac{0.5}{\delta^2}}.$$

   After doing the calculation that the (1,1) term minus the (2,2) term, and substituting $y_1 + y_3$ into it, we get

$$y_1 + y_3 = 0.$$

   This contradicts the above equation.

Consequently, if the equation (3.20) has a positive definite solution Y, it must satisfy $\frac{0.2}{\gamma_\infty^2} - \frac{0.5}{\delta^2} < 0$, implying the condition that $\gamma_\infty > \sqrt{0.4}\delta$.

3.2.3 Mixed LQG and H_∞ control problem

Upon the above derivations, we find that when we consider H_∞ control, we intend to design a quantum controller K to minimize $\|\Gamma(sI - M)^{-1}H\|_\infty$, which depends on matrices M, H and Γ, but these three matrices only depend on controller matrices A_k, B_{k3} and C_k. Then we use physical realizability constraints to design other matrices B_{k1} and B_{k2} to guarantee the controller is also a quantum system, but these will affect the LQG index, which depends on M, N, Ψ, so further depends on all matrices of the controller. That is, the LQG problem and the H_∞ problem are not independent.

According to the above analysis, we state the mixed LQG and H_∞ coherent feedback control problem for linear quantum systems.

Problem 3.3. *The mixed LQG and H_∞ coherent feedback control problem is to find a quantum controller K of form (3.2) that minimizes LQG and H_∞ performance indices simultaneously, while its coefficient matrices satisfy the physical realizability constraints (3.4).*

Lemma 3.3. *(Relaxed mixed problem (Bian et al., 2015)) Given Θ_k, $\gamma_l > 0$ and $\gamma_\infty > 0$, if there exist A_k, B_{k1}, B_{k2}, B_{k3}, C_k, Q, and symmetric matrices $P_L = P^{-1}$, P_H such that physical realizability constraints (3.4) and inequality constraints (3.14) and (3.17) hold, where P is the solution of equation (3.7), then the mixed LQG and H_∞ coherent feedback control problem admits a coherent feedback controller K of the form (3.2).*

3.3 Algorithms for mixed LQG and H_∞ coherent feedback control problem

In this section, the coherent feedback controllers for mixed LQG and H_∞ problems are constructed by using two different methods, the rank constrained LMI method and genetic-algorithm-based method.

3.3.1 Rank constrained LMI method

In Lemma 3.3 for the mixed problem, clearly, constraints (3.14) and (3.17) are non-linear matrix inequalities, and physical realizability conditions (3.4) are non-convex constraints. Therefore, it is difficult to obtain the optimal solution by existing optimization algorithms. By referring to Bian et al. (2015), Nurdin et al. (2009), Scherer et al. (1997), we could translate these non-convex and non-linear constraints to linear ones.

Firstly, we redefine the original plant (3.1) to a *modified plant* as follows

$$
\begin{aligned}
dx(t) &= Ax(t)dt + B_w d\tilde{w}_{cl}(t) + B_1\beta_w(t)dt \\
&\quad + B_2\beta_u(t)dt, \\
dy'(t) &= [dB_{vk1}^T(t)\ dB_{vk2}^T(t)\ dy^T(t)]^T \\
&= Cx(t)dt + D_w d\tilde{w}_{cl}(t) + D\beta_w(t)dt, \\
dz_\infty(t) &= C_1x(t)dt + D_\infty d\tilde{w}_{cl}(t) + D_{12}\beta_u(t)dt, \\
z_l(t) &= C_z(t) + D_z\beta_u(t),
\end{aligned}
\tag{3.21}
$$

where $B_w = [B_0\ B_1\ B_2\ 0]$, $C = [0\ 0\ C_2^T]^T$, $D = [0\ 0\ D_{12}^T]^T$, $D_\infty = [0\ 0\ D_{12}\ 0]$ and $D_w = \begin{bmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ D_{20} & D_{21} & 0 & 0 \end{bmatrix}$. Correspondingly, the *modified controller* equations become the following form

$$
\begin{aligned}
d\xi(t) &= A_k\xi(t)dt + B_{wk}dy'(t), \\
\beta_u(t) &= C_k\xi(t)
\end{aligned}
\tag{3.22}
$$

with $B_{wk} = [B_{k1}\ B_{k2}\ B_{k3}]$. It can be easily verified that the closed-loop system still has the same form as (3.3).

Assumption 3.2. *For simplicity we assume* $P_H = P_L = P^{-1}$.

We proceed to introduce matrix variables Ξ, Σ, X, Y, $Q \in \mathbb{R}^{n\times n}$, where X, Y and Q are symmetric. Then define the change of controller variables as follows

$$
\begin{aligned}
\hat{A} &:= \Xi A_k\Sigma^T + \Xi B_{wk}CX + YB_2C_k\Sigma^T + YAX, \\
\hat{B} &:= \Xi B_{wk}, \\
\hat{C} &:= C_k\Sigma^T,
\end{aligned}
\tag{3.23}
$$

where $\Sigma\Xi^T = I - XY$.

By using (3.23), LQG inequality constrains (3.14) can be transformed to (3.24) below.

$$\begin{bmatrix} AX + XA^T + B_2\hat{C} + (B_2\hat{C})^T & \hat{A}^T + A & B_w \\ \hat{A} + A^T & A^TY + YA + \hat{B}C + (\hat{B}C)^T & YB_w + \hat{B}D_w \\ B_w^T & (YB_w + \hat{B}D_w)^T & -I \end{bmatrix} < 0,$$

$$\begin{bmatrix} X & I & (C_zX + D_z\hat{C})^T \\ I & Y & C_z^T \\ (C_zX + D_z\hat{C}) & C_z & Q \end{bmatrix} > 0,$$

$$\mathrm{Tr}(Q) < \gamma_l. \tag{3.24}$$

Similarly, H_∞ inequality constraints (3.17) become (3.25) below.

$$\begin{bmatrix} AX + XA^T + B_2\hat{C} + (B_2\hat{C})^T & \hat{A}^T + A & * & * \\ \hat{A} + A^T & A^TY + YA + \hat{B}C + (\hat{B}C)^T & * & * \\ B_1^T & (YB_1 + \hat{B}D)^T & -\gamma_\infty I & * \\ C_1X + D_{12}\hat{C} & C_1 & 0 & -\gamma_\infty I \end{bmatrix} < 0. \tag{3.25}$$

It is clear that the above matrix inequalities are linear, so they can be easily solved by numerical algorithms.

From (3.23), we can obtain $C_k = \hat{C}\Sigma^{-T}$, $B_{wk} = \Xi^{-1}\hat{B}$, and $A_k = \Xi^{-1}(\hat{A} - \Xi B_{wk}CX - YB_2C_k\Sigma^T - YAX)\Sigma^{-T}$. After substituting A_k, B_{wk} and C_k into (3.4) and introducing new variables $\tilde{\Xi} = \Xi J_{N_\zeta}$, $\tilde{A}_k = \Xi A_k$, $\tilde{B}_{ki} = \Xi B_{ki}$, $i = 1, 2, 3$, physical realizability constraints (3.4) become

$$\begin{aligned} &(-\hat{A}\Sigma^{-T} + (\tilde{B}_{k3}C_2 + YA)X\Sigma^{-T} + YB_2C_k)\tilde{\Xi}^T \\ &+ \tilde{\Xi}(\hat{A}\Sigma^{-T} - (\tilde{B}_{k3}C_2 + YA)X\Sigma^{-T} - YB_2C_k)^T \\ &+ \sum_{i=1}^{3} \tilde{B}_{ki}J_{n_{vki}/2}\tilde{B}_{ki}^T = 0, \end{aligned} \tag{3.26a}$$

$$\tilde{B}_{k1} - \tilde{\Xi}C_k^TJ_{n_{vk1}/2} = 0. \tag{3.26b}$$

We get the following result for the mixed LQG and H_∞ coherent feedback control problem.

Lemma 3.4. *Given Θ_k, $\gamma_l > 0$ and $\gamma_\infty > 0$, if there exist matrices $\hat{A}$, $\tilde{B}_{k1}$, $\tilde{B}_{k2}$, $\tilde{B}_{k3}$, $\hat{C}$, X, Y, $\tilde{\Xi}$, Σ, Ξ, C_k such that the LMIs (3.24), (3.25) and equality constraints (3.26) hold, then the mixed LQG and H_∞ coherent feedback control problem admits a coherent feedback controller K of the form (3.2).*

Algorithm 3.1. *(Rank constrained LMI method (Bian et al., 2015))*

Firstly, introduce 13 basic matrix variables: $M_1 = \hat{A}$, $M_2 = \tilde{B}_{k1}$, $M_3 = \tilde{B}_{k2}$, $M_4 = \tilde{B}_{k3}$, $M_5 = \hat{C}$, $M_6 = X$, $M_7 = Y$, $M_8 = \tilde{\Xi}$, $M_9 = \Sigma$, $M_{10} = \Xi$, $M_{11} = C_k$, $M_{12} = \check{A} = \hat{A}\Sigma^{-T}$, $M_{13} = \check{X} = X\Sigma^{-T}$. And define 18 matrix lifting variables: $W_i = \tilde{B}_{ki}J_{N_{vki}}$ $(i = 1, 2, 3)$, $W_4 = YB_2$, $W_5 = \tilde{B}_{k3}C_2 + YA$, $W_6 = \tilde{\Xi}C_k^T$, $W_7 = \tilde{\Xi}\check{X}^T$, $W_8 = \check{A}\tilde{\Xi}^T$, $W_9 = YX$, $W_{10} = W_4W_6^T$, $W_{11} = W_5W_7^T$, $W_{12} = W_1\tilde{B}_{k1}^T$, $W_{13} = W_2\tilde{B}_{k2}^T$, $W_{14} = W_3\tilde{B}_{k3}^T$, $W_{15} = \Xi\Sigma^T = I - YX$, $W_{16} = \check{A}\Sigma^T = \hat{A}$, $W_{17} = \check{X}\Sigma^T = X$, $W_{18} = C_k\Sigma^T = \hat{C}$.

By defining

$$
\begin{aligned}
V &= [I \ Z_{m_1,1}^T \ \cdots \ Z_{m_{13},1}^T \ Z_{w_1,1}^T \ \cdots \ Z_{w_{18},1}^T]^T \\
&= [I \ M_1^T \ \cdots \ M_{13}^T \ W_1^T \ \cdots \ W_{18}^T]^T,
\end{aligned}
\tag{3.27}
$$

we could let Z be a $32n \times 32n$ symmetric matrix with $Z = VV^T$. It is clear that $Z_{m_i,w_i} = Z_{m_i,1}(Z_{w_i,1})^T$.

Meanwhile, because of relations between these 31 variables, we require the matrix

Z to satisfy the following constraints

$$\begin{aligned}
&& Z \geq 0, && \\
Z_{0,0} - I_{n\times n} &= 0, & Z_{w_7,1} - Z_{m_8,m_{13}} &= 0,\\
Z_{1,x_6} - Z_{m_6,1} &= 0, & Z_{w_8,1} - Z_{m_{12},m_8} &= 0\\
Z_{1,x_7} - Z_{m_7,1} &= 0, & Z_{w_9,1} - Z_{m_7,m_6} &= 0,\\
Z_{w_1,1} - Z_{m_2,1}J_{n_{vk1}/2} &= 0, & Z_{w_{10},1} - Z_{w_4,w_6} &= 0,\\
Z_{w_2,1} - Z_{m_3,1}J_{n_{vk2}/2} &= 0, & Z_{w_{11},1} - Z_{w_5,w_7} &= 0,\\
Z_{w_3,1} - Z_{m_4,1}J_{n_{vk3}/2} &= 0, & Z_{w_{12},1} - Z_{w_1,m_2} &= 0,\\
Z_{w_4,1} - Z_{m_7,1}B_2 &= 0, & Z_{w_{13},1} - Z_{w_2,m_3} &= 0,\\
Z_{w_5,1} - Z_{m_4,1}C_2 - Z_{m_7,1}A &= 0, & Z_{w_{14},1} - Z_{w_3,m_4} &= 0,\\
Z_{w_6,1} - Z_{m_8,m_{11}} &= 0, & Z_{w_{15},1} - Z_{m_{10},m_9} &= 0,\\
Z_{w_{16},1} - Z_{m_{12},m_9} &= 0, & Z_{w_{17},1} - Z_{m_{13},m_9} &= 0,\\
Z_{w_{18},1} - Z_{m_{11},m_9} &= 0, & Z_{w_{15},1} - I + Z_{w_9,1} &= 0,\\
Z_{m_1,1} - Z_{w_{16},1} &= 0, & Z_{m_6,1} - Z_{w_{17},1} &= 0,\\
Z_{m_8,1} - Z_{m_{10},1}J_{n_\xi/2} &= 0, & Z_{m_5,1} - Z_{w_{18},1} &= 0,
\end{aligned} \tag{3.28}$$

and moreover, Z satisfies a rank constraint, i.e. $rank(Z) \leqslant n$.

Then, we use $Z_{m_1,1}$, $[Z_{m_2,1}\ Z_{m_3,1}\ Z_{m_4,1}]$, $Z_{m_5,1}$, $Z_{m_6,1}$, $Z_{m_7,1}$ *to replace* $\hat{A}$, $\hat{B}$, $\hat{C}$, X, Y *in LMI constraints (3.24) and (3.25), and convert physical realizability conditions (3.26) to*

$$\begin{aligned}
&- Z_{w_8,1} + Z_{w_8,1}^T + Z_{w_{11},1} - Z_{w_{11},1}^T + Z_{w_{10},1} - Z_{w_{10},1}^T\\
&+ Z_{w_{12},1} + Z_{w_{13},1} + Z_{w_{14},1} = 0,
\end{aligned} \tag{3.29a}$$

$$Z_{m_2,1} - Z_{w_6,1}J_{N_{vk1}} = 0. \tag{3.29b}$$

We have transformed the mixed problem to a rank constrained problem, which could be solved by using Toolbox: Yalmip (Lofberg, 2004), SeDuMi and LMIRank

(Orsi et al., 2006).

Remark 3.4. *The above LMI-based approach solves a sub-optimal control problem for the mixed LQG and H_∞ coherent feedback control. Once a feasible solution is found by implementing Algorithm 3.1, we then know that the LQG index is bounded by γ_l from above, and simultaneously, the H_∞ index is bounded by γ_∞ from above.*

3.3.2 Genetic algorithm

Genetic algorithm is a search heuristic that mimics the process of natural selection in the field of artificial intelligence. This heuristic (sometimes called metaheuristic) is routinely used to generate useful solutions to optimization and search problems. Genetic algorithm belongs to the larger class of evolutionary algorithms, which get solutions using techniques inspired by natural evolution, such as inheritance, mutation, selection, and crossover, etc. Genetic algorithm is a useful method for controller design, see e.g., Campos-Delgado and Zhou (2003); Neumann and Araujo (2004); Pereira and Araujo (2004). In the field of quantum control, genetic algorithm methods are applied to design quantum coherent feedback controllers, see e.g., Zhang et al. (2012) and Harno and Petersen (2015).

We briefly introduce the procedures of genetic algorithm as follows.

Algorithm 3.2. *(Genetic algorithm)*

Step 1 *: Initialization for the population (the first generation), by using random functions, and binary strings denote controller parameters we want to design;*

Step 2 *: Transform binary strings to decimal numbers, and calculate the results of these parameters;*

Step 3 *: After obtaining coefficient matrices of the controller, we restrict one of the LQG or H_∞ indices in an interval, then minimize the other index (the* *fitness function in our problem). Since the lower bounds of these two indices can be calculated a priori, see Subsections 3.2.1 and 3.2.2, the above-mentioned interval can always be found. By the above procedure we get the best individual and corresponding performance index in this generation;*

Step 4 *: Perform the selection operation, for yielding new individuals;*

Step 5 *: Perform the crossover operation, for yielding new individuals;*

Step 6 *: Perform the mutation operation, for yielding new individuals;*

Step 7 *: Back to Step 2, recalculate all parameters and corresponding best fitness function result for new generation;*

Step 8 *: At the end of iterations, compare all best results of every generation, and get the optimal solution.*

Remark 3.5. *Algorithm 3.2 does not minimize both LQG and H_∞ performance indices simultaneously. More specifically, as can be seen in Step 3, one of the indices is first fixed, then the other one is minimized. This procedure is repeated as can be seen from Step 7. Therefore, Algorithm 3.2 is an iterative minimization algorithm.*

In our problem, because the coherent feedback controller K to be designed is a quantum system, it can be described by the (S, L, H) language introduced in Subsection 2.1.1. With this, physical realizability conditions are naturally satisfied. As a result, we apply the genetic algorithm to find K by simultaneously minimizing the LQG and H_∞ performance indices directly.

3.4 Numerical simulations and comparisons

In this section, we give two examples to illustrate the methods proposed in the previous section.

3.4.1 Numerical simulations

Example 1: This example is taken from Section VII of James et al. (2008). The plant is an optical cavity resonantly coupled to three optical channels.

The dynamics of this optical cavity system can be described by following equations

$$\begin{aligned} dx\left(t\right) &= -\frac{\gamma}{2} I_2 x\left(t\right) dt - \sqrt{\kappa_1} I_2 dv(t) - \sqrt{\kappa_2} I_2 dw\left(t\right) - \sqrt{\kappa_3} I_2 du(t), \\ dy(t) &= \sqrt{\kappa_2} I_2 x(t) dt + I_2 dw(t), \\ dz_\infty(t) &= \sqrt{\kappa_3} I_2 x(t) dt + I_2 du(t), \\ z_l(t) &= I_2 x(t) + I_2 \beta_u(t) \end{aligned} \tag{3.30}$$

with parameters $\gamma = \kappa_1 + \kappa_2 + \kappa_3$, $\kappa_1 = 2.6$, $\kappa_2 = \kappa_3 = 0.2$. In this example, $v(t)$ is quantum white noise, while $w(t)$ is a sum of quantum white noise and L_2 disturbance (See Assumption 2.1 for details). Therefore, there are two types of noises in this system. LQG control is used to suppress the influence of quantum white noise, while H_∞ control is used to attenuate the L_2 disturbance.

Example 2: In this example, we choose a DPA as our plant. For more details about DPA, one may refer to Leonhardt (2003). The QSDEs of DPA are

$$\begin{aligned} dx\left(t\right) &= -\frac{1}{2}\begin{bmatrix} \gamma-\epsilon & 0 \\ 0 & \gamma+\epsilon \end{bmatrix} x\left(t\right) dt - \sqrt{\kappa_3}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} dv(t) \\ &\quad - \sqrt{\kappa_1}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} dw\left(t\right) - \sqrt{\kappa_2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} du(t), \\ dy(t) &= \sqrt{\kappa_3}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x(t) dt + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} dv(t), \\ dz_\infty(t) &= \sqrt{\kappa_2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x(t) dt + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} du(t), \\ z_l(t) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \beta_u(t) \end{aligned} \tag{3.31}$$

with parameters $\gamma = \kappa_1 + \kappa_2 + \kappa_3$, $\kappa_1 = \kappa_2 = 0.2$, $\kappa_3 = 0.5$, $\epsilon = 0.01$.

According to Theorem 3.1, it is easy to find that lower bounds of the LQG index for both two examples are 1.

Firstly, we focus on the LQG performance index, and design two different types of controllers to minimize it by using the genetic algorithm. The results are shown in Table 3.1. For each case, we list two values obtained.

Table 3.1: Optimization results only for LQG index.

plant	controller	J_∞ (LQG index)
Cavity	Passive Controller	1.0005
		1.0000
	Non-passive Controller	1.0006
		1.0003
DPA	Passive Controller	1.0003
		1.0000
	Non-passive Controller	1.0002
		1.0000

Remark 3.6. *J_∞ in Table 3.1 is the LQG performance index defined in Eq. (3.6). In Theorem 3.1, a lower bound for J_∞ is proposed. This lower bound is obtained when both the plant and the controller are in the ground state, as stated in Theorem 3.1. In Table 3.1 above there are two systems, namely the optical cavity and DPA. For both of them, the lower bound in Theorem 3.1 satisfies $d_2 = 0$ and $c_1 = c_3 = 1$. Therefore, $J_\infty \geqslant 1$. From Table 3.1 we can see that our genetic algorithm finds controllers that yield the LQG performance which is almost optimal. And in this case, as guaranteed by Theorem 3.1, both the plant and the controller are almost in the ground state.*

Secondly, similarly to the LQG case, we focus on the H_∞ index and design controllers to minimize the objective, getting the following Table 3.2. For each case, we list two values obtained.

Remark 3.7. *Table 3.2 is for H_∞ performance index. For the cavity case, actually*

Table 3.2: Optimization results only for H_∞ index.

plant	controller	$\|G_{\beta_w \to z_\infty}\|_\infty$ (H_∞ index)
Cavity	Passive Controller	0.0134
		0.0050
	Non-passive Controller	0.0196
		0.0075
DPA	Passive Controller	0.0070
		0.0044
	Non-passive Controller	0.0057
		0.0045

it can be proved analytically that the H_∞ performance index can be made arbitrarily close to zero. On the other hand, by Remark 3.3, H_∞ index has a lower bound $\sqrt{0.4}\delta$. However, for the DPA studied in this example, $\delta = 0$, that is, the lower bound for H_∞ index is also zero. The simulation results in Table 3.2 confirmed this observation.

From above results we could see, if we only consider one performance index, either LQG index or H_∞ index, there are no significant differences between passive controllers and non-passive controllers, both of which can lead to a performance index close to the minimum.

In what follows, we apply two methods to the mixed control problem, to see whether we could make two performance indices close to the minima simultaneously. The results are shown in Table 3.3 and Table 3.4, respectively.

3.4.2 Comparisons of results

After getting numerical results shown in Tables 3.3 and 3.4, and doing comparison between the two proposed methods, we state the advantages of Algorithm 3.2:

1. Instead of single LQG or H_∞ optimal control for linear quantum systems in other literatures, the two proposed algorithms deal with the *mixed* LQG and H_∞ problem.

Table 3.3: Optimization results by rank constrained LMI method.

plant	constraints		results	
	γ_∞	γ_l	$\|G_{\beta_w\to z_\infty}\|_\infty$ (H_∞ index)	J_∞ (LQG index)
Cavity ($\gamma = \kappa_1+\kappa_2+\kappa_3$, $\kappa_1 = 2.6$, $\kappa_2 = \kappa_3 = 0.2$.)	0.1	2.5	0.039900	1.014555
	0.1	N/A	0.058805	1.039487
	N/A	2.5	0.134558	1.000577
	N/A	3	0.423970	1.379587
	2.8	3	0.444119	1.270835
DPA ($\gamma = \kappa_1+\kappa_2+\kappa_3$, $\kappa_1 = \kappa_2 = 0.2$, $\kappa_3 = 0.5, \epsilon = 0.01$.)	0.3	2.5	0.172385	1.175277
	0.5	3	0.447274	1.080976
	N/A	3	0.468007	1.149859
	1	5	0.647468	1.374547

Table 3.4: Optimization results by genetic algorithm.

plant	controller	results	
		$\|G_{\beta_w\to z_\infty}\|_\infty$ (H_∞ index)	J_∞ (LQG index)
Cavity ($\gamma = \kappa_1+\kappa_2+\kappa_3$, $\kappa_1 = 2.6$, $\kappa_2 = \kappa_3 = 0.2$.)	Passive Controller	0.003574	1.008917
		0.078977	1.000619
		0.146066	1.000009
	Non-passive Controller	0.071270	1.009303
		0.089383	1.002099
		0.123066	1.000283
DPA ($\gamma = \kappa_1+\kappa_2+\kappa_3$, $\kappa_1 = \kappa_2 = 0.2$, $\kappa_3 = 0.5, \epsilon = 0.01$.)	Passive Controller	0.428312	1.004787
		0.449534	1.000124
	Non-passive Controller	0.364979	1.009691
		0.387734	1.007164
	Passive Controller + Direct coupling	0.039183	1.000079
		0.042960	1.000002

2. Algorithm 3.1 relaxes two performance indices by introducing γ_l and γ_∞. When they are small, it will be quite difficult to solve the problem by Algorithm 3.1. But Algorithm 3.2 is able to minimize the two performance indices directly.

3. The solution of the differential evolution algorithm in Harno and Petersen (2015) involves a complex algebraic Riccati equation. On the contrary, all parameters of our Algorithm 3.2 are real. It might be easier to be solved by current computer softwares such as Matlab.

4. The numerical results showed that there seems to be a trend between these two indices, that sometimes one increases, while another decreases.

5. For a passive system (e.g. cavity), both the passive controller and the non-passive controller could let LQG and H_∞ indices go to the minima simultaneously, see Table 3.4.

6. For a non-passive system (e.g. DPA), neither the passive controller nor the non-passive controller can let these two indices go to the minima simultaneously. But when a direct coupling is added between the plant and the controller, we could use genetic algorithm to design a passive controller to minimize these two indices simultaneously, which is not achieved by the rank constrained LMI method, see Table 3.4.

7. Actually, the rank constrained LMI method could not be used to design specific passive controllers, or non-passive controllers, while this can be easily achieved using genetic algorithm, by setting all "plus" terms equal to 0.

8. Finally, from numerical simulations, the genetic algorithm often provides better results than the rank constrained LMI method. Both LQG and H_∞ indices are quite close to the minima, i.e. 1 for LQG index and 0 for H_∞ index, respectively. This means that by using these two methods to design a quantum controller K for the quantum plant P, we can succeed to minimize the influences on the closed-loop system, when there contain both external signals and quantum white noises. Furthermore, the genetic algorithm can do better than the rank constrained LMI method in minimizing these influences.

Chapter 4

Adaptive interaction diversity stabilizes the evolution of cooperation

The purpose of this chapter is to investigate the evolutionary game dynamics and stability in finite populations. Firstly, the standard game dynamics in finite populations is introduced, where each player has a choice between two strategies, cooperation (C) and defection (D). Secondly, we propose a model to investigate how tag-induced interaction diversity affects the evolution of cooperation. Finally, several examples are used for demonstration. Result that cooperation can get stabilized is confirmed, as has been reported in previous studies, as well as our study gives the root cause for the coexistence of cooperation and interaction diversity. Furthermore, the stationary distribution illustrates the effectiveness of the proposed method.

4.1 Standard evolutionary model in finite populations

In a one-shot two-strategy Prisoner's Dilemma game, two players simultaneously choose either to cooperate or to defect. As introduced in Subsection 2.2.2, it can be easily checked to defect is the best choice for the rational player who always attempt to pursue the possible maximal payoff. In the well-mixed populations of infinite size, different scenarios may arise, which provides us opportunity to choose strategies that allow cooperators to defeat and invade defectors successfully. We first introduce the standard game dynamics in finite populations (Nowak et al., 2004).

Consider a well-mixed population of finite size ($= N$). Two strategies are feasible to individuals, Cooperation (C) and Defection (D). The corresponding payoff matrix for the row player is

$$\begin{array}{cc} & \begin{array}{cc} C & D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{array}. \tag{4.1}$$

The evolutionary dynamics is formulated by a Moran process with frequency-dependent fitness. The payoffs of strategies C and D are, respectively, given by

$$\begin{aligned} p_c &= \frac{i-1}{N-1}a + \frac{N-i}{N-1}b, \\ p_d &= \frac{i}{N-1}c + \frac{N-i-1}{N-1}d, \end{aligned} \tag{4.2}$$

where i denotes the number of individuals using strategy C. The term $(i-1)/(N-1)$ is the probability that an individual with strategy C meets another player also using strategy C, and a is its payoff in this interaction. Following similar logic, other terms can easily obtained.

The parameter $w \in [0, 1]$ measures to which extent on the contribution of the payoff to fitness, we get the fitness of a cooperator and a defector as follows

$$\begin{aligned} f_i &= 1 - w + wp_c, \\ g_i &= 1 - w + wp_d. \end{aligned} \tag{4.3}$$

At each time step, an individual is chosen for reproduction proportional to its fitness. An offspring is produced and replaces another randomly chosen individual, so the population number N is constant over the evolutionary process. With the probability it randomly adopts strategy C or D, in which case we say a mutation occurs. For small mutation rate, at every step, the number of cooperators increase by one, fall by one, or stay the same. Other possibility happens with negligible probabilities. The probabilities of adding and reducing a cooperator in the population are $if_i/[if_i + (N-i)g_i]$ and $[(N-i)g_i]/[if_i + (N-i)g_i]$, respectively. Based on the Kolmogrov forward equation, we can get

$$
\begin{aligned}
P_{i,i+1} &= \frac{if_i}{if_i + (N-i)g_i} \, \frac{N-i}{N}, \\
P_{i,i-1} &= \frac{(N-i)g_i}{if_i + (N-i)g_i} \, \frac{i}{N}, \\
P_{i,i} &= 1 - P_{i,i+1} - P_{i,i-1}.
\end{aligned}
\tag{4.4}
$$

The process has two absorbing states, $i = 0$ and $i = N$. If the population has reached either of those states, then it will stay there forever. We denote x_i as the probability of ending up in state $i = N$ when starting in state i, and get

$$
x_i = P_{i,i+1}x_{i+1} + P_{i,i-1}x_{i-1} + P_{i,i}x_i, \tag{4.5}
$$

with boundary conditions $x_0 = 0$ and $x_N = 1$.

By denoting ρ_C as x_1, i.e. the probability that a single cooperation mutant invades and takes over the population of otherwise defectors, we obtain the the fixation probability as

$$
\rho_C = \frac{1}{1 + \sum_{k=1}^{N-1} \prod_{i=1}^{k} \frac{g_i}{f_i}}. \tag{4.6}
$$

4.2 Synthesis of finite-population evolutionary dynamics with tags

In this section, we formulate a kind of new evolutionary model, in which tags are added to individuals to express their characteristics. At each time step, every individual can choose the shown number of its tags, which will affect its payoff directly. Then we derive the evolutionary dynamics of the new model. In terms of the transition matrix of the Markov chain, stationary distribution can be calculated.

4.2.1 Evolutionary dynamics analysis with tags

In the evolutionary game, the population is well-mixed, and individuals interact with everyone else. The interactions are characterized by the simplified Prisoner's Dilemma game. We consider variation in the capacity of expressing different numbers of tags in the evolutionary game. Each individual possesses different numbers of tags. When a pair of individuals express the same number of tags, there will be an interaction between them. When they express different numbers of tags, there will be no interaction between them. When an individual possesses more tags, it will obtain higher probability to have common tags, which will make probabilistically higher payoff. Nevertheless, the players should also pay for the expressed tags. The cost of expressed tags increases with its numbers arising. Interested readers should also refer to Riolo et al. (2001).

The interaction outcome is dependent on their strategic behaviors. When both are cooperators, they each get the benefit $b - c$. When both are defectors, they get zero payoff each. When a cooperator meets a defector, the former gets the payoff $-c$, while the later reaps the payoff b. Similarly with payoff matrix (4.1), the interaction outcome matrix is shown as follows

$$\begin{array}{cc} & \begin{array}{cc} C & D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{pmatrix} b-c & -c \\ b & 0 \end{pmatrix} \end{array}, \tag{4.7}$$

which is called donation game, where both b and c are positive real number, satisfying the condition $b > c > 0$ (Nowak et al., 2004).

Firstly, we consider the case that a single individual with tags using strategy C invades the population of D-strategy individuals with tags.

We denote M as the total number of tags. $x > 0$, $y > 0$ are the numbers of expressed tags for C-strategy players and D-strategy players, respectively. It is clear that x, y and M are positive integers, and $x, y \in [1, M]$. k denotes the numbers of common tags between C-strategy and D-strategy players, so $max\{0, x+y-M\} \leq k \leq min\{x, y, M\}$.

When there are k common tags, we derive the payoff equations as follows

$$\begin{aligned} p_c &= r(\frac{x}{M})i(b-c) + r(\frac{k}{M})(N-i)(-c) - \delta x, \\ p_d &= r(\frac{k}{M})ib + r(\frac{y}{M})(N-i)0 - \delta y, \end{aligned} \tag{4.8}$$

where function $r(*)$ represents the interaction rate between two individuals, depending on the proportion of common-tags' numbers in the total tags' numbers, and δ is a penalty parameter for expressing the cost of each shown tag. N is the total population number, and i denotes the number of individuals using strategy C, which is the same as in Eq. (4.2).

We use exponential functions to specify the contribution of the game to fitness, shown as

$$f_i = e^{\beta p_c}, \quad g_i = e^{\beta p_d},$$

where β is the intensity of selection, which measures how much payoff contributes to fitness.

Based on Eq. (4.6), the fixation probability that a single C-strategy individual invades an N-number population of D-strategy individuals (the number of common tags between these two types of players is k), is given by

$$\rho^k_{C\to D} = \frac{1}{1 + \sum\limits_{l=1}^{N-1} \prod\limits_{i=1}^{l} \frac{g_i}{f_i}}. \tag{4.9}$$

As has been introduced above, $k \in [max\{0, x+y-M\}, min\{x, y, M\}]$, and the probability of k common tags between C-strategy players (with expressed tags x) and D-strategy players (with expressed tags y) is

$$p^k = \frac{\binom{y}{k}\binom{M-y}{x-k}}{\binom{M}{x}}. \tag{4.10}$$

Upon Eqs. (4.9) and (4.10), it can be easily derived that the transition rate of N-number D-strategy individuals by a single C-strategy individual is

$$\rho_{C\to D} = \sum_{k=max\{0,x+y-M\}}^{min\{x,y,M\}} p^k \rho^k_{C\to D}. \tag{4.11}$$

Similarly, it is easily to derive the transition rates of $\rho_{C\to C}$, $\rho_{D\to D}$ and $\rho_{D\to C}$. In summary, we could use a set of uniform symbols to express all these four issues, as shown in the following theorem.

Theorem 4.1. *For a two-strategy finite-population evolutionary game with tags, the transition rate of N-number Y-strategy individuals (namely residents) by a single X-strategy individual (namely mutant) is given by*

$$\rho_{X\to Y} = \sum_{k=max\{0,x+y-M\}}^{min\{x,y,M\}} p^k \rho^k_{X\to Y}, \tag{4.12}$$

where x, y are corresponding numbers of expressed tags for X-strategy player and Y-strategy player, respectively. M is the total number of tags, k is the number of common tags between mutant (X) and resident (Y).

The functions p^k and $\rho^k_{X\to Y}$ use the same form as in Eqs. (4.10) and (4.9), by replacing Eq. (4.8) with

$$\begin{aligned} p_X &= r(\frac{x}{M})i(b-c)s_x + r(\frac{k}{M})(N-i)(bs_y - cs_x) - \delta x, \\ p_Y &= r(\frac{k}{M})i(bs_x - cs_y) + r(\frac{y}{M})(N-i)(b-c)s_y - \delta y, \end{aligned} \tag{4.13}$$

where

$$s_x, s_y = \begin{cases} 1 & \textit{Player } X, Y \textit{ use Strategy } C, \\ 0 & \textit{Player } X, Y \textit{ use Strategy } D. \end{cases} \tag{4.14}$$

Proof. By referring to strategy symbols s_x and s_y in the Eq. (4.14), there are four cases, corresponding to $s_x = 1$ *or* 0 and $s_y = 1$ *or* 0. It can be easily verified that when $s_x = 1$ and $s_y = 0$, i.e. a mutant uses strategy C and residents use strategy D, Eq. (4.12) is equal to Eq. (4.11), as well as the other three transition rates. The proof is completed. □

4.2.2 Synthesis of stationary distribution

In this finite-population evolutionary model with tags, the transition matrix T has the following form,

$$\begin{array}{c c} & \begin{array}{ccccc} C_1 & D_1 & \cdots & C_M & D_M \end{array} \\ \begin{array}{c} C_1 \\ D_1 \\ \vdots \\ C_M \\ D_M \end{array} & \begin{pmatrix} \rho_{C_1\to C_1} & \rho_{C_1\to D_1} & \cdots & \rho_{C_1\to C_M} & \rho_{C_1\to D_M} \\ \rho_{D_1\to C_1} & \rho_{D_1\to D_1} & \cdots & \rho_{D_1\to C_M} & \rho_{D_1\to D_M} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \rho_{C_M\to C_1} & \rho_{C_M\to D_1} & \cdots & \rho_{C_M\to C_M} & \rho_{C_M\to D_M} \\ \rho_{D_M\to C_1} & \rho_{D_M\to D_1} & \cdots & \rho_{D_M\to C_M} & \rho_{D_M\to D_M} \end{pmatrix} \end{array}. \tag{4.15}$$

Notice that this is a $2M \times 2M$ matrix, and we use the traditional symbols (i.e. row and column numbers) to express it, as below

$$
\begin{array}{c c}
 & \begin{array}{ccccc} C_1 & D_1 & \cdots & C_M & D_M \end{array} \\
\begin{array}{c} C_1 \\ D_1 \\ \vdots \\ C_M \\ D_M \end{array} &
\begin{pmatrix}
\rho_{1,1} & \rho_{1,2} & \cdots & \rho_{1,2M-1} & \rho_{1,2M} \\
\rho_{2,1} & \rho_{2,2} & \cdots & \rho_{2,2M-1} & \rho_{2,2M} \\
\vdots & \vdots & \cdots & \vdots & \vdots \\
\rho_{2M-1,1} & \rho_{2M-1,2} & \cdots & \rho_{2M-1,2M-1} & \rho_{2M-1,2M} \\
\rho_{2M,1} & \rho_{2M,2} & \cdots & \rho_{2M,2M-1} & \rho_{2M,2M}
\end{pmatrix}_{2M \times 2M}
\end{array}
. \tag{4.16}
$$

By Theorem 4.1, we could calculate the non-diagonal transition rates in the transition matrix (4.16). Meanwhile, the individuals with the same number of tags, which use the same strategy, will not invade into each others. Therefore, by the method in Hauert et al. (2007) and Pacheco et al. (2015), the diagonal numbers are determined as

$$\rho_{j,j} = 1 - \sum_{i=1, i \neq j}^{2M} \rho_{i,j}, \quad j \in \{1, 2, \cdots, 2M\}.$$

As a result, the transition matrix T has the form

$$
\begin{array}{c c}
 & \begin{array}{ccccc} C_1 & D_1 & \cdots & C_M & D_M \end{array} \\
\begin{array}{c} C_1 \\ D_1 \\ \vdots \\ C_M \\ D_M \end{array} &
\begin{pmatrix}
1 - \sum_{i=2}^{2M} \rho_{i,1} & \rho_{1,2} & \cdots & \rho_{1,2M-1} & \rho_{1,2M} \\
\rho_{2,1} & 1 - \sum_{i=1, i \neq 2}^{2M} \rho_{i,2} & \cdots & \rho_{2,2M-1} & \rho_{2,2M} \\
\vdots & \vdots & \cdots & \vdots & \vdots \\
\rho_{2M-1,1} & \rho_{2M-1,2} & \cdots & 1 - \sum_{i=1, i \neq 2M-1}^{2M} \rho_{i,2M-1} & \rho_{2M-1,2M} \\
\rho_{2M,1} & \rho_{2M,2} & \cdots & \rho_{2M,2M-1} & 1 - \sum_{i=1}^{2M-1} \rho_{i,2M}
\end{pmatrix}
\end{array}
. \tag{4.17}
$$

It is clear that in the transition matrix T, the sum of each column is equal to 1, which means (4.17) is a transition matrix of a Markov chain, and the normalized eigenvector associated with the eigenvalue 1 of T provides the stationary distribution. For more details, please refer to Fudenberg and Imhof (2006) and Imhof et al. (2005).

4.3 Numerical simulations and analysis of results

In this section, a numerical model with population number $N = 20$ is used for the numerical simulation. Meanwhile, we set $\beta = 0.1$, $b = 1$ and $c = 0.3$. Then the payoff matrix (4.7) is

$$\begin{array}{cc} & \begin{array}{cc} C & D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{pmatrix} 0.7 & -0.3 \\ 1 & 0 \end{pmatrix} \end{array}.$$

Furthermore, since the transition matrix T in (4.17) is $2M \times 2M$, for the purpose of simulation, we set $M = 10$.

In a one-shot two-strategy game, defectors have higher fitness, so defectors always dominate over cooperators in any mixed populations. In the evolutionary model with tags, we consider different examples, by changing the interaction rate $r(x)$ and the penalty parameter δ, so as to obtain some cases in which C-strategy individuals get higher stationary probability than D-strategy players, i.e. cooperators have higher probability than defectors.

It is clear that the interaction rate $r(x)$ should obey conditions

$$r(x) \in [0, 1],\ \ x \in [0, 1];$$

$$r(0) = 0;\ \ r(1) = 1.$$

We choose three types of interaction rate $r(x)$, shown in the Eq. (4.18) and

Figure 4.1.

$$
\begin{aligned}
r_1(x) &= x,\ \ x \in [0,1] \\
r_2(x) &= \begin{cases} 2x^2, & x \in [0, \frac{1}{2}] \\ 1 - 2(1-x)^2, & x \in [\frac{1}{2}, 1] \end{cases} \\
r_3(x) &= 1 - \sqrt{1-x^2}.\ x \in [0,1]
\end{aligned}
\tag{4.18}
$$

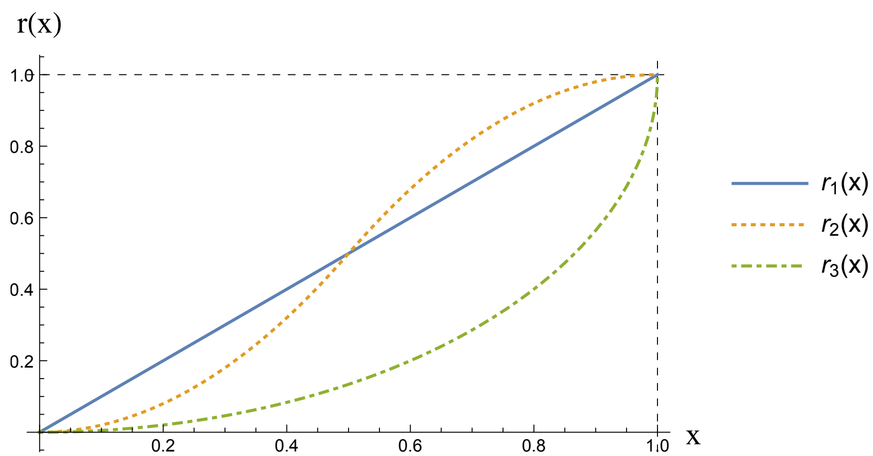
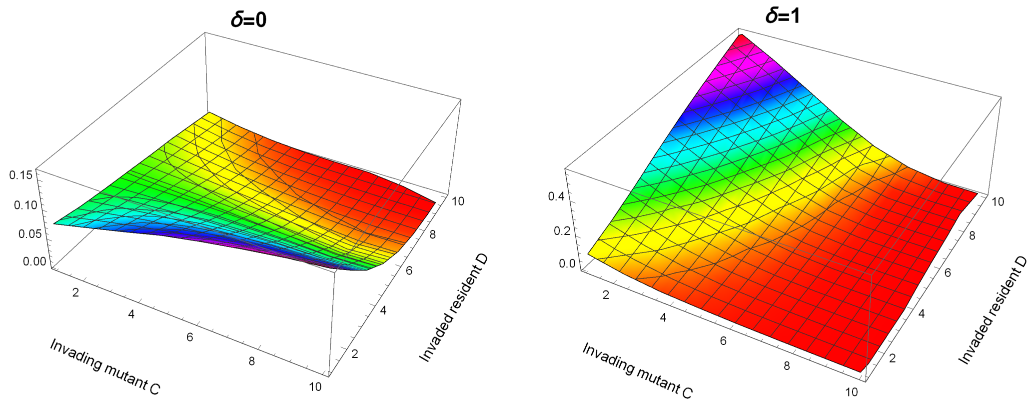


Figure 4.1: Three types of interaction rate $r(x)$.

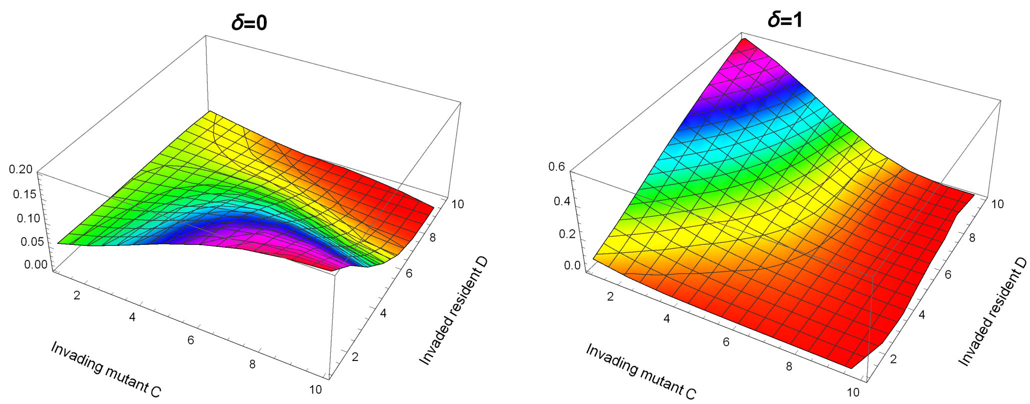
4.3.1 Evolutionary dynamics

According to Eq. (4.17), we obtain the transition matrix T once the interaction rate $r(x)$ and the penalty parameter δ are fixed. Since we mainly focus on the probability that C-strategy individuals invade the D-strategy population, we select the odd rows and even columns of the transition matrix T to compose a new matrix T_{CD}, as below

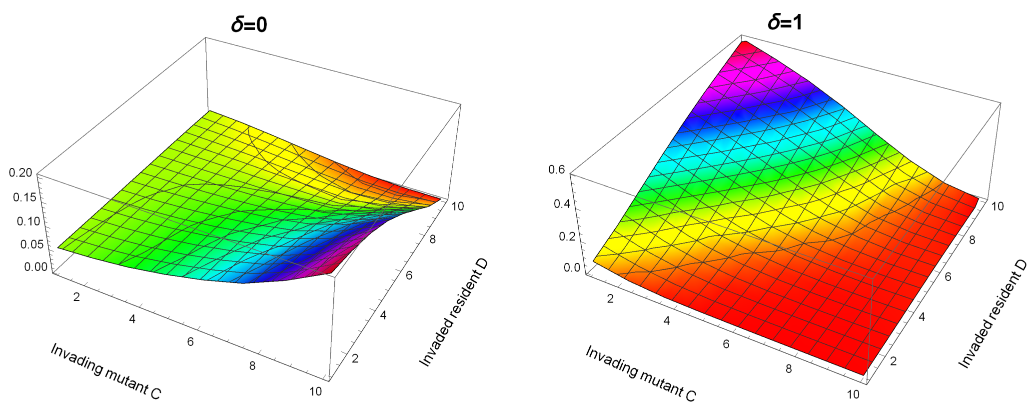
$$
T_{CD} = \begin{array}{c} \\ C_1 \\ C_2 \\ \vdots \\ C_M \end{array}
\begin{array}{c}
\begin{array}{cccc} D_1 & D_2 & \cdots & D_M \end{array} \\
\begin{pmatrix}
\rho_{C_1 \to D_1} & \rho_{C_1 \to D_2} & \cdots & \rho_{C_1 \to D_M} \\
\rho_{C_2 \to D_1} & \rho_{C_2 \to D_2} & \cdots & \rho_{C_2 \to D_M} \\
\vdots & \vdots & \cdots & \vdots \\
\rho_{C_M \to D_1} & \rho_{C_M \to D_2} & \cdots & \rho_{C_M \to D_M}
\end{pmatrix}
\end{array}.
\tag{4.19}
$$



(a) The invasion plots by using $r_1(x)$



(b) The invasion plots by using $r_2(x)$



(c) The invasion plots by using $r_3(x)$

Figure 4.2: The evolutionary dynamics for no penalty ($\delta = 0$) and large penalty ($\delta = 1$).

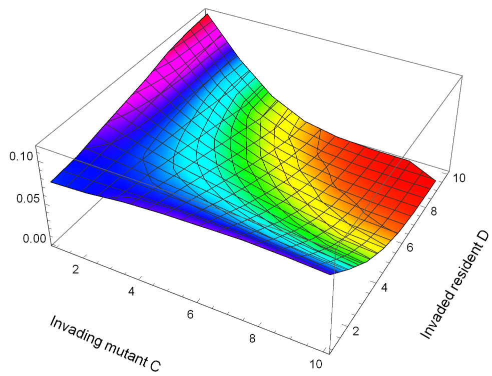
Figure 4.2 shows the transition rate with respect to three different interaction rates and two different penalty parameters, where the capital letter, C or D, along the x-axis, denotes the invading mutant's behavioral strategy, while the one along the y-axis denotes the invaded residents' behavioral strategy. The coordinate values in x-axis and y-axis denote the numbers of expressed tags for corresponding individuals, and the coordinate value in z-axis denotes the transition rate. Figure 4.2 indicates that, when there is no penalty (i.e. $\delta = 0$), or the penalty is very large (e.g. $\delta = 1$), the trends of transition rates subject to different interaction rates $r(x)$ are quite similar.

The dynamical scenarios are easily understandable. When there is no penalty ($\delta = 0$), both the fitness of invading mutant with strategy C and invaded resident with strategy D are determined by the interaction rate $r(x)$. It is well known that defectors always have higher fitness and dominate over cooperators without tags. Therefore, the cooperators express more tags, they will get higher interaction rates, meanwhile, the less tags defectors express, the less probabilities that cooperators play games with defectors, causing the transition rate to increase, as shown in left three figures in Figure 4.2.

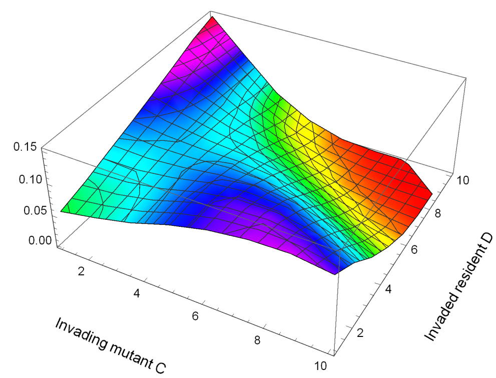
When the penalty parameter is very large ($\delta = 1$), the penalty cost is the significant factor of payoffs, while the interaction rate is negligible. Thus the highest value occurs in the point that defectors express most tags and cooperators express least tags, in which defectors have the highest interaction rate with other defectors and the lowest interaction rate with cooperators. The phenomena are shown in right three figures in Figure 4.2.

Moreover, we set a appropriate penalty parameter $\delta = 0.2$ and get the evolutionary dynamics in Figure 4.3, where the meanings of x-axis, y-axis and z-axis are the same as in Figure 4.2. It is easily seen that, for appropriate sets of interaction rate $r(x)$ and penalty parameter δ, the evolutionary dynamics is quite different.

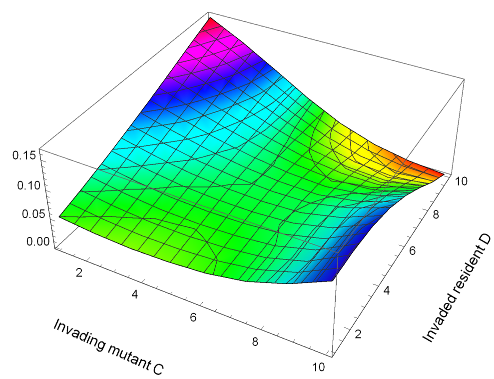
For the directly proportional function $r_1(x)$, its ratio is constant. When defectors express certain number of tags, larger numbers of expressed tags for cooperators



(a) The invasion plot by using $r_1(x)$



(b) The invasion plot by using $r_2(x)$



(c) The invasion plot by using $r_3(x)$

Figure 4.3: The evolutionary dynamics for $\delta = 0.2$.

cause higher penalty costs (subject to δ), which is more noticeable than higher interaction rates (subject to $r(x)$), thus the transition rates from cooperators to defectors generally decrease, shown in Figure 4.3(a). Then we consider the interaction rate $r_2(x)$, whose slope increases firstly and then decreases. Figure 4.3(b) shows that the transition rates increase firstly, since the effect of transition rate is higher than that of penalty. Nevertheless, with the slope of interaction rate $r_2(x)$ decreasing, the effect of penalty cost becomes more significant and the transition rates go down. Besides, the interaction rate $r_3(x)$ is similar to the part of $r_2(x)$ in $x \in [0, \frac{1}{2}]$. Therefore, the invasion plot in Figure 4.3(c) is similar to the part of that in Figure 4.3(b), where both expressed-tag numbers of C-strategy mutant and D-strategy resident are in $[1, 5]$.

4.3.2 Stationary distribution

Based on the analysis in Subsection 4.2.2, we study the stationary distribution of C-strategy mutant invading into D-strategy residents, which was obtained from the transition matrix T_{CD} in (4.19). This stationary distribution provides both the relative evolutionary advantage of each strategy, and the stationary fraction of cooperative acts (Pacheco et al., 2015).

As shown in Subsection 4.3.1, we know that both appropriate interaction rate $r(x)$ and penalty parameter δ have significant effect on the transition rates, when a single C-strategy mutant invades D-strategy population. We want to verify how $r(x)$ and δ affect the stationary distribution in long iterative time, as well as whether C-strategy individuals could have higher proportion in the stationary distribution.

For this goal, we continue to calculate the stationary distributions, subject to six different values of penalty parameter $\delta = 0, 0.1, 0.2, 0.3, 0.5, 1$ and three interaction rates $r_1(x)$, $r_2(x)$, $r_3(x)$ in Eq. (4.18), respectively.

For $r_1(x)$:

Figure 4.4 indicates the fraction of the N strains in the long run. In Figure 4.4, blue solid line denotes cooperator and orange dashed line for defector. The abscissa value represents how many tags individuals express, while the ordinate value denotes the fraction of strains.

From Figure 4.2(a) in Subsection 4.3.1, it is clear that when there is no penalty (i.e. $\delta = 0$), the transition rate only depends on interaction rate $r_1(x)$, which is a directly proportional function. Thus the fraction of defector is always higher than that of cooperator, as well as arises monotonically with the number of expressed tags increasing, shown in the first figure in Figure 4.4. When penalty parameter δ becomes large, the penalty cost becomes higher, until the effect of penalty is more noticeable than that of interaction rate and the stationary distributions of strains

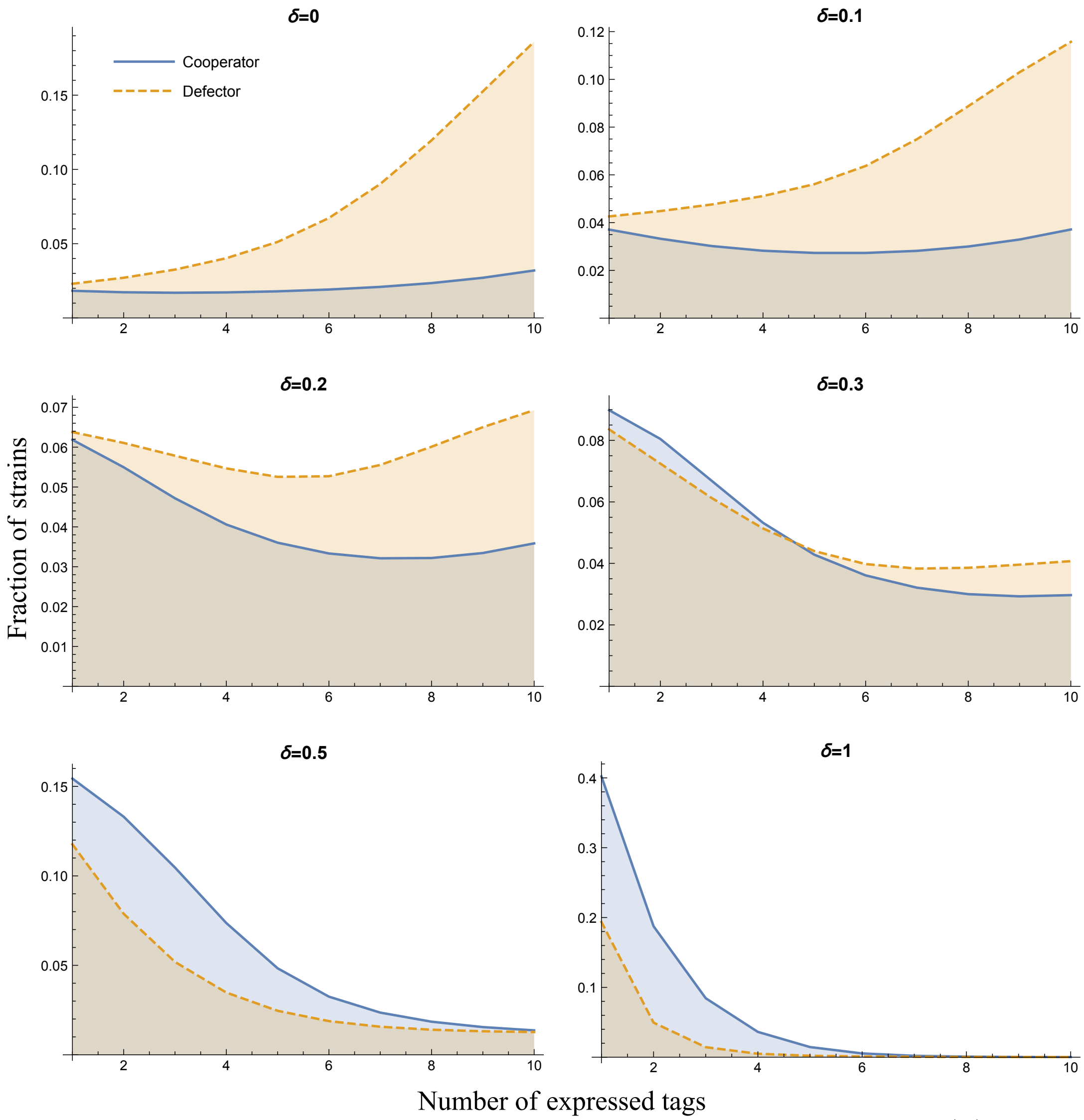


Figure 4.4: The stationary distributions for interaction rate $r_1(x)$.

decrease monotonically.

Since we focus on the transition from defectors to cooperators, we want to search for a kind of gaming strategy, and by using this strategy, the value of stationary distribution for cooperator could be higher (or higher in part) than the value for defector. From Figure 4.4 we notice that, when penalty parameter δ is larger than a point (around 0.3), the cooperator has higher overall stationary distribution.

For $r_2(x)$:

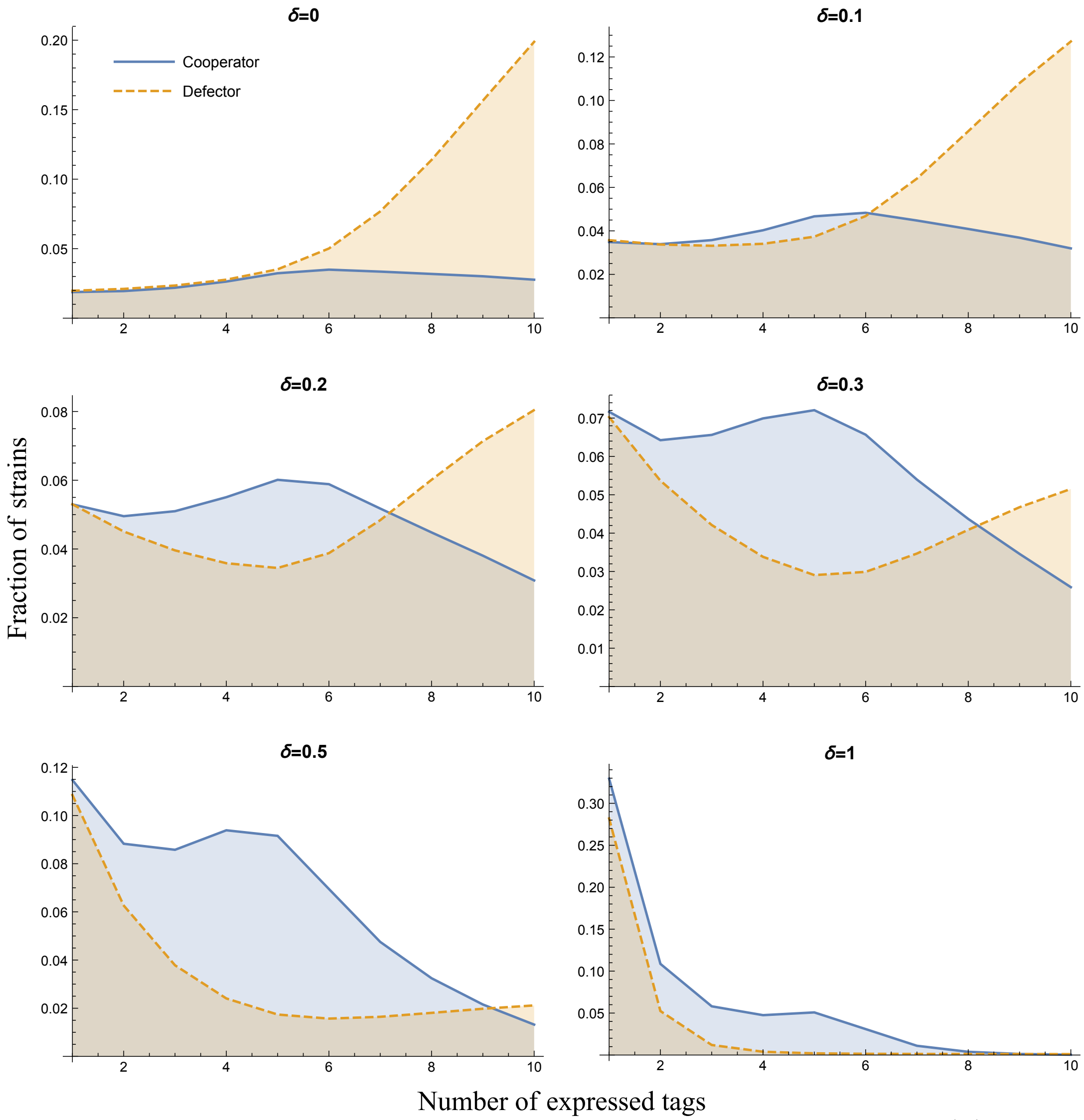


Figure 4.5: The stationary distributions for interaction rate $r_2(x)$.

We use interaction rate $r_2(x)$ and get the stationary distributions in terms of six different penalty parameters δ, as shown in Figure 4.5, where the meanings of x-axis and y-axis are the same as in Figure 4.4.

Similar to the evolutionary dynamics in Figure 4.3(b), when the penalty parameter δ is appropriate (e.g. $\delta = 0.3$), there are two local maximum point in the stationary distribution of cooperator. The reason is that: when the number of expressed tags is few, the value of interaction rate $r_2(x)$ increases very slowly. If cooperators express more tags, they will obtain very small increasing of payoffs, which is less significant than the penalty caused by the larger number of tags, so the stationary distribution decreases. However, the slope of interaction rate $r_2(x)$ rises more steeply in the middle, yielding the payoffs from interaction to increase. Thus the stationary distribution goes up until it reaches an inflection point. Then the stationary distribution decreases again, while the slope of interaction rate $r_2(x)$ becomes small.

Furthermore, which one of these two local maximum is the global maximum, depends on the specific parameter values. It is clearly seen from Figure 4.5, when $\delta = 0.2$, the second local maximum is also the global maximum, while for $\delta = 0.5$, the stationary distribution reaches the global maximum when cooperators express 1 tag.

For $r_3(x)$:

Figure 4.6 shows the stationary distributions by using interaction rate $r_3(x)$. Similar to the above analysis, since the slope of interaction rate $r_3(x)$ is small initially and increases monotonically, the stationary distribution of cooperators decreases in the beginning, and turns to increase when the payoffs from interaction become more noticeable.

Nevertheless, from Figure 4.6 we notice that, there is a significant advantage of interaction rate $r_3(x)$, the stationary fraction of cooperators is always higher than defectors. Inspired by this, we consider the overall cooperation level as a function of penalty parameter δ.

We set penalty parameter $\delta \in [0, 1]$ and use three different interaction rates $r(x)$, plot the overall cooperation level in Figure 4.7, where the abscissa value represents the value of penalty parameter δ, while the ordinate value denotes the overall cooperation level. The dashed line means that the cooperation level is 0.5.

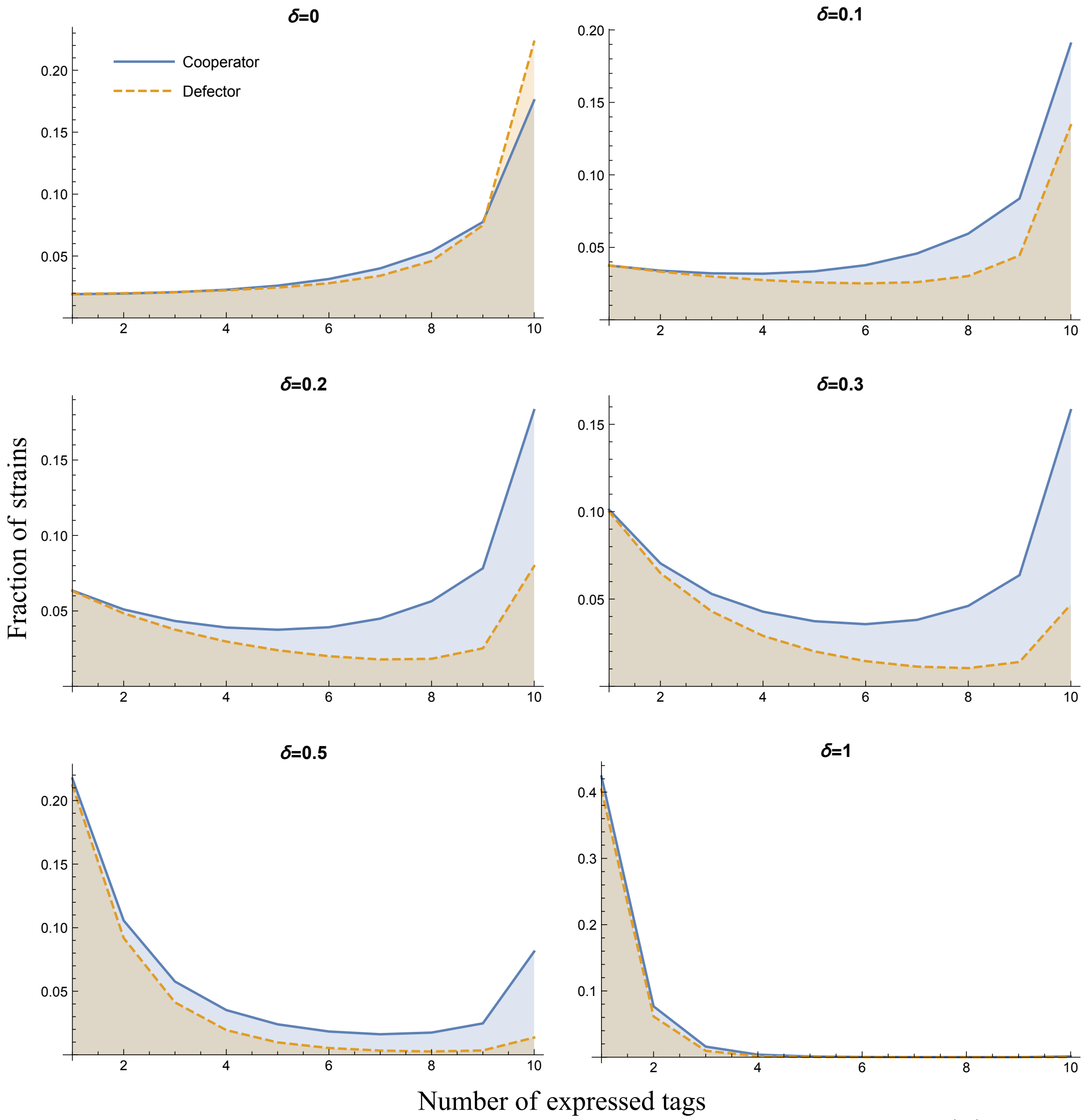


Figure 4.6: The stationary distributions for interaction rate $r_3(x)$.

From Figure 4.7, it is clear that in the range of $\delta \in [0, 1]$, the cooperation level by using interaction rate $r_1(x)$ increases monotonically, while that with interaction rate $r_2(x)$ has a maximum (approximate 0.688). The cooperation level with interaction rate $r_3(x)$ also has a maximum (approximate 0.646), when $\delta \approx 0.3$. We want to search for the value of stationary distribution for cooperator higher (or higher in part) than the value for defector, i.e. the strategy sets that cooperation level is larger than 0.5.

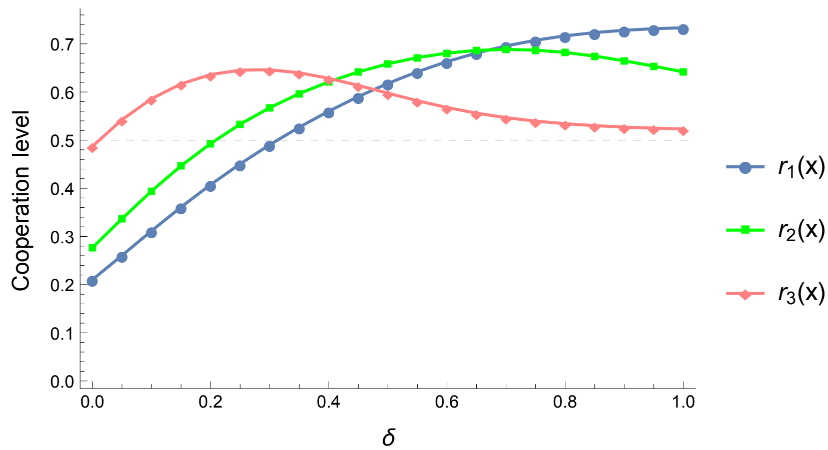


Figure 4.7: The cooperation level as a function of penalty parameter δ.

It is easily verified that there are three kinds of strategies: interaction rate $r_1(x)$ with $\delta > 0.3$, interaction rate $r_2(x)$ with $\delta > 0.2$, and interaction rate $r_3(x)$ with almost all $\delta \in [0, 1]$.

4.3.3 Analysis of results

We have used three different interaction rates $r(x)$ shown in Figure 4.1 with different penalty parameters δ as numerical examples, to do simulations and get the stationary distributions in Figures 4.4 - 4.6, as well as the cooperation level in terms of both $r(x)$ and δ, shown in Figure 4.7. After analyzing these numerical results in Subsection 4.3.2, we summarize the conclusion as follows.

1. We have introduced a new kind of strategy for classical iterated game theory in finite population by adding tags to the cooperators and defectors, and have derived the transition rate of N-number Y-strategy population by a single X-strategy individual, see Theorem 4.1.

2. For the new kind of finite-population evolutionary game theory model, we have focused on the case that a C-strategy mutant invades the D-strategy residents, and analyzed the evolutionary dynamics, as well as the stationary distributions.

In Subsection 4.3.1 and 4.3.2, the numerical results indicate the effectiveness of the proposed model.

3. Both the interaction rate $r(x)$ and penalty parameter δ are the important factors of the new model. When there is no penalty (i.e. $\delta = 0$), or the penalty is very large (e.g. $\delta = 1$), the transition rates and stationary distributions are mainly determined by one factor, interaction rate $r(x)$ or penalty parameter δ, respectively. By contrast, if both of them are set appropriate, they all have significant effect on transition rates and stationary distributions, which is illustrated by numerical results.

4. The numerical results demonstrate that the stationary distributions are quite different by using different rates $r(x)$ and parameters δ. We have set six values of δ and obtained the corresponding stationary distributions for three interaction rates $r(x)$, respectively. Result that cooperation can get stabilized is confirmed, as shown in Figure 4.7. Furthermore, we give the analyses and root causes for the numerical phenomena.

5. Finally, three kinds of feasible solutions are given: interaction rate $r_1(x)$ with $\delta > 0.3$, interaction rate $r_2(x)$ with $\delta > 0.2$, and interaction rate $r_3(x)$ with almost all $\delta \in [0, 1]$. Consequently, for a given example, we could succeed to get the feasible solutions, by choosing a suitable function $r(x)$ and parameter δ.

Chapter 5

Quantum Iterated Prisoner's Dilemma game and its zero-determinant strategies

In this chapter, we study a quantum two-player iterated game. Firstly in Section 5.1, a model of two-player Iterated Prisoner's Dilemma (IPD) game is introduced, in which the strategies used by two players are classical. By Press and Dyson (2012), there exist strategies whereby one player can enforce a unilateral claim to an unfair share of rewards. Secondly, by replacing the classical strategies with the quantum strategies, the game theory is generalized into the quantum domain. We state the principle and methods for the quantum strategy in Section 5.2. Then we consider a kind of two-player quantum/quantum-classical Iterated Prisoner's Dilemma games. Finally in Section 5.3, three kinds of quantum strategies are chosen for numerical studies. The numerical results demonstrate the effectiveness of the proposed quantum zero-determinant (ZD) strategies. Furthermore, the differences and advantages of the proposed quantum zero-determinant (ZD) strategies over the classical zero-determinant (ZD) strategies are also indicated in the numerical simulations.

5.1 Classical Iterated Prisoner's Dilemma and zero-determinant strategies

In Chapter 4, we considered a classical iterated game with tags where a C-strategy individual invades the N-number D-strategy population, and focused on the stationary distribution in a long time. In principle, in every generation, there are many games only occurred between two chosen individuals (determined by the number of shown-tags). If we consider the standard classical game without tags, the games will occur between every two individuals, yielding the iterated 2×2 game, where the Iterated Prisoner's Dilemma (IPD) game is a notable example.

It is generally assumed that there exists no simple ultimatum strategy whereby one player can enforce a unilateral claim to an unfair share of rewards. However, it has been proved in Press and Dyson (2012) that such strategies unexpectedly do exist. We introduce the principle of this type of game theory and these corresponding strategies firstly.

In Section 2.2, we discussed the general memory-one game. Firstly, the four outcomes of the previous move are labeled $1, 2, 3, 4$ for the respective outcomes $xy \in (CC, CD, DC, DD)$, where C and D denote cooperation and defection. Then we set the player X's strategy to be $\vec{p} = [p_1, p_2, p_3, p_4]^T$, the probabilities for cooperating under each of the previous outcomes, and the Y's strategy is analogously $\vec{q} = [q_1, q_2, q_3, q_4]^T$ for outcomes seen from Y's perspective, that is, in the order of $yx \in (CC, CD, DC, DD)$. Consequently, the outcome of this play is determined by a product of probabilities, as shown in Figure 5.1.

In Figure 5.1, X and Y denote the two players, C and D are the strategies for player X and Y respectively. $p_i q_j$, $p_i(1-q_j)$, $(1-p_i)q_j$ and $(1-p_i)(1-q_j)$ are the corresponding probabilities for the choice of current strategies $xy \in (CC, CD, DC, DD)$ subject to the previous move outcomes, where $i, j \in 1, 2, 3, 4$ and p_i, q_j are described

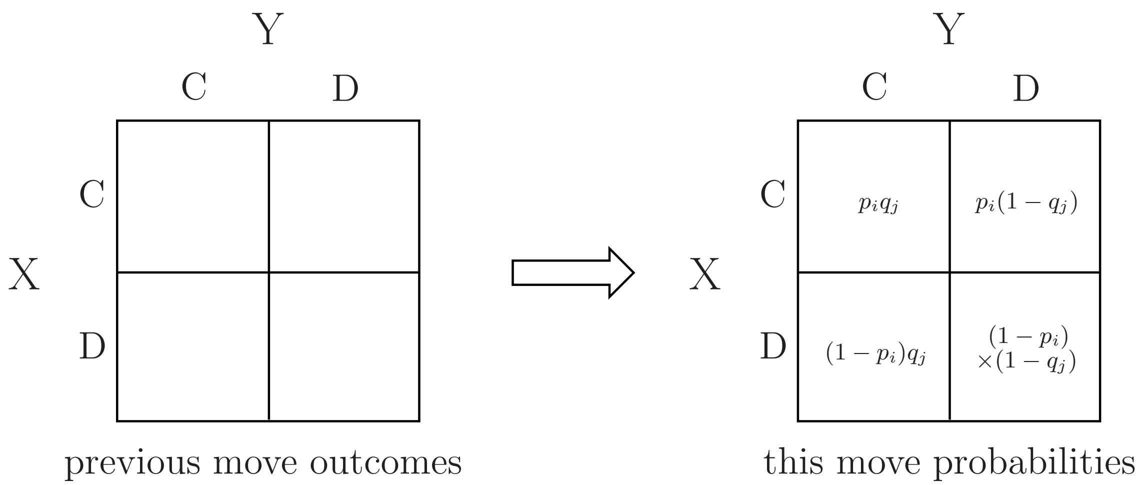


Figure 5.1: The two-player memory-one game.

as before. For example, if both players X and Y chose strategy C in the previous move, then in the current move, the probability that player X chooses strategy C is p_1 and the probability that player Y chooses strategy C is q_1. So the probabilities that the players X and Y choose strategy D are $(1 - p_1)$ and $(1 - q_1)$, respectively. Thus, the probabilities of strategies (CC, CD, DC, DD) in this move depending on the strategy CC in previous move is $[p_1q_1, p_1(1 - q_1), (1 - p_1)q_1, (1 - p_1)(1 - q_1)]^T$.

By the principle shown in Figure 5.1, $\vec{p}$ and $\vec{q}$ imply a Markov matrix whose stationary vector $\vec{v}$, combined with the respective payoff matrices, yields an expected outcome for each player. With rows and columns of the matrix in player X's order, the Markov transition matrix $M(\vec{p}, \vec{q})$ from one move to the next is shown in (5.1).

$$M(\vec{p},\vec{q}) = \begin{bmatrix} p_1q_1 & p_1(1-q_1) & (1-p_1)q_1 & (1-p_1)(1-q_1) \\ p_2q_3 & p_2(1-q_3) & (1-p_2)q_3 & (1-p_2)(1-q_3) \\ p_3q_2 & p_3(1-q_2) & (1-p_3)q_2 & (1-p_3)(1-q_2) \\ p_4q_4 & p_4(1-q_4) & (1-p_4)q_4 & (1-p_4)(1-q_4) \end{bmatrix} \tag{5.1}$$

It is clear to see that the sum of every row in the matrix M is equal to 1, so M has a unit eigenvalue, yielding that the matrix $M' \equiv M - I$ is singular. Consequently, the stationary vector $\vec{v}$ of the Markov matrix, or any vector proportional to it, satisfies

$$\vec{v}^T M = \vec{v}^T, \text{ or } \vec{v}^T M' = 0. \tag{5.2}$$

Besides, applying Cramer's rule to the matrix M' yields

$$\mathrm{Adj}(M')M' = \det(M')I = 0, \tag{5.3}$$

where $\mathrm{Adj}(M')$ is the adjugate matrix of the matrix M'. Eqs. (5.2) and (5.3) imply that every row of $\mathrm{Adj}(M')$ is proportional to $\vec{v}^T$. We denote $c_{ij} = (-1)^{i+j}\det(M'_{\bar{i}\bar{j}})$ as the algebraic cofactor of M'_{ij}, where M'_{ij} is the entry in the i-th row and j-th column of matrix M', $M'_{\bar{i}\bar{j}}$ means the 3×3 matrix from M' by deleting the i-th row and j-th column. Then we get

$$\mathrm{Adj}(M') = \begin{bmatrix} c_{11} & c_{21} & c_{31} & c_{41} \\ c_{12} & c_{22} & c_{32} & c_{42} \\ c_{13} & c_{23} & c_{33} & c_{43} \\ c_{14} & c_{24} & c_{34} & c_{44} \end{bmatrix}.$$

Therefore, by denoting $\vec{v}^T = [v_1, v_2, v_3, v_4]$ and choosing the fourth row of matrix $\mathrm{Adj}(M')$, we obtain

$$[v_1, v_2, v_3, v_4] \propto [c_{14}, c_{24}, c_{34}, c_{44}].$$

Moreover, the determinants $\det(M'_{\bar{i}\bar{j}})$ are unchanged if we add the first column of M' into the second and third columns.

Since

$$M' = M - I = \begin{bmatrix} -1 + p_1q_1 & p_1(1-q_1) & (1-p_1)q_1 & (1-p_1)(1-q_1) \\ p_2q_3 & p_2(1-q_3)-1 & (1-p_2)q_3 & (1-p_2)(1-q_3) \\ p_3q_2 & p_3(1-q_2) & (1-p_3)q_2-1 & (1-p_3)(1-q_2) \\ p_4q_4 & p_4(1-q_4) & (1-p_4)q_4 & p_4q_4 - p_4 - q_4 \end{bmatrix},$$

for an arbitrary vector $\vec{f} = [f_1, f_2, f_3, f_4]^T$, we can calculate the dot product of $\vec{f}$ and

$\vec{v}$ as follow

$$
\begin{aligned}
\vec{v} \cdot \vec{f} = \vec{v}^T \vec{f} &= [v_1, v_2, v_3, v_4] \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} \\
&= \det \begin{bmatrix} -1 + p_1 q_1 & p_1(1-q_1) & (1-p_1)q_1 & f_1 \\ p_2 q_3 & p_2(1-q_3) - 1 & (1-p_2)q_3 & f_2 \\ p_3 q_2 & p_3(1-q_2) & (1-p_3)q_2 - 1 & f_3 \\ p_4 q_4 & p_4(1-q_4) & (1-p_4)q_4 & f_4 \end{bmatrix} \\
&= \det \begin{bmatrix} -1 + p_1 q_1 & -1 + p_1 & -1 + q_1 & f_1 \\ p_2 q_3 & -1 + p_2 & q_3 & f_2 \\ p_3 q_2 & p_3 & -1 + q_2 & f_3 \\ p_4 q_4 & p_4 & q_4 & f_4 \end{bmatrix} \\
&:= D(\vec{p}, \vec{q}, \vec{f}).
\end{aligned}
$$

What is noteworthy about this formula for $\vec{v} \cdot \vec{f}$ is that it is a determinant whose second column,

$$\tilde{\vec{p}} := [-1 + p_1, -1 + p_2, p_3, p_4]^T, \tag{5.4}$$

is solely under the control of the player X; whose third column,

$$\tilde{\vec{q}} := [-1 + q_1, q_3, -1 + q_2, q_4]^T, \tag{5.5}$$

is solely under the control of the player Y; and whose fourth column is simply $\vec{f}$.

Meanwhile, in the order of the strategies $xy \in (CC, CD, DC, DD)$, the player X's payoff vector and Y's are $\vec{S_X} = [R, S, T, P]^T$ and $\vec{S_Y} = [R, T, S, P]^T$, respectively. Thus in the stationary state, their respective scores are

$$
\begin{aligned}
s_x &= \frac{\vec{v} \cdot \vec{S_X}}{\vec{v} \cdot \vec{1}} = \frac{D(\vec{p}, \vec{q}, \vec{S_X})}{D(\vec{p}, \vec{q}, \vec{1})}, \\
s_y &= \frac{\vec{v} \cdot \vec{S_Y}}{\vec{v} \cdot \vec{1}} = \frac{D(\vec{p}, \vec{q}, \vec{S_Y})}{D(\vec{p}, \vec{q}, \vec{1})},
\end{aligned} \tag{5.6}
$$

where $\vec{1}$ is the vector with all component 1. The denominators are needed because

$\vec{v}$ has not previously been normalized to have its components sum to 1 (as required for a stationary probability vector).

It can be easily verified that the scores s_x and s_y in Eq. (5.6) depend linearly on their corresponding payoff vectors S_X and S_Y, the same is true for any linear combination of scores, yielding

$$\alpha s_x + \beta s_y + \gamma = \frac{D(\vec{p}, \vec{q}, \alpha \vec{S_X} + \beta \vec{S_Y} + \gamma \vec{1})}{D(\vec{p}, \vec{q}, \vec{1})}. \tag{5.7}$$

By Eq. (5.7), we can see that both players X and Y have the possibility of choosing unilateral strategies that will make the determinant in the numerator vanish. More precisely, if player X chooses the strategy $\tilde{\vec{p}} = \alpha \vec{S_X} + \beta \vec{S_Y} + \gamma \vec{1}$, or player Y chooses the strategy $\tilde{\vec{q}} = \alpha \vec{S_X} + \beta \vec{S_Y} + \gamma \vec{1}$, the determinant in the numerator of Eq. (5.7) vanishes and a linear relation between these two scores s_x, s_y,

$$\alpha s_x + \beta s_y + \gamma = 0 \tag{5.8}$$

will be enforced, which are called *zero-determinant (ZD) strategies* (Press and Dyson, 2012).

By using ZD strategies, we will analyze how player X can deterministically set her opponent Y's score, independently of his strategy or response. However, it is important that not all zero-determinant strategies are feasible, with probabilities $\vec{p}$ and $\vec{q}$ all in the range $[0, 1]$, and whether they are feasible in any particular instance depends on the particulars of the application.

Case 1 (X unilaterally sets Y's score (Press and Dyson, 2012)):

Based on the above analysis, it is clear to see that using ZD strategies allows player X to unilaterally set player Y's score. To this goal, player X only needs to play a fixed strategy satisfying

$$\tilde{\vec{p}} = \beta \vec{S_Y} + \gamma \vec{1}, \tag{5.9}$$

that is, set $\alpha = 0$ in Eq. (5.8).

Then we get

$$\tilde{\vec{p}} = \begin{bmatrix} -1+p_1 \\ -1+p_2 \\ p_3 \\ p_4 \end{bmatrix} = \begin{bmatrix} \beta R+\gamma \\ \beta T+\gamma \\ \beta S+\gamma \\ \beta P+\gamma \end{bmatrix} = \beta \vec{S_Y} + \gamma \vec{1},$$

which are solved by eliminating the nuisance parameters β and γ, yielding the parameters p_2 and p_3 in terms of p_1 and p_4, as shown in Eq. (5.10) below:

$$\begin{aligned} p_2 &= \frac{(T-P)p_1 - (T-R)(1+p_4)}{R-P}, \\ p_3 &= \frac{(P-S)(1-p_1) + (R-S)p_4}{R-P}. \end{aligned} \tag{5.10}$$

Furthermore, we can calculate player Y's score, which is

$$s_y = \frac{P(1-p_1) + Rp_4}{(1-p_1)+p_4}. \tag{5.11}$$

Since there are conditions $T > R > P > S$ and $p_1, p_2, p_3, p_4 \in [0, 1]$ for all Iterated Prisoner's Dilemma, it is readily verified that the Eq. (5.10) has feasible solutions whenever $p_1 \to 1^-$ and $p_4 \to 0^+$. In this case, $p_2 \to 1^-$ and $p_3 \to 0^+$.

In Eq. (5.11), there is a weighted average of P and R with weights $(1 - p_1)$ and p_4, we see that all scores $P \leqslant s_y \leqslant R$ (and no others) can be forced by the player X. It means that the player X can set the player Y's score to any value in the range from the mutual noncooperation score to the mutual cooperation score by her own strategy $\vec{p}$, independent of the player Y's strategy $\vec{q}$. A consequence is that the player X can "spoof" any desired fitness landscape for the player Y that she wants, thereby guiding his evolutionary path.

Case 2 (X tries to set her own score (Press and Dyson, 2012)):

On the other side, could the player X set her own score by herself? In the ZD strategies, let the parameter $\beta = 0$ and the player X uses a strategy with

$$\vec{\tilde{p}} = \alpha \vec{S_X} + \gamma \vec{1}, \tag{5.12}$$

which is equivalent to

$$\vec{\tilde{p}} = \begin{bmatrix} -1+p_1 \\ -1+p_2 \\ p_3 \\ p_4 \end{bmatrix} = \begin{bmatrix} \alpha R+\gamma \\ \alpha S+\gamma \\ \alpha T+\gamma \\ \alpha P+\gamma \end{bmatrix} = \alpha \vec{S_X} + \gamma \vec{1}.$$

Then we use the analogous calculation as in case 1 to get

$$\begin{aligned} p_2 &= \frac{-(P-S)p_1 + (R-S)(1+p_4)}{R-P}, \\ p_3 &= \frac{-(T-P)(1-p_1) - (T-R)p_4}{R-P}. \end{aligned} \tag{5.13}$$

Since $T > R > P > S$ and $p_1, p_2, p_3, p_4 \in [0, 1]$, it is easily verified that $p_2 \geqslant 1$ and $p_3 \leqslant 0$, i.e. this strategy only has one feasible solution, namely the strategy $\vec{p} = [1, 1, 0, 0]^T$ (means that the player X always cooperate or never cooperate). Consequently, the player X can not use the ZD strategies to unilaterally set her own score in the Iterated Prisoner's Dilemma game.

5.2 Analysis of quantum two-player zero-determinant strategies

In this section, the game theory is generalized into the quantum domain. Firstly, the quantum two-player strategies are introduced. Then, similarly with the classical zero-determinant strategies, we synthesize the quantum zero-determinant strategies, which are used in the subsequent numerical simulations, yielding the properties of a certain quantum ZD strategy and its advantages.

5.2.1 Quantum strategies subject to quantum states

The physical model considered for a quantum two-player game consist of three elements (Eisert et al., 1999).

1. A source of two bits, one bit for each player;
2. A set of physical instruments that enables the player to manipulate her or his own bit in a strategic manner;
3. A physical measurement device which determines the players' payoff from the state of the two bits.

Furthermore, all these three ingredients, the source, the players' physical instruments, and the payoff physical measurement device are assumed to be perfectly known to both players.

The quantum formulation for the two-player game proceeds by assigning the possible outcomes of the classical strategies C and D to two basis vectors $|C\rangle$ and $|D\rangle$ in the Hilbert space of a one-qubit system, namely

$$|C\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ |D\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

At each instance, the state of the game is described by a vector in the tensor product space which is spanned by the game basis $|CC\rangle$, $|CD\rangle$, $|DC\rangle$ and $|DD\rangle$, where the first entry refers to the player X's qubit and the second for the player Y.

Figure 5.2 illustrates the general protocol of a quantum two-player game, where the initial states of both players X and Y are $|C\rangle$. The unitary operator $\hat{J}$ which is known to both two players is used to entangle their qubits. The strategies are executed on the two-player game's initial state $|\psi_0\rangle = \hat{J}|CC\rangle$. Strategic moves of the player X and the player Y are associated with unitary operators $\hat{U}_X$ and $\hat{U}_Y$, respectively. A disentangling operator $\hat{J}^\dagger$ is applied prior to making measurement on the final state $|\psi_f\rangle$, and the payoff is subsequently computed from the classical payoff matrix.

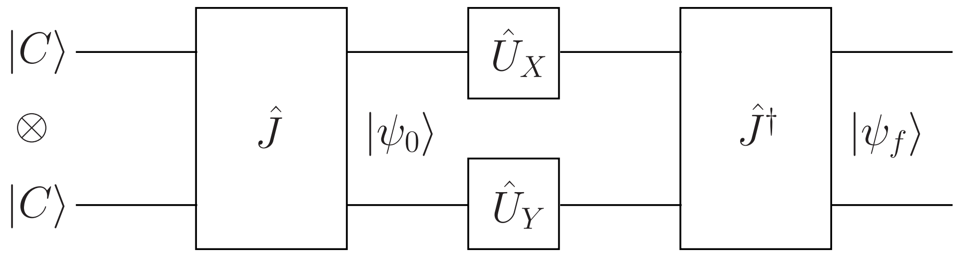


Figure 5.2: The general protocol of a two-player quantum game.

Remark 5.1. *For fair games, $\hat{J}$ must be symmetric with respect to the interchange of the two players.*

Therefore, the final state can be calculated by

$$|\psi_f\rangle = \hat{J}^\dagger(\hat{U}_X \otimes \hat{U}_Y)\hat{J}|\psi_0\rangle = \hat{J}^\dagger(\hat{U}_X \otimes \hat{U}_Y)\hat{J}|CC\rangle. \tag{5.14}$$

Because quantum mechanics is a fundamentally probabilistic theory, the only strategic notion of a payoff is the expected payoff (Eisert et al., 1999). By the quantum measurement theory (Wiseman and Milburn, 2010), the probabilities getting the choice of current strategies $xy \in (|CC\rangle, |CD\rangle, |DC\rangle, |DD\rangle)$ subject to the previous move outcomes (determined by $|\psi_f\rangle$) are calculated as

$$\begin{aligned} p_{CC} &= |\langle\psi_f|CC\rangle|^2, \quad p_{CD} = |\langle\psi_f|CD\rangle|^2, \\ p_{DC} &= |\langle\psi_f|DC\rangle|^2, \quad p_{DD} = |\langle\psi_f|DD\rangle|^2. \end{aligned} \tag{5.15}$$

Thus, the player X's expected payoff s_x is given by

$$s_x = Rp_{CC} + Sp_{CD} + Tp_{DC} + Pp_{DD}. \tag{5.16}$$

Similarly, the player Y's expected payoff s_y is

$$s_y = Rp_{CC} + Tp_{CD} + Sp_{DC} + Pp_{DD}. \tag{5.17}$$

Without loss of generality (Eisert et al., 1999; Flitney and Abbott, 2002), the entangling operator $\hat{J}$ is chosen as

$$\hat{J} = \exp(i\frac{\gamma}{2}\hat{\sigma} \otimes \hat{\sigma}), \tag{5.18}$$

where $i = \sqrt{-1}$ be the imaginary unit, $\hat{\sigma}$ is a unitary 2×2 matrix, the real parameter $\gamma \in [0, \frac{\pi}{2}]$ determines the degree of entanglement. Specifically, $\gamma = 0$ means that there has no entanglement and it degenerates to the classical case, while $\gamma = \frac{\pi}{2}$ implies the maximal entanglement (i.e. $|\psi_0\rangle$ is a maximally entangled state).

It has been proved in Benjamin and Hayden (2001) that, it is sufficient to restrict the strategic space to the two-parameter set of all unitary 2×2 matrices,

$$\hat{U}(\theta, \phi) = \begin{bmatrix} e^{i\phi} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & e^{-i\phi}\cos\frac{\theta}{2} \end{bmatrix},$$

with $\theta \in [0, \pi]$ and $\phi \in [0, \frac{\pi}{2}]$, where $\phi = 0$ means the classical strategies and $\phi \neq 0$ implies the quantum strategies. Thus, we choose the strategic move operators $\hat{U}_X$ and $\hat{U}_Y$ as, respectively,

$$\begin{aligned} \hat{U}_X(\alpha, \varphi) &= \begin{bmatrix} e^{i\varphi} \cos\frac{\alpha}{2} & \sin\frac{\alpha}{2} \\ -\sin\frac{\alpha}{2} & e^{-i\varphi}\cos\frac{\alpha}{2} \end{bmatrix}, \\ \hat{U}_Y(\beta, \psi) &= \begin{bmatrix} e^{i\psi} \cos\frac{\beta}{2} & \sin\frac{\beta}{2} \\ -\sin\frac{\beta}{2} & e^{-i\psi}\cos\frac{\beta}{2} \end{bmatrix}. \end{aligned} \tag{5.19}$$

It is clear that $0 \leqslant \alpha, \beta \leqslant \pi$ and $0 \leqslant \varphi, \psi \leqslant \frac{\pi}{2}$.

By referring to the classical case introduced in Section 5.1, in the order of the respective previous outcomes $xy \in (|CC\rangle, |CD\rangle, |DC\rangle, |DD\rangle)$, the two-player quantum strategies are set as $\vec{p} = [\alpha_1, \alpha_2, \alpha_3, \alpha_4; \varphi]^T$ and $\vec{q} = [\beta_1, \beta_2, \beta_3, \beta_4; \psi]^T$ for the players X and Y, respectively.

Remark 5.2. *Generally speaking, for different outcomes* $xy \in (|CC\rangle, |CD\rangle, |DC\rangle, |DD\rangle)$, *there are 8 parameters for each player's strategy, i.e.*

$$\vec{p} = [\alpha_1; \varphi_1, \alpha_2; \varphi_2, \alpha_3; \varphi_3, \alpha_4; \varphi_4]^T, \quad \vec{q} = [\beta_1; \psi_1, \beta_2; \psi_2, \beta_3; \psi_3, \beta_4; \psi_4]^T.$$

For simplicity, we assume $\varphi_1 = \varphi_2 = \varphi_3 = \varphi_4 := \varphi$, $\psi_1 = \psi_2 = \psi_3 = \psi_4 := \psi$, *yielding the strategies* $\vec{p} = [\alpha_1, \alpha_2, \alpha_3, \alpha_4; \varphi]^T$ *and* $\vec{q} = [\beta_1, \beta_2, \beta_3, \beta_4; \psi]^T$.

Finally, for better understanding, we give a simple example. If the previous outcome is $xy = |CC\rangle$, the player X's strategy is $\hat{U}_X(\alpha_1, \varphi)$, while $\hat{U}_Y(\beta_1, \psi)$ is the player Y's strategy, then we calculate the state

$$|\psi_f\rangle = \hat{J}^\dagger(\hat{U}_X(\alpha_1, \varphi) \otimes \hat{U}_Y(\beta_1, \psi))\hat{J}|CC\rangle.$$

The probabilities p_{CC}, p_{CD}, p_{DC} and p_{DD} are given by Eq. (5.15).

5.2.2 Synthesis of quantum zero-determinant strategies

Firstly, we choose two different types of the entangling operator $\hat{J}$ subject to

$$\hat{\sigma}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \hat{\sigma}_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{5.20}$$

and set $\gamma = \frac{\pi}{2}$, which means we only concern the case of maximal entanglement. By Eq. (5.18), we could get

$$\hat{J}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & i \\ 0 & 1 & i & 0 \\ 0 & i & 1 & 0 \\ i & 0 & 0 & 1 \end{bmatrix}, \quad \hat{J}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & i \\ 0 & 1 & -i & 0 \\ 0 & -i & 1 & 0 \\ i & 0 & 0 & 1 \end{bmatrix}. \tag{5.21}$$

The probabilities p_{CC}, p_{CD}, p_{DC} and p_{DD} can be calculated by Eq. (5.15). Similar to Eq. (5.1), the transition matrices are given by

For $\hat{\sigma}_1$:

$$M_1 = \begin{bmatrix}
\cos^2\frac{\alpha_1}{2}\cos^2\frac{\beta_1}{2}\cos^2(\varphi+\psi) & (\sin\frac{\alpha_1}{2}\cos\frac{\beta_1}{2}\cos\psi-\cos\frac{\alpha_1}{2}\sin\frac{\beta_1}{2}\sin\varphi)^2 & (\cos\frac{\alpha_1}{2}\sin\frac{\beta_1}{2}\cos\varphi-\sin\frac{\alpha_1}{2}\cos\frac{\beta_1}{2}\sin\psi)^2 & (\sin\frac{\alpha_1}{2}\sin\frac{\beta_1}{2}+\cos\frac{\alpha_1}{2}\cos\frac{\beta_1}{2}\sin(\varphi+\psi))^2 \\
\cos^2\frac{\alpha_2}{2}\cos^2\frac{\beta_3}{2}\cos^2(\varphi+\psi) & (\sin\frac{\alpha_2}{2}\cos\frac{\beta_3}{2}\cos\psi-\cos\frac{\alpha_2}{2}\sin\frac{\beta_3}{2}\sin\varphi)^2 & (\cos\frac{\alpha_2}{2}\sin\frac{\beta_3}{2}\cos\varphi-\sin\frac{\alpha_2}{2}\cos\frac{\beta_3}{2}\sin\psi)^2 & (\sin\frac{\alpha_2}{2}\sin\frac{\beta_3}{2}+\cos\frac{\alpha_2}{2}\cos\frac{\beta_3}{2}\sin(\varphi+\psi))^2 \\
\cos^2\frac{\alpha_3}{2}\cos^2\frac{\beta_2}{2}\cos^2(\varphi+\psi) & (\sin\frac{\alpha_3}{2}\cos\frac{\beta_2}{2}\cos\psi-\cos\frac{\alpha_3}{2}\sin\frac{\beta_2}{2}\sin\varphi)^2 & (\cos\frac{\alpha_3}{2}\sin\frac{\beta_2}{2}\cos\varphi-\sin\frac{\alpha_3}{2}\cos\frac{\beta_2}{2}\sin\psi)^2 & (\sin\frac{\alpha_3}{2}\sin\frac{\beta_2}{2}+\cos\frac{\alpha_3}{2}\cos\frac{\beta_2}{2}\sin(\varphi+\psi))^2 \\
\cos^2\frac{\alpha_4}{2}\cos^2\frac{\beta_4}{2}\cos^2(\varphi+\psi) & (\sin\frac{\alpha_4}{2}\cos\frac{\beta_4}{2}\cos\psi-\cos\frac{\alpha_4}{2}\sin\frac{\beta_4}{2}\sin\varphi)^2 & (\cos\frac{\alpha_4}{2}\sin\frac{\beta_4}{2}\cos\varphi-\sin\frac{\alpha_4}{2}\cos\frac{\beta_4}{2}\sin\psi)^2 & (\sin\frac{\alpha_4}{2}\sin\frac{\beta_4}{2}+\cos\frac{\alpha_4}{2}\cos\frac{\beta_4}{2}\sin(\varphi+\psi))^2
\end{bmatrix}. \tag{5.22}$$

For $\hat{\sigma}_2$:

$$M_2 = \begin{bmatrix}
\cos^2\frac{\alpha_1}{2}\cos^2\frac{\beta_1}{2}\cos^2(\varphi+\psi) & (\cos\frac{\alpha_1}{2}\sin\frac{\beta_1}{2}\cos\varphi-\sin\frac{\alpha_1}{2}\cos\frac{\beta_1}{2}\sin\psi)^2 & (\sin\frac{\alpha_1}{2}\cos\frac{\beta_1}{2}\cos\psi-\cos\frac{\alpha_1}{2}\sin\frac{\beta_1}{2}\sin\varphi)^2 & (\sin\frac{\alpha_1}{2}\sin\frac{\beta_1}{2}+\cos\frac{\alpha_1}{2}\cos\frac{\beta_1}{2}\sin(\varphi+\psi))^2 \\
\cos^2\frac{\alpha_2}{2}\cos^2\frac{\beta_3}{2}\cos^2(\varphi+\psi) & (\cos\frac{\alpha_2}{2}\sin\frac{\beta_3}{2}\cos\varphi-\sin\frac{\alpha_2}{2}\cos\frac{\beta_3}{2}\sin\psi)^2 & (\sin\frac{\alpha_2}{2}\cos\frac{\beta_3}{2}\cos\psi-\cos\frac{\alpha_2}{2}\sin\frac{\beta_3}{2}\sin\varphi)^2 & (\sin\frac{\alpha_2}{2}\sin\frac{\beta_3}{2}+\cos\frac{\alpha_2}{2}\cos\frac{\beta_3}{2}\sin(\varphi+\psi))^2 \\
\cos^2\frac{\alpha_3}{2}\cos^2\frac{\beta_2}{2}\cos^2(\varphi+\psi) & (\cos\frac{\alpha_3}{2}\sin\frac{\beta_2}{2}\cos\varphi-\sin\frac{\alpha_3}{2}\cos\frac{\beta_2}{2}\sin\psi)^2 & (\sin\frac{\alpha_3}{2}\cos\frac{\beta_2}{2}\cos\psi-\cos\frac{\alpha_3}{2}\sin\frac{\beta_2}{2}\sin\varphi)^2 & (\sin\frac{\alpha_3}{2}\sin\frac{\beta_2}{2}+\cos\frac{\alpha_3}{2}\cos\frac{\beta_2}{2}\sin(\varphi+\psi))^2 \\
\cos^2\frac{\alpha_4}{2}\cos^2\frac{\beta_4}{2}\cos^2(\varphi+\psi) & (\cos\frac{\alpha_4}{2}\sin\frac{\beta_4}{2}\cos\varphi-\sin\frac{\alpha_4}{2}\cos\frac{\beta_4}{2}\sin\psi)^2 & (\sin\frac{\alpha_4}{2}\cos\frac{\beta_4}{2}\cos\psi-\cos\frac{\alpha_4}{2}\sin\frac{\beta_4}{2}\sin\varphi)^2 & (\sin\frac{\alpha_4}{2}\sin\frac{\beta_4}{2}+\cos\frac{\alpha_4}{2}\cos\frac{\beta_4}{2}\sin(\varphi+\psi))^2
\end{bmatrix}. \tag{5.23}$$

It is easily verified that for both M_1 and M_2, the sum of every row is equal to 1, so both of them are Markov transition matrices, and have a unit eigenvalue.

Therefore, we could do the similar derivation as in Section 5.1. Let $\vec{v}_1$ and $\vec{v}_2$ be the stationary vectors of the Markov transition matrices M_1 and M_2, respectively. The function $D(\vec{\alpha}, \vec{\beta}, \varphi, \psi, \vec{f})$ are defined as

$$D(\vec{\alpha}, \vec{\beta}, \varphi, \psi, \vec{f}) = \vec{v}_i \cdot \vec{f},$$

where $\vec{v}_i \in (\vec{v}_1, \vec{v}_2)$, $\vec{f} = [f_1, f_2, f_3, f_4]^T$ is an arbitrary vector, $\vec{\alpha} = [\alpha_1, \alpha_2, \alpha_3, \alpha_4]^T$ and $\vec{\beta} = [\beta_1, \beta_2, \beta_3, \beta_4]^T$.

Furthermore, the respective scores of the players X and Y are

$$\begin{aligned} s_x &= \frac{\vec{v}_i \cdot \vec{S_X}}{\vec{v}_i \cdot \vec{1}} = \frac{D(\vec{\alpha}, \vec{\beta}, \varphi, \psi, \vec{S_X})}{D(\vec{\alpha}, \vec{\beta}, \varphi, \psi, \vec{1})}, \\ s_y &= \frac{\vec{v}_i \cdot \vec{S_Y}}{\vec{v}_i \cdot \vec{1}} = \frac{D(\vec{\alpha}, \vec{\beta}, \varphi, \psi, \vec{S_Y})}{D(\vec{\alpha}, \vec{\beta}, \varphi, \psi, \vec{1})}, \end{aligned} \tag{5.24}$$

It is easy to see that the scores s_x and s_y in Eq. (5.24) depend linearly on their corresponding payoff vector S_X and S_Y. So we have

$$\alpha s_x + \beta s_y + \gamma = \frac{D(\vec{\alpha}, \vec{\beta}, \varphi, \psi, \alpha\vec{S_X} + \beta\vec{S_Y} + \gamma\vec{1})}{D(\vec{\alpha}, \vec{\beta}, \varphi, \psi, \vec{1})}. \tag{5.25}$$

Consequently, by choosing appropriate parameters and corresponding strategies, we can also derive $\alpha s_x + \beta s_y + \gamma = 0$, which are named *quantum zero-determinant (ZD) strategies*.

5.3 Numerical simulations and analysis of results

In this section, three numerical examples are considered.

5.3.1 Numerical simulations

Example 1 (subject to operator $\hat{\sigma}_1$ in Eq. (5.20)):

By using $\hat{\sigma}_1$, we get the entangling operator $\hat{J}_1$ as shown in Eq. (5.21). Then a game between the quantum strategy (the player X with $\varphi \neq 0$) and the classical strategy (the player Y with $\psi = 0$) is considered.

Since $\psi = 0$, the Markov transition matrix M_1 in Eq. (5.22) becomes

$$\begin{bmatrix} \cos^2\frac{\alpha_1}{2}\cos^2\frac{\beta_1}{2}\cos^2\varphi & (\sin\frac{\alpha_1}{2}\cos\frac{\beta_1}{2}-\cos\frac{\alpha_1}{2}\sin\frac{\beta_1}{2}\sin\varphi)^2 \\ \cos^2\frac{\alpha_2}{2}\cos^2\frac{\beta_3}{2}\cos^2\varphi & (\sin\frac{\alpha_2}{2}\cos\frac{\beta_3}{2}-\cos\frac{\alpha_2}{2}\sin\frac{\beta_3}{2}\sin\varphi)^2 \\ \cos^2\frac{\alpha_3}{2}\cos^2\frac{\beta_2}{2}\cos^2\varphi & (\sin\frac{\alpha_3}{2}\cos\frac{\beta_2}{2}-\cos\frac{\alpha_3}{2}\sin\frac{\beta_2}{2}\sin\varphi)^2 \\ \cos^2\frac{\alpha_4}{2}\cos^2\frac{\beta_4}{2}\cos^2\varphi & (\sin\frac{\alpha_4}{2}\cos\frac{\beta_4}{2}-\cos\frac{\alpha_4}{2}\sin\frac{\beta_4}{2}\sin\varphi)^2 \end{matrix}$$

$$\begin{matrix} \cos^2\frac{\alpha_1}{2}\sin^2\frac{\beta_1}{2}\cos^2\varphi & (\sin\frac{\alpha_1}{2}\sin\frac{\beta_1}{2}+\cos\frac{\alpha_1}{2}\cos\frac{\beta_1}{2}\sin\varphi)^2 \\ \cos^2\frac{\alpha_2}{2}\sin^2\frac{\beta_3}{2}\cos^2\varphi & (\sin\frac{\alpha_2}{2}\sin\frac{\beta_3}{2}+\cos\frac{\alpha_2}{2}\cos\frac{\beta_3}{2}\sin\varphi)^2 \\ \cos^2\frac{\alpha_3}{2}\sin^2\frac{\beta_2}{2}\cos^2\varphi & (\sin\frac{\alpha_3}{2}\sin\frac{\beta_2}{2}+\cos\frac{\alpha_3}{2}\cos\frac{\beta_2}{2}\sin\varphi)^2 \\ \cos^2\frac{\alpha_4}{2}\sin^2\frac{\beta_4}{2}\cos^2\varphi & (\sin\frac{\alpha_4}{2}\sin\frac{\beta_4}{2}+\cos\frac{\alpha_4}{2}\cos\frac{\beta_4}{2}\sin\varphi)^2 \end{bmatrix}.$$

By calculating $M_1' = M_1 - I$ and adding the first column of $\det(M_1')$ into the third column, we get

$$D(\vec{\alpha},\vec{\beta},\varphi,0,\vec{f}) = \vec{v}_1\cdot\vec{f}$$

$$=\det\begin{bmatrix} \cos^2\frac{\alpha_1}{2}\cos^2\frac{\beta_1}{2}\cos^2\varphi-1 & (\sin\frac{\alpha_1}{2}\cos\frac{\beta_1}{2}-\cos\frac{\alpha_1}{2}\sin\frac{\beta_1}{2}\sin\varphi)^2 & \cos^2\frac{\alpha_1}{2}\cos^2\varphi-1 & f_1 \\ \cos^2\frac{\alpha_2}{2}\cos^2\frac{\beta_3}{2}\cos^2\varphi & (\sin\frac{\alpha_2}{2}\cos\frac{\beta_3}{2}-\cos\frac{\alpha_2}{2}\sin\frac{\beta_3}{2}\sin\varphi)^2-1 & \cos^2\frac{\alpha_2}{2}\cos^2\varphi & f_2 \\ \cos^2\frac{\alpha_3}{2}\cos^2\frac{\beta_2}{2}\cos^2\varphi & (\sin\frac{\alpha_3}{2}\cos\frac{\beta_2}{2}-\cos\frac{\alpha_3}{2}\sin\frac{\beta_2}{2}\sin\varphi)^2 & \cos^2\frac{\alpha_3}{2}\cos^2\varphi-1 & f_3 \\ \cos^2\frac{\alpha_4}{2}\cos^2\frac{\beta_4}{2}\cos^2\varphi & (\sin\frac{\alpha_4}{2}\cos\frac{\beta_4}{2}-\cos\frac{\alpha_4}{2}\sin\frac{\beta_4}{2}\sin\varphi)^2 & \cos^2\frac{\alpha_4}{2}\cos^2\varphi & f_4 \end{bmatrix}. \tag{5.26}$$

We notice that, the third column of the determinant in Eq. (5.26),

$$\vec{\tilde{p}}_1 := \begin{bmatrix} \cos^2\frac{\alpha_1}{2}\cos^2\varphi-1 \\ \cos^2\frac{\alpha_2}{2}\cos^2\varphi \\ \cos^2\frac{\alpha_3}{2}\cos^2\varphi-1 \\ \cos^2\frac{\alpha_4}{2}\cos^2\varphi \end{bmatrix} \tag{5.27}$$

only depends on $\vec{\alpha}$ and φ, i.e. solely under the control of the player X. Thus we could use the quantum ZD strategies to do the simulations.

Case 1 (Whether the quantum-strategy player X could unilaterally set the classical-strategy player Y's score):

Let the parameter α in Eq. (5.25) be 0, and X chooses the strategy

$$\vec{\tilde{p}}_1 = \begin{bmatrix} \cos^2\frac{\alpha_1}{2}\cos^2\varphi-1 \\ \cos^2\frac{\alpha_2}{2}\cos^2\varphi \\ \cos^2\frac{\alpha_3}{2}\cos^2\varphi-1 \\ \cos^2\frac{\alpha_4}{2}\cos^2\varphi \end{bmatrix} = \begin{bmatrix} \beta R+\gamma \\ \beta T+\gamma \\ \beta S+\gamma \\ \beta P+\gamma \end{bmatrix} = \beta\vec{S_Y}+\gamma\vec{1}.$$

By eliminating the nuisance parameters β and γ, the parameters α_2 and α_3 in terms of α_1, α_4 and φ are shown as

$$\begin{aligned}\cos^2\frac{\alpha_2}{2} &= \frac{(T-P)(\cos^2\frac{\alpha_1}{2}-\sec^2\varphi)-(T-R)\cos^2\frac{\alpha_4}{2}}{R-P},\\ \cos^2\frac{\alpha_3}{2} &= \frac{(P-S)(\sec^2\varphi-\cos^2\frac{\alpha_1}{2})+(R-S)\cos^2\frac{\alpha_4}{2}+(R-P)\sec^2\varphi}{R-P}.\end{aligned} \tag{5.28}$$

Since $T > R > P > S$, $0 \leqslant \cos^2\frac{\alpha_i}{2} \leqslant 1$ and $\sec^2\varphi \geqslant 1$, there is only one feasible solution

$$\cos^2\frac{\alpha_2}{2} = 0$$

for the first equation of Eq. (5.28), while

$$\begin{aligned}\cos^2\frac{\alpha_4}{2} &= 0,\\ \cos^2\frac{\alpha_1}{2}-\sec^2\varphi = 0 &\leftrightarrow \cos^2\frac{\alpha_1}{2}\cos^2\varphi = 1.\end{aligned} \tag{5.29}$$

By substituting (5.29) into the second equation of Eq. (5.28), we obtain

$$\cos^2\frac{\alpha_3}{2} = \sec^2\varphi \leftrightarrow \cos^2\frac{\alpha_3}{2}\cos^2\varphi = 1.$$

That is, this case has only one feasible solution, given by $\alpha_1 = \alpha_3 = 0$, $\alpha_2 = \alpha_4 = \pi$ and $\varphi = 0$. Thus in this case, the quantum-strategy player X can not unilaterally set the classical-strategy player Y's score. (Actually, the feasible solution indicates that the player X's strategy degenerates to a classical strategy, because of $\varphi = 0$.)

Case 2 (Whether the quantum-strategy player X could unilaterally set her own score):

Similarly, the player X sets the parameter β in Eq. (5.25) to be 0, and chooses the strategy

$$\vec{p}_1 = \begin{bmatrix}\cos^2\frac{\alpha_1}{2}\cos^2\varphi-1\\ \cos^2\frac{\alpha_2}{2}\cos^2\varphi\\ \cos^2\frac{\alpha_3}{2}\cos^2\varphi-1\\ \cos^2\frac{\alpha_4}{2}\cos^2\varphi\end{bmatrix} = \begin{bmatrix}\alpha R+\gamma\\ \alpha S+\gamma\\ \alpha T+\gamma\\ \alpha P+\gamma\end{bmatrix} = \alpha\vec{S_X}+\gamma\vec{1}.$$

For the simplicity of the derivation, we denote $p_i := \cos^2\frac{\alpha_i}{2}\cos^2\varphi$, where $i = 1, 2, 3, 4$. Then we can obtain

$$\begin{aligned} p_2 &= \frac{(P-S)(1-p_1)+(R-S)p_4}{R-P}, \\ p_3 &= \frac{(T-P)p_1-(T-R)(1+p_4)}{R-P}. \end{aligned} \tag{5.30}$$

By Eq. (5.10) and the derivation of case 1 in Section 5.1, Eq. (5.30) has feasible solutions that $p_2 \to 0^+$ and $p_3 \to 1^-$ subject to $p_1 \to 1^-$ and $p_4 \to 0^+$, i.e.

$$\cos^2\frac{\alpha_1}{2}\cos^2\varphi \to 1^-, \quad \cos^2\frac{\alpha_2}{2}\cos^2\varphi \to 0^+,$$
$$\cos^2\frac{\alpha_3}{2}\cos^2\varphi \to 1^-, \quad \cos^2\frac{\alpha_4}{2}\cos^2\varphi \to 0^+.$$

Moreover, the player X's score

$$s_x = \frac{P(1-p_1)+Rp_4}{(1-p_1)+p_4} \in [P, R].$$

A consequence is that the quantum-strategy player X can unilaterally set her own score $s_x \in [P, R]$ in this case.

Example 2 (subject to operator $\hat{\sigma}_2$ in Eq. (5.20)):

By using $\hat{\sigma}_2$, we get the entangling operator $\hat{J}_2$ as shown in Eq. (5.21). Similarly with *Example 1*, we consider a game between the quantum-strategy player X and the classical-strategy player Y.

Since $\psi = 0$, the Markov transition matrix M_2 in Eq. (5.23) becomes

$$\begin{bmatrix} \cos^2\frac{\alpha_1}{2}\cos^2\frac{\beta_1}{2}\cos^2\varphi & \cos^2\frac{\alpha_1}{2}\sin^2\frac{\beta_1}{2}\cos^2\varphi & (\sin\frac{\alpha_1}{2}\cos\frac{\beta_1}{2}-\cos\frac{\alpha_1}{2}\sin\frac{\beta_1}{2}\sin\varphi)^2 & (\sin\frac{\alpha_1}{2}\sin\frac{\beta_1}{2}+\cos\frac{\alpha_1}{2}\cos\frac{\beta_1}{2}\sin\varphi)^2 \\ \cos^2\frac{\alpha_2}{2}\cos^2\frac{\beta_3}{2}\cos^2\varphi & \cos^2\frac{\alpha_2}{2}\sin^2\frac{\beta_3}{2}\cos^2\varphi & (\sin\frac{\alpha_2}{2}\cos\frac{\beta_3}{2}-\cos\frac{\alpha_2}{2}\sin\frac{\beta_3}{2}\sin\varphi)^2 & (\sin\frac{\alpha_2}{2}\sin\frac{\beta_3}{2}+\cos\frac{\alpha_2}{2}\cos\frac{\beta_3}{2}\sin\varphi)^2 \\ \cos^2\frac{\alpha_3}{2}\cos^2\frac{\beta_2}{2}\cos^2\varphi & \cos^2\frac{\alpha_3}{2}\sin^2\frac{\beta_2}{2}\cos^2\varphi & (\sin\frac{\alpha_3}{2}\cos\frac{\beta_2}{2}-\cos\frac{\alpha_3}{2}\sin\frac{\beta_2}{2}\sin\varphi)^2 & (\sin\frac{\alpha_3}{2}\sin\frac{\beta_2}{2}+\cos\frac{\alpha_3}{2}\cos\frac{\beta_2}{2}\sin\varphi)^2 \\ \cos^2\frac{\alpha_4}{2}\cos^2\frac{\beta_4}{2}\cos^2\varphi & \cos^2\frac{\alpha_4}{2}\sin^2\frac{\beta_4}{2}\cos^2\varphi & (\sin\frac{\alpha_4}{2}\cos\frac{\beta_4}{2}-\cos\frac{\alpha_4}{2}\sin\frac{\beta_4}{2}\sin\varphi)^2 & (\sin\frac{\alpha_4}{2}\sin\frac{\beta_4}{2}+\cos\frac{\alpha_4}{2}\cos\frac{\beta_4}{2}\sin\varphi)^2 \end{bmatrix}.$$

By calculating $M_2' = M_2 - I$ and adding the first column of $\det(M_2')$ into the second column, we get

$$D(\vec{\alpha}, \vec{\beta}, \varphi, 0, \vec{f}) = \vec{v}_2 \cdot \vec{f}$$

$$=\det\begin{bmatrix} \cos^2\frac{\alpha_1}{2}\cos^2\frac{\beta_1}{2}\cos^2\varphi - 1 & \cos^2\frac{\alpha_1}{2}\cos^2\varphi - 1 & (\sin\frac{\alpha_1}{2}\cos\frac{\beta_1}{2} - \cos\frac{\alpha_1}{2}\sin\frac{\beta_1}{2}\sin\varphi)^2 & f_1 \\ \cos^2\frac{\alpha_2}{2}\cos^2\frac{\beta_3}{2}\cos^2\varphi & \cos^2\frac{\alpha_2}{2}\cos^2\varphi - 1 & (\sin\frac{\alpha_2}{2}\cos\frac{\beta_3}{2} - \cos\frac{\alpha_2}{2}\sin\frac{\beta_3}{2}\sin\varphi)^2 & f_2 \\ \cos^2\frac{\alpha_3}{2}\cos^2\frac{\beta_2}{2}\cos^2\varphi & \cos^2\frac{\alpha_3}{2}\cos^2\varphi & (\sin\frac{\alpha_3}{2}\cos\frac{\beta_2}{2} - \cos\frac{\alpha_3}{2}\sin\frac{\beta_2}{2}\sin\varphi)^2 - 1 & f_3 \\ \cos^2\frac{\alpha_4}{2}\cos^2\frac{\beta_4}{2}\cos^2\varphi & \cos^2\frac{\alpha_4}{2}\cos^2\varphi & (\sin\frac{\alpha_4}{2}\cos\frac{\beta_4}{2} - \cos\frac{\alpha_4}{2}\sin\frac{\beta_4}{2}\sin\varphi)^2 & f_4 \end{bmatrix}. \tag{5.31}$$

We notice that, the second column of the determinant in Eq. (5.31),

$$\vec{\tilde{p}}_2 := \begin{bmatrix} \cos^2\frac{\alpha_1}{2}\cos^2\varphi - 1 \\ \cos^2\frac{\alpha_2}{2}\cos^2\varphi - 1 \\ \cos^2\frac{\alpha_3}{2}\cos^2\varphi \\ \cos^2\frac{\alpha_4}{2}\cos^2\varphi \end{bmatrix} \tag{5.32}$$

only depends on $\vec{\alpha}$ and φ, i.e. solely under the control of the player X. Thus we could use the quantum ZD strategies to do the simulations.

Case 1 (Whether the quantum-strategy player X could unilaterally set the classical-strategy player Y's score):

Let the parameter α in Eq. (5.25) be 0, and X chooses the strategy

$$\vec{\tilde{p}}_2 = \begin{bmatrix} \cos^2\frac{\alpha_1}{2}\cos^2\varphi - 1 \\ \cos^2\frac{\alpha_2}{2}\cos^2\varphi - 1 \\ \cos^2\frac{\alpha_3}{2}\cos^2\varphi \\ \cos^2\frac{\alpha_4}{2}\cos^2\varphi \end{bmatrix} = \begin{bmatrix} \beta R + \gamma \\ \beta T + \gamma \\ \beta S + \gamma \\ \beta P + \gamma \end{bmatrix} = \beta \vec{S_Y} + \gamma \vec{1}.$$

We use the same symbol $p_i = \cos^2\frac{\alpha_i}{2}\cos^2\varphi$, where $i = 1, 2, 3, 4$. Then we get

$$\begin{aligned} p_2 &= \frac{(T-P)p_1 - (T-R)(1+p_4)}{R-P}, \\ p_3 &= \frac{(P-S)(1-p_1) + (R-S)p_4}{R-P}. \end{aligned} \tag{5.33}$$

It is noticed that Eq. (5.33) is same as Eq. (5.10). Consequently, in this case, the quantum-strategy player X could unilaterally set the classical-strategy player Y's score, where the player X's strategy is

$$p_1 = \cos^2\frac{\alpha_1}{2}\cos^2\varphi \to 1^-, \quad p_2 = \cos^2\frac{\alpha_2}{2}\cos^2\varphi \to 1^-,$$

$$p_3 = \cos^2\frac{\alpha_3}{2}\cos^2\varphi \to 0^+, \quad p_4 = \cos^2\frac{\alpha_4}{2}\cos^2\varphi \to 0^+,$$

and the player Y's score

$$s_y = \frac{P(1-p_1)+Rp_4}{(1-p_1)+p_4} \in [P, R].$$

Case 2 (Whether the quantum-strategy player X could unilaterally set her own score):

The player X chooses $\beta = 0$ and the strategy is shown as

$$\vec{\tilde{p}}_2 = \begin{bmatrix} \cos^2\frac{\alpha_1}{2}\cos^2\varphi - 1 \\ \cos^2\frac{\alpha_2}{2}\cos^2\varphi - 1 \\ \cos^2\frac{\alpha_3}{2}\cos^2\varphi \\ \cos^2\frac{\alpha_4}{2}\cos^2\varphi \end{bmatrix} = \begin{bmatrix} \alpha R + \gamma \\ \alpha S + \gamma \\ \alpha T + \gamma \\ \alpha P + \gamma \end{bmatrix} = \alpha\vec{S_X} + \gamma\vec{1}.$$

By using $p_i (i = 1, 2, 3, 4)$ as above, we obtain

$$\begin{aligned} p_2 &= \frac{-(P-S)p_1 + (R-S)(1+p_4)}{R-P} \geqslant 1, \\ p_3 &= \frac{-(T-P)(1-p_1) - (T-R)p_4}{R-P} \leqslant 0. \end{aligned} \tag{5.34}$$

The same as Eq. (5.13), Eq. (5.34) has only one feasible solution

$$p_1 = \cos^2\frac{\alpha_1}{2}\cos^2\varphi = 1, \quad p_2 = \cos^2\frac{\alpha_2}{2}\cos^2\varphi = 1,$$

$$p_3 = \cos^2\frac{\alpha_3}{2}\cos^2\varphi = 0, \quad p_4 = \cos^2\frac{\alpha_4}{2}\cos^2\varphi = 0,$$

i.e. $\alpha_1 = \alpha_2 = 0$, $\alpha_3 = \alpha_4 = \pi$ and $\varphi = 0$. Therefore, the player X cannot use the quantum ZD strategies to unilaterally set her own score.

Example 3 (quantum strategy vs quantum strategy):

In this example, we consider the case that both two players choose quantum strategies, which are achieved by setting $\varphi = \psi = \frac{\pi}{2}$. Then the Markov transition matrices in Eq. (5.22) and (5.23) become Eq. (5.35) and (5.36), respectively.

$$M_1 = \begin{bmatrix} \cos^2\frac{\alpha_1}{2}\cos^2\frac{\beta_1}{2} & \cos^2\frac{\alpha_1}{2}\sin^2\frac{\beta_1}{2} & \sin^2\frac{\alpha_1}{2}\cos^2\frac{\beta_1}{2} & \sin^2\frac{\alpha_1}{2}\sin^2\frac{\beta_1}{2} \\ \cos^2\frac{\alpha_2}{2}\cos^2\frac{\beta_3}{2} & \cos^2\frac{\alpha_2}{2}\sin^2\frac{\beta_3}{2} & \sin^2\frac{\alpha_2}{2}\cos^2\frac{\beta_3}{2} & \sin^2\frac{\alpha_2}{2}\sin^2\frac{\beta_3}{2} \\ \cos^2\frac{\alpha_3}{2}\cos^2\frac{\beta_2}{2} & \cos^2\frac{\alpha_3}{2}\sin^2\frac{\beta_2}{2} & \sin^2\frac{\alpha_3}{2}\cos^2\frac{\beta_2}{2} & \sin^2\frac{\alpha_3}{2}\sin^2\frac{\beta_2}{2} \\ \cos^2\frac{\alpha_4}{2}\cos^2\frac{\beta_4}{2} & \cos^2\frac{\alpha_4}{2}\sin^2\frac{\beta_4}{2} & \sin^2\frac{\alpha_4}{2}\cos^2\frac{\beta_4}{2} & \sin^2\frac{\alpha_4}{2}\sin^2\frac{\beta_4}{2} \end{bmatrix}. \tag{5.35}$$

$$M_2 = \begin{bmatrix} \cos^2\frac{\alpha_1}{2}\cos^2\frac{\beta_1}{2} & \sin^2\frac{\alpha_1}{2}\cos^2\frac{\beta_1}{2} & \cos^2\frac{\alpha_1}{2}\sin^2\frac{\beta_1}{2} & \sin^2\frac{\alpha_1}{2}\sin^2\frac{\beta_1}{2} \\ \cos^2\frac{\alpha_2}{2}\cos^2\frac{\beta_3}{2} & \sin^2\frac{\alpha_2}{2}\cos^2\frac{\beta_3}{2} & \cos^2\frac{\alpha_2}{2}\sin^2\frac{\beta_3}{2} & \sin^2\frac{\alpha_2}{2}\sin^2\frac{\beta_3}{2} \\ \cos^2\frac{\alpha_3}{2}\cos^2\frac{\beta_2}{2} & \sin^2\frac{\alpha_3}{2}\cos^2\frac{\beta_2}{2} & \cos^2\frac{\alpha_3}{2}\sin^2\frac{\beta_2}{2} & \sin^2\frac{\alpha_3}{2}\sin^2\frac{\beta_2}{2} \\ \cos^2\frac{\alpha_4}{2}\cos^2\frac{\beta_4}{2} & \sin^2\frac{\alpha_4}{2}\cos^2\frac{\beta_4}{2} & \cos^2\frac{\alpha_4}{2}\sin^2\frac{\beta_4}{2} & \sin^2\frac{\alpha_4}{2}\sin^2\frac{\beta_4}{2} \end{bmatrix}. \tag{5.36}$$

By adding the first column of $\det(M_1')$ or $\det(M_2')$ into the corresponding second and third columns, we will get

$$\vec{v}_1 \cdot \vec{f} = \det \begin{bmatrix} \cos^2\frac{\alpha_1}{2}\cos^2\frac{\beta_1}{2} - 1 & \cos^2\frac{\alpha_1}{2} - 1 & \cos^2\frac{\beta_1}{2} - 1 & f_1 \\ \cos^2\frac{\alpha_2}{2}\cos^2\frac{\beta_3}{2} & \cos^2\frac{\alpha_2}{2} - 1 & \cos^2\frac{\beta_3}{2} & f_2 \\ \cos^2\frac{\alpha_3}{2}\cos^2\frac{\beta_2}{2} & \cos^2\frac{\alpha_3}{2} & \cos^2\frac{\beta_2}{2} - 1 & f_3 \\ \cos^2\frac{\alpha_4}{2}\cos^2\frac{\beta_4}{2} & \cos^2\frac{\alpha_4}{2} & \cos^2\frac{\beta_4}{2} & f_4 \end{bmatrix},$$

$$\vec{v}_2 \cdot \vec{f} = \det \begin{bmatrix} \cos^2\frac{\alpha_1}{2}\cos^2\frac{\beta_1}{2} - 1 & \cos^2\frac{\beta_1}{2} - 1 & \cos^2\frac{\alpha_1}{2} - 1 & f_1 \\ \cos^2\frac{\alpha_2}{2}\cos^2\frac{\beta_3}{2} & \cos^2\frac{\beta_3}{2} - 1 & \cos^2\frac{\alpha_2}{2} & f_2 \\ \cos^2\frac{\alpha_3}{2}\cos^2\frac{\beta_2}{2} & \cos^2\frac{\beta_2}{2} & \cos^2\frac{\alpha_3}{2} - 1 & f_3 \\ \cos^2\frac{\alpha_4}{2}\cos^2\frac{\beta_4}{2} & \cos^2\frac{\beta_4}{2} & \cos^2\frac{\alpha_4}{2} & f_4 \end{bmatrix}.$$

It can be easily verified that the second column and the third column for both $\vec{v}_1 \cdot \vec{f}$ and $\vec{v}_2 \cdot \vec{f}$ only depend on $\vec{\alpha}$ or $\vec{\beta}$, i.e. they are solely under the control of the player X or Y. This property indicates that these two cases degenerate to the classical ones, and can be solved by a derivation similar to that in Section 5.1. Therefore, these kinds of strategies are named *quasi-classical zero-determinant (ZD) strategies.*

5.3.2 Analysis of results

Based on the numerical results in Subsection 5.3.1, and the comparison between the quantum zero-determinant strategies and the classical zero-determinant strategies, we state the summary as follows.

1. A type of quantum two-player zero-determinant (ZD) strategies is constructed.
2. By choosing two different operators $\hat{\sigma}_1$ and $\hat{\sigma}_2$, two numerical examples indicate the effectiveness of the proposed quantum zero-determinant strategies. More specifically, a quantum-strategy player X could succeed to unilaterally set her opponent's or her own score, by choosing different parameters of the quantum zero-determinant strategies.
3. It can be seen in Eq. (5.26) and (5.31) that, we could not obtain a column which only depends on the control of the classical-strategy player Y, no matter how we do linear transformation on columns. These results demonstrate the advantage of the quantum zero-determinant strategies.
4. Finally, we considered the case of a game between two quantum-strategy players. As shown in *Example 3*, what is interesting is that the quantum zero-determinant strategies degenerate to the classical zero-determinant ones. Thus we name these kinds of quantum zero-determinant strategies as *quasi-classical zero-determinant (ZD) strategies.*

Chapter 6

Conclusions and future work

This chapter draws conclusions on the thesis, and points out some possible research directions related to the work done in this thesis.

6.1 Conclusions

The focus of the thesis has been placed on two parts: coherent feedback control problem for linear quantum systems and two-strategy evolutionary game theory. Specifically, three research problems have been investigated in detail.

1. A *mixed* LQG and H_∞ coherent feedback control problem for a linear quantum system is considered. In this mixed control problem, LQG and H_∞ performances are not independent. Moreover, as the controller to be designed is another quantum-mechanical system, the "physical realizability conditions" should be satisfied, yielding the intricate polynomial matrix equality. A result for the lower bound of LQG index is proved. To solve this mixed problem, we propose two algorithms, rank constrained LMI method and genetic-algorithm-based method, by which the feasible solutions are attained in numerical simulations. Furthermore, the superiority of genetic algorithm (GA) is verified by the comparison between the numerical results of these two proposed methods.

2. In the static two-strategy game (e.g. Prisoner's Dilemma), defectors have higher fitness, thus in any mixed populations, defectors always dominate cooperators. Therefore, we focus on a class of evolutionary game with finite populations, and investigate a novel evolutionary model by adding tags to the game players. Then for given numerical examples, the corresponding evolutionary dynamics and stationary distributions are synthesized. Finally, three kinds of feasible strategies are obtained, yielding that C-strategy individual can succeed to invade into D-strategy population and cooperation can get stabilized. Moreover, the stationary distributions state the effectiveness of the proposed model.

3. To solve the problem that defectors always dominate cooperators in the static two-strategy game, we present another model, by using quantum strategies. Inspired by the classical zero-determinant (ZD) strategies in Press and Dyson (2012), a class of two-player quantum zero-determinant (ZD) strategies is proposed. Three kinds of numerical examples are given, which show that quantum zero-determinant strategies have significant advantage over the classical zero-determinant strategies. When a quantum-strategy player X plays a game with a classical-strategy player Y, X could choose her own strategies to unilaterally set her own score or Y's score, whereas Y could never to do so. What's interesting is that, when both the players X and Y choose quantum strategy, it is the same as the classical case, which are named as quasi-classical zero-determinant (ZD) strategies.

6.2 Future Work

Related topics for the future research work are listed below.

1. In Chapter 3, the proposed mixed control problem is for continuous quantum

systems, while the discrete system theory in both classical and quantum cases has been developed in the literature. Therefore, in the future work, considering the mixed LQG and H_∞ feedback control problem for discrete quantum systems is worthwhile and challenging. Moreover, the "physical realizable conditions" may not always be reached by a practical Hamiltonian, so the conditions of reaching physical realizability for a practical Hamiltonian will also be considered in the future.

2. For the classical evolutionary game theory with tags presented in Chapter 4, it assumed that a game could only occurred between two players, with only two strategies C (cooperation)and D (defection). But many results have been attained for multi-strategy and multi-player game theory, respectively. Thus in the future, investigating the multi-strategy/multi-player evolutionary game theory with tags is meaningful and interesting.

3. Both Chapter 4 and Chapter 5 focus on seeking new kinds of strategies to solve the problem that defectors always dominate cooperators in the static two-strategy game, which inspires us to combine these two proposed strategies together, so as to yield a class of quantum zero-determinant strategies with tags. It will also be a topic of my future research.

4. Depending on the classical zero-determinant strategies, a kind of classical extortion strategy is investigated in Press and Dyson (2012). Similarly as this, it is worthy to discuss potential results about quantum extortion strategy in the future.

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