BOUNDARY LAYER PROBLEMS IN CHEMOTAXIS MODELS

QIANQIAN HOU

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The Hong Kong Polytechnic University

Department of Applied Mathematics

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Qianqian HOU

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Dedicate to my parents.

Certificate of Originality

I hereby declare that this thesis is my own work and that, to the best of my knowledge and belief, it reproduces no material previously published or written, nor material that has been accepted for the award of any other degree or diploma, except where due acknowledgement has been made in the text.

_____________________(Signed)

______HOU Qianqian___(Name of student)

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Abstract

This thesis is concerned with the zero-diffusion limit and boundary layers of a viscous hyperbolic system transformed via a Cole-Hopf transformation from a singular chemotactic system modeling numerous biological processes, such as traveling waves of bacterial chemotaxis[36], boundary movement of bacterial population in response to chemotaxis by Nossal (cf. [64]) and the initiation of tumor angiogenesis proposed in [41].

It was numerically found in [44] that when prescribed with Dirichlet boundary conditions, the considered system exhibits boundary layer phenomena at the boundaries in a bounded interval $(0,1)$ as the chemical diffusion rate (denoted by $\varepsilon > 0$) is small, while the rigorous justification still remains open. The purpose of this thesis will be to develop some mathematical theories for the boundary layer solutions of chemotaxis models in one and multi-dimensions and hence to justify the numerical findings of [44] with further development in multi-dimensions. We first show the existence of boundary layers (BLs) in one dimension, where outside the BLs the solution with $\varepsilon > 0$ converges to the one with $\varepsilon = 0$, but inside the BLs the convergence no longer holds. We then proceed to prove the stability of boundary layer solutions and identify its precise structure. Roughly speaking, we justify that the solution with $\varepsilon > 0$ converges to the solution with $\varepsilon = 0$ (outer layer) plus the (inner) boundary layer solutions with the optimal rate at order of $O(\varepsilon^{1/2})$, where the outer and inner layer solutions are well determined by explicit equations.

For the multi-dimensional case, motivated from the study in one dimension, we first study the boundary layer problem for radial solutions in an annulus and show the existence of boundary layers. Then we study the system in a half-plane of $\mathbb{R}^2$ subject to Dirichlet boundary conditions and prove the stability of boundary layer solutions with explicit outer and inner layer profiles. Finally, we covert the result for the transformed system to the original pre-transformed chemotaxis system and discuss the biological implications of our results.

Boundary layer formation in chemotaxis has been observed in the real experiment [78] and its theoretical study is just in its infant stage. This thesis develops the first theoretical results on the boundary layers of chemotaxis models and will pave the road for the further studies on the boundary layer theories of general/different chemotaxis models to explain the experimental observations of boundary layer phenomena of chemotaxis such as the one [78].

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Notations

Ω	$(0,1)$ in Chapter 2 and Chapter 3; $\{\vec{x}=(x_1,x_2,\cdots,x_d)\in\mathbb{R}^d \mid 0<a<\|\vec{x}\|<b\}$ in Chapter 4; $\mathbb{R}^2_+=\{(x,y)\in\mathbb{R}^2 \mid y>0\}$ in Chapter 5.
L^p	$L^p(0,1)$ with respect to $x\in(0,1)$ in Chapter 2 and Chapter 3; $L^p(a,b)$ with respect to $r\in(a,b)$ in Chapter 4; $L^p(\mathbb{R}^2_+)$ with respect to $(x,y)\in\mathbb{R}^2_+$ in Chapter 5.
H^k	$W^{k,2}(0,1)$ with respect to $x\in(0,1)$ in Chapter 2 and Chapter 3; $W^{k,2}(a,b)$ with respect to $r\in(a,b)$ in Chapter 4; $W^{k,2}(\mathbb{R}^2_+)$ with respect to $(x,y)\in\mathbb{R}^2_+$ in Chapter 5.
z	$\frac{x}{\sqrt{\varepsilon}}$ with $x\in[0,1]$.
ξ	$\frac{x-1}{\sqrt{\varepsilon}}$ with $x\in[0,1]$.
η	$\frac{y}{\sqrt{\varepsilon}}$ with $y\in[0,\infty)$.
$\langle z\rangle$	$\sqrt{z^2+1}$.
$\langle \xi\rangle$	$\sqrt{\xi^2+1}$.
$\langle \eta\rangle$	$\sqrt{\eta^2+1}$.
L^p_x	$L^p(\mathbb{R})$ with respect to $x\in\mathbb{R}$.
L^p_z	$L^p(0,\infty)$ with respect to $z\in(0,\infty)$.
L^p_ξ	$L^p(-\infty,0)$ with respect to $\xi\in(-\infty,0)$.
H^k_x	$W^{k,2}(\mathbb{R})$ with respect to $x\in\mathbb{R}$
H^k_z	$W^{k,2}(0,\infty)$ with respect to $z\in(0,\infty)$.
H^k_ξ	$W^{k,2}(-\infty,0)$ with respect to $\xi\in(-\infty,0)$.
$L^p_{x\eta}$	$L^p(\mathbb{R}^2_+)$ with respect to $(x,\eta)\in\mathbb{R}^2_+$.
$H^k_{x\eta}$	$W^{k,2}(\mathbb{R}^2_+)$ with respect to $(x,\eta)\in\mathbb{R}^2_+$.
$H^k_x H^m_\eta$	$\{f(x,\eta)\in L^2(\mathbb{R}^2_+) \mid \sum_{0\le k'\le k, 0\le m'\le m} \|\partial_x^{k'}\partial_z^{m'} f(x,\eta)\|_{L^2_{x\eta}}<\infty\}$.
$\|\cdot\|_{L^q_T\mathbf{X}}$	$\|\cdot\|_{L^q(0,T;\mathbf{X})}$ for Banach space $\mathbf{X}$.
C	a generic positive constant independent of ε, depending on time variable T.
C_0	a generic positive constant independent of ε and time T.
ε	is assumed less than 1 for we consider the diffusion limit problem as $\varepsilon\to 0$.

Chapter 1

Introduction

1.1 Chemotaxis and the Mathematical Model

Chemotaxis, in contrast to random walk describes the oriented movement of an organism/species in response to a chemical stimulus spread in their living environment. In particular, it is classified into attractive and repulsive chemotaxis by the nature of the motion towards or away from the higher concentration of the stimulus, respectively. Chemotaxis has been an important mechanism of various biological processes. For instance, in the bacterial aggregation process, the E-coli secretes aspartate to guide other E-coli to move towards the region with high density of population where more aspartate is excreted to form pattern, cf. [62, 79]. In the slime mould formation, the amoebae cells direct their motion towards the increasing direction of cyclic adenosine $3^{'}5^{'}$-monophosphate, cf. [25]. The white stripes on the skin of angelfish is a consequence of the repulsion of the dark pigment to iridophores, which is the reservoir of the white pigment, cf. [67]. In the early stage of avian embryo development, large amount of cells migrate in response to a chemotactic attractant to produce a trilaminar blastoderm, cf. [68]. In the blood vessel formation, the connected vascular network results from the chemotactic motion of blood vessel cells towards the higher concentration of a soluble factor, cf. [19]. In the wound healing process, the clot near the wound contains a grow factor, which recruits circulating inflammatory cells moving to the wound site to cue the wound, cf. [70]. Chemotaxis also plays an important role in tumor angiogenesis, cf. [8, 10, 11].

The mathematical chemotaxis model, known as Keller-Segel (KS) model was first proposed by Keller and Segel in their seminal works [34–36] and reads in its general form as

$$\begin{cases} u_t = \nabla \cdot (D\nabla u - \chi u \nabla \phi(c)), \\ c_t = \varepsilon \Delta c + g(u,c), \end{cases} \tag{1.1}$$

where $u(\vec{x},t)$ and $c(\vec{x},t)$ denote the cell density and chemical (signal) concentration at position $\vec{x}$ and time t, respectively. The sensitivity function $\phi(c)$ accounts for the signal response mechanism and $g(u,c)$ is the chemical kinetics (birth and death). $D > 0$ and $\varepsilon \geq 0$ are respectively the coefficients of cell and chemical diffusion. $\chi \neq 0$ denotes the chemotactic coefficient with $|\chi|$ measuring the strength of the chemotactic sensitivity, where $\chi > 0$ and $\chi < 0$ indicates the chemotaxis is attractive and repulsive, respectively. The application of (1.1) generically depends on the specific forms of $\phi(c)$ and $g(u,c)$. There are two major classes of chemotactic sensitivity functions: linear law $\phi(c) = c$ and logarithmic law $\phi(c) = \ln c$. The former was originally derived in [34, 35] to model the self-aggregation of *Dictyostelium discoideum* in response to cyclic adenosine monophosphate (cAMP), while the latter was first employed in [36] to model the wave propagation of bacterial chemotaxis though it has many other prominent applications in biology (cf. [2, 3, 12, 32, 40, 66]). Compared with massive well-known results on the KS system with linear chemotactic sensitivity (cf. [4, 5, 24, 28]), not much results are available for the logarithmic sensitivity due to its singularity nature (at $c = 0$). We aim to study the following attractive chemotaxis model with logarithmic sensitivity function:

$$\begin{cases} u_t = \nabla \cdot [D\nabla u - \chi u \nabla(\ln c)], \\ c_t = \varepsilon \Delta c - \mu u c, \end{cases} \tag{1.2}$$

which was originally proposed by Keller and Segel in [36] to describe the bacterial wave propagation and then applied to model the boundary movement of bacterial population in response to chemotaxis by Nossal (cf. [64]). Levine and his collaborators later employ (1.2) (cf. [41]) to interpret the initiation of tumor angiogenesis, where the chemotactic motion of the vascular endothelial cells (VECs) denoted by u, is stimulated by the vascular endothelial growth factor (VEGF), denoted by c. Though bearing specific applications, (1.2) is of great challenge to be analyzed directly, due to the logarithmic singularity at $c = 0$. The successful way to overcome this singularity is to apply a Cole-Hopf type transformation (cf. [40, 49])

$$\vec{v} = -\frac{\sqrt{\chi\mu}}{\mu}\nabla \ln c = -\frac{\sqrt{\chi\mu}}{\mu}\frac{\nabla c}{c}, \tag{1.3}$$

and transforms (1.2) into

$$\begin{cases} u_t - \nabla \cdot (u\vec{v}) = \Delta u, \qquad (\vec{x},t) \in \Omega \times (0,\infty), \\ \vec{v}_t - \nabla(u - \frac{\varepsilon}{\chi}|\vec{v}|^2) = \frac{\varepsilon}{D}\Delta \vec{v}, \\ (u,\vec{v})(\vec{x},0) = (u_0,\vec{v}_0)(\vec{x}), \end{cases} \tag{1.4}$$

with domain $\Omega \subset \mathbb{R}^d$. The transformed system (1.4) has attracted extensive studies from both numerical and analytic aspects. However its well-posedness in multi-dimension is merely confined to local and global small solutions. With $\Omega = \mathbb{R}^d$ ($d \geq 2$) and $\varepsilon = 0$, Li et al. (cf. [42]) derived the unique global solution with small initial data in $H^s \times H^s$ ($s > \frac{d}{2} + 1$), which was later improved by Deng and Li (cf. [14]) by only assuming the smallness of initial data in space $L^2 \times H^1$. Peng et al. (cf. [69]) further improved this result (with $\Omega = \mathbb{R}^3$) by replacing the smallness assumption space $L^2 \times H^1$ with $L^2 \times L^2$ and proved the global well-posedness in space $H^3 \times H^3$ for both $\varepsilon = 0$ and $\varepsilon > 0$. The existence space for solution was recently extended to $H^k \times H^k$ with $k \geq 2$ for $d = 2,3$ in [82]. The global well-posedness in Besov space was also studied by Hao (cf. [23]) with small data for $\varepsilon = 0$. With bounded domain $\Omega \subset \mathbb{R}^d$ ($d = 2,3$), problem (1.4) subject to the Neumann-Dirichlet boundary condition was investigated in [46] and the unique global small solution in space $H^3 \times H^3$ was derived when $\varepsilon = 0$.

In contrast to the multi-dimensional case, (1.4) in dimension one is well-understood and the unique global large solutions are obtained with various options of Ω. When $\Omega = \mathbb{R}$, the solution is proved in space $H^2 \times H^2$ and decay to zero as $t \to \infty$ by Li et al. (cf. [43]) for $\varepsilon = 0$ and by Martinez et al. (cf. [59]) for $\varepsilon > 0$. With $\Omega = (0,1)$, the solution subject to the Neumann-Dirichlet boundary conditions is still well-posed in space $H^2 \times H^2$ and further satisfies an exponential decay in time, cf. [46] for $\varepsilon = 0$ and [83] for small $\varepsilon > 0$. This result is extended to arbitrary $\varepsilon > 0$ by Tao et al. (cf. [76]), where the global solution is derived in space $C^{2,1}([0,1] \times (0,\infty))$ and decays to 0 in space $L^\infty \times L^\infty$ as $t \to \infty$. When subject to the Dirichlet boundary conditions, Zhang and Zhu (cf. [87]) obtained the global solution with small initial data in $H^2 \times H^2$, which was later extended to arbitrary large initial data for $\varepsilon \geq 0$ in [44]. Moreover, the existence and stability of traveling wave solutions have been studied in [7, 31, 45, 47–50].

Except the above results on the well-posedness, the zero-diffusion (inviscid) limit of (1.2) is a particularly relevant issue, since it is pointed out in [41] that the magnitude of the diffusion rate ε of the chemical VEGF can be negligible compared to the diffusion of VECs in the initiation of tumor angiogenesis. Moreover, the diffusion rate ε was assumed to be zero in the analysis of [36] and many subsequent works for simplicity (cf. [81]). In particular, the solution is justified to be convergent in ε when $\Omega = \mathbb{R}^d$ in [69] for $d = 3$, [82] for $d = 2$ and [59] for $d = 1$. With $\Omega = (0,1)$, the solution subject to Neumann-Dirichlet boundary conditions is still convergent in ε (cf. [83]).

1.2 History of Boundary Layers

The concept of boundary layers was first proposed by Prandtl in 1904 (cf. [71]), to analyze viscous flows about a solid body. In the field of fluid dynamics, the well-know mathematical model used to describe the evolution of viscous incompressible flows is the Navier-Stokes equations, which reads as

$$\begin{cases} \vec{w}_t + \vec{w} \cdot \nabla \vec{w} + \nabla p = \varepsilon \Delta \vec{w}, \quad (\vec{x},t) \in \ \Omega \times (0,\infty) \\ \nabla \cdot \vec{w} = 0, \\ \vec{w}(\vec{x},0) = \vec{w}_0(\vec{x}), \end{cases} \tag{1.5}$$

where $\vec{w} : \Omega \times \mathbb{R}^+ \to \mathbb{R}^d$ and $p : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ represent the velocity and pressure of the fluid respectively, with domain $\Omega \subset \mathbb{R}^d$. The first equation comes from the conversation of momentum, where the term $\varepsilon \Delta \vec{w}$ on the right-hand side represents the friction force due to the viscosity of the fluid, and $\varepsilon \geq 0$ is viscosity coefficient of the fluid. The second equation is the incompressible conditions, which implies that during the evolution of flow, the volume of individual fluid element never change. When $\varepsilon > 0$, this system models the evolution of viscous fluid and is called the Navier-Stokes equations. When $\varepsilon = 0$, this system describes the evolution of inviscid fluid and is called the Euler equations.

Most of theoretical studies in fluid dynamics are based on the inviscid fluid and it provides in many circumstances a satisfactory approximation of real motions, in a high degree of completeness. Due to the absence of viscosity, in the motion of such inviscid fluid, the drag force (friction force) between each individual fluid element is zero, which means the fluid element does not experience any forces that parallel to the velocity. Hence, when the inviscid fluid flows along a motionless solid wall, there is no friction between them, and in general there exists a difference in relative tangential velocity. However, the real fluids even those with very small viscosity would adhere to the solid wall at the boundary due to the intermolecular attractions. Thus for the Euler equations (model of the inviscid fluid) one should not prescribe boundary condition on the tangential direction of the boundary. In contrast the Navier-Stokes equations should subject to the no slip boundary condition. Hence the boundary conditions (e.g. see [15]) ought to be:

$$\begin{cases} \vec{w}|_{\partial\Omega} = 0, & \text{if } \varepsilon > 0; \\ \vec{w}\cdot\vec{n}|_{\partial\Omega} = 0, & \text{if } \varepsilon = 0. \end{cases} \tag{1.6}$$

Here $\vec{n}$ is the unit outer normal vector at $\partial\Omega$.

As we can see on the boundary, for Navier-Stokes equations (the real fluid) the tangential velocity is zero, while for Euler equations the tangential velocity is determined by the system itself and may not equal to zero. Hence, one can not use the solution of the Euler equations to approximate the solutions of the Navier-Stokes equations on the whole domain Ω. This conforms to the above argument of the difference between perfect and real fluid near the boundary. Nevertheless, in the region away from boundary, the Navier-Stokes equations with small ε are well approximated by the Euler equations.

Actually, Prandtl (cf. [71]) proposed the concept of boundary layer to analyse this question. Combining both theoretical analysis and experimental results, he showed that the flow along a solid wall consists of two regions: a very thin layer in the neighbourhood of the boundary (boundary layer) where friction is crucial, and the remaining region outside the layer, where friction may be neglected. A great deal of mathematical investigations concerning the vanishing diffusion limit for the Navier-Stokes equations with boundaries have been conducted since Prandtl's boundary layer theory was introduced. However, this theory was merely rigorously verified under some specific circumstances, since the well-posedness on the Prandtl equations (boundary layer solutions) with general initial data still remains unjustified. In particular, the boundary layer theory for the Navier-Stokes was strictly proved in [73, 74] with analytic initial data, in [6, 37, 54, 55, 60, 61] with circularly symmetric domain and initial data, in [33, 37, 38, 80] for L^2-convergence of the solutions and in [56, 57] with initial data satisfying curl-free near the boundary. Moreover, for $\Omega = \{(x,y,z) \in \mathbb{R}^3 : z > 0\}$, by replacing the diffusion term $\varepsilon\Delta\vec{w}$ with $\kappa(\partial_{xx}\vec{w}+\partial_{yy}\vec{w})+\varepsilon\partial_{zz}\vec{w}$, Liu and Wang (cf. [52]) proved the convergence of solutions in space L^∞ as $\varepsilon \to 0$.

1.3 Contributions and Organization of the Thesis

In this thesis, we shall investigate the zero-diffusion (inviscid) limit of problem (1.4) subject to the Dirichlet boundary conditions with $\Omega = (0,1)$, $\Omega = \{\vec{x} \in \mathbb{R}^d \mid a < |\vec{x}| < b\}$ and $\Omega = \mathbb{R}^2_+$, respectively. This work is originally motivated by the boundary layer phenomenon discovered by Li and Zhao (cf. [44]) when studying asymptotic behavior of solutions of (1.4) with $\Omega = (0,1)$ subject to the following Dirichlet boundary conditions:

$$\begin{cases} u|_{x=0,1} = \bar{u} \geq 0, \quad v|_{x=0,1} = \bar{v}, & \text{if } \varepsilon > 0, \\ u|_{x=0,1} = \bar{u} \geq 0, & \text{if } \varepsilon = 0; \end{cases} \tag{1.7}$$

where the boundary value for v with $\varepsilon = 0$ is not prescribed, since it is intrinsically determined by the second equation of (1.4). Due to this mismatched boundary values of v between $\varepsilon > 0$ and $\varepsilon = 0$, the solution component v would diverge in a thin layer (boundary layer) near the boundary as $\varepsilon \to 0$, but still converge outside the boundary layer, which is in a similar scenario as the aforementioned emergence of boundary layer theory in the fluid dynamics. The observation of [44] marked the starting point for the study of boundary layer theory in chemotaxis and moreover they numerically illustrated the presence of boundary layers for solution component v, however the rigorous justification is left open.

In Chapter 2, we shall rigorously prove the existence of boundary layers (see Theorem 2.2) for the solution component v of (1.4), (1.7) as $\varepsilon \to 0$ and complement the numerical findings of [44] with analytical verifications. Chapter 3 is devoted to exploiting the structure of v^ε (solution component with $\varepsilon > 0$) inside the boundary layers to derive a uniform approximation for v^ε in the entire interval $[0,1]$ and justify the stability of boundary layers. Roughly speaking, we prove that the solution component v^ε converges to v^0 (outer layer) plus the boundary (inner) layer solutions $v^{B,0}$ and $v^{b,0}$ (see Theorem 3.1) with the optimal rate at order of $O(\varepsilon^{1/2})$ as $\varepsilon \to 0$, where the inner layer solutions are well determined by explicit equations and the thickness of boundary layers is strictly justified as $O(\varepsilon^{1/2})$. We then transfer the results to the original pre-transformed chemotaxis system and discuss the implications of our results.

Inspired by the one dimensional case, we proceed to investigate the boundary layer problem of (1.4) in multi-dimensions in Chapter 4 and Chapter 5. When $\Omega = \{\vec{x} = (x_1, x_2, \cdots, x_d) \mid 0 < a < |\vec{x}| < b\}$, we prove in Chapter 4 that the radial solutions of (1.4) subject to Dirichlet boundary conditions also possess boundary layers near $|\vec{x}| = a$ and $|\vec{x}| = b$. Moreover, with $\Omega = \mathbb{R}^2_+ = \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$, Chapter 5 is devoted to studying the boundary layer problem of (1.4) subject to the following boundary conditions:

$$\begin{cases} u|_{y=0} = \bar{u}(x,t),\ \nabla \times \vec{v}|_{y=0} = 0,\ v_2|_{y=0} = \bar{v}(x,t), & \text{if } \varepsilon > 0; \\ u|_{y=0} = \bar{u}(x,t), & \text{if } \varepsilon = 0, \end{cases} \tag{1.8}$$

where the condition $\nabla \times \vec{v}|_{y=0} = 0$ is the compatible condition to guarantee that $\nabla \times \vec{v}(x,y,t) \equiv 0$ holds for $(x,y,t) \in \mathbb{R}^2_+ \times (0,\infty)$ so that the result of (1.4) can be converted back to the pre-transformed model (1.2) via (1.3). Due to the mismatched boundary values for v_2, the boundary layer phenomenon would emerge for system (1.4), (1.8). Actually, in Chapter 5 we prove that the solution component v_2^ε ($(u^\varepsilon, \vec{v}^\varepsilon) = (u^\varepsilon, v_1^\varepsilon, v_2^\varepsilon)$ denotes the solution with $\varepsilon > 0$) converges to the outer layer solution v_2^0 ($(u^0, \vec{v}^0)$ represents the solution with $\varepsilon = 0$) plus the boundary layer solution $v_2^{B,0}$ as $\varepsilon \to 0$ (see Theorem 5.2), while the solution components u^ε and v_1^ε converge to u^0 and v_1^0, respectively.

The challenges encountered and the main ideas employed in proving the above results of this thesis will be specified at the beginning of each chapter, refer to the discussions below Remark 2.2 (in Chapter 2), Theorem 3.2 (in Chapter 3), Proposition 4.1 (in Chapter 4) and Proposition 5.4 (in Chapter 5) for detail.

We clarify that the results of Chapter 2 have been published as part of our paper [29].

Chapter 2

Existence of Boundary Layers in One Dimension

In this chapter, we shall investigate the boundary layer problem of (1.4) in one-dimension with $\Omega = (0,1)$. For brevity, we take $D = \chi = \mu = 1$ in (1.4) to derive the following initial-boundary problem

$$\begin{cases} u_t - (uv)_x = u_{xx}, & (x,t) \in (0,1) \times (0,\infty), \\ v_t - (u - \varepsilon |v|^2)_x = \varepsilon v_{xx}, \\ (u,v)(x,0) = (u_0, v_0)(x), \end{cases} \tag{2.1}$$

subject to the Dirichlet boundary conditions

$$\begin{cases} u|_{x=0,1} = \bar{u} \geq 0, \quad v|_{x=0,1} = \bar{v}, & \text{if } \varepsilon > 0, \\ u|_{x=0,1} = \bar{u} \geq 0, & \text{if } \varepsilon = 0. \end{cases} \tag{2.2}$$

We emphasize that the results of this thesis hold for general values of $D > 0$, $\chi > 0$ and $\mu > 0$. For illustration, let us denote by $(u^\varepsilon, v^\varepsilon)$ and (u^0, v^0) the solutions of (2.1)-(2.2) with $\varepsilon > 0$ and $\varepsilon = 0$, respectively. Due to the mismatched boundary values for v between $\varepsilon > 0$ and $\varepsilon = 0$, the boundary layer phenomenon would appear for the above system (2.1)-(2.2) as $\varepsilon \to 0$. Indeed, we rigorously prove the existence of boundary layers for the solution component v and the uniform convergence for u in Theorem 2.2, which complement the numerical findings of [44] by analytical justifications. The main ideas to prove Theorem 2.2 are given below Remark 2.2. Section 2.2 and Section 2.3 are devoted to the proof.

Li and Zhao (cf. [44]) proved the global well-posedness of classical solutions to system (2.1)-(2.2) with $\varepsilon \geq 0$. We cite the results below for later use.

Lemma 2.1 ([44]). *Suppose that $(u_0, v_0) \in H^2 \times H^2$ with $u_0 \geq 0$ satisfy the compatibility conditions $(u_0, v_0)(0) = (u_0, v_0)(1) = (\bar{u}, \bar{v})$. Then for any $\varepsilon \geq 0$, the initial-boundary value problem* (2.1)-(2.2) *has unique global classical solution $(u^\varepsilon, v^\varepsilon)$ satisfying the following properties:*

(i) If $\varepsilon > 0$, then $(u^\varepsilon - \bar{u}, v^\varepsilon - \bar{v}) \in C([0,\infty); H^2 \times H^2) \cap L^2(0,\infty; H^3 \times H^3)$ such that

$$\|(u^\varepsilon - \bar{u})(t)\|_{L^2}^2 + \|(v^\varepsilon - \bar{v})(t)\|_{L^2}^2 + \int_0^t (\|u_x^\varepsilon(\tau)\|_{L^2}^2 + \varepsilon \|v_x^\varepsilon(\tau)\|_{L^2}^2)\, d\tau \le C,$$

where C is a positive constant independent of ε.
(ii) If $\varepsilon = 0$, then $(u^0 - \bar{u}, v^0) \in C([0,\infty); H^2 \times H^2) \cap L^2(0,\infty; H^3 \times H^2)$.

2.1 Results on Existence of Boundary Layers

In this section, we establish the existence results of boundary layers for initial-boundary problem (2.1)- (2.2). To this end, we need the following uniform-in-ε bound of solutions with $\varepsilon > 0$, which is the key to show the existence of boundary layer solutions.

Theorem 2.1 (uniform-in-ε estimates)**.** *Assume that $(u_0, v_0) \in H^2$ and satisfies the compatible condition $(u_0, v_0)(0) = (\bar{u}, \bar{v})$. Let $(u^\varepsilon, v^\varepsilon)$ be the unique global solution of the system* (2.1)-(2.2) *with $\varepsilon > 0$ obtained in Lemma 2.1. Then for any $0 < T < \infty$, the following estimates hold*

$$\sup_{0 \le t \le T} \left(\|u_x^\varepsilon\|_{L^2}^2 + \varepsilon^{1/2} \|v_x^\varepsilon\|_{L^2}^2 \right)(t) + \int_0^T \left(\varepsilon^{1/2} \|u_{xx}^\varepsilon\|_{L^2}^2 + \varepsilon^{3/2} \|v_{xx}^\varepsilon\|_{L^2}^2 \right) dt \le C, \tag{2.3}$$

where C is a positive constant independent of ε.

Then the results on the existence of boundary layers for the transformed problem (2.1)-(2.2) are given in the following theorem.

Theorem 2.2. *Assume the conditions of Theorem 2.1 hold. Let $(u^\varepsilon, v^\varepsilon)$ and (u^0, v^0) be the solution of system* (2.1)-(2.2) *corresponding to $\varepsilon > 0$ and $\varepsilon = 0$, respectively. Then for any non-negative function $\delta(\varepsilon)$ satisfying*

$$\delta(\varepsilon) \to 0 \text{ and } \varepsilon^{1/2}/\delta(\varepsilon) \to 0, \text{ as } \varepsilon \to 0$$

and for any $0 < T < \infty$, we have

$$\|u^\varepsilon - u^0\|_{L^\infty(0,T;C[0,1])}^2 < C\varepsilon^{1/2} \tag{2.4}$$

and

$$\|v^\varepsilon - v^0\|_{L^\infty(0,T;C[\delta,1-\delta])}^2 < C\delta^{-1}\varepsilon^{1/2}, \tag{2.5}$$

$$\liminf_{\varepsilon \to 0} \|v^\varepsilon - v^0\|_{L^\infty(0,T;C[0,1])} > 0, \tag{2.6}$$

if and only if

$$\int_0^t u_x^0(0,\tau)\,d\tau \neq 0, \quad \text{or} \quad \int_0^t u_x^0(1,\tau)\,d\tau \neq 0, \quad \text{for some } t \in [0,T], \tag{2.7}$$

where the constant C *is independent of* ε*. That is the problem* (2.1)-(2.2) *has a boundary layer solution as* $\varepsilon \to 0$ *iff (2.7) holds.*

Remark 2.1. *If* $u_{0x}(0) \neq 0$ *or* $u_{0x}(1) \neq 0$*, then the condition (2.7) in Theorem 2.2 is satisfied.*

Remark 2.2. *Following the nomenclature of [18], the function* $\delta(\varepsilon)$ *in Theorem 2.2 is called the BL-thickness (to measure the thickness of boundary layer), which however does not uniquely determine the thickness of the boundary layer (as stated in [18]), since the function* $\delta(\varepsilon) = \varepsilon^{\alpha}$ *with any* $0 < \alpha < \frac{1}{2}$ *is also a BL-thickness. Indeed, the thickness was showed being exactly of order* $O(\varepsilon^{1/2})$ *in our paper [29, Appendix] by performing a formal asymptotic analysis to solutions* $(u^{\varepsilon}, v^{\varepsilon})$ *(with small* $\varepsilon > 0$*) based on WKB method, and we shall study the boundary layer stability in Chapter 3 and hence justify that BL-thickness of the problem* (2.1)-(2.2) *is* $O(\varepsilon^{1/2})$*.*

Before proceeding, we outline the main ideas employed to prove Theorem 2.2. The uniform-in-ε estimate (2.3) is the key for the proof of Theorem 2.2. The standard energy method as employed in [44] only can give the estimates depending on ε due to appearance of the boundary term $\varepsilon(v_x^{\varepsilon} u_x^{\varepsilon})|_{x=0}^{x=1}$. For example, the following estimates was obtained in [44, Lemma 2.3]):

$$\|u_x^{\varepsilon}\|_{L^\infty(0,T;L^2)}^2 + \|v_x^{\varepsilon}\|_{L^\infty(0,T;L^2)}^2 + \|u_{xx}^{\varepsilon}\|_{L^2(0,T;L^2)}^2 + \varepsilon \|v_{xx}^{\varepsilon}\|_{L^2(0,T;L^2)}^2 \leq C\varepsilon^{-1}$$

where C is a constant independent of ε. Thus to derive the solution convergence as $\varepsilon \to 0$, one needs new approach to get the estimates of the boundary term $\varepsilon(v_x^{\varepsilon} u_x^{\varepsilon})|_{x=0}^{x=1}$. Observing that by integrating $(3.18)_1$ with respect to x, $u_x^{\varepsilon}|_{x=0,x=1}$ can be expressed in terms of u_t^{ε}, and hence bounded by $\|u_t^{\varepsilon}\|_{L^2}$ and other controllable terms, where $\|u_t^{\varepsilon}\|_{L^2}$ can be estimated by the routine L^2-energy estimate thanks to the condition $u_t^{\varepsilon}|_{x=0,x=1} = v_t^{\varepsilon}|_{x=0,x=1} = 0$ (see Lemma 2.2). Based on this crucial observation, we undertake a refined estimates for $\varepsilon(v_x^{\varepsilon} u_x^{\varepsilon})|_{x=0}^{x=1}$, which readily gives rise to (2.3) by employing various inequalities (see the proof of Lemma 2.3). With the key estimates (2.3), we prove Theorem 2.2 by exploiting the weighted L^2-method, inspired from a work [30]. By a delicate computation, we succeed in deriving the weighted L^2-estimate (see Lemma 2.6):

$$\int_0^1 \omega(x)|(v^{\varepsilon} - v^0)_x|^2(x,t)\,dx + \varepsilon \int_0^t \int_0^1 \omega(x)|(v^{\varepsilon} - v^0)_{xx}|^2(x,\tau)\,dxd\tau \leq C\varepsilon^{1/2},$$

where $\omega(x) := x^2(1-x)^2$, $x \in [0,1]$. Then we can readily derive (2.5) based on the above estimates.

2.2 Proof of Theorem 2.1.

Suppose that $(u^\varepsilon, v^\varepsilon)$ is the unique global solution to the system (2.1)-(2.2) with $\varepsilon > 0$ given in Lemma 2.1. In this section we are devoted to deriving the uniform estimate (2.3) for u^ε and v^ε, and thus prove Theorem 2.1. Let $\tilde{u} = u^\varepsilon - \bar{u}$, $\tilde{v} = v^\varepsilon - \bar{v}$. Substituting $\tilde{u}$ and $\tilde{v}$ into (2.1) and (2.2), we can reformulate the problem (2.1)-(2.2) as

$$\begin{cases} \tilde{u}_t = \tilde{u}_{xx} + (\tilde{u}\tilde{v})_x + \bar{u}\tilde{v}_x + \bar{v}\tilde{u}_x, \\ \tilde{v}_t = \varepsilon\tilde{v}_{xx} - \varepsilon[(\tilde{v})^2]_x - 2\varepsilon\bar{v}\tilde{v}_x + \tilde{u}_x, \\ (\tilde{u},\tilde{v})(x,0) = (u_0 - \bar{u}, v_0 - \bar{v})(x), \\ \tilde{u}|_{x=0,x=1} = 0, \quad \tilde{v}|_{x=0,x=1} = 0. \end{cases} \tag{2.8}$$

With the uniform L^2 estimates in Lemma 2.1, we proceed to derive the higher order estimates in the following Lemma 2.2 and Lemma 2.3.

Lemma 2.2. *Suppose that the assumptions in Theorem 2.1 hold. Then for any* $0 < T < \infty$, *there exists a positive constant C, independent of* ε *but dependent on* T, *such that*

$$\sup_{0\le t\le T} \left(\|u_t^\varepsilon(t)\|_{L^2}^2 + \|v_t^\varepsilon(t)\|_{L^2}^2\right) + \int_0^T \left(\|u_{xt}^\varepsilon\|_{L^2}^2 + \varepsilon\|v_{xt}^\varepsilon\|_{L^2}^2\right) dt \le C.$$

Proof. Differentiating $(2.8)_1$ with respect to t, we have

$$\tilde{u}_{tt} = \tilde{u}_{xxt} + (\tilde{u}\tilde{v})_{xt} + \bar{u}\tilde{v}_{xt} + \bar{v}\tilde{u}_{xt}.$$

Taking the L^2 inner product of this equation with $\tilde{u}_t$, integrating the result by parts over $(0,1)$, and using the boundary conditions, we arrive at

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\tilde{u}_t\|_{L^2}^2 + \|\tilde{u}_{xt}\|_{L^2}^2 \\ &\quad = -\int_0^1 (\tilde{u}\tilde{v})_t\tilde{u}_{xt}\,dx - \bar{u}\int_0^1 \tilde{v}_t\tilde{u}_{xt}\,dx \\ &\quad = -\int_0^1 \tilde{u}_t\tilde{v}\tilde{u}_{xt}\,dx - \int_0^1 \tilde{u}\tilde{v}_t\tilde{u}_{xt}\,dx - \bar{u}\int_0^1 \tilde{v}_t\tilde{u}_{xt}\,dx \\ &\quad := I_1 + I_2 + I_3. \end{aligned} \tag{2.9}$$

Observing that $\tilde{u}_t|_{x=0,x=1} = 0$, then by Hölder and Gagliardo-Nirenberg interpolation inequalities, we have

$$\begin{aligned} I_1 &\le \|\tilde{u}_t\|_{L^\infty}\|\tilde{u}_{xt}\|_{L^2}\|\tilde{v}\|_{L^2} \\ &\le C\|\tilde{u}_t\|_{L^2}^{1/2}\|\tilde{u}_{xt}\|_{L^2}^{3/2}\|\tilde{v}\|_{L^2} \\ &\le \frac{1}{8}\|\tilde{u}_{xt}\|_{L^2}^2 + C\|\tilde{v}\|_{L^2}^4\|\tilde{u}_t\|_{L^2}^2. \end{aligned}$$

Due to the boundary conditions and Sobolev embedding inequality, I_2 is estimated as follows:

$$\begin{aligned}
I_2 &\leq \|\tilde{u}\|_{L^\infty}\|\tilde{v}_t\|_{L^2}\|\tilde{u}_{xt}\|_{L^2} \\
&\leq C\|\tilde{u}\|_{H^1}\|\tilde{v}_t\|_{L^2}\|\tilde{u}_{xt}\|_{L^2} \\
&\leq C\|\tilde{u}_x\|_{L^2}\|\tilde{v}_t\|_{L^2}\|\tilde{u}_{xt}\|_{L^2} \\
&\leq \frac{1}{8}\|\tilde{u}_{xt}\|_{L^2}^2 + C\|\tilde{u}_x\|_{L^2}^2\|\tilde{v}_t\|_{L^2}^2.
\end{aligned}$$

Moreover, the Cauchy-Schwarz inequality, yields

$$I_3 \leq \frac{1}{4}\|\tilde{u}_{xt}\|_{L^2}^2 + \bar{u}^2\|\tilde{v}_t\|_{L^2}^2.$$

Substituting above estimates for I_1-I_3 into (2.9), we obtain

$$\frac{d}{dt}\|\tilde{u}_t\|_{L^2}^2 + \|\tilde{u}_{xt}\|_{L^2}^2 \leq C\|\tilde{v}\|_{L^2}^4\|\tilde{u}_t\|_{L^2}^2 + C(\|\tilde{u}_x\|_{L^2}^2 + \bar{u}^2)\|\tilde{v}_t\|_{L^2}^2. \tag{2.10}$$

We next estimate $\|\tilde{v}_t\|_{L^2}$. Differentiating $(2.8)_2$ with respect to t, gives

$$\tilde{v}_{tt} = \varepsilon\tilde{v}_{xxt} - \varepsilon[(\tilde{v})^2]_{xt} - 2\varepsilon\bar{v}\tilde{v}_{xt} + \tilde{u}_{xt},$$

which, multiplied by $\tilde{v}_t$ and integrated by parts with respect to x over $(0,1)$, results in

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\tilde{v}_t\|_{L^2}^2 + \varepsilon\|\tilde{v}_{xt}\|_{L^2}^2 &= \varepsilon\int_0^1 [(\tilde{v})^2]_t\tilde{v}_{xt}\,dx + \int_0^1 \tilde{u}_{xt}\tilde{v}_t\,dx \\
&= 2\varepsilon\int_0^1 \tilde{v}\tilde{v}_t\tilde{v}_{xt}\,dx + \int_0^1 \tilde{u}_{xt}\tilde{v}_t\,dx \\
&:= I_4 + I_5.
\end{aligned} \tag{2.11}$$

Upon using Hölder, Poincaré and Sobolev embedding inequalities, we estimate I_4 as

$$\begin{aligned}
I_4 &\leq 2\varepsilon\|\tilde{v}\|_{L^\infty}\|\tilde{v}_t\|_{L^2}\|\tilde{v}_{xt}\|_{L^2} \\
&\leq C\varepsilon\|\tilde{v}\|_{H^1}\|\tilde{v}_t\|_{L^2}\|\tilde{v}_{xt}\|_{L^2} \\
&\leq C(\varepsilon^{1/2}\|\tilde{v}_x\|_{L^2})\|\tilde{v}_t\|_{L^2}(\varepsilon^{1/2}\|\tilde{v}_{xt}\|_{L^2}) \\
&\leq \frac{1}{2}\varepsilon\|\tilde{v}_{xt}\|_{L^2}^2 + C(\varepsilon\|\tilde{v}_x\|_{L^2}^2)\|\tilde{v}_t\|_{L^2}^2.
\end{aligned}$$

With Cauchy-Schwarz inequality, I_5 can be easily estimated as

$$I_5 \leq \frac{1}{4}\|\tilde{u}_{xt}\|_{L^2}^2 + \|\tilde{v}_t\|_{L^2}^2.$$

Inserting above estimates for I_4 and I_5 into (2.11), we obtain

$$\frac{d}{dt}\|\tilde{v}_t\|_{L^2}^2 + \varepsilon\|\tilde{v}_{xt}\|_{L^2}^2 \leq \frac{1}{2}\|\tilde{u}_{xt}\|_{L^2}^2 + C(\varepsilon\|\tilde{v}_x\|_{L^2}^2 + 1)\|\tilde{v}_t\|_{L^2}^2,$$

which, combined with (2.10), yields

$$\begin{aligned}&\frac{d}{dt}(\|\tilde{u}_t\|_{L^2}^2+\|\tilde{v}_t\|_{L^2}^2)+(\|\tilde{u}_{xt}\|_{L^2}^2+\varepsilon\|\tilde{v}_{xt}\|_{L^2}^2)\\&\le C(\|\tilde{v}\|_{L^2}^4+\|\tilde{u}_x\|_{L^2}^2+\varepsilon\|\tilde{v}_x\|_{L^2}^2+\bar{u}^2+1)(\|\tilde{u}_t\|_{L^2}^2+\|\tilde{v}_t\|_{L^2}^2).\end{aligned}$$

This, along with Gronwall's inequality and Lemma 2.1, gives

$$\|\tilde{u}_t(t)\|_{L^2}^2+\|\tilde{v}_t(t)\|_{L^2}^2+\int_0^t\left(\|\tilde{u}_{xt}\|_{L^2}^2+\varepsilon\|\tilde{v}_{xt}\|_{L^2}^2\right)d\tau\le C,$$

where the constant C is independent of ε but depends on t. The proof of Lemma 2.2 is thus finished.

□

Lemma 2.3. *Suppose that the assumptions in Theorem 2.1 hold. Then for any* $0<T<\infty$, *there exists a positive constant C, independent of* ε *but dependent on* T, *such that*

$$\sup_{0\le t\le T}\left(\|u_x^\varepsilon(t)\|_{L^2}^2+\varepsilon^{1/2}\|v_x^\varepsilon(t)\|_{L^2}^2\right)+\int_0^T\left(\varepsilon^{1/2}\|u_{xx}^\varepsilon\|_{L^2}^2+\varepsilon^{3/2}\|v_{xx}^\varepsilon\|_{L^2}^2\right)dt\le C.$$

Proof. Taking the L^2 inner product of $(2.8)_1$ with $(-\varepsilon\tilde{u}_{xx})$, integrating the result by parts over $(0,1)$, and using the boundary conditions, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\varepsilon\|\tilde{u}_x\|_{L^2}^2\right)+\varepsilon\|\tilde{u}_{xx}\|_{L^2}^2\\&=-\varepsilon\int_0^1\tilde{u}_x\tilde{v}\tilde{u}_{xx}\,dx-\varepsilon\int_0^1\tilde{u}\tilde{v}_x\tilde{u}_{xx}\,dx\\&\quad-\varepsilon\bar{u}\int_0^1\tilde{v}_x\tilde{u}_{xx}\,dx-\varepsilon\bar{v}\int_0^1\tilde{u}_x\tilde{u}_{xx}\,dx\\&:=J_1+J_2+J_3+J_4.\end{aligned}\tag{2.12}$$

We next estimate J_1 - J_4. First by the boundary conditions and Hölder and Sobolev embedding inequalities, we infer that

$$\begin{aligned}J_1&\le\varepsilon\|\tilde{u}_x\|_{L^2}\|\tilde{v}\|_{L^\infty}\|\tilde{u}_{xx}\|_{L^2}\\&\le C\|\tilde{u}_x\|_{L^2}(\varepsilon^{1/2}\|\tilde{v}_x\|_{L^2})(\varepsilon^{1/2}\|\tilde{u}_{xx}\|_{L^2})\\&\le\frac{1}{8}\varepsilon\|\tilde{u}_{xx}\|_{L^2}^2+C\|\tilde{u}_x\|_{L^2}^2(\varepsilon\|\tilde{v}_x\|_{L^2}^2)\end{aligned}$$

and

$$\begin{aligned}J_2&\le\varepsilon\|\tilde{u}\|_{L^\infty}\|\tilde{v}_x\|_{L^2}\|\tilde{u}_{xx}\|_{L^2}\\&\le C\|\tilde{u}_x\|_{L^2}(\varepsilon^{1/2}\|\tilde{v}_x\|_{L^2})(\varepsilon^{1/2}\|\tilde{u}_{xx}\|_{L^2})\\&\le\frac{1}{8}\varepsilon\|\tilde{u}_{xx}\|_{L^2}^2+C\|\tilde{u}_x\|_{L^2}^2(\varepsilon\|\tilde{v}_x\|_{L^2}^2).\end{aligned}$$

Furthermore, the Cauchy-Schwarz inequality gives

$$
\begin{aligned}
J_3+J_4 &\le \left(\bar{u}\varepsilon^{1/2}\|\tilde{v}_x\|_{L^2}+|\bar{v}|\varepsilon^{1/2}\|\tilde{u}_x\|_{L^2}\right)(\varepsilon^{1/2}\|\tilde{u}_{xx}\|_{L^2})\\
&\le \frac{1}{4}\varepsilon\|\tilde{u}_{xx}\|_{L^2}^2+2\bar{u}^2(\varepsilon\|\tilde{v}_x\|_{L^2}^2)+2\bar{v}^2(\varepsilon\|\tilde{u}_x\|_{L^2}^2).
\end{aligned}
$$

Then it follows from (2.12) that

$$
\begin{aligned}
&\frac{d}{dt}(\varepsilon\|\tilde{u}_x\|_{L^2}^2)+\varepsilon\|\tilde{u}_{xx}\|_{L^2}^2\\
\le& C(\|\tilde{u}_x\|_{L^2}^2+\bar{u}^2+\bar{v}^2)(\varepsilon\|\tilde{u}_x\|_{L^2}^2+\varepsilon\|\tilde{v}_x\|_{L^2}^2).
\end{aligned}
\tag{2.13}
$$

We are now in a position to estimate $\|\tilde{v}_x\|_{L^2}$. Taking the L^2 inner product of $(2.8)_2$ with $(-\varepsilon\tilde{v}_{xx})$, and integrating the result by parts, we derive

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\varepsilon\|\tilde{v}_x\|_{L^2}^2\right)+\varepsilon^2\|\tilde{v}_{xx}\|_{L^2}^2\\
=&2\varepsilon^2\int_0^1\tilde{v}\tilde{v}_x\tilde{v}_{xx}\,dx+2\varepsilon^2\bar{v}\int_0^1\tilde{v}_x\tilde{v}_{xx}\,dx\\
&+\varepsilon\int_0^1\tilde{u}_{xx}\tilde{v}_x\,dx-\varepsilon(\tilde{u}_x\tilde{v}_x)|_{x=0}^{x=1}\\
:=&J_5+J_6+J_7+J_8.
\end{aligned}
\tag{2.14}
$$

We proceed to estimate J_5 - J_8. Using Hölder and Sobolev embedding inequalities and the boundary conditions, we deduce

$$
\begin{aligned}
J_5 &\le 2\varepsilon^2\|\tilde{v}\|_{L^\infty}\|\tilde{v}_x\|_{L^2}\|\tilde{v}_{xx}\|_{L^2}\\
&\le C\varepsilon^2\|\tilde{v}\|_{H^1}\|\tilde{v}_x\|_{L^2}\|\tilde{v}_{xx}\|_{L^2}\\
&\le C(\varepsilon\|\tilde{v}_x\|_{L^2}^2)(\varepsilon\|\tilde{v}_{xx}\|_{L^2})\\
&\le \frac{1}{8}\varepsilon^2\|\tilde{v}_{xx}\|_{L^2}^2+C(\varepsilon\|\tilde{v}_x\|_{L^2}^2)^2.
\end{aligned}
$$

By Cauchy-Schwarz inequality and the assumption that $0<\varepsilon<1$, we obtain

$$
\begin{aligned}
J_6 &\le 2\varepsilon^2\bar{v}\|\tilde{v}_x\|_{L^2}\|\tilde{v}_{xx}\|_{L^2}\\
&\le \frac{1}{8}\varepsilon^2\|\tilde{v}_{xx}\|_{L^2}^2+C\varepsilon\bar{v}^2(\varepsilon\|\tilde{v}_x\|_{L^2}^2)\\
&\le \frac{1}{8}\varepsilon^2\|\tilde{v}_{xx}\|_{L^2}^2+C\bar{v}^2(\varepsilon\|\tilde{v}_x\|_{L^2}^2)
\end{aligned}
$$

and

$$
J_7 \le \frac{1}{4}\varepsilon\|\tilde{u}_{xx}\|_{L^2}^2+\varepsilon\|\tilde{v}_x\|_{L^2}^2.
$$

To estimate J_8, we rewrite $\tilde{u}_x|_{x=0,x=1}$ as follows. First, integrating $(2.8)_1$ over $(x,1)$ and using the boundary conditions $\tilde{u}(1,t)=\tilde{v}(1,t)=0$, we have

$$\begin{aligned}
\tilde{u}_x(1,t) =&\tilde{u}_x(x,t)+\int_x^1 \tilde{u}_{yy}\,dy \\
=&\tilde{u}_x(x,t)+\int_x^1 \tilde{u}_t\,dy-\int_x^1 (\tilde{u}\tilde{v})_y\,dy-\bar{u}\int_x^1 \tilde{v}_y\,dy-\bar{v}\int_x^1 \tilde{u}_y\,dy \\
=&\tilde{u}_x(x,t)+\int_x^1 \tilde{u}_t\,dy-[(\tilde{u}\tilde{v})(1,t)-(\tilde{u}\tilde{v})(x,t)] \\
&-\bar{u}[\tilde{v}(1,t)-\tilde{v}(x,t)]-\bar{v}[\tilde{u}(1,t)-\tilde{u}(x,t)] \\
=&\tilde{u}_x(x,t)+\int_x^1 \tilde{u}_t\,dy+(\tilde{u}\tilde{v})(x,t)+\bar{u}\tilde{v}(x,t)+\bar{v}\tilde{u}(x,t).
\end{aligned} \tag{2.15}$$

Then integrating (2.15) over $(0,1)$ with respect to x, and using the boundary conditions again, we end up with

$$\begin{aligned}
\tilde{u}_x(1,t)=&\int_0^1 \tilde{u}_x(x,t)\,dx+\int_0^1\int_x^1 \tilde{u}_t\,dydx \\
&+\int_0^1 (\tilde{u}\tilde{v})(x,t)\,dx+\bar{u}\int_0^1 \tilde{v}(x,t)\,dx+\bar{v}\int_0^1 \tilde{u}(x,t)\,dx \\
=&\int_0^1\int_x^1 \tilde{u}_t\,dydx+\int_0^1 (\tilde{u}\tilde{v})(x,t)\,dx \\
&+\bar{u}\int_0^1 \tilde{v}(x,t)\,dx+\bar{v}\int_0^1 \tilde{u}(x,t)\,dx,
\end{aligned}$$

which, upon the application of Hölder inequality, gives

$$|\tilde{u}_x(1,t)|\le \|\tilde{u}_t\|_{L^2}+\|\tilde{u}\|_{L^2}\|\tilde{v}\|_{L^2}+\bar{u}\|\tilde{v}\|_{L^2}+|\bar{v}|\|\tilde{u}\|_{L^2}. \tag{2.16}$$

In a similar fashion as to obtain (2.16), we derive

$$|\tilde{u}_x(0,t)|\le \|\tilde{u}_t\|_{L^2}+\|\tilde{u}\|_{L^2}\|\tilde{v}\|_{L^2}+\bar{u}\|\tilde{v}\|_{L^2}+|\bar{v}|\|\tilde{u}\|_{L^2}. \tag{2.17}$$

Combination of (2.16), (2.17) and Gagliardo-Nirenberg interpolation inequality, gives

$$\begin{aligned}
J_8 \le&\varepsilon\|\tilde{v}_x\|_{L^\infty}(|\tilde{u}_x(0,t)|+|\tilde{u}_x(1,t)|) \\
\le&2\varepsilon\|\tilde{v}_x\|_{L^\infty}(\|\tilde{u}_t\|_{L^2}+\|\tilde{u}\|_{L^2}\|\tilde{v}\|_{L^2}+\bar{u}\|\tilde{v}\|_{L^2}+|\bar{v}|\|\tilde{u}\|_{L^2}) \\
\le&C\varepsilon(\|\tilde{v}_x\|_{L^2}+\|\tilde{v}_x\|_{L^2}^{1/2}\|\tilde{v}_{xx}\|_{L^2}^{1/2}) \\
&\times(\|\tilde{u}_t\|_{L^2}+\|\tilde{u}\|_{L^2}\|\tilde{v}\|_{L^2}+\bar{u}\|\tilde{v}\|_{L^2}+|\bar{v}|\|\tilde{u}\|_{L^2}) \\
\le&\frac{1}{4}\varepsilon^2\|\tilde{v}_{xx}\|_{L^2}^2+\varepsilon\|\tilde{v}_x\|_{L^2}^2 \\
&+C\varepsilon^{1/2}(\|\tilde{u}_t\|_{L^2}^2+\|\tilde{u}\|_{L^2}^2\|\tilde{v}\|_{L^2}^2+\bar{u}^2\|\tilde{v}\|_{L^2}^2+\bar{v}^2\|\tilde{u}\|_{L^2}^2),
\end{aligned}$$

where the assumption that $0 < \varepsilon < 1$ has been used. Substituting above estimates for J_5-J_8 into (2.14), we obtain

$$\begin{aligned}
&\frac{d}{dt}\left(\varepsilon\|\tilde{v}_x\|_{L^2}^2\right)+\varepsilon^2\|\tilde{v}_{xx}\|_{L^2}^2\\
&\leq \frac{1}{2}\varepsilon\|\tilde{u}_{xx}\|_{L^2}^2+C(\varepsilon\|\tilde{v}_x\|_{L^2}^2+\bar{v}^2+1)(\varepsilon\|\tilde{v}_x\|_{L^2}^2)\\
&\quad+C\varepsilon^{1/2}(\|\tilde{u}_t\|_{L^2}^2+\|\tilde{u}\|_{L^2}^2\|\tilde{v}\|_{L^2}^2+\bar{u}^2\|\tilde{v}\|_{L^2}^2+\bar{v}^2\|\tilde{u}\|_{L^2}^2),
\end{aligned}$$

which, added to (2.13), yields

$$\begin{aligned}
&\frac{d}{dt}(\varepsilon\|\tilde{u}_x\|_{L^2}^2+\varepsilon\|\tilde{v}_x\|_{L^2}^2)+(\varepsilon\|\tilde{u}_{xx}\|_{L^2}^2+\varepsilon^2\|\tilde{v}_{xx}\|_{L^2}^2)\\
&\leq C(\|\tilde{u}_x\|_{L^2}^2+\varepsilon\|\tilde{v}_x\|_{L^2}^2+\bar{u}^2+\bar{v}^2+1)(\varepsilon\|\tilde{u}_x\|_{L^2}^2+\varepsilon\|\tilde{v}_x\|_{L^2}^2)\\
&\quad+C\varepsilon^{1/2}(\|\tilde{u}_t\|_{L^2}^2+\|\tilde{u}\|_{L^2}^2\|\tilde{v}\|_{L^2}^2+\bar{u}^2\|\tilde{v}\|_{L^2}^2+\bar{v}^2\|\tilde{u}\|_{L^2}^2).
\end{aligned}$$

This, combined with Lemma 2.1, Lemma 2.2 and Gronwall's inequality, gives

$$\varepsilon\|\tilde{u}_x(t)\|_{L^2}^2+\varepsilon\|\tilde{v}_x(t)\|_{L^2}^2+\int_0^t\left(\varepsilon\|\tilde{u}_{xx}\|_{L^2}^2+\varepsilon^2\|\tilde{v}_{xx}\|_{L^2}^2\right)d\tau\leq C\varepsilon^{1/2},$$

where the constant C is independent of ε but depends on t, which implies

$$\varepsilon^{1/2}\|\tilde{v}_x(t)\|_{L^2}^2+\int_0^t\left(\varepsilon^{1/2}\|\tilde{u}_{xx}\|_{L^2}^2+\varepsilon^{3/2}\|\tilde{v}_{xx}\|_{L^2}^2\right)d\tau\leq C. \tag{2.18}$$

As a consequence of Lemma 2.1 and Lemma 2.2, we get

$$\|\tilde{u}_x\|_{W^{1,2}(0,T;L^2)}^2\leq C,$$

which, along with the Sobolev embedding inequality, yields

$$\|\tilde{u}_x\|_{L^\infty(0,T;L^2)}^2\leq C\|\tilde{u}_x\|_{W^{1,2}(0,T;L^2)}^2\leq C.$$

This, combined with (2.18) completes the proof.

□

Proof of Theorem 2.1. By Lemma 2.3, we derive estimate (2.3), which finishes the proof of Theorem 2.1.

□

2.3 Proof of Theorem 2.2.

Recall that $(u^\varepsilon, v^\varepsilon)$ denote the global solution of (2.1)-(2.2) with $\varepsilon \geq 0$. For convenience, we set

$$\hat{u}=u^\varepsilon-u^0,\ \hat{v}=v^\varepsilon-v^0 \tag{2.19}$$

Then from system (2.1)-(2.2), we deduce that $(\hat{u},\hat{v})$ satisfies the following initial-boundary value problem:

$$\begin{cases} \hat{u}_t = \hat{u}_{xx} + (u^\varepsilon \hat{v} + \hat{u} v^0)_x, \\ \hat{v}_t = \varepsilon \hat{v}_{xx} + \varepsilon v^0_{xx} - \varepsilon[(v^\varepsilon)^2]_x + u_x, \\ \hat{u}|_{x=0,x=1} = 0, \quad \hat{v}|_{x=0,x=1} = (\bar{v} - v^0)|_{x=0,x=1}, \\ \hat{u}(x,0) = 0, \quad \hat{v}(x,0) = 0. \end{cases} \tag{2.20}$$

Based on the reformulated problem (2.20), we shall derive a series of results below.

Lemma 2.4. *Suppose that the assumptions in Theorem 2.2 hold. Then for any* $0 < T < \infty$, *there exists a positive constant C, independent of* ε *but dependent on* T, *such that*

$$\begin{aligned} &\sup_{0\le t\le T} \left(\|(u^\varepsilon - u^0)(t)\|_{L^2}^2 + \|(v^\varepsilon - v^0)(t)\|_{L^2}^2 \right) \\ &\quad + \int_0^T \left(\|(u^\varepsilon - u^0)_x\|_{L^2}^2 + \varepsilon \|(v^\varepsilon - v^0)_x\|_{L^2}^2 \right) dt \le C\varepsilon^{1/2}. \end{aligned}$$

Proof. Testing $(2.20)_1$ with $\hat{u}$, integrating the result by parts, with Hölder and Sobolev embedding inequalities we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\hat{u}\|_{L^2}^2 + \|\hat{u}_x\|_{L^2}^2 \\ &\quad = -\int_0^1 u^\varepsilon \hat{v} \hat{u}_x \, dx - \int_0^1 \hat{u} v^0 \hat{u}_x \, dx \\ &\quad \le \|u^\varepsilon\|_{L^\infty} \|\hat{v}\|_{L^2} \|\hat{u}_x\|_{L^2} + \|\hat{u}\|_{L^2} \|v^0\|_{L^\infty} \|\hat{u}_x\|_{L^2} \\ &\quad \le \frac{1}{4}\|\hat{u}_x\|_{L^2}^2 + C\|u^\varepsilon\|_{H^1}^2 \|\hat{v}\|_{L^2}^2 + C\|v^0\|_{H^1}^2 \|\hat{u}\|_{L^2}^2. \end{aligned} \tag{2.21}$$

Taking the L^2 inner product of $(2.20)_2$ with $\hat{v}$, and using the integration by parts again, we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\hat{v}\|_{L^2}^2 + \varepsilon\|\hat{v}_x\|_{L^2}^2 =& \varepsilon \int_0^1 v^0_{xx} \hat{v} \, dx - 2\varepsilon \int_0^1 v^\varepsilon v^\varepsilon_x v \, dx \\ &+ \int_0^1 \hat{u}_x v \, dx + \varepsilon (\hat{v}\hat{v}_x)|_{x=0}^{x=1} \\ :=& K_1 + K_2 + K_3 + K_4. \end{aligned} \tag{2.22}$$

By Hölder and Sobolev embedding inequalities, we estimate K_1 - K_3 as follows:

$$K_1 \le \varepsilon^2 \|v^0_{xx}\|_{L^2}^2 + \|\hat{v}\|_{L^2}^2,$$

$$\begin{aligned} K_2 &\le 2\varepsilon \|v^\varepsilon\|_{L^\infty} \|v^\varepsilon_x\|_{L^2} \|\hat{v}\|_{L^2} \\ &\le C\varepsilon \left(\|v^\varepsilon\|_{L^2} + \|v^\varepsilon_x\|_{L^2} \right) \|v^\varepsilon_x\|_{L^2} \|\hat{v}\|_{L^2} \\ &\le C\varepsilon \|v^\varepsilon_x\|_{L^2}^2 + C\left(\varepsilon \|v^\varepsilon\|_{L^2}^2 + \varepsilon \|v^\varepsilon_x\|_{L^2}^2 \right) \|\hat{v}\|_{L^2}^2 \end{aligned}$$

and

$$K_3 \le \frac{1}{8}\|\hat{u}_x\|_{L^2}^2 + 2\|\hat{v}\|_{L^2}^2.$$

With boundary conditions in (2.20), we rewrite K_4 as follows:

$$\begin{aligned} K_4 =& \varepsilon\left[(\bar{v}-v^0)(v_x^\varepsilon - v_x^0)\right]|_{x=0}^{x=1} \\ =& \varepsilon\left[(\bar{v}-v^0)v_x^\varepsilon\right]|_{x=0}^{x=1} - \varepsilon\left[(\bar{v}-v^0)v_x^0\right]|_{x=0}^{x=1} \\ :=& M_1 + M_2. \end{aligned}$$

By Hölder and Gagliardo-Nirenberg interpolation inequalities, we deduce

$$\begin{aligned} M_1 \le& 2\varepsilon\left(\bar{v} + \|v^0\|_{L^\infty}\right)\|v_x^\varepsilon\|_{L^\infty} \\ \le& C\varepsilon\left(\bar{v} + \|v^0\|_{H^1}\right)\left(\|v_x^\varepsilon\|_{L^2} + \|v_x^\varepsilon\|_{L^2}^{1/2}\|v_{xx}^\varepsilon\|_{L^2}^{1/2}\right) \\ =& C\varepsilon^{1/2}(\bar{v} + \|v^0\|_{H^1})\left(\varepsilon^{1/2}\|v_x^\varepsilon\|_{L^2} + (\varepsilon^{1/8}\|v_x^\varepsilon\|_{L^2}^{1/2})(\varepsilon^{3/8}\|v_{xx}^\varepsilon\|_{L^2}^{1/2})\right) \\ \le& C\varepsilon^{1/2}\left((\bar{v} + \|v^0\|_{H^1})^2 + \varepsilon\|v_x^\varepsilon\|_{L^2}^2 + \varepsilon^{1/2}\|v_x^\varepsilon\|_{L^2}^2 + \varepsilon^{3/2}\|v_{xx}^\varepsilon\|_{L^2}^2\right) \end{aligned}$$

and

$$\begin{aligned} M_2 \le& 2\varepsilon\left(\bar{v} + \|v^0\|_{L^\infty}\right)\|v_x^0\|_{L^\infty} \\ \le& C\varepsilon\left(\bar{v} + \|v^0\|_{H^2}\right)^2. \end{aligned}$$

With the above estimates for M_1 and M_2, and keeping in mind that $0 < \varepsilon < 1$, we get

$$K_4 \le C\varepsilon^{1/2}\left((\bar{v} + \|v^0\|_{H^2})^2 + \varepsilon^{1/2}\|v_x^\varepsilon\|_{L^2}^2 + \varepsilon^{3/2}\|v_{xx}^\varepsilon\|_{L^2}^2\right),$$

which, combined with the above estimates for K_1-K_3 and (2.22), leads to

$$\begin{aligned} &\frac{d}{dt}\|\hat{v}\|_{L^2}^2 + \varepsilon\|\hat{v}_x\|_{L^2}^2 \\ \le& \frac{1}{4}\|\hat{u}_x\|_{L^2}^2 + C\left(\|v^\varepsilon\|_{L^2}^2 + \varepsilon\|v_x^\varepsilon\|_{L^2}^2 + 1\right)\|\hat{v}\|_{L^2}^2 \\ &+ C\varepsilon^{1/2}\left((\bar{v} + \|v^0\|_{H^2})^2 + \varepsilon^{1/2}\|v_x^\varepsilon\|_{L^2}^2 + \varepsilon^{3/2}\|v_{xx}^\varepsilon\|_{L^2}^2\right). \end{aligned}$$

This, along with (2.21) gives

$$\begin{aligned} &\frac{d}{dt}\left(\|\hat{u}\|_{L^2}^2 + \|\hat{v}\|_{L^2}^2\right) + \left(\|\hat{u}_x\|_{L^2}^2 + \varepsilon\|\hat{v}_x\|_{L^2}^2\right) \\ \le& C\left(\|u^\varepsilon\|_{H^1}^2 + \|v^0\|_{H^1}^2 + \|v^\varepsilon\|_{L^2}^2 + \varepsilon\|v_x^\varepsilon\|_{L^2}^2 + 1\right)\left(\|\hat{u}\|_{L^2}^2 + \|\hat{v}\|_{L^2}^2\right) \\ &+ C\varepsilon^{1/2}\left((\bar{v} + \|v^0\|_{H^2})^2 + \varepsilon^{1/2}\|v_x^\varepsilon\|_{L^2}^2 + \varepsilon^{3/2}\|v_{xx}^\varepsilon\|_{L^2}^2\right). \end{aligned}$$

Applying Gronwall's inequality to this, and using Part (ii) of Lemma 2.1, Theorem 2.1 and Lemma 2.1, we arrive at

$$\|\hat{u}(t)\|_{L^2}^2 + \|\hat{v}(t)\|_{L^2}^2 + \int_0^t \left(\|\hat{u}_x\|_{L^2}^2 + \varepsilon\|\hat{v}_x\|_{L^2}^2\right) d\tau \le C\varepsilon^{1/2}, \tag{2.23}$$

where the constant C is independent of ε but depends on t. This, along with the convention (2.19), completes the proof.

□

Lemma 2.5. *Suppose that the assumptions in Theorem 2.2 hold. Then for any* $0 < T < \infty$*, there exists a positive constant C, independent of* ε *but dependent on T, such that*

$$\varepsilon \sup_{0\le t\le T} \|(v^\varepsilon - v^0)_x(t)\|_{L^2}^2 + \int_0^T \|(v^\varepsilon - v^0)_t\|_{L^2}^2\, dt \le C\varepsilon^{1/2} \tag{2.24}$$

and

$$\sup_{0\le t\le T} \|(u^\varepsilon - u^0)_x(t)\|_{L^2}^2 + \int_0^T \|(u^\varepsilon - u^0)_t\|_{L^2}^2\, dt \le C\varepsilon^{1/2}. \tag{2.25}$$

Proof. We first estimate (2.24). Taking the L^2 inner product of $(2.20)_2$ with $\hat{v}_t$ and integrating the result by parts, we derive

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\varepsilon\|\hat{v}_x\|_{L^2}^2\right) + \|\hat{v}_t\|_{L^2}^2\\
=&\varepsilon\int_0^1 v_{xx}^0 \hat{v}_t\, dx - 2\varepsilon\int_0^1 v^\varepsilon v_x^\varepsilon \hat{v}_t\, dx\\
&+\int_0^1 \hat{u}_x\hat{v}_t\, dx + \varepsilon(\hat{v}_x\hat{v}_t)|_{x=0}^{x=1}\\
:=&K_5 + K_6 + K_7 + K_8.
\end{aligned} \tag{2.26}$$

First by Cauchy-Schwarz inequality and Part (ii) of Lemma 2.1, we obtain

$$\begin{aligned}
K_5 \le& 2\varepsilon^2\|v_{xx}^0\|_{L^2}^2 + \frac{1}{8}\|\hat{v}_t\|_{L^2}^2\\
\le& C\varepsilon^2 + \frac{1}{8}\|\hat{v}_t\|_{L^2}^2.
\end{aligned}$$

Then using Hölder and Sobolev embedding inequalities, we estimate K_6 and K_7 as follows:

$$\begin{aligned}
K_6 \le& 2\varepsilon\|v^\varepsilon\|_{L^\infty}\|v_x^\varepsilon\|_{L^2}\|\hat{v}_t\|_{L^2}\\
\le& C\varepsilon\left(\|v^\varepsilon\|_{L^2} + \|v_x^\varepsilon\|_{L^2}\right)\|v_x^\varepsilon\|_{L^2}\|\hat{v}_t\|_{L^2}\\
\le& \frac{1}{4}\|\hat{v}_t\|_{L^2}^2 + C\varepsilon\left(\varepsilon\|v_x^\varepsilon\|_{L^2}^2\|v^\varepsilon\|_{L^2}^2 + \varepsilon\|v_x^\varepsilon\|_{L^2}^4\right)
\end{aligned}$$

and

$$K_7 \le \frac{1}{8}\|\hat{v}_t\|_{L^2}^2 + 2\|\hat{u}_x\|_{L^2}^2.$$

With the boundary conditions, Sobolev embedding $W^{1,2}(0,1) \hookrightarrow C([0,1])$, and Gagliardo-Nirenberg interpolation inequality, K_8 is estimated as follows:

$$
\begin{aligned}
K_8 =& \varepsilon[(v_x^\varepsilon - v_x^0)(-v_t^0)]|_{x=0}^{x=1} \\
\leq& 2\varepsilon\|v_x^\varepsilon\|_{L^\infty}\|v_t^0\|_{L^\infty} + 2\varepsilon\|v_x^0\|_{L^\infty}\|v_t^0\|_{L^\infty} \\
\leq& C\varepsilon(\|v_x^\varepsilon\|_{L^2} + \|v_x^\varepsilon\|_{L^2}^{1/2}\|v_{xx}^\varepsilon\|_{L^2}^{1/2})\|v_t^0\|_{H^1} + C\varepsilon\|v^0\|_{H^2}\|v_t^0\|_{H^1} \\
=& C\varepsilon^{1/2}\left((\varepsilon^{1/2}\|v_x^\varepsilon\|_{L^2}) + (\varepsilon^{1/8}\|v_x^\varepsilon\|_{L^2}^{1/2})(\varepsilon^{3/8}\|v_{xx}^\varepsilon\|_{L^2}^{1/2})\right)\|v_t^0\|_{H^1} \\
&+ C\varepsilon\|v^0\|_{H^2}\|v_t^0\|_{H^1} \\
\leq& C\varepsilon^{1/2}\left(\varepsilon^{1/2}\|v_x^\varepsilon\|_{L^2}^2 + \varepsilon^{3/2}\|v_{xx}^\varepsilon\|_{L^2}^2 + \|v_t^0\|_{H^1}^2\right) \\
&+ C\varepsilon\|v^0\|_{H^2}\|v_t^0\|_{H^1},
\end{aligned}
\tag{2.27}
$$

where the assumption $0 < \varepsilon < 1$ has been used. We proceed to estimate $\|v_t^0\|_{H^1}$ in the right-hand side of (2.27). By the second equation of (2.1) with $\varepsilon = 0$ and Part (ii) of Lemma 2.1, we derive

$$
\|v_t^0\|_{H^1} = \|u_x^0\|_{H^1} \leq \|u^0\|_{H^2} \leq C.
$$

Putting the above estimates into (2.27), and using Part (ii) of Lemma 2.1 again, we obtain that for $0 < \varepsilon < 1$

$$
K_8 \leq C\varepsilon^{1/2}\left(\varepsilon^{1/2}\|v_x^\varepsilon\|_{L^2}^2 + \varepsilon^{3/2}\|v_{xx}^\varepsilon\|_{L^2}^2 + 1\right).
$$

Substituting the above estimates for K_5-K_8 into (2.26), using Theorem 2.1 and Lemma 2.1 we deduce

$$
\begin{aligned}
&\frac{d}{dt}\left(\varepsilon\|\hat{v}_x\|_{L^2}^2\right) + \|\hat{v}_t\|_{L^2}^2 \\
\leq& 4\|\hat{u}_x\|_{L^2}^2 + C\varepsilon\|v_x^\varepsilon\|_{L^2}^2\left(\|v^\varepsilon\|_{L^2}^2 + \varepsilon^{1/2}\|v_x^\varepsilon\|_{L^2}^2\right) \\
&+ C\varepsilon^{1/2}\left(\varepsilon^{1/2}\|v_x^\varepsilon\|_{L^2}^2 + \varepsilon^{3/2}\|v_{xx}^\varepsilon\|_{L^2}^2 + 1\right) \\
\leq& 4\|\hat{u}_x\|_{L^2}^2 + C\varepsilon^{1/2}\left(\varepsilon^{1/2}\|v_x^\varepsilon\|_{L^2}^2 + \varepsilon^{3/2}\|v_{xx}^\varepsilon\|_{L^2}^2 + 1\right),
\end{aligned}
$$

where the assumption that $0 < \varepsilon < 1$ has been used. Integrating this inequality over $(0,t)$ and using Theorem 2.1 and Lemma 2.4, we obtain

$$
\varepsilon\|\hat{v}_x(t)\|_{L^2}^2 + \int_0^t \|\hat{v}_t\|_{L^2}^2\, d\tau \leq C\varepsilon^{1/2}, \tag{2.28}
$$

where the constant C is independent of ε but depends on t. The above estimate completes the proof of (2.24).

We next prove (2.25). Testing $(2.20)_1$ with $\hat{u}_t$, integrating the result by parts, and using the boundary conditions, we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\hat{u}_x\|_{L^2}^2+\|\hat{u}_t\|_{L^2}^2 =& -\int_0^1 u^\varepsilon v\hat{u}_{xt}\,dx+\int_0^1(\hat{u}v^0)_x\hat{u}_t\,dx\\
=& -\frac{d}{dt}\int_0^1 u^\varepsilon\hat{v}\hat{u}_x\,dx+\int_0^1(u^\varepsilon\hat{v})_t\hat{u}_x\,dx+\int_0^1(\hat{u}v^0)_x\hat{u}_t\,dx\\
=& -\frac{d}{dt}\int_0^1\left(\left(u^\varepsilon\hat{v}+\frac{\hat{u}_x}{2}\right)^2-(u^\varepsilon\hat{v})^2-\frac{u_x^2}{4}\right)dx\\
&+\int_0^1(u^\varepsilon\hat{v})_t\hat{u}_x\,dx+\int_0^1(\hat{u}v^0)_x\hat{u}_t\,dx\\
=& -\frac{d}{dt}\int_0^1\left(u^\varepsilon\hat{v}+\frac{\hat{u}_x}{2}\right)^2dx+\frac{d}{dt}\int_0^1(u^\varepsilon\hat{v})^2\,dx+\frac{1}{4}\frac{d}{dt}\int_0^1 u_x^2\,dx\\
&+\int_0^1(u^\varepsilon\hat{v})_t\hat{u}_x\,dx+\int_0^1(\hat{u}v^0)_x\hat{u}_t\,dx,
\end{aligned}$$

which, gives

$$\begin{aligned}
&\frac{1}{4}\frac{d}{dt}\|\hat{u}_x\|_{L^2}^2+\frac{d}{dt}\left\|\left(u^\varepsilon\hat{v}+\frac{\hat{u}_x}{2}\right)\right\|_{L^2}^2+\|\hat{u}_t\|_{L^2}^2\\
&=\frac{d}{dt}\int_0^1(u^\varepsilon\hat{v})^2\,dx+\int_0^1(u^\varepsilon\hat{v})_t\hat{u}_x\,dx+\int_0^1(\hat{u}v^0)_x\hat{u}_t\,dx.
\end{aligned}$$

For fixed $t\in(0,T]$, integrating this equation over $(0,t)$ and using the initial conditions, we deduce

$$\begin{aligned}
&\frac{1}{4}\|\hat{u}_x(t)\|_{L^2}^2+\left\|\left(u^\varepsilon\hat{v}+\frac{\hat{u}_x}{2}\right)(t)\right\|_{L^2}^2+\int_0^t\|\hat{u}_t\|_{L^2}^2\,d\tau\\
=&\|(u^\varepsilon\hat{v})(t)\|_{L^2}^2+\int_0^t\int_0^1(u^\varepsilon\hat{v})_t\hat{u}_x\,dx\,d\tau+\int_0^t\int_0^1(\hat{u}v^0)_x\hat{u}_t\,dx\,d\tau\\
:=&K_9+K_{10}+K_{11}.
\end{aligned}\tag{2.29}$$

Let us estimate K_9 - K_{11}. First by Theorem 2.1, Lemma 2.1, Lemma 2.4 and the Sobolev embedding inequality, we get

$$K_9\le\|u^\varepsilon(t)\|_{L^\infty}^2\|\hat{v}(t)\|_{L^2}^2\le C\left(\|u^\varepsilon(t)\|_{L^2}^2+\|u_x^\varepsilon(t)\|_{L^2}^2\right)\|\hat{v}(t)\|_{L^2}^2\le C\varepsilon^{1/2}.$$

With Hölder and Sobolev embedding inequalities, we have

$$\begin{aligned}
K_{10}=&\int_0^t\int_0^1 u_t^\varepsilon v\hat{u}_x\,dx\,d\tau+\int_0^t\int_0^1 u^\varepsilon\hat{v}_t\hat{u}_x\,dx\,d\tau\\
\le& C(\|u_t^\varepsilon\|_{L^2(0,T;L^2)}+\|u_{xt}^\varepsilon\|_{L^2(0,T;L^2)})\|\hat{v}\|_{L^\infty(0,T;L^2)}\|\hat{u}_x\|_{L^2(0,T;L^2)}\\
&+C(\|u^\varepsilon\|_{L^\infty(0,T;L^2)}+\|u_x^\varepsilon\|_{L^\infty(0,T;L^2)})\|\hat{v}_t\|_{L^2(0,T;L^2)}\|\hat{u}_x\|_{L^2(0,T;L^2)}\\
\le& C\varepsilon^{1/2},
\end{aligned}$$

where we have used Theorem 2.1, Lemma 2.1, Lemma 2.2, Lemma 2.4 and (2.28). It follows from Poincaré and Sobolev embedding inequalities that

$$
\begin{aligned}
K_{11} = & \int_0^t \int_0^1 \hat{u}_x v^0 \hat{u}_t \, dx \, d\tau + \int_0^t \int_0^1 u v_x^0 \hat{u}_t \, dx \, d\tau \\
\leq & \|\hat{u}_x\|_{L^2(0,T;L^2)} \|v^0\|_{L^\infty(0,T;L^\infty)} \|\hat{u}_t\|_{L^2(0,t;L^2)} \\
& + \|u\|_{L^2(0,T;L^\infty)} \|v_x^0\|_{L^\infty(0,T;L^2)} \|\hat{u}_t\|_{L^2(0,t;L^2)} \\
\leq & \frac{1}{4} \|\hat{u}_t\|^2_{L^2(0,t;L^2)} + C\|v^0\|^2_{L^\infty(0,T;H^1)} \|\hat{u}\|^2_{L^2(0,T;H^1)} \\
\leq & \frac{1}{4} \|\hat{u}_t\|^2_{L^2(0,t;L^2)} + C\|v^0\|^2_{L^\infty(0,T;H^1)} \|\hat{u}_x\|^2_{L^2(0,T;L^2)} \\
\leq & \frac{1}{4} \|\hat{u}_t\|^2_{L^2(0,t;L^2)} + C\varepsilon^{1/2},
\end{aligned}
$$

where Lemma 2.4 and Part (ii) of Lemma 2.1 have been used . Substituting the above estimates for K_9-K_{11} into (2.29), we obtain

$$
\|\hat{u}_x(t)\|^2_{L^2} + \int_0^t \|\hat{u}_t\|^2_{L^2} \, d\tau \leq C\varepsilon^{1/2},
$$

where the constant C is independent of ε but depends on t. Thus, the proof of (2.25) is completed.

□

Lemma 2.6. *Suppose that the assumptions in Theorem 2.2 hold. Define* $\omega(x) = x^2(1-x)^2$ *for* $0 \leq x \leq 1$*. Then for any* $0 < T < \infty$*, there exists a positive constant C, independent of* ε *but dependent on T, such that*

$$
\sup_{0 \leq t \leq T} \left(\int_0^1 \omega(x) |(v^\varepsilon - v^0)_x|^2(x,t) \, dx \right) + \varepsilon \int_0^T \int_0^1 \omega(x) |(v^\varepsilon - v^0)_{xx}|^2(x,t) \, dx dt \leq C\varepsilon^{1/2}.
$$

Proof. Differentiating $(2.20)_2$ with respect to x, we have

$$
\hat{v}_{xt} = \varepsilon \hat{v}_{xxx} + \varepsilon v^0_{xxx} - \varepsilon [(v^\varepsilon)^2]_{xx} + \hat{u}_{xx}.
$$

Multiplying the above equation by $x^2(1-x)^2 \hat{v}_x$, integrating the resulting equation with respect to x by parts, and using the fact that $\Gamma(x)|_{x=0,x=1} = 0$, we get

$$
\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|x(1-x)\hat{v}_x\|^2_{L^2} + \varepsilon \|x(1-x)\hat{v}_{xx}\|^2_{L^2} \\
= & -\varepsilon \int_0^1 2(1-2x)x(1-x)\hat{v}_x \hat{v}_{xx} \, dx + \varepsilon \int_0^1 x^2(1-x)^2 \hat{v}_x v^0_{xxx} \, dx \\
& - 2\varepsilon \int_0^1 x^2(1-x)^2 (v_x^\varepsilon)^2 \hat{v}_x \, dx - 2\varepsilon \int_0^1 x^2(1-x)^2 v^\varepsilon \hat{v}_x v^\varepsilon_{xx} \, dx \\
& + \int_0^1 x^2(1-x)^2 \hat{v}_x \hat{u}_{xx} \, dx \\
:= & K_{12} + K_{13} + K_{14} + K_{15} + K_{16}.
\end{aligned}
\tag{2.30}
$$

We proceed to estimate K_{12} - K_{16}. Starting with Cauchy-Schwarz inequality, we first have

$$\begin{aligned}K_{12} \leq& 2\varepsilon\|x(1-x)\hat{v}_{xx}\|_{L^2}\|(1-2x)\hat{v}_x\|_{L^2}\\ \leq& 2\left(\varepsilon^{1/2}\|x(1-x)\hat{v}_{xx}\|_{L^2}\right)\left(\varepsilon^{1/2}\|\hat{v}_x\|_{L^2}\right)\\ \leq& \frac{1}{8}\varepsilon\|x(1-x)\hat{v}_{xx}\|_{L^2}^2+8\varepsilon\|\hat{v}_x\|_{L^2}^2.\end{aligned}$$

The integration by parts with Hölder inequality yields

$$\begin{aligned}K_{13} =& -\varepsilon\int_0^1 x^2(1-x)^2\hat{v}_{xx}v^0_{xx}\,dx-\varepsilon\int_0^1 2(1-2x)x(1-x)\hat{v}_x v^0_{xx}\,dx\\ \leq& \varepsilon\|v^0_{xx}\|_{L^2}\|x(1-x)\hat{v}_{xx}\|_{L^2}+2\varepsilon\|\hat{v}_x\|_{L^2}\|v^0_{xx}\|_{L^2}\\ \leq& \frac{1}{8}\varepsilon\|x(1-x)\hat{v}_{xx}\|_{L^2}^2+2\left(\varepsilon\|\hat{v}_x\|_{L^2}^2+\varepsilon\|v^0_{xx}\|_{L^2}^2\right).\end{aligned}$$

By the assumption that $0<\varepsilon<1$ and Hölder and Gagliardo-Nirenberg interpolation inequalities, we derive

$$\begin{aligned}K_{14} \leq& 2\varepsilon\|v^\varepsilon_x\|_{L^2}\|v^\varepsilon_x\|_{L^\infty}\|x(1-x)\hat{v}_x\|_{L^2}\\ \leq& C\varepsilon\|v^\varepsilon_x\|_{L^2}\left(\|v^\varepsilon_x\|_{L^2}+\|v^\varepsilon_x\|_{L^2}^{1/2}\|v^\varepsilon_{xx}\|_{L^2}^{1/2}\right)\|x(1-x)\hat{v}_x\|_{L^2}\\ =& C\left(\varepsilon\|v^\varepsilon_x\|_{L^2}^2\right)\|x(1-x)\hat{v}_x\|_{L^2}\\ &+C\left(\varepsilon^{1/2}\|v^\varepsilon_x\|_{L^2}^{3/2}\right)\left(\varepsilon^{1/2}\|v^\varepsilon_{xx}\|_{L^2}^{1/2}\right)\|x(1-x)\hat{v}_x\|_{L^2}\\ \leq& \|x(1-x)\hat{v}_x\|_{L^2}^2+C\varepsilon^{1/2}\left((\varepsilon^{1/2}\|v^\varepsilon_x\|_{L^2}^2)^2+(\varepsilon^{1/2}\|v^\varepsilon_x\|_{L^2}^2)^3+\varepsilon^{3/2}\|v^\varepsilon_{xx}\|_{L^2}^2\right).\end{aligned}$$

Noting that $v^\varepsilon=\hat{v}+v^0$, we have

$$\begin{aligned}K_{15} =& 2\varepsilon\int_0^1 x^2(1-x)^2v^\varepsilon\hat{v}_x\hat{v}_{xx}\,dx+2\varepsilon\int_0^1 x^2(1-x)^2v^\varepsilon\hat{v}_x v^0_{xx}\,dx\\ \leq& 2\varepsilon\|v^\varepsilon\|_{L^\infty}\|x(1-x)\hat{v}_x\|_{L^2}\|x(1-x)\hat{v}_{xx}\|_{L^2}\\ &+2\varepsilon\|v^\varepsilon\|_{L^\infty}\|x(1-x)\hat{v}_x\|_{L^2}\|v^0_{xx}\|_{L^2}\\ \leq& C\varepsilon\|v^\varepsilon\|_{H^1}\|x(1-x)\hat{v}_x\|_{L^2}\|x(1-x)\hat{v}_{xx}\|_{L^2}\\ &+C\varepsilon\|v^\varepsilon\|_{H^1}\|x(1-x)\hat{v}_x\|_{L^2}\|v^0_{xx}\|_{L^2}\\ \leq& \frac{1}{8}\varepsilon\|x(1-x)\hat{v}_{xx}\|_{L^2}^2+C\left(\varepsilon\|v^\varepsilon\|_{H^1}^2\right)\|x(1-x)\hat{v}_x\|_{L^2}^2+\varepsilon\|v^0_{xx}\|_{L^2}^2,\end{aligned}$$

where we have used the Sobolev embedding $H^1\hookrightarrow L^\infty$. For K_{16}, we use equation $(2.20)_1$ to rewrite it as

$$\begin{aligned}K_{16} =& \int_0^1 x^2(1-x)^2\hat{v}_x\hat{u}_t\,dx-\int_0^1 x^2(1-x)^2\hat{v}_x u^\varepsilon_x v\,dx-\int_0^1 x^2(1-x)^2\hat{v}_x u^\varepsilon\hat{v}_x\,dx\\ &-\int_0^1 x^2(1-x)^2\hat{v}_x\hat{u}_x v^0\,dx-\int_0^1 x^2(1-x)^2\hat{v}_x u v^0_x\,dx\\ :=& R_1+R_2+R_3+R_4+R_5.\end{aligned}$$

To bound K_{16}, we estimate R_1 - R_5 below. First Cauchy-Schwarz inequality leads to

$$R_1 \le \|x(1-x)\hat{v}_x\|_{L^2}^2 + \|\hat{u}_t\|_{L^2}^2.$$

By Hölder and Sobolev embedding inequalities, we estimate R_2 - R_5 as follows:

$$\begin{aligned}
R_2 \le& \|x(1-x)\hat{v}\|_{L^\infty}\|x(1-x)\hat{v}_x\|_{L^2}\|u_x^\varepsilon\|_{L^2}\\
\le& C\|x(1-x)\hat{v}\|_{H^1}\|x(1-x)\hat{v}_x\|_{L^2}\|u_x^\varepsilon\|_{L^2}\\
\le& C(\|x(1-x)\hat{v}\|_{L^2} + \|[x(1-x)\hat{v}]_x\|_{L^2})\|x(1-x)\hat{v}_x\|_{L^2}\|u_x^\varepsilon\|_{L^2}\\
\le& C(\|\hat{v}\|_{L^2} + \|x(1-x)\hat{v}_x\|_{L^2})\|x(1-x)\hat{v}_x\|_{L^2}\|u_x^\varepsilon\|_{L^2}\\
\le& C(\|u_x^\varepsilon\|_{L^2} + \|u_x^\varepsilon\|_{L^2}^2)\|x(1-x)\hat{v}_x\|_{L^2}^2 + \|\hat{v}\|_{L^2}^2,
\end{aligned}$$

$$\begin{aligned}
R_3 \le& \|x(1-x)\hat{v}_x\|_{L^2}^2\|u^\varepsilon\|_{L^\infty}\\
\le& C\left(\|u^\varepsilon\|_{L^2} + \|u_x^\varepsilon\|_{L^2}\right)\|x(1-x)\hat{v}_x\|_{L^2}^2,
\end{aligned}$$

$$\begin{aligned}
R_4 \le& \|x(1-x)\hat{v}_x\|_{L^2}\|\hat{u}_x\|_{L^2}\|v^0\|_{L^\infty}\\
\le& C\|v^0\|_{H^1}^2\|x(1-x)\hat{v}_x\|_{L^2}^2 + \|\hat{u}_x\|_{L^2}^2
\end{aligned}$$

and

$$\begin{aligned}
R_5 \le& \|x(1-x)\hat{v}_x\|_{L^2}\|\hat{u}\|_{L^2}\|v_x^0\|_{L^\infty}\\
\le& C\|v^0\|_{H^2}^2\|x(1-x)\hat{v}_x\|_{L^2}^2 + \|\hat{u}\|_{L^2}^2.
\end{aligned}$$

With above estimates in hand, we obtain

$$\begin{aligned}
K_{16} \le& C\left(\|u^\varepsilon\|_{L^2} + \|u_x^\varepsilon\|_{L^2} + \|u_x^\varepsilon\|_{L^2}^2 + \|v^0\|_{H^2}^2 + 1\right)\|x(1-x)\hat{v}_x\|_{L^2}^2\\
&+ \left(\|\hat{v}\|_{L^2}^2 + \|\hat{u}\|_{L^2}^2 + \|\hat{u}_x\|_{L^2}^2 + \|\hat{u}_t\|_{L^2}^2\right).
\end{aligned}$$

Substituting the above estimates for K_{12}-K_{16} into (2.30), we derive that for $0 < \varepsilon < 1$

$$\begin{aligned}
&\frac{d}{dt}\|x(1-x)\hat{v}_x\|_{L^2}^2 + \varepsilon\|x(1-x)\hat{v}_{xx}\|_{L^2}^2\\
\le& C\left(\|u^\varepsilon\|_{L^2} + \|u_x^\varepsilon\|_{L^2} + \|u_x^\varepsilon\|_{L^2}^2 + \varepsilon^{1/2}\|v^\varepsilon\|_{H^1}^2 + \|v^0\|_{H^2}^2 + 1\right)\|x(1-x)\hat{v}_x\|_{L^2}^2\\
&+ C\left(\|\hat{v}\|_{L^2}^2 + \|\hat{u}\|_{L^2}^2 + \|\hat{u}_x\|_{L^2}^2 + \|\hat{u}_t\|_{L^2}^2 + \varepsilon\|\hat{v}_x\|_{L^2}^2\right)\\
&+ C\varepsilon^{1/2}\left(\left(\varepsilon^{1/2}\|v_x^\varepsilon\|_{L^2}^2\right)^2 + \left(\varepsilon^{1/2}\|v_x^\varepsilon\|_{L^2}^2\right)^3 + \varepsilon^{3/2}\|v_{xx}^\varepsilon\|_{L^2}^2 + \|v^0\|_{H^2}^2\right).
\end{aligned}$$

Applying Gronwall's inequality to this, and using Part (ii) of Lemma 2.1, Theorem 2.1, Lemma 2.1, Lemma 2.4 and Lemma 2.5, we obtain

$$\int_0^1 x^2(1-x)^2|\hat{v}_x|^2(x,t)\,dx + \varepsilon\int_0^t\int_0^1 x^2(1-x)^2|\hat{v}_{xx}|^2(x,\tau)\,dxd\tau \le C\varepsilon^{1/2},$$

where the constant C is independent of ε but depends on t. Thus, the proof Lemma 2.6 is completed.

$\square$

With the help of Lemma 2.4, Lemma 2.5 and Lemma 2.6, we can estimate the thickness of boundary layers.

Proof of Theorem 2.2. By Lemma 2.4, Lemma 2.5 and Sobolev embedding inequality, we have that for any $t \in [0,T]$,

$$\|(u^\varepsilon - u^0)(t)\|^2_{C[0,1]} \leq C\left(\|(u^\varepsilon - u^0)(t)\|^2_{L^2(0,1)} + \|(u^\varepsilon - u^0)_x(t)\|^2_{L^2(0,1)}\right) \leq C\varepsilon^{1/2}.$$

Thus, we obtain (2.4) and

$$\|(u^\varepsilon - u^0)\|^2_{L^\infty(0,T;C[0,1])} \to 0, \quad \text{as } \varepsilon \to 0.$$

Next, we prove (2.5). First one can prove that for any $\delta \in (0,1/2)$,

$$\delta^2 \leq 4x^2(1-x)^2, \quad \forall x \in (\delta, 1-\delta).$$

This, along with Lemma 2.6 gives

$$\delta^2 \int_\delta^{1-\delta} \hat{v}_x^2(x,t)\,dx \leq 4\int_\delta^{1-\delta} x^2(1-x)^2\hat{v}_x^2(x,t)\,dx \leq C\varepsilon^{1/2},$$

from which we derive that for any $\delta \in (0,1/2)$,

$$\|(v^\varepsilon - v^0)_x(t)\|_{L^2(\delta,1-\delta)} \leq C\delta^{-1}\varepsilon^{1/4}, \quad \forall t \in [0,T].$$

Combining this with Lemma 2.4, we have by Gagliardo-Nirenberg interpolation inequality that for any $t \in [0,T]$,

$$\begin{aligned}
&\|(v^\varepsilon - v^0)(t)\|^2_{C[\delta,1-\delta]} \\
&\quad \leq C(\|(v^\varepsilon - v^0)(t)\|^2_{L^2(\delta,1-\delta)} + \|(v^\varepsilon - v^0)(t)\|_{L^2(\delta,1-\delta)}\|(v^\varepsilon - v^0)_x(t)\|_{L^2(\delta,1-\delta)}) \\
&\quad \leq C(\varepsilon^{1/2} + \varepsilon^{1/2}\delta^{-1}) \\
&\quad \leq C\varepsilon^{1/2}\delta^{-1},
\end{aligned}$$

where the constant C is independent of ε but depends on T. Hence, we obtain (2.5) and

$$\|(v^\varepsilon - v^0)\|^2_{L^\infty(0,T;C[\delta,1-\delta])} \to 0, \quad \text{as } \varepsilon \to 0,$$

provided that $\delta = \delta(\varepsilon)$ satisfies

$$\delta(\varepsilon) \to 0 \quad \text{and} \quad \varepsilon^{1/2}/\delta(\varepsilon) \to 0, \quad \text{as } \varepsilon \to 0.$$

Next we turn to show that (2.6) and (2.7) are equivalent. For this, we integrate $(2.1)_2$ with $\varepsilon = 0$ over $(0,t)$ and set $x = 0$ in the resulting integral equation to obtain

$$v^0(0,t) = v^0(0,0) + \int_0^t u_x^0(0,\tau)d\tau = \hat{v}_0(0) + \int_0^t u_x^0(0,\tau)d\tau = \bar{v} + \int_0^t u_x^0(0,\tau)d\tau, \quad (2.31)$$

where we have used the compatible condition $\hat{v}_0(0) = \bar{v}$. Then it follows from (2.31) that $v^0(0,t) - \bar{v} = \int_0^t u_x^0(0,\tau)d\tau$, which, along with the boundary condition $v^\varepsilon(0,t) = \bar{v}$ gives for any $\varepsilon > 0$ that

$$v^0(0,t) - v^\varepsilon(0,t) = \int_0^t u_x^0(0,\tau)d\tau. \quad (2.32)$$

If we assume that $\int_0^t u_x^0(0,\tau)d\tau \neq 0$ for some $t \in [0,T]$, then (2.6) holds and the boundary layer appears at $x = 0$. Similarly if we assume that $\int_0^t u_x^0(1,\tau)d\tau \neq 0$, then (2.6) holds and the boundary layer appears at $x = 1$. Thus, we have proved that (2.7) implies (2.6). It remains to show (2.6) implies (2.7) by argument of contradiction. Indeed if we assume (2.7) is false, that is

$$\int_0^t p_x^0(0,\tau)\,d\tau \cdot \int_0^t p_x^0(0,\tau)\,d\tau = 0, \quad \forall\, t \in [0,T],$$

then it follows from (2.32) that

$$v^0(0,t) - v^\varepsilon(0,t) = v^0(1,t) - v^\varepsilon(1,t) = 0, \quad \forall\, t \in [0,T]. \quad (2.33)$$

We shall show below that under (2.33) the boundary terms for v in the proof of Lemma 2.4 and Lemma 2.5 will vanish and hence lead to a estimates violating (2.6). In fact, with (2.33), we have $K_4 = 0$ in (2.22) and K_2 can be estimated in a more delicate way by

$$\begin{aligned}
K_2 \leq& 2\varepsilon\|v^\varepsilon\|_{L^\infty}\|v_x^\varepsilon\|_{L^2}\|v\|_{L^2} \\
\leq& C\varepsilon\left(\|v^\varepsilon\|_{L^2} + \|v^\varepsilon\|_{L^2}^{1/2}\|v_x^\varepsilon\|_{L^2}^{1/2}\right)\|v_x^\varepsilon\|_{L^2}\|v\|_{L^2} \\
\leq& C\varepsilon^2\|v^\varepsilon\|_{L^2}^2\|v_x^\varepsilon\|_{L^2}^2 + C\varepsilon^2\|v^\varepsilon\|_{L^2}\|v_x^\varepsilon\|_{L^2}^3 + \|v\|_{L^2}^2 \\
\leq& C\varepsilon^{5/4}\left(\|v^\varepsilon\|_{L^2} + \|v^\varepsilon\|_{L^2}^2\right)\left(\varepsilon^{1/2}\|v_x^\varepsilon\|_{L^2}^2 + \varepsilon^{3/4}\|v_x^\varepsilon\|_{L^2}^3\right) + \|v\|_{L^2}^2,
\end{aligned} \quad (2.34)$$

where the assumption $\varepsilon < 1$ has been used. Now we modify the proof of Lemma 2.4 directly by using $K_4 = 0$ and replacing K_2 in (2.22) with (2.34) and get by a similar argument as deriving (2.23) that

$$\sup_{0\leq t\leq T}\left(\|u(t)\|_{L^2}^2 + \|v(t)\|_{L^2}^2\right) + \int_0^T \left(\|\hat{u}_x\|_{L^2}^2 + \varepsilon\|\hat{v}_x\|_{L^2}^2\right)dt \leq C\varepsilon^{5/4}. \quad (2.35)$$

Similarly we can modify the proof of Lemma 2.5 directly to get a better estimates for $\hat{v}_x$. First, differentiating (2.33) with respect to t gives $\hat{v}_t|_{x=0,x=1} = 0$, which leads to $K_8 = 0$ in

(2.26). Then using a similar argument as obtaining (2.34), we find

$$
\begin{aligned}
K_6 \leq & 2\varepsilon \|v^\varepsilon\|_{L^\infty} \|v_x^\varepsilon\|_{L^2} \|\hat{v}_t\|_{L^2} \\
\leq & \frac{1}{4}\|\hat{v}_t\|_{L^2}^2 + C\varepsilon^{5/4}\left(\|v^\varepsilon\|_{L^2} + \|v^\varepsilon\|_{L^2}^2\right)\left(\varepsilon^{1/2}\|v_x^\varepsilon\|_{L^2}^2 + \varepsilon^{3/4}\|v_x^\varepsilon\|_{L^2}^3\right).
\end{aligned}
$$

Now substituting the above estimate for K_6 into (2.26), keeping the estimates of K_5, K_7 unchanged, and using the same arguments as deriving (2.28), one easily gets that

$$
\varepsilon \sup_{0\leq t\leq T} \|\hat{v}_x(t)\|_{L^2}^2 + \int_0^T \|\hat{v}_t\|_{L^2}^2 \, dt \leq C\varepsilon^{5/4},
$$

which, entails that for $t \in [0,T]$

$$
\|\hat{v}_x(t)\|_{L^2}^2 \leq C\varepsilon^{1/4}. \tag{2.36}
$$

Then from (2.35) and (2.36), we deduce for $t \in [0,T]$ that

$$
\begin{aligned}
& \|(v^\varepsilon - v^0)(t)\|_{C[0,1]}^2 \\
\leq & C(\|(v^\varepsilon - v^0)(t)\|_{L^2(0,1)}^2 + \|(v^\varepsilon - v^0)(t)\|_{L^2(0,1)} \|(v^\varepsilon - v^0)_x(t)\|_{L^2(0,1)}) \\
\leq & C(\varepsilon^{5/4} + \varepsilon^{5/8} \cdot \varepsilon^{1/8}) \\
\leq & C\varepsilon^{3/4},
\end{aligned}
$$

which, yields

$$
\liminf_{\varepsilon\to 0} \|v^\varepsilon - v^0\|_{L^\infty(0,T;C[0,1])} = \lim_{\varepsilon\to 0} \|v^\varepsilon - v^0\|_{L^\infty(0,T;C[0,1])} = 0.
$$

This contradicts (2.6) and hence (2.7) holds by argument of contradiction. The proof is completed.

□

Note that in general the condition (2.7) in Theorem 2.2 is hardly checkable unless the term $u_x^0(0,\tau)$ or $u_x^0(1,\tau)$ is known. Below we shall show that the condition (2.7) can be ensured by assuming $u_{0x}(0) \neq 0$ or $u_{0x}(1) \neq 0$. For example without loss of generality, we assume that $u_{0x}(0) \neq 0$ and furthermore $u_{0x}(0) > 0$. By part (ii) of Lemma 2.1, we know that $u^0 \in C([0,\infty);H^2)$, which along with Sobolev embedding theorem, entails for any $T \in (0,\infty)$ that

$$
u_x^0(x,t) \in C([0,1]\times[0,T]),
$$

which implies

$$
u_x^0(0,t) \in C([0,T]). \tag{2.37}
$$

We know from the initial conditions that $u^0(x,0) = u_0(x)$. Differentiating this equation with respect to x and then setting $x = 0$, we obtain

$$u^0_x(0,0) = u_{0x}(0) > 0. \tag{2.38}$$

Combing (2.37) and (2.38), we conclude that there exists a suitably small $T^* > 0$, such that

$$u^0_x(0,\tau) > 0, \quad \forall\, \tau \in [0,T^*]. \tag{2.39}$$

Then from (2.32) and (2.39), we have for any $\varepsilon > 0$ that

$$\|v^\varepsilon - v^0\|_{L^\infty(0,T;C[0,1])} \geq \int_0^T u^0_x(0,t)dt > 0, \quad \forall 0 < T < T^*$$

and

$$\|v^\varepsilon - v^0\|_{L^\infty(0,T;C[0,1])} \geq \int_0^{T^*} u^0_x(0,t)dt > 0, \quad \forall T \geq T^*.$$

From the above two inequalities, we conclude that there exists a positive constant $C(T,T^*)$ independent of ε but dependent on T and T^*, such that for any $\varepsilon > 0$

$$\|v^\varepsilon - v^0\|_{L^\infty(0,T;C[0,1])} \geq C(T,T^*) > 0.$$

Hence, for any $0 < T < \infty$

$$\liminf_{\varepsilon\to 0} \|v^\varepsilon - v^0\|_{L^\infty(0,T;C[0,1])} > 0.$$

Thus, we obtain (2.6) under the assumption that $u_{0x}(0) > 0$. The result can be extended to the case $u_{0x}(0) < 0$ similarly. In a similar fashion as above, one can derive (2.6) for the case $u_{0x}(1) \neq 0$. This yields the results of Remark 2.1.

Chapter 3

Stability of Boundary Layers in One Dimension

Theorem 2.2 only showed the existence of boundary layers for (2.1)-(2.2) and proved the convergence of the solution component v^ε as $\varepsilon \to 0$ outside the boundary layers. However, the structure of v^ε as $\varepsilon \to 0$ inside the boundary layers remains unknown.

In this chapter, we shall exploit the structure of v^ε inside the boundary layers and justify the stability of boundary layer solutions of (2.1)-(2.2) in the entire interval $(0,1)$. With the general boundary layer theory [71, 75] applied to (2.1)-(2.2), the solution profile $(u^\varepsilon, v^\varepsilon)$ of (2.1) for small $\varepsilon > 0$ is composed of two parts: outer layer profile and inner (boundary) layer profile. Since u^ε converges uniformly in ε and hence the inner layer profile part will be absent, $(u^\varepsilon, v^\varepsilon)$ is anticipated to possess the form:

$$\begin{aligned} &u^\varepsilon = u^0 + O(\varepsilon^\alpha); \\ &v^\varepsilon = v^0 + v^L\big(\tfrac{x}{\sqrt{\varepsilon}}, t\big) + v^R\big(\tfrac{x-1}{\sqrt{\varepsilon}}, t\big) + O(\varepsilon^\alpha) \end{aligned} \tag{3.1}$$

for some $\alpha \leq 1/2$, where (u^0, v^0) is the outer layer profile which is the solution of non-diffusive problem of (2.1)-(2.2) with $\varepsilon = 0$, and the inner (boundary) layer profile v^L/v^R adjust rapidly from a value away from the boundary to a different value on the left/right end point. Outside the boundary layer, the non-diffusive problem dominates. Inside the boundary layer, diffusion becomes important.

We shall first explicitly derive the outer/inner layer profiles in Section 3.1 and give the main results in Section 3.2 (see Theorem 3.1), which states that (3.1) holds as $\varepsilon \to 0$ for $\alpha = 1/2$, which is the optimal convergence rate since the magnitude of boundary layer thickness is of order $\varepsilon^{1/2}$. We then convert the results of (2.1)-(2.2) back to the original chemotaxis model (1.2) in Theorem 3.2 and find that the chemical concentration has no boundary layer but its gradient does. The regularity on outer/inner layer profiles and the proof of the main results will be given in Section 3.3 - Section 3.5. Finally, Section 3.6 is devoted to the formal derivation of the outer/inner layer profiles of Section 3.1.

3.1 Equations for Outer/Inner Layer Profiles

In this section, we are devoted to using formal asymptotic analysis to find the equations of boundary layer profiles of (2.1) with small $\varepsilon > 0$. The boundary layer thickness has been formally justified as $O(\varepsilon^{1/2})$ in appendix of [29]. Thus based on the WKB method (cf. [26, 22, 72]), solutions of (2.1) have the following expansions for $j \in \mathbb{N}$:

$$\begin{aligned} u^{\varepsilon}(x,t) &= \sum_{j=0}^{\infty} \varepsilon^{j/2} \left(u^{I,j}(x,t) + u^{B,j}(z,t) + u^{b,j}(\xi,t) \right), \\ v^{\varepsilon}(x,t) &= \sum_{j=0}^{\infty} \varepsilon^{j/2} \left(v^{I,j}(x,t) + v^{B,j}(z,t) + v^{b,j}(\xi,t) \right), \end{aligned} \tag{3.2}$$

with boundary layer coordinates (or stretching transformations) defined as:

$$z = \frac{x}{\sqrt{\varepsilon}}, \qquad \xi = \frac{x-1}{\sqrt{\varepsilon}}, \ x \in [0,1], \tag{3.3}$$

where each term in (3.2) is assumed to be smooth, and the boundary layer profiles $(u^{B,j}, v^{B,j})$ and $(u^{b,j}, v^{b,j})$ enjoy the following basic hypothesis (cf. [26, Chapter 4], [22], [72]):

(H) $u^{B,j}$ and $v^{B,j}$ decay to zero exponentially as $z \to \infty$, while $u^{b,j}$ and $v^{b,j}$ decay to zero exponentially as $\xi \to -\infty$ for all $j \geq 0$.

To derive the equations of boundary layer profiles in (3.2), we split our analysis into three steps. We first insert expansions (3.2) into the initial data in (2.1) and into (2.2) to obtain the initial and boundary values of outer and inner layer profiles. Then in the second and third steps, equations for both outer and inner layer solutions will be derived by substituting (3.2) into the first and second equations of (2.1) successively. Proceeding with these procedures by the asymptotic matching method (details are given in Section 3.6), we derive that the leading-order outer layer solution pair $(u^{I,0}, v^{I,0})(x,t)$ satisfies the following problem:

$$\begin{cases} u_t^{I,0} = (u^{I,0} v^{I,0})_x + u_{xx}^{I,0}, \qquad (x,t) \in (0,1) \times (0,\infty), \\ v_t^{I,0} = u_x^{I,0}, \\ (u^{I,0}, v^{I,0})(x,0) = (u_0, v_0)(x), \\ u^{I,0}(0,t) = u^{I,0}(1,t) = \bar{u}. \end{cases} \tag{3.4}$$

The leading-order inner layer solution $v^{B,0}(z,t)$ near the left end point of $(0,1)$ satisfies

$$\begin{cases} v_t^{B,0} = -\bar{u} v^{B,0} + v_{zz}^{B,0}, \quad (z,t) \in (0,\infty) \times (0,\infty), \\ v^{B,0}(z,0) = 0, \\ v^{B,0}(0,t) = \bar{v} - v^{I,0}(0,t), \end{cases} \tag{3.5}$$

and $u^{B,0}(z,t) \equiv 0$, and the first-order inner layer solution $u^{B,1}(z,t)$ is determined by $v^{B,0}(z,t)$ through

$$u^{B,1}(z,t) = \bar{u} \int_z^\infty v^{B,0}(s,t)\, ds, \quad z \in [0,\infty). \tag{3.6}$$

The leading-order inner layer solution $v^{b,0}(\xi,t)$ near the right end point of $(0,1)$ satisfies

$$\begin{cases} v_t^{b,0} = -\bar{u} v^{b,0} + v_{\xi\xi}^{b,0}, \quad (\xi,t) \in (-\infty,0) \times (0,\infty), \\ v^{b,0}(\xi,0) = 0, \\ v^{b,0}(0,t) = \bar{v} - v^{I,0}(1,t), \end{cases} \tag{3.7}$$

and $u^{b,0}(\xi,t) \equiv 0$, and the corresponding first-order inner layer solution $u^{b,1}(\xi,t)$ is given by

$$u^{b,1}(\xi,t) = \bar{u} \int_\xi^{-\infty} v^{b,0}(s,t)\, ds, \quad \xi \in (-\infty,0]. \tag{3.8}$$

To carry out our desired results, we need the estimates of the first-order outer layer solution pair $(u^{I,1}, v^{I,1})(x,t)$ which satisfies the following problem:

$$\begin{cases} u_t^{I,1} = (u^{I,0} v^{I,1})_x + (u^{I,1} v^{I,0})_x + u_{xx}^{I,1}, \quad (x,t) \in (0,1) \times (0,\infty), \\ v_t^{I,1} = u_x^{I,1}, \\ (u^{I,1}, v^{I,1})(x,0) = (0,0), \\ u^{I,1}(0,t) = -\bar{u} \int_0^\infty v^{B,0}(z,t)\, dz, \\ u^{I,1}(1,t) = -\bar{u} \int_0^{-\infty} v^{b,0}(\xi,t)\, d\xi. \end{cases} \tag{3.9}$$

Moreover the inner layer profile $(u^{B,2}, v^{B,1})(z,t)$ satisfies

$$\begin{cases} v_t^{B,1} = -\bar{u} v^{B,1} + v_{zz}^{B,1} - 2(v^{I,0}(0,t) + v^{B,0}) v_z^{B,0} + \int_z^\infty \Phi(s,t)\, ds, \\ v^{B,1}(z,0) = 0, \\ v^{B,1}(0,t) = -v^{I,1}(0,t), \end{cases} \tag{3.10}$$

and

$$u^{B,2}(z,t) = \bar{u} \int_z^\infty v^{B,1}(s,t)\, ds - \int_z^\infty \int_s^\infty \Phi(\zeta,t)\, d\zeta ds, \tag{3.11}$$

where $\Phi(z,t) := (u^{I,1}(0,t) + u^{B,1}) v_z^{B,0} + u_x^{I,0}(0,t) v^{B,0} + u_z^{B,1}(v^{I,0}(0,t) + v^{B,0}) + z u_x^{I,0}(0,t) v_z^{B,0}$. Correspondingly the inner layer profile $(u^{b,2}, v^{b,1})(\xi,t)$ satisfies

$$\begin{cases} v_t^{b,1} = -\bar{u} v^{b,1} + v_{\xi\xi}^{b,1} - 2(v^{I,0}(1,t) + v^{b,0}) v_\xi^{b,0} + \int_\xi^{-\infty} \Psi(s,t)\, ds, \\ v^{b,1}(\xi,0) = 0, \\ v^{b,1}(0,t) = -v^{I,1}(1,t), \end{cases} \tag{3.12}$$

and

$$u^{b,2}(\xi,t) = \bar{u}\int_{\xi}^{-\infty} v^{b,1}(s,t)\,ds - \int_{\xi}^{-\infty}\int_{s}^{-\infty}\Psi(\zeta,t)\,d\zeta ds, \tag{3.13}$$

where

$$\Psi(\xi,t) := (u^{I,1}(1,t)+u^{b,1})v^{b,0}_{\xi} + u^{I,0}_{x}(1,t)v^{b,0} + u^{b,1}_{\xi}(v^{I,0}(1,t)+v^{b,0}) + \xi u^{I,0}_{x}(1,t)v^{b,0}_{\xi}.$$

One can derive the initial-boundary value problems for higher-order layer profiles $(u^{I,j}, v^{I,j})$, $(u^{B,j+1}, v^{B,j})$ and $(u^{b,j+1}, v^{b,j})$ for $j \geq 2$. But the equations (3.4)-(3.13) have been sufficient for our purpose. The detailed derivations of above equations are postponed to be given in Section 3.6, since it is a little lengthy. The global solutions of (3.4) have been achieved in [44] (see Lemma 2.1) and their regularities will be shown in Section 3.3. The existence of global solutions of (3.5)-(3.13) with regularities will be detailed also in Section 3.3.

3.2 Results on Stability of Boundary Layers

In order to prove the stability of boundary layer solutions of (2.1)-(2.2), we need some further compatibility conditions on boundaries and higher regularity for the initial data (u_0, v_0) to gain necessary estimates for solutions of equations (3.4)-(3.12). Precisely, we postulate that the initial data $(u_0, v_0) \in H^3 \times H^3$ satisfy

$$(A) \qquad \begin{cases} (u_0, v_0)|_{x=0,1} = (\bar{u}, \bar{v}), \\ u_{0x}|_{x=0,1} = 0, \\ [(u_0 v_0)_x + u_{0xx}]|_{x=0,1} = 0. \end{cases}$$

We underline that the condition (A) can be fulfilled by many functions, for instance $u_0(x) = \bar{u} + ax^4(x-1)^4$, $v_0(x) = \bar{v} + bx^2(x-1)^2$ with $a \geq 0$ and $b \in \mathbb{R}$.

Now we are in a position to state the main results of this chapter as follows.

Theorem 3.1. *Assume that $(u_0, v_0) \in H^3 \times H^3$ with $u_0 \geq 0$ satisfy the compatibility conditions (A). Denote by $v^{B,0}$ and $v^{b,0}$ the solutions of (3.5) and (3.7), respectively. Let $(u^{\varepsilon}, v^{\varepsilon})$ be the global solution of (2.1)-(2.2) with $\varepsilon \geq 0$. Then as $\varepsilon \to 0$, the following asymptotic expansions hold in space $L^{\infty}([0,1] \times [0,T])$ for any fixed $0 < T < \infty$:*

$$\begin{aligned} u^{\varepsilon}(x,t) &= u^0(x,t) + O(\varepsilon^{1/2}), \\ v^{\varepsilon}(x,t) &= v^0(x,t) + v^{B,0}\Big(\frac{x}{\sqrt{\varepsilon}}, t\Big) + v^{b,0}\Big(\frac{x-1}{\sqrt{\varepsilon}}, t\Big) + O(\varepsilon^{1/2}), \end{aligned} \tag{3.14}$$

where $(u^0, v^0) = (u^{I,0}, v^{I,0})$ and

$$v^{B,0}(z,t) := \int_0^t \int_{-\infty}^0 \frac{1}{\sqrt{\pi(t-s)}} e^{-\left(\frac{(z-\zeta)^2}{4(t-s)} + \bar{u}(t-s)\right)} \left[\bar{u}(\bar{v} - v^0(0,s)) - v^0_s(0,s)\right] d\zeta\, ds$$

and

$$v^{b,0}(\xi,t) := \int_0^t \int_0^\infty \frac{1}{\sqrt{\pi(t-s)}} e^{-\left(\frac{(\xi-\zeta)^2}{4(t-s)}+\bar{u}(t-s)\right)} \left[\bar{u}(\bar{v}-v^0(1,s))-v_s^0(1,s)\right] d\zeta\, ds.$$

A numerical simulation of the boundary layer solution component $v^\varepsilon(x,t)$ is plotted in Fig.3.1, where the structure of $v^\varepsilon(x,t)$ is graphically demonstrated.

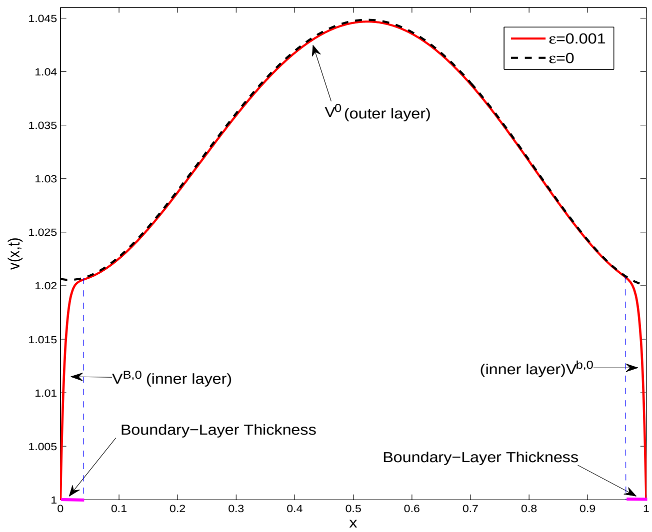

Fig. 3.1 A numerical simulation of the boundary layer profile $v^\varepsilon(x,t)$ of the system (2.1)-(2.2) solved by the Matlab PDE solver based on the finite difference scheme with time step size $\Delta t = 0.01$ and spatial step size $\Delta x = 0.001$ where initial data $u_0(x) = 1 + x^4(x-1)^4$, $v_0(x) = 1 + x^2(x-1)^2$ and boundary data $\bar{u} = \bar{v} = 1$. The profile consists of two parts: outer layer profile v^0 and inner layer profiles $v^{B,0}$ and $v^{b,0}$ near left and right end points, respectively. Outside the boundary layer the profile $v^\varepsilon(x,t)$ matches well with the outer layer profile $v^0(x,t)$, whereas there is a rapid transition inside the boundary layer.

The counterpart of the original system (1.2) in $[0,1]$ corresponding to the initial-boundary value problem of the transformed system (2.1)-(2.2) reads as follows:

$$\begin{cases} u_t = [u_x - u(\ln c)_x]_x, \\ c_t = \varepsilon c_{xx} - uc, \\ (u,c)(x,0) = (u_0,c_0)(x), \quad x \in [0,1], \\ u|_{x=0,1} = \bar{u}, \quad \frac{c_x}{c}|_{x=0,1} = -\bar{c}, \qquad \text{if } \varepsilon > 0, \\ u|_{x=0,1} = \bar{u}, \qquad \text{if } \varepsilon = 0. \end{cases} \tag{3.15}$$

With the results obtained for the transformed system (2.1)-(2.2), we have the following assertions for the initial-boundary value problem (3.15).

Theorem 3.2. *Suppose that the initial data* $(u_0, \ln c_0) \in H^3 \times H^4$ *satisfy* $u_0(x) \geq 0$, $c_0(x) > 0$ *and the compatibility conditions* (A) *with* $v_0 = -(\ln c_0)_x$ *and* $\bar{v} = \bar{c}$. *Let* $(u^\varepsilon, c^\varepsilon)$ *be the unique global solution of* (3.15) *with* $\varepsilon \geq 0$. *Then for any fixed* $0 < T < \infty$, *we have in space* $L^\infty([0,1] \times [0,T])$ *that*

$$u^\varepsilon(x,t) = u^0(x,t) + O(\varepsilon^{1/2}), \qquad c^\varepsilon(x,t) = c^0(x,t) + O(\varepsilon^{1/2}) \tag{3.16}$$

and

$$c_x^\varepsilon(x,t) = c_x^0(x,t) - c^0(x,t)\Big[v^{B,0}\Big(\frac{x}{\sqrt{\varepsilon}},t\Big) + v^{b,0}\Big(\frac{x-1}{\sqrt{\varepsilon}},t\Big)\Big] + O(\varepsilon^{1/2}). \tag{3.17}$$

In view of model (1.2) and the transformation (1.3), we see that the quantity $\vec{v}$ represents the velocity of chemotactic flux crossing the boundary (in the tumor angiogenesis the blood vessel wall can be understood as a boundary). Therefore the results in Theorem 3.2 assert that although both cell density and chemical concentration will have no boundary layer as chemical diffusion ε goes to zero, the chemotactic flux, namely the term $u(\ln c)_x = -uv$, has a sharp transition near the boundary (i.e. the endothelial cells cross the blood vessel wall quickly). Hence our results indicate that the diffusion of chemical signal (i.e. vascular endothelial growth factor) plays an essential role in the transition of cell mass from boundaries to the field away from boundaries during the initiation of tumor angiogenesis.

3.3 Regularity of Outer/Inner Layer Profiles

In this section, we shall devote ourselves to deriving some regularities for solutions of (3.4)-(3.13) for later use. We depart with a basic regularity result.

Let functions $f_1(x,t)$, $f_2(x,t)$, $f(x,t)$ and $g(x,t)$ defined on $[0,1] \times [0,\infty)$ satisfy the following regularity properties for any $m \in \mathbb{N}_+$ and $0 < T < \infty$:

$$\begin{aligned} &\partial_t^k f_1 \in L^2(0,T;H^{2m-1-2k}), \quad \partial_t^k f_2 \in L^2(0,T;H^{2m-1-2k}), \\ &\partial_t^k f \in L^2(0,T;H^{2m-2-2k}), \quad \partial_t^k g \in L^2(0,T;H^{2m-1-2k}), \end{aligned}$$

where $k = 0,1,\cdots,m-1$. To solve the outer layer solution pairs $(u^{I,j}, v^{I,j})(x,t), j = 0,1$ from problems (3.4) and (3.9), we first consider the following auxiliary initial-boundary value problem

$$\begin{cases} h_t = (f_1 h)_x + (f_2 w)_x + h_{xx} + f, & (x,t) \in (0,1) \times (0,T), \\ w_t = h_x + g, \\ (h,w)(x,0) = (h_0, w_0)(x), \\ h(0,t) = h(1,t) = 0. \end{cases} \tag{3.18}$$

To derive the desired regularity (3.20) for solutions (h,w) of (3.18) (see Proposition 3.1 below), we require that $\partial_t^k h|_{t=0}=0$ on boundaries for $0\le k\le m-1$ (cf. [39, page 319]). First for $k=0,1$, $h|_{t=0}$ and $\partial_t h|_{t=0}$ can be determined by initial data (h_0,w_0), functions f_1,f_2 and f through the first equation of (3.18):

$$h|_{t=0}=h_0(x),\ \partial_t h|_{t=0}=(f_1(x,0)h_0)_x+(f_2(x,0)w_0)_x+h_{0xx}+f(x,0). \tag{3.19}$$

Moreover, for $2\le k\le m-1$, applying ∂_t^{k-1} and ∂_t^{k-2} to the first and second equations of (3.18), respectively, then combining the results with (3.19), one finds by mathematical induction that $\partial_t^k h|_{t=0}$ $(0\le k\le m-1)$ can be determined by h_0, w_0, f_1, f_2, f, g and their i-th order time derivatives with $0\le i\le m-2$. In the sequel, by "h_0, w_0, f_1, f_2, f and g satisfy the compatibility conditions up to order $(m-1)$ for the problem (3.18)", we mean that $\partial_t^k h|_{t=0}$, which is determined by h_0, w_0, f_1, f_2, f and g through the equations in (3.18), are equal to zeros on boundaries for $0\le k\le m-1$ (cf. [39, page 319]).

Then the solution of (3.18) has the following regularity properties.

Proposition 3.1. *Suppose that $(h_0,w_0)\in H^{2m-1}\times H^{2m-1}$, f_1, f_2, f and g satisfy the compatibility conditions up to order $(m-1)$ for the problem* (3.18). *Then there exists a unique solution (h,w) to* (3.18) *for any $0<T<\infty$ such that*

$$\begin{aligned}&\partial_t^k h\in L^2(0,T;H^{2m-2k}),\quad k=0,1,\cdots,m;\\&w\in L^\infty(0,T;H^{2m-1});\quad \partial_t^k w\in L^2(0,T;H^{2m+1-2k}),\quad k=1,\cdots,m.\end{aligned} \tag{3.20}$$

Proof. The global existence and uniqueness of solutions to (3.18) is standard (see Lemma 2.1). We prove the regularity given in (3.20) by mathematical induction. We first prove it is true for $m=1$. Assume that $(h_0,w_0)\in H^1\times H^1$ with $h_0(0)=h_0(1)=0$ and that f_1, f_2, f and g satisfy

$$f_1,f_2,g\in L^2(0,T;H^1),\quad f\in L^2(0,T;L^2). \tag{3.21}$$

We aim to prove that

$$\partial_t^k h\in L^2(0,T;H^{2-2k}),\ k=0,1;\quad w\in L^\infty(0,T;H^1),\quad w_t\in L^2(0,T;H^1). \tag{3.22}$$

Taking the L^2 inner products of the first and second equations of (3.18) with $2h$ and $2w$ respectively, using integration by parts and then adding the results, we find that

$$\begin{aligned}&\frac{d}{dt}(\|h(t)\|_{L^2}^2+\|w(t)\|_{L^2}^2)+2\|h_x(t)\|_{L^2}^2\\&=-2\int_0^1(f_1h+f_2w)h_x\,dx+2\int_0^1(fh+h_xw+gw)\,dx\\&:=I_1+I_2.\end{aligned} \tag{3.23}$$

By the Sobolev embedding inequality and the Cauchy-Schwarz inequality, I_1 and I_2 are estimated as follows:

$$\begin{aligned}
I_1 \leq & \|f_1(t)\|_{L^\infty}\|h(t)\|_{L^2}\|h_x(t)\|_{L^2} + \|f_2(t)\|_{L^\infty}\|w(t)\|_{L^2}\|h_x(t)\|_{L^2} \\
\leq & \frac{1}{2}\|h_x(t)\|_{L^2}^2 + C_0\|f_1(t)\|_{H^1}^2\|h(t)\|_{L^2}^2 + C_0\|f_2(t)\|_{H^1}^2\|w(t)\|_{L^2}^2, \\
I_2 \leq & \|f(t)\|_{L^2}^2 + \|h(t)\|_{L^2}^2 + \frac{1}{2}\|h_x(t)\|_{L^2}^2 + 3\|w(t)\|_{L^2}^2 + \|g(t)\|_{L^2}^2.
\end{aligned}$$

Now feeding (3.23) on the above estimates and using Gronwall's inequality and (3.21), one gets

$$\|h\|_{L^\infty(0,T;L^2)}^2 + \|w\|_{L^\infty(0,T;L^2)}^2 + \|h_x\|_{L^2(0,T;L^2)}^2 \leq C. \tag{3.24}$$

We proceed with the derivation of higher regularities. Differentiating the second equation of (3.18) with respect to x and using the first equation of (3.18), we derive

$$w_{xt} = h_t - (f_1 h)_x - (f_2 w)_x - f + g_x,$$

which, multiplied by $2w_x$ in L^2 gives

$$\begin{aligned}
\frac{d}{dt}\|w_x(t)\|_{L^2}^2 = & 2\int_0^1 h_t w_x\, dx - 2\int_0^1 [(f_1 h)_x + (f_2 w)_x] w_x\, dx \\
& + 2\int_0^1 (g_x - f) w_x\, dx \\
:= & I_3 + I_4 + I_5.
\end{aligned} \tag{3.25}$$

We first rewrite I_3 as follows:

$$I_3(t) = 2\frac{d}{dt}\int_0^1 h w_x\, dx - 2\int_0^1 h w_{xt}\, dx := M_1 + M_2,$$

where M_1 can be written as

$$\begin{aligned}
M_1 = & 2\frac{d}{dt}\int_0^1 [w_x^2/4 + h^2 - (w_x/2 - h)^2]\, dx \\
= & \frac{1}{2}\frac{d}{dt}\|w_x(t)\|_{L^2}^2 - 2\frac{d}{dt}\|(w_x/2 - h)(t)\|_{L^2}^2 + 2\frac{d}{dt}\|h(t)\|_{L^2}^2,
\end{aligned}$$

and M_2 can be estimated as

$$M_2 = -2\int_0^1 h h_{xx}\, dx - 2\int_0^1 h g_x\, dx \leq 2\|h_x(t)\|_{L^2}^2 + \|h(t)\|_{L^2}^2 + \|g_x(t)\|_{L^2}^2,$$

by the second equation of (3.18) and integration by parts. We turn to estimating terms I_4 and I_5 by the Sobolev embedding inequality and the Cauchy-Schwarz inequality:

$$\begin{aligned}
I_4 \leq & C_0(\|f_1(t)\|_{H^1}^2 + \|f_2(t)\|_{H^1}^2)\|w_x(t)\|_{L^2}^2 + \|h(t)\|_{H^1}^2 + \|w(t)\|_{H^1}^2, \\
I_5 \leq & 2\|w_x(t)\|_{L^2}^2 + \|f(t)\|_{L^2}^2 + \|g_x(t)\|_{L^2}^2.
\end{aligned}$$

Substituting the above estimates for I_3, I_4 and I_5 into (3.25), we end up with

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|w_x(t)\|_{L^2}^2+2\frac{d}{dt}\|(w_x/2-h)(t)\|_{L^2}^2\\
\le& C_0(\|f_1(t)\|_{H^1}^2+\|f_2(t)\|_{H^1}^2+1)\|w_x(t)\|_{L^2}^2+2\frac{d}{dt}\|h(t)\|_{L^2}^2\\
&+C_0(\|h(t)\|_{H^1}^2+\|w(t)\|_{L^2}^2+\|f(t)\|_{L^2}^2+\|g(t)\|_{H^1}^2),
\end{aligned}
$$

which, along with Gronwall's inequality, (3.24) and (3.21) entails that

$$
\|w_x\|_{L^\infty(0,T;L^2)}^2 \le C. \tag{3.26}
$$

Taking the L^2 inner product of the first equation of (3.18) with $2h_t$, one gets

$$
\begin{aligned}
&\frac{d}{dt}\|h_x(t)\|_{L^2}^2+2\|h_t(t)\|_{L^2}^2\\
=&2\int_0^1 (f_1h)_x h_t\,dx+2\int_0^1 (f_2w)_x h_t\,dx+2\int_0^1 f h_t\,dx\\
\le&\|h_t(t)\|_{L^2}^2+C_0\|f_1(t)\|_{H^1}^2\|h_x(t)\|_{L^2}^2+C_0(\|f_2(t)\|_{H^1}^2\|w(t)\|_{H^1}^2+\|f(t)\|_{L^2}^2),
\end{aligned}
\tag{3.27}
$$

where the Sobolev embedding inequality and the Cauchy-Schwarz inequality have been used. Then applying Gronwall's inequality to (3.27) and using (3.24), (3.26) and (3.21) we conclude that

$$
\|h_t\|_{L^2(0,T;L^2)}^2+\|h_x\|_{L^\infty(0,T;L^2)}^2 \le C, \tag{3.28}
$$

which, in conjunction with the first equation of (3.18), (3.24), (3.26) and (3.21) gives

$$
\|h_{xx}\|_{L^2(0,T;L^2)}^2 \le C. \tag{3.29}
$$

Collecting (3.24),(3.26), (3.28) and (3.29) and using the second equation of (3.18) we obtain (3.22). Thus the conclusion of Proposition 3.1 holds true with $m=1$. The remaining procedure of mathematical induction is quite routine (e.g. see details in [16, page 387-388]) and will be omitted for brevity. $\square$

To solve inner layer profiles $v^{B,0}(z,t)$ and $v^{B,1}(z,t)$ from (3.5) and (3.10), we need the following result.

Proposition 3.2. *Let $m\in\mathbb{N}_+$ and $0<T<\infty$. Suppose $\rho(z,t)$ satisfies for any $l\in\mathbb{N}$ that*

$$
\langle z\rangle^l\,\partial_t^k\rho \in L^2(0,T;H_z^{2m-2-2k}),\quad k=0,1,\cdots,m-1,
$$

and the compatibility conditions up to order $(m-1)$ for the following problem:

$$
\begin{cases}
\varphi_t=-\bar{u}\varphi+\varphi_{zz}+\rho, & (z,t)\in(0,\infty)\times(0,T),\\
\varphi(z,0)=0,\\
\varphi(0,t)=0.
\end{cases}
\tag{3.30}
$$

Then there exists a unique solution φ *to (3.30) such that for any* $l \in \mathbb{N}$,

$$\langle z\rangle^l \partial_t^k \varphi \in L^2(0,T;H_z^{2m-2k}), \quad k=0,1,\cdots,m.$$

Proposition 3.2 follows directly from the standard energy method, and we hence omit the proof. We proceed to introduce the following well-known result for later use.

Proposition 3.3. *[77, Lemma 1.2.] Let* V, H, V' *be three Hilbert spaces, satisfying* $V \subset H \subset V'$ *with* V' *being the dual of* V. *If a function* u *belongs to* $L^2(0,T;V)$ *and its time derivative* u_t *belongs to* $L^2(0,T;V')$, *then*

$$u \in C([0,T];H) \quad \text{and } \|u\|_{L^\infty(0,T;H)} \le C(\|u\|_{L^2(0,T;V)} + \|u_t\|_{L^2(0,T;V')}),$$

where the constant C *depends on* T.

Remark 3.1. *Let* $m \in \mathbb{N}$. *Suppose that* $u \in L^2(0,T;H^{m+2})$ *and* $u_t \in L^2(0,T;H^m)$. *Then it follows from Proposition 3.3 that*

$$u \in C([0,T];H^{m+1}) \quad \text{and } \|u\|_{L^\infty(0,T;H^{m+1})} \le C(\|u\|_{L^2(0,T;H^{m+2})} + \|u_t\|_{L^2(0,T;H^m)}).$$

Based on above preliminaries, we can establish the regularities of solutions to (3.4)-(3.13). First for the problem (3.4), the existence of global solution has been available (see Lemma 2.1). We prove the following regularity results.

Lemma 3.1. *Let* $(u_0,v_0) \in H^3 \times H^3$ *satisfy the assumptions in Theorem 3.1. Then the unique solution* $(u^{I,0}, v^{I,0})$ *of* (3.4) *satisfies that*

$$\begin{aligned}
&\partial_t^k u^{I,0} \in L^2(0,T;H^{4-2k}), \quad k=0,1,2;\\
&\partial_t^k v^{I,0} \in L^2(0,T;H^{5-2k}), \quad k=1,2;\\
&v^{I,0} \in L^\infty(0,T;H^3).
\end{aligned}$$

Proof. We shall prove this lemma by Proposition 3.1 and Lemma 2.1. Differentiating the first and second equations of (3.4) with respect to t respectively, and setting $\tilde{u}^{I,0} = u_t^{I,0}$, $\tilde{v}^{I,0} = v_t^{I,0}$, one gets

$$\begin{cases}
\tilde{u}_t^{I,0} = (f_1 \tilde{u}^{I,0})_x + (f_2 \tilde{v}^{I,0})_x + \tilde{u}_{xx}^{I,0},\\
\tilde{v}_t^{I,0} = \tilde{u}_x^{I,0},\\
(\tilde{u}^{I,0}, \tilde{v}^{I,0})(x,0) = (\tilde{u}_0, \tilde{v}_0)(x),\\
\tilde{u}^{I,0}(0,t) = \tilde{u}^{I,0}(1,t) = 0,
\end{cases} \tag{3.31}$$

where $f_1 := v^{I,0}$, $f_2 := u^{I,0}$, $\tilde{u}_0 := (u_0 v_0)_x + u_{0xx}$, $\tilde{v}_0 := u_{0x}$ and the first and second equations of (3.4) have been used to determine initial data $\tilde{u}_0$ and $\tilde{v}_0$, respectively. We next verify that $\tilde{u}_0$, $\tilde{v}_0$, f_1 and f_2 fulfill the assumptions in Proposition 3.1 with $m=1$. First, by the assumptions in Theorem 3.1 one finds

$$\begin{aligned}\|\tilde{u}_0\|_{H^1} \leq & C_0\|u_0\|_{H^2}\|v_0\|_{H^2}+\|u_0\|_{H^3} \leq C_0,\\ \|\tilde{v}_0\|_{H^1} \leq & \|u_0\|_{H^2} \leq C_0.\end{aligned} \tag{3.32}$$

Lemma 2.1 leads to

$$\|f_1\|_{L^2(0,T;H^1)}+\|f_2\|_{L^2(0,T;H^1)}=\|v^{I,0}\|_{L^2(0,T;H^1)}+\|u^{I,0}\|_{L^2(0,T;H^1)} \leq C. \tag{3.33}$$

Noting that the compatibility condition of order zero for (3.31) is satisfied under assumption (A), thus using (3.32) and (3.33), we apply Proposition 3.1 with $m=1$ to system (3.31) and conclude that

$$\begin{aligned}&\partial_t^k u^{I,0} \in L^2(0,T;H^{4-2k}), \quad k=1,2\\ &\partial_t^k v^{I,0} \in L^2(0,T;H^{5-2k}), \quad k=2,\end{aligned} \tag{3.34}$$

where $\tilde{u}^{I,0} := u_t^{I,0}$ and $\tilde{v}^{I,0} := v_t^{I,0}$ have been used. It only remains to prove

$$u^{I,0} \in L^2(0,T;H^4), \quad v^{I,0} \in L^\infty(0,T;H^3), \quad v_t^{I,0} \in L^2(0,T;H^3). \tag{3.35}$$

To this end, we apply the differential operator ∂_x^3 to the second equation of (3.4), and use the first equation of (3.4) to get

$$v_{xxxt}^{I,0}=u_{xxxx}^{I,0}=u_{xxt}^{I,0}-(u^{I,0}v^{I,0})_{xxx}, \tag{3.36}$$

which, multiplied by $2v_{xxx}^{I,0}$ in L^2 gives

$$\begin{aligned}\frac{d}{dt}\|v_{xxx}^{I,0}(t)\|_{L^2}^2 \leq & 2\|u_{xxt}^{I,0}(t)\|_{L^2}\|v_{xxx}^{I,0}(t)\|_{L^2}+C_0\|u^{I,0}(t)\|_{H^3}\|v^{I,0}(t)\|_{H^3}^2\\ \leq & C_0(1+\|u^{I,0}(t)\|_{H^3})\|v_{xxx}^{I,0}(t)\|_{L^2}^2\\ & +C_0(\|u_t^{I,0}(t)\|_{H^2}^2+\|u^{I,0}(t)\|_{H^3}\|v^{I,0}(t)\ |_{H^2}^2).\end{aligned}$$

Thus it follows from Gronwall's inequality, Lemma 2.1 and (3.34) that

$$\|v_{xxx}^{I,0}\|_{L^\infty(0,T;L^2)}^2 \leq C. \tag{3.37}$$

Furthermore, using (3.36), (3.34), (3.37) and Lemma 2.1, one has

$$\begin{aligned}\|u_{xxxx}^{I,0}\|_{L^2(0,T;L^2)} \leq & \|u_t^{I,0}\|_{L^2(0,T;H^2)}+C_0\|u^{I,0}\|_{L^2(0,T;H^3)}\|v^{I,0}\|_{L^\infty(0,T;H^3)}\\ \leq & C.\end{aligned} \tag{3.38}$$

Finally, the second equation of (3.4) along with (3.38) and Lemma 2.1 yields

$$\|v_t^{I,0}\|_{L^2(0,T;H^3)} \leq \|u^{I,0}\|_{L^2(0,T;H^4)} \leq C. \tag{3.39}$$

Collecting (3.37), (3.38), (3.39) and using Lemma 2.1 we obtain (3.35), which in conjunction with (3.34) finishes the proof.

□

Lemma 3.2. *Let* $(u^{I,0}, v^{I,0})$ *be the solution obtained in Lemma 3.1. Then*

$$v^{B,0}(z,t) := \int_0^t \int_{-\infty}^0 \frac{1}{\sqrt{\pi(t-s)}} e^{-\left(\frac{(z-\zeta)^2}{4(t-s)} + \bar{u}(t-s)\right)} \left[\bar{u}(\bar{v} - v^{I,0}(0,s)) - v_s^{I,0}(0,s)\right] d\zeta\, ds \quad (3.40)$$

is the unique solution of (3.5). *Moreover, for any* $0 < T < \infty$ *and* $l \in \mathbb{N}$, *it holds that*

$$\langle z\rangle^l \, \partial_t^k v^{B,0} \in L^2(0,T; H_z^{4-2k}) \quad \text{for } k = 0,1,2. \quad (3.41)$$

Consequently it follows from (3.6) *that*

$$\langle z\rangle^l \, \partial_t^k u^{B,1} \in L^2(0,T; H_z^{4-2k}) \quad \text{for } k = 0,1,2.$$

Proof. We first prove (3.40) by setting $w(z,t) := e^{\bar{u}t}[v^{B,0}(z,t) - (\bar{v} - v^{I,0}(0,t))]$. Then from (3.5) we derive the following heat equation subject to homogeneous Dirichlet boundary condition

$$\begin{cases} w_t - w_{zz} = -[e^{\bar{u}t}(\bar{v} - v^{I,0}(0,t))]_t, \quad (z,t) \in (0,\infty) \times (0,\infty) \\ w(z,0) = 0, \\ w(0,t) = 0, \end{cases}$$

which can be solved explicitly by the reflection method with odd extensions (cf. [39]) as follows:

$$w(z,t) = 2\int_0^t \int_{-\infty}^0 \Gamma(z-\zeta, t-s)[e^{\bar{u}s}(\bar{v} - v^{I,0}(0,s))]_s \, d\zeta ds - e^{\bar{u}t}(\bar{v} - v^{I,0}(0,t)),$$

with the heat kernel $\Gamma(z,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{z^2}{4t}}$. Hence (3.40) follows by substituting the above equality into the definition of $w(z,t)$. We proceed to prove (3.41). Let $\theta(z)$ be a smooth function defined on $[0,\infty)$ satisfying

$$\theta(0) = 1, \quad \theta(z) = 0 \text{ for} z > 1. \quad (3.42)$$

Let $b(t) := \bar{v} - v^{I,0}(0,t)$ and $\tilde{v}^{B,0} := v^{B,0} - \theta(z)b(t)$. Then from (3.5) we deduce that $\tilde{v}^{B,0}$ satisfies

$$\begin{cases} \tilde{v}_t^{B,0} = -\bar{u}\tilde{v}^{B,0} + \tilde{v}_{zz}^{B,0} + \tilde{\rho}, \\ \tilde{v}^{B,0}(z,0) = 0, \\ \tilde{v}^{B,0}(0,t) = 0, \end{cases} \quad (3.43)$$

where $\tilde{\rho}(z,t) := \theta_{zz}(z)b(t) - \bar{u}\theta(z)b(t) - \theta(z)b_t(t)$, and the compatibility condition $\bar{v} = v_0(0)$ has been used to determine the initial value of $\tilde{v}^{B,0}$. We shall apply Proposition 3.2 to

(3.43) to derive the desired regularity for $v^{B,0}$. To this end, we need to verify that $\tilde{\rho}$ satisfies the assumptions in Proposition 3.2 with $m=2$. First, it is easy to check that $\tilde{\rho}$ satisfies the compatibility conditions up to order one for problem (3.43) under assumption (A). Then noticing that for any $G(x,t)\in L^p(0,T;H^1)$ with $1\le p\le\infty$, it follows from the Sobolev embedding inequality that

$$\|G(0,t)\|_{L^p(0,T)}\le\|G\|_{L^p(0,T;L^\infty)}\le C_0\|G\|_{L^p(0,T;H^1)}. \tag{3.44}$$

By (3.44) and Lemma 3.1, one finds for $k=1,2$ that

$$\|\partial_t^k v^{I,0}(0,t)\|_{L^2(0,T)}\le C_0\|\partial_t^k v^{I,0}\|_{L^2(0,T;H^{5-2k})}\le C, \tag{3.45}$$

and

$$\|v^{I,0}(0,t)\|_{L^2(0,T)}\le C\|v^{I,0}\|_{L^\infty(0,T;H^1)}\le C. \tag{3.46}$$

Collecting (3.42), (3.45) and (3.46), one deduces for $k=0,1$ and $l\in\mathbb{N}$ that

$$\langle z\rangle^l\,\partial_t^k\tilde{\rho}=\langle z\rangle^l\,\theta_{zz}\,\partial_t^k b-\bar{u}\langle z\rangle^l\,\theta\,\partial_t^k b-\langle z\rangle^l\,\theta\,\partial_t^{k+1}b\in L^2(0,T;H_z^{2-2k}),$$

which, along with Proposition 3.2 entails for $k=0,1,2$ and $l\in\mathbb{N}$ that

$$\langle z\rangle^l\,\partial_t^k\tilde{v}^{B,0}\in L^2(0,T;H_z^{4-2k}).$$

Thus (3.41) follows from the definition of $\tilde{v}^{B,0}$, (3.42), (3.45) and (3.46). By (3.41), we use (3.6) and Hölder inequality to get for $k=0,1,2$ and $l\in\mathbb{N}$ that

$$\begin{aligned}&\|\langle z\rangle^l\partial_t^k u^{B,1}\|^2_{L^2(0,T;H_z^{4-2k})}\\ \le& C_0\bar{u}^2\Big(1+\int_0^\infty\int_z^\infty\langle s\rangle^{-4}\,dsdz\Big)\|\langle z\rangle^{l+2}\partial_t^k v^{B,0}\|^2_{L^2(0,T;H_z^{4-2k})}\\ \le& C,\end{aligned}$$

which completes the proof. □

By a similar procedure as proving Lemma 3.2, we have the following results.

Lemma 3.3. *Let* $(u^{I,0},v^{I,0})$ *be the solution obtained in Lemma 3.1. Then the unique solution* $v^{b,0}(\xi,t)$ *of* (3.7) *is as follows:*

$$v^{b,0}(\xi,t):=\int_0^t\int_0^\infty\frac{1}{\sqrt{\pi(t-s)}}e^{-\left(\frac{(\xi-\zeta)^2}{4(t-s)}+\bar{u}(t-s)\right)}\left[\bar{u}(\bar{v}-v^{I,0}(1,s))-v_s^{I,0}(1,s)\right]d\zeta\,ds. \tag{3.47}$$

Furthermore, for any $0<T<\infty$ *and* $l\in\mathbb{N}$, *the following holds true:*

$$\langle\xi\rangle^l\,\partial_t^k v^{b,0},\ \langle\xi\rangle^l\,\partial_t^k u^{b,1}\in L^2(0,T;H_\xi^{4-2k})\quad\text{for } k=0,1,2.$$

Based on Lemma 3.2 and Lemma 3.3, we proceed to solve (3.9).

Lemma 3.4. *Let $v^{B,0}$ and $v^{b,0}$ be the solution obtained in Lemma 3.2 and Lemma 3.3, respectively. Then there exists a unique solution $(u^{I,1}, v^{I,1})$ to (3.9) on $[0,T]$ for any $0 < T < \infty$, such that*

$$\partial_t^k u^{I,1} \in L^2(0,T;H^{4-2k}), \quad k = 0,1,2;$$
$$v^{I,1} \in L^\infty(0,T;H^3); \quad \partial_t^k v^{I,1} \in L^2(0,T;H^{5-2k}), \quad k = 1,2.$$

Proof. Let $b_1(t) := \bar{u}\int_0^\infty v^{B,0}(z,t)\,dz$, $b_2(t) := \bar{u}\int_0^{-\infty} v^{b,0}(\xi,t)\,d\xi$, $b(x,t) := xb_2(t) + (1-x)b_1(t)$ and $\tilde{u}^{I,1} := u^{I,1} + b(x,t)$. Then from (3.9), we deduce that $(\tilde{u}^{I,1}, v^{I,1})$ satisfy

$$\begin{cases} \tilde{u}_t^{I,1} = (f_1\tilde{u}^{I,1})_x + (f_2 v^{I,1})_x + \tilde{u}_{xx}^{I,1} + f, \\ v_t^{I,1} = \tilde{u}_x^{I,1} + g, \\ (\tilde{u}^{I,1}, v^{I,1})(x,0) = (0,0), \\ \tilde{u}^{I,1}(0,t) = \tilde{u}^{I,1}(1,t) = 0, \end{cases} \tag{3.48}$$

where $f_1 := v^{I,0}$, $f_2 := u^{I,0}$, $f := -(bv^{I,0})_x + b_t$, $g := b_1(t) - b_2(t)$, and $v^{B,0}(z,0) = v^{b,0}(\xi,0) = 0$ has been used in deriving the initial data for $\tilde{u}^{I,1}$. We next verify that f_1, f_2, f and g fulfill the assumptions in Proposition 3.1 with $m = 2$. Indeed, it follows from Lemma 3.1 that

$$\partial_t^k u^{I,0},\ \partial_t^k v^{I,0} \in L^2(0,T;H^{3-2k}), \quad k = 0,1. \tag{3.49}$$

Lemma 3.2 gives for $k = 0,1,2$ that

$$\|\partial_t^k b_1\|_{L^2(0,T)}^2 \le \bar{u}^2 \int_0^\infty \langle z\rangle^{-2}\,dy \cdot \|\langle z\rangle\,\partial_t^k v^{B,0}\|_{L^2(0,T;L_z^2)}^2 \le C \tag{3.50}$$

and similarly Lemma 3.3 implies for $k = 0,1,2$ that

$$\|\partial_t^k b_2\|_{L^2(0,T)}^2 \le C. \tag{3.51}$$

Thus from (3.50), (3.51) and the definition of g, we have

$$\partial_t^k g \in L^2(0,T;H^{3-2k}), \quad k = 0,1. \tag{3.52}$$

To estimate f, we use (3.50)-(3.51), Lemma 3.1 and Proposition 3.3 and get for $k = 0,1$ that

$$\begin{aligned} &\|\partial_t^k (bv^{I,0})_x\|_{L^2(0,T;H^{2-2k})} \\ \le &\sum_{j=0}^k (\|\partial_t^j b_1\|_{L^2(0,T)} + \|\partial_t^j b_2\|_{L^2(0,T)})\|\partial_t^{k-j} v^{I,0}\|_{L^\infty(0,T;H^{3-2(k-j)})} \\ \le &C, \end{aligned}$$

which, in conjunction with the definition of f, (3.50) and (3.51) entails that

$$\partial_t^k f \in L^2(0,T;H^{2-2k}), \quad k = 0,1. \tag{3.53}$$

Noting that for (3.48), compatibility conditions up to order one are fulfilled under assumption (A), thus by (3.49), (3.52) and (3.53), we apply Proposition 3.1 with $m=2$ to (3.48) and get

$$\begin{aligned}
&\partial_t^k \tilde{u}^{I,1} \in L^2(0,T;H^{4-2k}), \quad k=0,1,2;\\
&\partial_t^k v^{I,1} \in L^2(0,T;H^{5-2k}), \quad k=1,2;\\
&v^{I,1} \in L^\infty(0,T;H^3).
\end{aligned} \tag{3.54}$$

The first estimate in (3.54) along with the definition of $\tilde{u}^{I,1}$, (3.50) and (3.51) gives rise to

$$\partial_t^k u^{I,1} \in L^2(0,T;H^{4-2k}), \quad k=0,1,2. \tag{3.55}$$

Thus the combination of (3.54) and (3.55) completes the proof. □

We next turn to the regularity of solutions to (3.10) and (3.11):

Lemma 3.5. *Let $(u^{I,1},v^{I,1})$ be the solution obtained in Lemma 3.4. Then there exists a unique solution $v^{B,1}$ to* (3.10) *on $[0,T]$ for any $0<T<\infty$, such that for any $l\in\mathbb{N}$,*

$$\langle z\rangle^l\, \partial_t^k v^{B,1} \in L^2(0,T;H_z^{4-2k}), \quad k=0,1,2.$$

Consequently, it follows from (3.11) *that*

$$\langle z\rangle^l\, \partial_t^k u^{B,2} \in L^2(0,T;H_z^{4-2k}), \quad k=0,1.$$

Proof. Let $\tilde{v}^{B,1} := v^{B,1} + \theta(z)v^{I,1}(0,t)$ with θ defined in (3.42). Then from (3.10), we deduce that $\tilde{v}^{B,1}$ satisfies

$$\begin{cases}
\tilde{v}_t^{B,1} = -\bar{u}\tilde{v}^{B,1} + \tilde{v}_{zz}^{B,1} + \rho,\\
\tilde{v}^{B,1}(z,0)=0,\\
\tilde{v}^{B,1}(0,t)=0,
\end{cases} \tag{3.56}$$

where $\rho := \bar{u}\theta v^{I,1}(0,t) + \theta v_t^{I,1}(0,t) - \theta_{zz}v^{I,1}(0,t) - 2(v^{I,0}(0,t)+v^{B,0})v_z^{B,0} + \int_z^\infty \Phi(s,t)\,ds$. We shall apply Proposition 3.2 with $m=2$ to (3.56) to prove this lemma by verifying that ρ satisfies the assumptions in Proposition 3.2. Let us start by dividing ρ into three parts:

$$\begin{aligned}
\rho =& (\bar{u}\theta v^{I,1}(0,t) + \theta v_t^{I,1}(0,t) - \theta_{zz}v^{I,1}(0,t))\\
&- 2(v^{I,0}(0,t)+v^{B,0})v_z^{B,0} + \int_z^\infty \Phi(s,t)\,ds\\
:=& I_1 + I_2 + I_3.
\end{aligned} \tag{3.57}$$

We next estimate I_1, I_2 and I_3. First it follows from (3.44) and Lemma 3.4 that

$$\|\partial_t^k v^{I,1}(0,t)\|_{L^2(0,T)} \le C_0 \|\partial_t^k v^{I,1}\|_{L^2(0,T;H^1)} \le C, \quad k=0,1,2,$$

which, along with the definition of θ in (3.42) implies that

$$\langle z\rangle^l\, \partial_t^k I_1 \in L^2(0,T;H_z^{2-2k}), \quad l\in\mathbb{N},\ k=0,1. \tag{3.58}$$

Then applying (3.44) to $\partial_t^j v^{I,0}$ and using Lemma 3.1, Lemma 3.2 and Remark 3.1, we have for $k=0,1$ and $l\in\mathbb{N}$ that

$$\begin{aligned}
&\|\langle z\rangle^l\, \partial_t^k I_2\|_{L^2(0,T;H_z^{2-2k})}\\
\le& C_0\sum_{j=0}^{k}(\|\partial_t^j v^{I,0}\|_{L^\infty(0,T;H^{3-2j})}+\|\partial_t^j v^{B,0}\|_{L^\infty(0,T;H_z^{3-2j})})\times\|\partial_t^{k-j}v^{B,0}\|_{L^2(0,T;H_z^{4-2(k-j)})}\\
\le& C.
\end{aligned} \tag{3.59}$$

For I_3, the estimate is a little more complicated, since it involves several terms. The Hölder inequality entails for $k=0,1$ and $l\in\mathbb{N}$ that

$$\begin{aligned}
&\|\langle z\rangle^l\, \partial_t^k I_3\|^2_{L^2(0,T;H_z^{2-2k})}\\
\le& C_0\Big(1+\int_0^\infty\int_z^\infty \langle s\rangle^{-4}\,dsdz\Big)\,\|\langle z\rangle^{l+2}\,\partial_t^k\Phi\|^2_{L^2(0,T;H_z^{2-2k})}.
\end{aligned} \tag{3.60}$$

Noting that the integration term in parentheses of the above inequality is finite, we only need to estimate the remaining term. By the definition of Φ below (3.11), one gets for $l\in\mathbb{N}$ that

$$\begin{aligned}
\langle z\rangle^l\,\partial_t^k\Phi =& \langle z\rangle^l\,\partial_t^k[u^{I,1}(0,t)v_z^{B,0}]+\langle z\rangle^l\,\partial_t^k[u^{B,1}v_z^{B,0}]+\langle z\rangle^l\,\partial_t^k[u_x^{I,0}(0,t)v^{B,0}]\\
&+\langle z\rangle^l\,\partial_t^k[v^{I,0}(0,t)u_z^{B,1}]+\langle z\rangle^l\,\partial_t^k[u_z^{B,1}v^{B,0}]+z\,\langle z\rangle^l\,\partial_t^k[u_x^{I,0}(0,t)v_z^{B,0}]\\
:=& M_1+M_2+M_3+M_4+M_5+M_6.
\end{aligned} \tag{3.61}$$

Applying (3.44) to $\partial_t^j u^{I,1}$, by Lemma 3.2, Lemma 3.4 and Remark 3.1, we obtain for $k=0,1$, that

$$\begin{aligned}
\|M_1\|_{L^2(0,T;H_z^{2-2k})}\le& \sum_{j=0}^{k}\|\partial_t^j u^{I,1}(0,t)\|_{L^\infty(0,T)}\|\langle z\rangle^l\,\partial_t^{k-j}v^{B,0}\|_{L^2(0,T;H_z^{3-2k})}\\
\le& C_0\sum_{j=0}^{k}\|\partial_t^j u^{I,1}\|_{L^\infty(0,T;H^{3-2j})}\|\langle z\rangle^l\,\partial_t^{k-j}v^{B,0}\|_{L^2(0,T;H_z^{3-2(k-j)})}\\
\le& C.
\end{aligned}$$

Similar arguments further give the estimate for $\{M_i\}_{2\le i\le 6}$:

$$\|M_i\|_{L^2(0,T;H_z^{2-2k})}\le C, \quad 2\le i\le 6,\ k=0,1.$$

Plugging the above estimates into (3.61), we conclude for any $l\in\mathbb{N}$ that

$$\langle z\rangle^l\,\partial_t^k\Phi\in L^2(0,T;H_z^{2-2k}), \quad k=0,1, \tag{3.62}$$

which, along with (3.60) gives rise to

$$\langle z\rangle^l \partial_t^k I_3 \in L^2(0,T;H_z^{2-2k}), \quad k=0,1. \tag{3.63}$$

Then it follows from (3.57), (3.58), (3.59) and (3.63) that

$$\langle z\rangle^l \partial_t^k \rho \in L^2(0,T;H_z^{2-2k}), \quad k=0,1,\ l\in\mathbb{N}. \tag{3.64}$$

Moreover for (3.56) it is easy to check that ρ fulfills the compatibility conditions up to order one under assumption (A). Thus by (3.64), we apply Proposition 3.2 with $m=2$ to (3.56) and have

$$\langle z\rangle^l \partial_t^k \tilde{v}^{B,1} \in L^2(0,T;H_z^{4-2k}), \qquad k=0,1,2,\ l\in\mathbb{N}. \tag{3.65}$$

To convert the result in (3.65) back to $v^{B,1}$, we note that

$$\langle z\rangle^l \partial_t^k v^{B,1} = \langle z\rangle^l \partial_t^k \tilde{v}^{B,1} - \langle z\rangle^l \theta(z)\partial_t^k v^{I,1}(0,t), \tag{3.66}$$

where the second term on the right-hand side is estimated by the definition of θ, (3.44) and Lemma 3.4 for $k=1,2$ and $l\in\mathbb{N}$ as:

$$\begin{aligned}\|\langle z\rangle^l \theta(z)\partial_t^k v^{I,1}(0,t)\|_{L^2(0,T;H_z^{4-2k})} \leq & C_0\|\partial_t^k v^{I,1}(0,t)\|_{L^2(0,T)} \\ \leq & C_0\|\partial_t^k v^{I,1}\|_{L^2(0,T;H^{5-2k})} \leq C\end{aligned}$$

and for $k=0$ and $l\in\mathbb{N}$ as:

$$\|\langle z\rangle^l \theta(z) v^{I,1}(0,t)\|_{L^2(0,T;H_z^4)} \leq C_0\|v^{I,1}(0,t)\|_{L^2(0,T)} \leq C\|v^{I,1}\|_{L^\infty(0,T;H^1)} \leq C.$$

Inserting the above two estimates with (3.65) into (3.66), one derives for $l\in\mathbb{N}$ that

$$\langle z\rangle^l \partial_t^k v^{B,1} \in L^2(0,T;H_z^{4-2k}), \quad k=0,1,2, \tag{3.67}$$

which gives the desired estimate for $v^{B,1}$. It remains to estimate $u^{B,2}$. (3.11) implies for $l\in\mathbb{N}$ that

$$\begin{aligned}\langle z\rangle^l \partial_t^k u^{B,2} =& \bar{u}\langle z\rangle^l \int_z^\infty \partial_t^k v^{B,1}(s,t)\,ds - \langle z\rangle^l \int_z^\infty\int_s^\infty \partial_t^k \Phi(\zeta,t)\,d\zeta ds \\ :=& I_4+I_5,\end{aligned} \tag{3.68}$$

where I_4 with $k=0,1$ is estimated by the Hölder inequality and (3.67) as follows:

$$\begin{aligned}&\|I_4\|^2_{L^2(0,T;H_z^{4-2k})} \\ \leq & C_0\Big(1+\int_0^\infty\int_z^\infty \langle s\rangle^{-4}\,dsdz\Big)\|\langle z\rangle^{l+2}\partial_t^k v^{B,1}\|^2_{L^2(0,T;H_z^{4-2k})} \\ \leq & C.\end{aligned}$$

Noting that I_5 is a double integral of $\partial_t^k\Phi$, we employ (3.62) and have for $k=0,1$ that

$$\begin{aligned}\|I_5\|^2_{L^2(0,T;H_z^{4-2k})} \leq & C_0\Big(1+\int_0^\infty\int_z^\infty \langle s\rangle^{-4}\,dsdz+\int_0^\infty\Big\{\int_z^\infty\Big[\int_s^\infty\langle\zeta\rangle^{-6}d\zeta\Big]^{\frac{1}{2}}ds\Big\}^2dz\Big)\\ & \times\|\langle z\rangle^{l+3}\partial_t^k\Phi\|^2_{L^2(0,T;H_z^{2-2k})}\\ \leq & C.\end{aligned}$$

Substituting the above estimates for I_4 and I_5 into (3.68) one gets for any $l\in\mathbb{N}$ that

$$\langle z\rangle^l\,\partial_t^k u^{B,2}\in L^2(0,T;H_z^{4-2k}),\quad k=0,1,$$

which, along with (3.67) completes the proof.

□

Noticing the similarity between (3.10) and (3.12), by analogous arguments as proving Lemma 3.5, one gets that

Lemma 3.6. *Let $(u^{I,1},v^{I,1})$ be the solution obtained in Lemma 3.4. Then there exists a unique solution $v^{b,1}$ to* (3.12) *on $[0,T]$ for any $0<T<\infty$, such that for any $l\in\mathbb{N}$,*

$$\langle\xi\rangle^l\,\partial_t^k v^{b,1}\in L^2(0,T;H_\xi^{4-2k}),\quad k=0,1,2,$$

and

$$\langle\xi\rangle^l\,\partial_t^k u^{b,2}\in L^2(0,T;H_\xi^{4-2k}),\quad k=0,1.$$

3.4 Proof of Theorem 3.1

To prove Theorem 3.1, if we decompose the solution $(u^\varepsilon,v^\varepsilon)$ as:

$$\begin{aligned}&u^\varepsilon(x,t)=u^{I,0}(x,t)+R_1^\varepsilon(x,t),\\&v^\varepsilon(x,t)=v^{I,0}(x,t)+v^{B,0}\Big(\frac{x}{\sqrt{\varepsilon}},t\Big)+v^{b,0}\Big(\frac{x-1}{\sqrt{\varepsilon}},t\Big)+R_2^\varepsilon(x,t),\end{aligned}\tag{3.69}$$

then it remains to derive the equations satisfied by $R_i^\varepsilon(x,t)\,(i=1,2)$, and to show

$$\|R_i^\varepsilon\|_{L^\infty([0,1]\times[0,T])}=O(\varepsilon^{1/2}).$$

But if we substitute (3.69) into equations (2.1), we shall find that the equations of R_i^ε have source terms containing a singular quantity of order $\varepsilon^{-1/2}$, which brings the difficulty to derive the uniform-in-ε boundedness of $\|R_i^\varepsilon\|_{L^\infty([0,1]\times[0,T])}\,(i=1,2)$. Therefore we invoke the higher order terms in the expansion of $(u^\varepsilon,v^\varepsilon)$ to overcome this difficulty motivated by a work [52]. To this end, we employ (3.2)-(3.3) to write $R_i^\varepsilon(x,t)(i=1,2)$ as:

$$\begin{aligned}
R_1^\varepsilon(x,t) =& \varepsilon^{1/2}[u^{I,1}(x,t)+u^{B,1}(z,t)+u^{b,1}(\xi,t)]+\varepsilon[u^{B,2}(z,t)+u^{b,2}(\xi,t)]\\
&+b_1^\varepsilon(x,t)+\varepsilon^{1/2}U^\varepsilon(x,t),\\
R_2^\varepsilon(x,t) =& \varepsilon^{1/2}[v^{I,1}(x,t)+v^{B,1}(z,t)+v^{b,1}(\xi,t)]\\
&+b_2^\varepsilon(x,t)+\varepsilon^{1/2}V^\varepsilon(x,t),
\end{aligned}$$

where the perturbation functions $(U^\varepsilon,V^\varepsilon)(x,t)$ are to be determined, and the auxiliary functions $b_i(x,t)\,(i=1,2)$ are constructed as follows to homogenize the boundary conditions of $(U^\varepsilon,V^\varepsilon)(x,t)$:

$$\begin{aligned}
b_1^\varepsilon(x,t) =& -(1-x)[\varepsilon^{1/2}u^{b,1}(-\varepsilon^{-1/2},t)+\varepsilon u^{B,2}(0,t)+\varepsilon u^{b,2}(-\varepsilon^{-1/2},t)]\\
&-x[\varepsilon^{1/2}u^{B,1}(\varepsilon^{-1/2},t)+\varepsilon u^{b,2}(0,t)+\varepsilon u^{B,2}(\varepsilon^{-1/2},t)],\\
b_2^\varepsilon(x,t) =& -(1-x)[v^{b,0}(-\varepsilon^{-1/2},t)+\varepsilon^{1/2}v^{b,1}(-\varepsilon^{-1/2},t)]\\
&-x[v^{B,0}(\varepsilon^{-1/2},t)+\varepsilon^{1/2}v^{B,1}(\varepsilon^{-1/2},t)].
\end{aligned}$$

We should remark that the term $u^{I,2}$ has been intentionally omitted in the expression of $R_1^\varepsilon(x,t)$ since we find it is unnecessary for our purpose. Indeed if we include the term $u^{I,2}$ in $R_1^\varepsilon(x,t)$, then a higher regularity $L^2(0,T;H^4)$ will be required on $u^{I,2}$ in the proof of Lemma 3.7 when estimating f^ε. This demands a higher regularity on initial data $(u_0,\,v_0)$ so that $(u_0,\,v_0)\in H^5\times H^5$. Therefore, to reduce the regularity of (u_0,v_0), we deliberately omit $u^{I,2}$ in $R_1^\varepsilon(x,t)$, which is a trick we employed.

For simplicity of presentation, we define new functions

$$\begin{aligned}
\tilde U^\varepsilon(x,t) :=& u^{I,0}(x,t)+\varepsilon^{1/2}[u^{I,1}(x,t)+u^{B,1}(z,t)+u^{b,1}(\xi,t)]\\
&+\varepsilon[u^{B,2}(z,t)+u^{b,2}(\xi,t)]+b_1^\varepsilon(x,t),\\
\tilde V^\varepsilon(x,t) :=& v^{I,0}(x,t)+v^{B,0}(z,t)+v^{b,0}(\xi,t)\\
&+\varepsilon^{1/2}[v^{I,1}(x,t)+v^{B,1}(z,t)+v^{b,1}(\xi,t)]+b_2^\varepsilon(x,t),
\end{aligned}$$

and then the perturbation functions $(U^\varepsilon,V^\varepsilon)(x,t)$ can be written as

$$U^\varepsilon=\varepsilon^{-1/2}(u^\varepsilon-\tilde U^\varepsilon),\qquad V^\varepsilon=\varepsilon^{-1/2}(v^\varepsilon-\tilde V^\varepsilon). \tag{3.70}$$

Substituting (3.70) into (2.1)-(2.2) and using the initial-boundary conditions in (3.4)-(3.12), one finds that $(U^\varepsilon,V^\varepsilon)$ satisfies

$$\begin{cases}
U_t^\varepsilon=\varepsilon^{1/2}(U^\varepsilon V^\varepsilon)_x+(U^\varepsilon\tilde V^\varepsilon)_x+(V^\varepsilon\tilde U^\varepsilon)_x+U_{xx}^\varepsilon+\varepsilon^{-1/2}f^\varepsilon,\\
V_t^\varepsilon=-2\varepsilon^{3/2}V^\varepsilon V_x^\varepsilon-2\varepsilon(V^\varepsilon\tilde V^\varepsilon)_x+U_x^\varepsilon+\varepsilon V_{xx}^\varepsilon+\varepsilon^{-1/2}g^\varepsilon,\\
(U^\varepsilon,V^\varepsilon)(x,0)=(0,0),\\
(U^\varepsilon,V^\varepsilon)(0,t)=(U^\varepsilon,V^\varepsilon)(1,t)=(0,0),
\end{cases} \tag{3.71}$$

with

$$f^\varepsilon = \tilde U^\varepsilon_{xx} + (\tilde U^\varepsilon \tilde V^\varepsilon)_x - \tilde U^\varepsilon_t, \quad g^\varepsilon = \varepsilon \tilde V^\varepsilon_{xx} + \tilde U^\varepsilon_x - \tilde V^\varepsilon_t - 2\varepsilon \tilde V^\varepsilon \tilde V^\varepsilon_x. \tag{3.72}$$

Now the key is to give the L^∞-estimates for the solution $(U^\varepsilon, V^\varepsilon)$ of (3.71)-(3.72), which will be gradually achieved in the sequel by the method of energy estimates.

We shall develop various delicate energy estimates in this subsection to attain the L^∞ estimates of $(U^\varepsilon, V^\varepsilon)$ to (3.71)-(3.72). Before proceeding, we introduce some basic facts for later use. First for any $G_1(z,t) \in H^m_z$ and $G_2(\xi,t) \in H^m_\xi$ with $m \in \mathbb{N}$, we have from the change of variables in (3.3) that

$$\left\| \partial_x^m G_1\Big(\frac{x}{\sqrt{\varepsilon}}, t\Big) \right\|_{L^2} = \varepsilon^{\frac14 - \frac m2} \|\partial_z^m G_1(z,t)\|_{L^2_z}, \tag{3.73}$$

and

$$\left\| \partial_x^m G_2\Big(\frac{x-1}{\sqrt{\varepsilon}}, t\Big) \right\|_{L^2} = \varepsilon^{\frac14 - \frac m2} \|\partial_\xi^m G_2(\xi,t)\|_{L^2_\xi}. \tag{3.74}$$

For $h(\cdot,t) \in H^1$ with $h|_{x=0,1} = 0$, we have $h^2(x,t) = 2\int_0^x h h_y \, dy \le 2\|h(\cdot,t)\|_{L^2} \|h_x(\cdot,t)\|_{L^2}$. Thus

$$\|h(\cdot,t)\|_{L^\infty} \le \sqrt{2} \|h(\cdot,t)\|^{1/2}_{L^2} \|h_x(\cdot,t)\|^{1/2}_{L^2}, \qquad \|h(\cdot,t)\|_{L^\infty} \le C_0 \|h_x(\cdot,t)\|_{L^2}, \tag{3.75}$$

thanks to the Poincaré inequality

$$\|h(\cdot,t)\|_{L^2} \le C_0 \|h_x(\cdot,t)\|_{L^2}.$$

We start with estimating f^ε and g^ε.

Lemma 3.7. *Let* $0 < T < \infty$, $0 < \varepsilon < 1$ *and* f^ε *be as defined in* (3.72). *Then there is a constant* C *independent of* ε, *such that*

$$\|f^\varepsilon\|_{L^2(0,T;L^2)} \le C\varepsilon^{3/4}.$$

Proof. First applying the definitions of $\tilde U^\varepsilon$ and $\tilde V^\varepsilon$ into the expression of f^ε in (3.72) and using the first equations in (3.4) and in (3.9), we end up with

$$\begin{aligned} f^\varepsilon =& \varepsilon^{1/2} u^{B,1}_{xx} + \varepsilon^{1/2} u^{b,1}_{xx} + \varepsilon u^{B,2}_{xx} + \varepsilon u^{b,2}_{xx} + \varepsilon (u^{I,1} v^{I,1})_x \\ &+ \Big[(u^{I,0} + \varepsilon^{1/2} u^{I,1})(v^{B,0} + v^{b,0} + \varepsilon^{1/2} v^{B,1} + \varepsilon^{1/2} v^{b,1}) \Big]_x \\ &+ \Big[(\varepsilon^{1/2} u^{B,1} + \varepsilon^{1/2} u^{b,1} + \varepsilon u^{B,2} + \varepsilon u^{b,2}) \\ &\times (v^{I,0} + v^{B,0} + v^{b,0} + \varepsilon^{1/2} v^{I,1} + \varepsilon^{1/2} v^{B,1} + \varepsilon^{1/2} v^{b,1}) \Big]_x \\ &- \varepsilon^{1/2} u^{B,1}_t - \varepsilon^{1/2} u^{b,1}_t - \varepsilon u^{B,2}_t - \varepsilon u^{b,2}_t + F^\varepsilon, \end{aligned} \tag{3.76}$$

where

$$
\begin{aligned}
F^{\varepsilon} := & \Big[b_1^{\varepsilon}(v^{I,0}+v^{B,0}+v^{b,0}+\varepsilon^{1/2}v^{I,1}+\varepsilon^{1/2}v^{B,1}+\varepsilon^{1/2}v^{b,1})\Big]_x \\
& +\Big[b_2^{\varepsilon}(u^{I,0}+\varepsilon^{1/2}u^{I,1}+\varepsilon^{1/2}u^{B,1}+\varepsilon^{1/2}u^{b,1}+\varepsilon u^{B,2}+\varepsilon u^{b,2})\Big]_x \\
& +(b_1^{\varepsilon}b_2^{\varepsilon})_x-b_{1t}^{\varepsilon}.
\end{aligned}
\tag{3.77}
$$

By the transformation (3.3), one gets from (3.114), (3.116), (3.120) and (3.121) (see Appendix) that

$$
\begin{aligned}
\varepsilon^{1/2}u_{xx}^{B,1} =& -u^{I,0}(0,t)v_x^{B,0}, \\
\varepsilon u_{xx}^{B,2} =& -xu_x^{I,0}(0,t)v_x^{B,0}-\varepsilon^{1/2}u^{I,1}(0,t)v_x^{B,0}-u_x^{I,0}(0,t)v^{B,0} \\
& -\varepsilon^{1/2}u^{I,0}(0,t)v_x^{B,1}-\varepsilon^{1/2}u_x^{B,1}v^{I,0}(0,t)-\varepsilon^{1/2}(u^{B,1}v^{B,0})_x,
\end{aligned}
$$

and

$$
\begin{aligned}
\varepsilon^{1/2}u_{xx}^{b,1} =& -u^{I,0}(1,t)v_x^{b,0}, \\
\varepsilon u_{xx}^{b,2} =& -(x-1)u_x^{I,0}(1,t)v_x^{b,0}-\varepsilon^{1/2}u^{I,1}(1,t)v_x^{b,0}-u_x^{I,0}(1,t)v^{b,0} \\
& -\varepsilon^{1/2}u^{I,0}(1,t)v_x^{b,1}-\varepsilon^{1/2}u_x^{b,1}v^{I,0}(1,t)-\varepsilon^{1/2}(u^{b,1}v^{b,0})_x.
\end{aligned}
$$

Then feeding (3.76) on the above four expressions and rearranging the results, we arrive at

$$
\begin{aligned}
f^{\varepsilon} =& \left[(u^{I,0}(x,t)-u^{I,0}(0,t)-xu_x^{I,0}(0,t))v_x^{B,0}\right] \\
& +\Big[(u^{I,0}(x,t)-u^{I,0}(1,t)-(x-1)u_x^{I,0}(1,t))v_x^{b,0}\Big] \\
& +\Big[(u_x^{I,0}(x,t)-u_x^{I,0}(0,t))v^{B,0}+(u_x^{I,0}(x,t)-u_x^{I,0}(1,t))v^{b,0}\Big] \\
& +\varepsilon^{1/2}\Big[(u^{I,0}(x,t)-u^{I,0}(0,t))v_x^{B,1}+(u^{I,1}(x,t)-u^{I,1}(0,t))v_x^{B,0} \\
& \qquad +u_x^{B,1}(v^{I,0}(x,t)-v^{I,0}(0,t))\Big] \\
& +\varepsilon^{1/2}\Big[(u^{I,0}(x,t)-u^{I,0}(1,t))v_x^{b,1}+(u^{I,1}(x,t)-u^{I,1}(1,t))v_x^{b,0} \\
& \qquad +u_x^{b,1}(v^{I,0}(x,t)-v^{I,0}(1,t))\Big] \\
& +\varepsilon^{1/2}\Big[u_x^{I,0}(v^{B,1}+v^{b,1})+u_x^{I,1}(v^{B,0}+v^{b,0})+(u^{B,1}+u^{b,1})v_x^{I,0}\Big] \\
& +\varepsilon^{1/2}\Big[u^{B,1}v^{b,0}+u^{b,1}v^{B,0}\Big]_x \\
& +\varepsilon\Big[(u^{I,1}u^{B,1}+u^{b,1})(v^{I,1}+v^{B,1}+v^{b,1})\Big]_x \\
& +\varepsilon\Big[(u^{B,2}+u^{b,2})(v^{I,0}+v^{B,0}+v^{b,0}+\varepsilon^{1/2}v^{I,1}+\varepsilon^{1/2}v^{B,1}+\varepsilon^{1/2}v^{b,1})\Big]_x \\
& -\Big[\varepsilon^{1/2}u_t^{B,1}+\varepsilon^{1/2}u_t^{b,1}+\varepsilon u_t^{B,2}+\varepsilon u_t^{b,2}\Big]+F^{\varepsilon} \\
:=& \sum_{i=1}^{10}K_i+F^{\varepsilon}.
\end{aligned}
\tag{3.78}
$$

We proceed to estimate K_i $(1 \le i \le 10)$. Recalling that $x = \varepsilon^{1/2} z$, then by Taylor's formula, (3.73) and Lemma 3.1-Lemma 3.2, we estimate K_1 as follows:

$$
\begin{aligned}
\|K_1\|_{L^2(0,T;L^2)} =&\varepsilon \left\| \frac{u^{I,0}(x,t) - u^{I,0}(0,t) - x u_x^{I,0}(0,t)}{x^2} \cdot z^2 v_x^{B,0} \right\|_{L^2(0,T;L^2)} \\
\le& \varepsilon \|u_{xx}^{I,0}\|_{L^2(0,T;L^\infty)} \|z^2 v_x^{B,0}\|_{L^\infty(0,T;L^2)} \\
\le& C_0 \varepsilon^{3/4} \|u^{I,0}\|_{L^2(0,T;H^3)} \|z^2 v_z^{B,0}\|_{L^\infty(0,T;L_z^2)} \\
\le& C\varepsilon^{3/4}.
\end{aligned}
$$

Similarly, by using (3.74) we have

$$
\begin{aligned}
\|K_2\|_{L^2(0,T;L^2)} \le& \varepsilon \left\| \frac{u^{I,0}(x,t) - u^{I,0}(1,t) - (x-1) u_x^{I,0}(1,t)}{(x-1)^2} \cdot \xi^2 v_x^{b,0} \right\|_{L^2(0,T;L^2)} \\
\le& C_0 \varepsilon^{3/4} \|u^{I,0}\|_{L^2(0,T;H^3)} \|\xi^2 v_\xi^{b,0}\|_{L^\infty(0,T;L_\xi^2)} \\
\le& C\varepsilon^{3/4}
\end{aligned}
$$

and

$$
\begin{aligned}
\|K_3\|_{L^2(0,T;L^2)} =& \varepsilon^{1/2} \left\| \frac{u_x^{I,0}(x,t) - u_x^{I,0}(0,t)}{x} \cdot z v^{B,0} \right\|_{L^2(0,T;L^2)} \\
&+ \varepsilon^{1/2} \left\| \frac{u_x^{I,0}(x,t) - u_x^{I,0}(1,t)}{x-1} \cdot \xi v^{b,0} \right\|_{L^2(0,T;L^2)} \le C\varepsilon^{3/4}.
\end{aligned}
$$

Similar arguments further give

$$
\|K_i\|_{L^2(0,T;L^2)} \le C\varepsilon^{3/4}, \quad i = 4, 5.
$$

By the Sobolev embedding inequality, (3.73)-(3.74) and Lemma 3.1-Lemma 3.6 we obtain

$$
\begin{aligned}
\|K_6\|_{L^2(0,T;L^2)} \le& \varepsilon^{1/2} \|u_x^{I,0}\|_{L^\infty(0,T;L^\infty)} \left(\|v^{B,1}\|_{L^2(0,T;L^2)} + \|v^{b,1}\|_{L^2(0,T;L^2)} \right) \\
&+ \varepsilon^{1/2} \|u_x^{I,1}\|_{L^\infty(0,T;L^\infty)} \left(\|v^{B,0}\|_{L^2(0,T;L^2)} + \|v^{b,0}\|_{L^2(0,T;L^2)} \right) \\
&+ \varepsilon^{1/2} \|v_x^{I,0}\|_{L^\infty(0,T;L^\infty)} \left(\|u^{B,1}\|_{L^2(0,T;L^2)} + \|u^{b,1}\|_{L^2(0,T;L^2)} \right) \\
\le& C_0 \varepsilon^{3/4} \|u^{I,0}\|_{L^\infty(0,T;H^2)} \left(\|v^{B,1}\|_{L^2(0,T;L_z^2)} + \|v^{b,1}\|_{L^2(0,T;L_\xi^2)} \right) \\
&+ C_0 \varepsilon^{3/4} \|u^{I,1}\|_{L^\infty(0,T;H^2)} \left(\|v^{B,0}\|_{L^2(0,T;L_z^2)} + \|v^{b,0}\|_{L^2(0,T;L_\xi^2)} \right) \\
&+ C_0 \varepsilon^{3/4} \|v^{I,0}\|_{L^\infty(0,T;H^2)} \left(\|u^{B,1}\|_{L^2(0,T;L_z^2)} + \|u^{b,1}\|_{L^2(0,T;L_\xi^2)} \right) \\
\le& C\varepsilon^{3/4}.
\end{aligned}
$$

Then using a similar argument as estimating K_6 and recalling $0 < \varepsilon < 1$, one infers that

$$
\|K_i\|_{L^2(0,T;L^2)} \le C\varepsilon^{3/4}, \quad 8 \le i \le 10.
$$

To bound K_7, we first rewrite it as:

$$K_7 = \varepsilon^{1/2}\left(u_x^{B,1}v^{b,0} + u^{B,1}v_x^{b,0} + u_x^{b,1}v^{B,0} + u^{b,1}v_x^{B,0}\right). \tag{3.79}$$

We next estimate each term on the right-hand side of (3.79). Indeed for $0<x<1/2$, it follows that $-\infty<\xi=\frac{x-1}{\sqrt{\varepsilon}}<-\frac{1}{2\sqrt{\varepsilon}}$. Thus, by transformation (3.3) and the Sobolev embedding inequality, one deduces for fixed $t\in[0,T]$ and $m\in\mathbb{N}_+$ that

$$\begin{aligned}
\int_0^{\frac12}(u_x^{B,1}v^{b,0})^2\,dx &= \int_0^{\frac12}\left[u_x^{B,1}\Big(\frac{x}{\sqrt{\varepsilon}},t\Big)v^{b,0}\Big(\frac{x-1}{\sqrt{\varepsilon}},t\Big)\right]^2dx\\
&\le \int_0^{\frac12}\left[u_x^{B,1}\Big(\frac{x}{\sqrt{\varepsilon}},t\Big)\right]^2dx\cdot(2\sqrt{\varepsilon})^{2m}\left\|\Big(-\frac{1}{2\sqrt{\varepsilon}}\Big)^m v^{b,0}(\xi,t)\right\|^2_{L^\infty_\xi(-\infty,-\frac{1}{2\sqrt{\varepsilon}})}\\
&\le \varepsilon^{-1/2}\int_0^{\frac{1}{2\sqrt{\varepsilon}}}[u_z^{B,1}(z,t)]^2\,dz\cdot(2\sqrt{\varepsilon})^{2m}\|\langle\xi\rangle^m v^{b,0}(\xi,t)\|^2_{L^\infty_\xi(-\infty,0)}\\
&\le C_0\varepsilon^{(2m-1)/2}\|u_z^{B,1}(z,t)\|^2_{L^2_z}\|\langle\xi\rangle^m v^{b,0}(\xi,t)\|^2_{H^1_\xi}\\
&\le C\varepsilon^{1/2},
\end{aligned}$$

where Lemma 3.2 and Lemma 3.3 have been used. Similarly, for $\frac12<x<1$ one has that $\frac{1}{2\sqrt{\varepsilon}}<z=\frac{x}{\sqrt{\varepsilon}}<\infty$ and for $m\in\mathbb{N}_+$ that

$$\begin{aligned}
&\int_{\frac12}^1(u_x^{B,1}v^{b,0})^2\,dx\\
&\le \int_{\frac12}^1\left[v^{b,0}\left(\frac{x-1}{\sqrt{\varepsilon}},t\right)\right]^2dx\cdot\left\|u_x^{B,1}\left(\frac{x}{\sqrt{\varepsilon}},t\right)\right\|^2_{L^\infty_x(\frac12,1)}\\
&\le \varepsilon^{1/2}\int_{-\infty}^0[v^{b,0}(\xi,t)]^2\,d\xi\cdot\varepsilon^{-1}\left\|u_z^{B,1}(z,t)\right\|^2_{L^\infty_z(\frac{1}{2\sqrt{\varepsilon}},\infty)}\\
&\le \varepsilon^{1/2}\|v^{b,0}(\xi,t)\|^2_{L^2_\xi}\cdot\varepsilon^{-1}(2\sqrt{\varepsilon})^{2m}\left\|z^m u_z^{B,1}(z,t)\right\|^2_{L^\infty_z(0,\infty)}\\
&\le C_0\varepsilon^{(2m-1)/2}\|v^{b,0}(\xi,t)\|^2_{L^2_\xi}\left(\|\langle z\rangle^m u^{B,1}(z,t)\|^2_{H^2_z}+\|\langle z\rangle^{m-1}u^{B,1}(z,t)\|^2_{H^2_z}\right)\\
&\le C\varepsilon^{1/2}.
\end{aligned}$$

Combining the above two estimates, we end up with

$$\|u_x^{B,1}v^{b,0}\|_{L^2(0,T;L^2)}\le C\varepsilon^{1/4}. \tag{3.80}$$

With similar arguments as above (3.80), one derives that

$$\|u^{B,1}v_x^{b,0}\|_{L^2(0,T;L^2)}+\|u_x^{b,1}v^{B,0}\|_{L^2(0,T;L^2)}+\|u^{b,1}v_x^{B,0}\|_{L^2(0,T;L^2)}\le C\varepsilon^{1/4}. \tag{3.81}$$

Substituting (3.81) and (3.80) into (3.79), we get the estimate for K_7:

$$\|K_7\|_{L^2(0,T;L^2)}\le C\varepsilon^{3/4}.$$

For the last term F^ε, we first note for any integer $m \geq 2$ that

$$\begin{aligned}
\|u^{b,1}(-\varepsilon^{-1/2},t)\|_{L^\infty(0,T)} =&\varepsilon^{m/2}\|(-\varepsilon^{-1/2})^m u^{b,1}(-\varepsilon^{-1/2},t)\|_{L^\infty(0,T)}\\
\leq&\varepsilon^{m/2}\|\langle\xi\rangle^m u^{b,1}(\xi,t)\|_{L^\infty(0,T;L^\infty_\xi)}\\
\leq&C_0\varepsilon^{m/2}\|\langle\xi\rangle^m u^{b,1}(\xi,t)\|_{L^\infty(0,T;H^1_\xi)}\\
\leq&C\varepsilon^{m/2},\\
\|u_t^{b,1}(-\varepsilon^{1/2},t)\|_{L^2(0,T)} \leq&C_0\varepsilon^{m/2}\|\langle\xi\rangle^m u_t^{b,1}(\xi,t)\|_{L^2(0,T;H^1_\xi)}\\
\leq&C\varepsilon^{m/2},
\end{aligned}$$

and

$$\begin{aligned}
\|u^{B,2}(0,t)\|_{L^\infty(0,T)} \leq&C_0\|u^{B,2}\|_{L^\infty(0,T;H^1_z)} \leq C,\\
\|u_t^{B,2}(0,t)\|_{L^2(0,T)} \leq&C_0\|u_t^{B,2}\|_{L^2(0,T;H^1_z)} \leq C.
\end{aligned}$$

By similar arguments, we can estimate other terms in b_1^ε, b_2^ε and conclude that

$$\|b_1^\varepsilon\|_{L^\infty(0,T;H^1)} + \|b_{1t}^\varepsilon\|_{L^2(0,T;H^1)} \leq C(\varepsilon^{(m+1)/2}+\varepsilon) \leq C\varepsilon \tag{3.82}$$

and

$$\|b_2^\varepsilon\|_{L^\infty(0,T;H^1)} + \|b_{2t}^\varepsilon\|_{L^2(0,T;H^1)} \leq C\varepsilon^{m/2}, \tag{3.83}$$

where $m \geq 2$. Then substituting (3.82)-(3.83) into the definition of F^ε in (3.77) and using $0<\varepsilon<1$ and (3.73)-(3.74), one has

$$\begin{aligned}
&\|F^\varepsilon\|_{L^2(0,T;L^2)}\\
\leq&C_0\|b_1^\varepsilon\|_{L^\infty(0,T;H^1)}\Big\{\|v^{I,0}\|_{L^2(0,T;H^1)} + \varepsilon^{-1/4}\|v^{B,0}\|_{L^2(0,T;H^1_z)} + \varepsilon^{-1/4}\|v^{b,0}\|_{L^2(0,T;H^1_\xi)}\\
&\qquad + \varepsilon^{1/2}\|v^{I,1}\|_{L^2(0,T;H^1)} + \varepsilon^{1/4}\|v^{B,1}\|_{L^2(0,T;H^1_z)} + \varepsilon^{1/4}\|v^{b,1}\|_{L^2(0,T;H^1_\xi)}\Big\}\\
&+C_0\|b_2^\varepsilon\|_{L^\infty(0,T;H^1)}\Big\{\|u^{I,0}\|_{L^2(0,T;H^1)} + \varepsilon^{1/2}\|u^{I,1}\|_{L^2(0,T;H^1)} + \varepsilon^{1/4}\|u^{B,1}\|_{L^2(0,T;H^1_z)}\\
&\qquad + \varepsilon^{1/4}\|u^{b,1}\|_{L^2(0,T;H^1_\xi)} + \varepsilon^{3/4}\|u^{B,2}\|_{L^2(0,T;H^1_z)} + \varepsilon^{3/4}\|u^{b,2}\|_{L^2(0,T;H^1_\xi)}\Big\}\\
&+T^{1/2}\|b_1^\varepsilon\|_{L^\infty(0,T;H^1)}\|b_2^\varepsilon\|_{L^\infty(0,T;H^1)} + \|b_{1t}^\varepsilon\|_{L^2(0,T;L^2)}\\
\leq&C\varepsilon^{3/4}.
\end{aligned}$$

Collecting the above estimates for K_i $(1 \leq i \leq 10)$ and F^ε, from (3.78) we conclude that

$$\|f^\varepsilon\|_{L^2(0,T;L^2)} \leq C\varepsilon^{3/4},$$

which finishes the proof. □

Lemma 3.8. *Let* $0<T<\infty$, $0<\varepsilon<1$ *and* g^ε *be as defined in* (3.72). *Then*

$$\|g^\varepsilon\|_{L^2(0,T;L^2)} \leq C\varepsilon^{3/4}.$$

Proof. Substituting the definition for $\tilde{U}^\varepsilon$ and $\tilde{V}^\varepsilon$ into g^ε in (3.72), and using the second equations of (3.4) and (3.9), (3.126) with $j=0$ and the first equation of (3.127) (see Appendix), we have

$$
\begin{aligned}
g^\varepsilon =& \left(\varepsilon v_{xx}^{I,0} + \varepsilon^{3/2} v_{xx}^{I,1}\right) + \varepsilon^{3/2}\left(v_{xx}^{B,1} + v_{xx}^{b,1}\right) \\
&+ \varepsilon\left(u_x^{B,2} + u_x^{b,2}\right) - \varepsilon^{1/2}\left(v_t^{B,1} + v_t^{b,1}\right) - 2\varepsilon \tilde{V}^\varepsilon \tilde{V}_x^\varepsilon + (b_{1x}^\varepsilon - b_{2t}^\varepsilon) \\
:=& \sum_{i=13}^{18} K_i.
\end{aligned}
\tag{3.84}
$$

We proceed to estimate K_i $(13 \le i \le 18)$. First it follows from Lemma 3.1 and Lemma 3.4 that

$$
\begin{aligned}
\|K_{13}\|_{L^2(0,T;L^2)} \le& \varepsilon \|v^{I,0}\|_{L^2(0,T;H^2)} + \varepsilon^{3/2}\|v^{I,1}\|_{L^2(0,T;H^2)} \\
\le& C(\varepsilon + \varepsilon^{3/2}).
\end{aligned}
$$

Using (3.3), (3.73)-(3.74) and Lemma 3.5-Lemma 3.6, we estimate K_{14} as follows:

$$
\begin{aligned}
\|K_{14}\|_{L^2(0,T;L^2)} \le& \varepsilon^{3/2}\left(\|v_{xx}^{B,1}\|_{L^2(0,T;L^2)} + \|v_{xx}^{b,1}\|_{L^2(0,T;L^2)}\right) \\
\le& \varepsilon^{3/4}\left(\|v_{zz}^{B,1}\|_{L^2(0,T;L_z^2)} + \|v_{\xi\xi}^{b,1}\|_{L^2(0,T;L_\xi^2)}\right) \\
\le& C\varepsilon^{3/4}.
\end{aligned}
$$

Similarly, it follows from Lemma 3.5 and Lemma 3.6 that

$$
\|K_i\|_{L^2(0,T;L^2)} \le C\varepsilon^{3/4}, \quad i = 15, 16.
$$

To bound K_{17}, we first estimate $\|\tilde{V}^\varepsilon\|_{L^\infty([0,1]\times[0,T])}$ and $\|\tilde{V}_x^\varepsilon\|_{L^\infty(0,T;L^2)}$. For any $G_1(z,t) \in L^p(0,T;H_z^1)$, $G_2(\xi,t) \in L^p(0,T;H_\xi^1)$ with $1 \le p \le \infty$, it follows from the Sobolev embedding inequality that

$$
\begin{aligned}
&\left\|G_1\left(\frac{x}{\sqrt{\varepsilon}},t\right)\right\|_{L^p(0,T;L^\infty)} \le \|G_1(z,t)\|_{L^p(0,T;L_z^\infty)} \le C_0\|G_1\|_{L^p(0,T;H_z^1)} \le C, \\
&\left\|G_2\left(\frac{x-1}{\sqrt{\varepsilon}},t\right)\right\|_{L^p(0,T;L^\infty)} \le \|G_2(\xi,t)\|_{L^p(0,T;L_\xi^\infty)} \le C_0\|G_2\|_{L^p(0,T;H_\xi^1)} \le C.
\end{aligned}
\tag{3.85}
$$

Then by the definition of $\tilde{V}^\varepsilon$, (3.85), Lemma 3.1-Lemma 3.6 and (3.83), we deduce that

$$
\begin{aligned}
&\|\tilde{V}^\varepsilon\|_{L^\infty([0,1]\times[0,T])} \\
\le& \|v^{I,0}\|_{L^\infty([0,1]\times[0,T])} + \|v^{B,0}\|_{L^\infty(0,T;L_z^\infty)} \\
&+ \|v^{b,0}\|_{L^\infty(0,T;L_\xi^\infty)} + C_0\|b_2^\varepsilon\|_{L^\infty(0,T;H^1)} \\
&+ \varepsilon^{1/2}\left(\|v^{I,1}\|_{L^\infty([0,1]\times[0,T])} + \|v^{B,1}\|_{L^\infty(0,T;L_z^\infty)} + \|v^{b,1}\|_{L^\infty(0,T;L_\xi^\infty)}\right) \\
\le& C\left(1 + \varepsilon^{1/2} + \varepsilon^{m/2}\right) \le C,
\end{aligned}
\tag{3.86}
$$

where the assumption $0<\varepsilon<1$ has been used. Moreover (3.73), (3.74) and (3.83) lead to

$$\begin{aligned}
&\|\tilde{V}_x^\varepsilon\|_{L^\infty(0,T;L^2)}\\
\le&\|v_x^{I,0}\|_{L^\infty(0,T;L^2)}+\varepsilon^{-1/4}\left(\|v_z^{B,0}\|_{L^\infty(0,T;L_z^2)}+\|v_\xi^{b,0}\|_{L^\infty(0,T;L_\xi^2)}\right)+C\varepsilon^{m/2}\\
&+\varepsilon^{1/2}\|v_x^{I,1}\|_{L^\infty(0,T;L^2)}+\varepsilon^{1/4}\left(\|v_z^{B,1}\|_{L^\infty(0,T;L_z^2)}+\|v_\xi^{b,1}\|_{L^\infty(0,T;L_\xi^2)}\right)\\
\le&C\varepsilon^{-1/4}.
\end{aligned} \tag{3.87}$$

Thus the above two estimates indicate that

$$\|K_{17}\|_{L^2(0,T;L^2)}\le C\varepsilon\|\tilde{V}^\varepsilon\|_{L^\infty([0,1]\times[0,T])}\|\tilde{V}_x^\varepsilon\|_{L^\infty(0,T;L^2)}\le C\varepsilon^{3/4}.$$

Finally, the estimate for K_{18} follows from (3.82), (3.83) and the assumption $0<\varepsilon<1$ that

$$\|K_{18}\|_{L^2(0,T;L^2)}\le\|b_1^\varepsilon\|_{L^2(0,T;H^1)}+\|b_{2t}^\varepsilon\|_{L^2(0,T;L^2)}\le C\varepsilon\le C\varepsilon^{3/4}.$$

Then inserting the above estimates for K_i ($13\le i\le 18$) into (3.84) yields the desired estimate for g^ε.

□

Next lemma gives the estimate for U^ε, V^ε in $L^\infty(0,T;L^2)$.

Lemma 3.9. *Let $0<T<\infty$ and $0<\varepsilon<1$. Then there exists a constant C, independent of ε, such that*

$$\|U^\varepsilon\|^2_{L^\infty(0,T;L^2)}+\|V^\varepsilon\|^2_{L^\infty(0,T;L^2)}+\|U_x^\varepsilon\|^2_{L^2(0,T;L^2)}+\varepsilon\|V_x^\varepsilon\|^2_{L^2(0,T;L^2)}\le C\varepsilon^{1/2}.$$

Proof. Taking the L^2 inner product of the first equation of (3.71) with $2U^\varepsilon$, then using integration by parts to have

$$\begin{aligned}
&\frac{d}{dt}\|U^\varepsilon(t)\|^2_{L^2}+2\|U_x^\varepsilon(t)\|^2_{L^2}\\
=&-2\varepsilon^{1/2}\int_0^1 U^\varepsilon V^\varepsilon U_x^\varepsilon\,dx-2\int_0^1(U^\varepsilon\tilde{V}^\varepsilon+V^\varepsilon\tilde{U}^\varepsilon)U_x^\varepsilon\,dx\\
&+2\varepsilon^{-1/2}\int_0^1 f^\varepsilon U^\varepsilon\,dx\\
:=&M_1+M_2+M_3.
\end{aligned} \tag{3.88}$$

We next estimate M_i ($i=1,2,3$). First, (3.75) gives

$$\begin{aligned}
M_1\le&2\varepsilon^{1/2}\|U^\varepsilon(t)\|_{L^\infty}\|V^\varepsilon(t)\|_{L^2}\|U_x^\varepsilon(t)\|_{L^2}\\
\le&C_0\varepsilon^{1/2}\|U^\varepsilon(t)\|^{1/2}_{L^2}\|U_x^\varepsilon(t)\|^{3/2}_{L^2}\|V^\varepsilon(t)\|_{L^2}\\
\le&\frac{1}{4}\|U_x^\varepsilon(t)\|^2_{L^2}+C_0\varepsilon^2\|V^\varepsilon(t)\|^4_{L^2}\|U^\varepsilon(t)\|^2_{L^2}.
\end{aligned}$$

For the term $\|V^\varepsilon(t)\|_{L^2}^4$, we use the definition of V^ε, Lemma 2.1 and (3.86) to get

$$\begin{aligned}\|V^\varepsilon\|_{L^\infty(0,T;L^2)} \le &\varepsilon^{-1/2}\left(\|v^\varepsilon\|_{L^\infty(0,T;L^2)} + \|\tilde{V}^\varepsilon\|_{L^\infty(0,T;L^2)}\right)\\ \le &C\varepsilon^{-1/2},\end{aligned} \tag{3.89}$$

which, substituted into the above estimate for M_1 gives rise to

$$M_1 \le \frac{1}{4}\|U_x^\varepsilon(t)\|_{L^2}^2 + C\|U^\varepsilon(t)\|_{L^2}^2.$$

By a similar argument as deriving (3.86), one infers that

$$\|\tilde{U}^\varepsilon\|_{L^\infty([0,1]\times[0,T])} \le C, \tag{3.90}$$

which along with (3.86) leads to

$$\begin{aligned}M_2 &\le \frac{1}{4}\|U_x^\varepsilon(t)\|_{L^2}^2 + 8\|U^\varepsilon(t)\|_{L^2}^2\|\tilde{V}^\varepsilon(t)\|_{L^\infty}^2 + 8\|V^\varepsilon(t)\|_{L^2}^2\|\tilde{U}^\varepsilon(t)\|_{L^\infty}^2\\ &\le \frac{1}{4}\|U_x^\varepsilon(t)\|_{L^2}^2 + C\left(\|U^\varepsilon(t)\|_{L^2}^2 + \|V^\varepsilon(t)\|_{L^2}^2\right).\end{aligned}$$

For the last term M_3, we have by the Cauchy-Schwarz inequality that

$$M_3 \le \|U^\varepsilon(t)\|_{L^2}^2 + \varepsilon^{-1}\|f^\varepsilon(t)\|_{L^2}^2.$$

Substituting the above estimates of M_i $(1 \le i \le 3)$ into (3.88), we arrive at

$$\frac{d}{dt}\|U^\varepsilon(t)\|_{L^2}^2 + \frac{3}{2}\|U_x^\varepsilon(t)\|_{L^2}^2 \le C\left(\|U^\varepsilon(t)\|_{L^2}^2 + \|V^\varepsilon(t)\|_{L^2}^2\right) + \varepsilon^{-1}\|f^\varepsilon(t)\|_{L^2}^2. \tag{3.91}$$

We turn to estimate V^ε. Multiplying the second equation of (3.71) by $2V^\varepsilon$ in L^2 and using the integration by parts to derive

$$\begin{aligned}&\frac{d}{dt}\|V^\varepsilon(t)\|_{L^2}^2 + 2\varepsilon\|V_x^\varepsilon(t)\|_{L^2}^2\\ &= -4\varepsilon^{3/2}\int_0^1 V^\varepsilon V^\varepsilon V_x^\varepsilon\,dx + 4\varepsilon\int_0^1 V^\varepsilon\tilde{V}^\varepsilon V_x^\varepsilon\,dx\\ &\quad + 2\int_0^1 U_x^\varepsilon V^\varepsilon\,dx + 2\varepsilon^{-1/2}\int_0^1 g^\varepsilon V^\varepsilon\,dx\\ &:= M_4 + M_5 + M_6 + M_7.\end{aligned} \tag{3.92}$$

We proceed to bound M_i $(4 \le i \le 7)$. Applying (3.75) to V^ε with (3.89) leads to

$$\begin{aligned}M_4 \le &C_0\varepsilon^{3/2}\|V_x^\varepsilon(t)\|_{L^2}^{3/2}\|V^\varepsilon(t)\|_{L^2}^{3/2}\\ \le &\frac{1}{4}\varepsilon\|V_x^\varepsilon(t)\|_{L^2}^2 + C_0\varepsilon^3\|V^\varepsilon(t)\|_{L^2}^6\\ \le &\frac{1}{4}\varepsilon\|V_x^\varepsilon(t)\|_{L^2}^2 + C\varepsilon\|V^\varepsilon(t)\|_{L^2}^2.\end{aligned}$$

We employ the Cauchy-Schwarz inequality and (3.86) to deduce that

$$M_5 \leq \frac{1}{4}\varepsilon\|V_x^\varepsilon(t)\|_{L^2}^2 + 16\varepsilon\|\tilde{V}^\varepsilon(t)\|_{L^\infty}^2\|V^\varepsilon(t)\|_{L^2}^2 \leq \frac{1}{4}\varepsilon\|V_x^\varepsilon(t)\|_{L^2}^2 + C\varepsilon\|V^\varepsilon(t)\|_{L^2}^2.$$

Finally, the estimates for M_6 and M_7 follow from the Cauchy-Schwarz inequality that

$$M_6 \leq \frac{1}{4}\|U_x^\varepsilon(t)\|_{L^2}^2 + 4\|V^\varepsilon(t)\|_{L^2}^2, \quad M_7(t) \leq \|V^\varepsilon(t)\|_{L^2}^2 + \varepsilon^{-1}\|g^\varepsilon(t)\|_{L^2}^2.$$

Plugging the above estimates for M_i $(4 \leq i \leq 7)$ into (3.92) and using $0 < \varepsilon < 1$ give

$$\frac{d}{dt}\|V^\varepsilon(t)\|_{L^2}^2 + \varepsilon\|V_x^\varepsilon(t)\|_{L^2}^2 \leq \frac{1}{4}\|U_x^\varepsilon(t)\|_{L^2}^2 + C\|V^\varepsilon(t)\|_{L^2}^2 + \varepsilon^{-1}\|g^\varepsilon(t)\|_{L^2}^2,$$

which added to (3.91) yields

$$\begin{aligned}&\frac{d}{dt}(\|U^\varepsilon(t)\|_{L^2}^2 + \|V^\varepsilon(t)\|_{L^2}^2) + \|U_x^\varepsilon(t)\|_{L^2}^2 + \varepsilon\|V_x^\varepsilon(t)\|_{L^2}^2 \\ \leq& C(\|U^\varepsilon(t)\|_{L^2}^2 + \|V^\varepsilon(t)\|_{L^2}^2) + \varepsilon^{-1}\|f^\varepsilon(t)\|_{L^2}^2 + \varepsilon^{-1}\|g^\varepsilon(t)\|_{L^2}^2.\end{aligned}$$

Applying Gronwall's inequality to above inequality along with Lemma 3.7 and Lemma 3.8, one gets the desired estimates. The proof is completed. □

Lemma 3.10. *Let $0 < T < \infty$ and $0 < \varepsilon < 1$. Then there is a constant C, independent of ε, such that*

$$\|U_x^\varepsilon\|_{L^\infty(0,T;L^2)}^2 + \|V_x^\varepsilon\|_{L^\infty(0,T;L^2)}^2 + \|U_{xx}^\varepsilon\|_{L^2(0,T;L^2)}^2 + \varepsilon\|V_{xx}^\varepsilon\|_{L^2(0,T;L^2)}^2 \leq C\varepsilon^{-1/2}.$$

Proof. Taking the L^2 inner product of the second equation of (3.71) with $-2\varepsilon V_{xx}^\varepsilon$, and using integration by parts, we obtain

$$\begin{aligned}\frac{d}{dt}(\varepsilon\|V_x^\varepsilon(t)\|_{L^2}^2) + 2\varepsilon^2\|V_{xx}^\varepsilon(t)\|_{L^2}^2 =& 4\varepsilon^{5/2}\int_0^1 V^\varepsilon V_x^\varepsilon V_{xx}^\varepsilon\,dx + 4\varepsilon^2\int_0^1 (V^\varepsilon\tilde{V}^\varepsilon)_x V_{xx}^\varepsilon\,dx \\ &- 2\varepsilon\int_0^1 U_x^\varepsilon V_{xx}^\varepsilon\,dx - 2\varepsilon^{1/2}\int_0^1 g^\varepsilon V_{xx}^\varepsilon\,dx \\ :=& R_1 + R_2 + R_3 + R_4.\end{aligned} \tag{3.93}$$

We proceed to estimate R_i for $1 \leq i \leq 4$. By (3.75) we deduce that

$$R_1 \leq C_0\varepsilon^{5/2}\|V_x^\varepsilon(t)\|_{L^2}^2\|V_{xx}^\varepsilon(t)\|_{L^2} \leq \frac{1}{4}\varepsilon^2\|V_{xx}^\varepsilon(t)\|_{L^2}^2 + C_0\varepsilon^3\|V_x^\varepsilon(t)\|_{L^2}^4.$$

Similarly, it follows from (3.75), (3.86) and (3.87) that

$$\begin{aligned}R_2 \leq& C_0\varepsilon^2\|V_x^\varepsilon(t)\|_{L^2}\|\tilde{V}^\varepsilon(t)\|_{H^1}\|V_{xx}^\varepsilon(t)\|_{L^2} \\ \leq& \frac{1}{4}\varepsilon^2\|V_{xx}^\varepsilon(t)\|_{L^2}^2 + C_0\varepsilon^2\left(\|\tilde{V}^\varepsilon(t)\|_{L^2}^2 + \|\tilde{V}_x^\varepsilon(t)\|_{L^2}^2\right)\|V_x^\varepsilon(t)\|_{L^2}^2 \\ \leq& \frac{1}{4}\varepsilon^2\|V_{xx}^\varepsilon(t)\|_{L^2}^2 + C\left(\varepsilon^2 + \varepsilon^{3/2}\right)\|V_x^\varepsilon(t)\|_{L^2}^2.\end{aligned}$$

For R_3 and R_4, we employ the Cauchy-Schwarz inequality to have

$$R_3 \le \frac{1}{4}\varepsilon^2\|V_{xx}^\varepsilon(t)\|_{L^2}^2 + 4\|U_x^\varepsilon(t)\|_{L^2}^2 \quad \text{and } R_4 \le \frac{1}{4}\varepsilon^2\|V_{xx}^\varepsilon(t)\|_{L^2}^2 + 4\varepsilon^{-1}\|g^\varepsilon(t)\|_{L^2}^2.$$

Collecting the above estimates of R_i ($1 \le i \le 4$) and using (3.93), we end up with

$$\begin{aligned}&\frac{d}{dt}(\varepsilon\|V_x^\varepsilon(t)\|_{L^2}^2) + \varepsilon^2\|V_{xx}^\varepsilon(t)\|_{L^2}^2\\ \le& C(\varepsilon^2\|V_x^\varepsilon(t)\|_{L^2}^2 + \varepsilon + \varepsilon^{1/2})(\varepsilon\|V_x^\varepsilon(t)\|_{L^2}^2) + 4(\|U_x^\varepsilon(t)\|_{L^2}^2 + \varepsilon^{-1}\|g^\varepsilon(t)\|_{L^2}^2),\end{aligned}$$

which, along with Gronwall's inequality, Lemma 3.8-Lemma 3.9 and $0 < \varepsilon < 1$ yields

$$\varepsilon\|V_x^\varepsilon\|_{L^\infty(0,T;L^2)}^2 + \varepsilon^2\|V_{xx}^\varepsilon\|_{L^2(0,T;L^2)}^2 \le C\varepsilon^{1/2}. \tag{3.94}$$

We turn to estimate U_x^ε. Taking the L^2 inner product of the first equation of (3.71) against $-2U_{xx}^\varepsilon$ and using integration by parts to get

$$\begin{aligned}&\frac{d}{dt}\|U_x^\varepsilon(t)\|_{L^2}^2 + 2\|U_{xx}^\varepsilon(t)\|_{L^2}^2\\ =&-2\varepsilon^{1/2}\int_0^1 (U^\varepsilon V^\varepsilon)_x U_{xx}^\varepsilon\,dx - 2\int_0^1 (U^\varepsilon \tilde{V}^\varepsilon)_x U_{xx}^\varepsilon\,dx\\ &-2\int_0^1 (V^\varepsilon \tilde{U}^\varepsilon)_x U_{xx}^\varepsilon\,dx - 2\varepsilon^{-1/2}\int_0^1 f^\varepsilon U_{xx}^\varepsilon\,dx\\ :=& R_5 + R_6 + R_7 + R_8.\end{aligned} \tag{3.95}$$

By (3.75) and (3.94), we estimate R_5 as

$$\begin{aligned}R_5 \le& \frac{1}{4}\|U_{xx}^\varepsilon(t)\|_{L^2}^2 + C_0\varepsilon\|V_x^\varepsilon(t)\|_{L^2}^2\|U_x^\varepsilon(t)\|_{L^2}^2\\ \le& \frac{1}{4}\|U_{xx}^\varepsilon(t)\|_{L^2}^2 + C\varepsilon^{1/2}\|U_x^\varepsilon(t)\|_{L^2}^2.\end{aligned}$$

Similarly, we estimate $R_6(t)$ from (3.75), (3.86) and (3.87) as

$$\begin{aligned}R_6 \le& \frac{1}{4}\|U_{xx}^\varepsilon(t)\|_{L^2}^2 + C_0(\|\tilde{V}^\varepsilon(t)\|_{L^2}^2 + \|\tilde{V}_x^\varepsilon(t)\|_{L^2}^2)\|U_x^\varepsilon(t)\|_{L^2}^2\\ \le& \frac{1}{4}\|U_{xx}^\varepsilon(t)\|_{L^2}^2 + C(1+\varepsilon^{-1/2})\|U_x^\varepsilon(t)\|_{L^2}^2.\end{aligned}$$

To bound R_7, we use the definition of $\tilde{U}^\varepsilon$ and a similar argument as deriving (3.87) to get

$$\|\tilde{U}_x^\varepsilon\|_{L^\infty(0,T;L^2)} \le C\left(1 + \varepsilon^{1/2} + \varepsilon^{1/4} + \varepsilon^{3/4} + \varepsilon\right) \le C,$$

where $0 < \varepsilon < 1$ has been used. The above estimate in conjunction with (3.90) and (3.94) gives

$$R_7 \le \frac{1}{4}\|U_{xx}^\varepsilon(t)\|_{L^2}^2 + C_0\left(\|\tilde{U}^\varepsilon(t)\|_{L^2}^2 + \|\tilde{U}_x^\varepsilon(t)\|_{L^2}^2\right)\|V_x^\varepsilon(t)\|_{L^2}^2 \le \frac{1}{4}\|U_{xx}^\varepsilon(t)\|_{L^2}^2 + C\varepsilon^{-1/2}.$$

Lastly, the Cauchy-Schwarz inequality yields the estimate for R_8 as

$$R_8 \leq \frac{1}{4}\|U^\varepsilon_{xx}(t)\|^2_{L^2} + 4\varepsilon^{-1}\|f^\varepsilon(t)\|^2_{L^2}.$$

Feeding (3.95) on the above estimates of R_i ($5 \leq i \leq 8$) leads to

$$\frac{d}{dt}\|U^\varepsilon_x(t)\|^2_{L^2} + \|U^\varepsilon_{xx}(t)\|^2_{L^2} \leq C\left(\varepsilon^{-1/2} + 1 + \varepsilon^{1/2}\right)\|U^\varepsilon_x(t)\|^2_{L^2} + C\varepsilon^{-1/2} + 4\varepsilon^{-1}\|f^\varepsilon(t)\|^2_{L^2},$$

which, upon integration over $(0,t)$ with $t \leq T$ gives rise to

$$\|U^\varepsilon_x\|^2_{L^\infty(0,T;L^2)} + \|U^\varepsilon_{xx}\|^2_{L^2(0,T;L^2)} \leq C\varepsilon^{-1/2},$$

where Lemma 3.7, Lemma 3.9 and $0 < \varepsilon < 1$ have been used. The above estimate along with (3.94) completes the proof.

□

Proof of Theorem 3.1. To prove Theorem 3.1, it suffices to estimate $\|R^\varepsilon_1\|_{L^\infty([0,1]\times[0,T])}$ and $\|R^\varepsilon_2\|_{L^\infty([0,1]\times[0,T])}$. For this, we first estimate U^ε and V^ε in $L^\infty([0,1]\times[0,T])$ by (3.75), Lemma 3.9-Lemma 3.10 and get

$$\begin{aligned} \|U^\varepsilon\|_{L^\infty([0,1]\times[0,T])} &\leq C_0\|U^\varepsilon\|^{1/2}_{L^\infty(0,T;L^2)}\|U^\varepsilon_x\|^{1/2}_{L^\infty(0,T;L^2)} \leq C, \\ \|V^\varepsilon\|_{L^\infty([0,1]\times[0,T])} &\leq C_0\|V^\varepsilon\|^{1/2}_{L^\infty(0,T;L^2)}\|V^\varepsilon_x\|^{1/2}_{L^\infty(0,T;L^2)} \leq C. \end{aligned} \tag{3.96}$$

Then the estimate for R^ε_1 follows from (3.85), Lemma 3.2-Lemma 3.6, (3.82) and (3.96) that

$$\begin{aligned} &\|R^\varepsilon_1\|_{L^\infty([0,1]\times[0,T])} \\ \leq &C_0\varepsilon^{1/2}\left(\|u^{I,1}\|_{L^\infty(0,T;H^1)} + \|u^{B,1}\|_{L^\infty(0,T;H^1_z)} + \|u^{b,1}\|_{L^\infty(0,T;H^1_\xi)}\right) \\ &+ C_0\varepsilon\left(\|u^{B,2}\|_{L^\infty(0,T;H^1_z)} + \|u^{b,2}\|_{L^\infty(0,T;H^1_\xi)}\right) \\ &+ C_0\|b^\varepsilon_1\|_{L^\infty(0,T;H^1)} + \varepsilon^{1/2}\|U^\varepsilon\|_{L^\infty([0,1]\times[0,T])} \\ \leq &C\varepsilon^{1/2}, \end{aligned}$$

where $0 < \varepsilon < 1$ has been used. Similarly, by (3.85), Lemma 3.4-Lemma 3.6, (3.83), (3.96) and $0 < \varepsilon < 1$, we have

$$\begin{aligned} \|R^\varepsilon_2\|_{L^\infty([0,1]\times[0,T])} \leq &C_0\varepsilon^{1/2}\left(\|v^{I,1}\|_{L^\infty(0,T;H^1)} + \|v^{B,1}\|_{L^\infty(0,T;H^1_z)} + \|v^{b,1}\|_{L^\infty(0,T;H^1_\xi)}\right) \\ &+ C_0\|b^\varepsilon_2\|_{L^\infty(0,T;H^1)} + \varepsilon^{1/2}\|V^\varepsilon\|_{L^\infty([0,1]\times[0,T])} \\ \leq &C\varepsilon^{1/2}. \end{aligned}$$

The above two estimates along with (3.69) imply (3.14). Moreover, the explicit formulas for $v^{B,0}$ and $v^{b,0}$ follow from Lemma 3.2 and Lemma 3.3. The proof is completed. □

3.5 Proof of Theorem 3.2

We are now in a position to prove Theorem 3.2 by converting the result of Theorem 3.1 to the pre-transformed model (3.15).

Proof of Theorem 3.2. Let $(u^\varepsilon, c^\varepsilon)$ and (u^0, c^0) be solutions of (3.15) with $\varepsilon > 0$ and $\varepsilon = 0$, respectively. The convergence rate in (3.16) between u^ε and u^0 is a direct consequence of Theorem 3.1. We are left to prove the convergence for c^ε in (3.16) and for c_x^ε in (3.17). Indeed from the second equation of (3.15) one deduces that

$$\begin{cases} (\ln c^\varepsilon)_t = \varepsilon(v^\varepsilon)^2 - \varepsilon v_x^\varepsilon - u^\varepsilon, \\ (\ln c^0)_t = -u^0, \end{cases}$$

where $v^\varepsilon = -(\ln c^\varepsilon)_x$. We consider the difference of the two equations:

$$(\ln c^\varepsilon - \ln c^0)_t = \varepsilon(v^\varepsilon)^2 - \varepsilon v_x^\varepsilon - (u^\varepsilon - u^0),$$

which, upon integration with respect to t, gives rise to

$$\frac{c^\varepsilon(x,t)}{c^0(x,t)} = \frac{c^\varepsilon(x,0)}{c^0(x,0)} \exp\left\{ \int_0^t [-(u^\varepsilon - u^0) + \varepsilon(v^\varepsilon)^2 - \varepsilon v_x^\varepsilon]\, d\tau \right\}.$$

It follows from the initial condition $c^\varepsilon(x,0) = c^0(x,0) = c_0(x)$ that

$$\begin{aligned} &|c^\varepsilon(x,t) - c^0(x,t)| \\ =&|c^0(x,t)| \cdot \left| \exp\left\{ \int_0^t [-(u^\varepsilon - u^0) + \varepsilon(v^\varepsilon)^2 - \varepsilon v_x^\varepsilon]\, d\tau \right\} - 1 \right| \\ =&|c^0(x,t)| \cdot \left| \exp\left\{ G_1^\varepsilon(x,t) + G_2^\varepsilon(x,t) + G_3^\varepsilon(x,t) \right\} - 1 \right|, \end{aligned} \tag{3.97}$$

with $G_1^\varepsilon(x,t) := -\int_0^t (u^\varepsilon - u^0)\, d\tau$, $G_2^\varepsilon(x,t) := \varepsilon \int_0^t (v^\varepsilon)^2\, d\tau$ and $G_3^\varepsilon(x,t) := -\varepsilon \int_0^t v_x^\varepsilon\, d\tau$.

We next estimate $G_1^\varepsilon(x,t)$, $G_2^\varepsilon(x,t)$ and $G_3^\varepsilon(x,t)$. First, Theorem 3.1 gives

$$\begin{aligned} |G_1^\varepsilon(x,t)| \leq& T \|u^\varepsilon - u^0\|_{L^\infty([0,1]\times[0,T])} \\ \leq& CT\varepsilon^{1/2}. \end{aligned} \tag{3.98}$$

Using Theorem 3.1, (3.85) and Lemma 3.1-Lemma 3.3, we estimate $G_2^\varepsilon(x,t)$ as

$$\begin{aligned} |G_2^\varepsilon(x,t)| \leq& \varepsilon T \|v^\varepsilon\|_{L^\infty([0,1]\times[0,T])}^2 \\ \leq& T\varepsilon \left(\|v^{I,0}\|_{L^\infty(0,T;H^1)}^2 + \|v^{B,0}\|_{L^\infty(0,T;H_z^1)}^2 \right) \\ &+ T\varepsilon \left(\|v^{b,0}\|_{L^\infty(0,T;H_\xi^1)}^2 + C\varepsilon^{1/2} \right) \\ \leq& CT\varepsilon \left(1 + \varepsilon^{1/2} \right). \end{aligned} \tag{3.99}$$

For any integer $m \geq 2$, similar arguments as deriving (3.83) entail that

$$\|b_{2x}^{\varepsilon}\|_{L^2(0,T;L^\infty)} \leq C\varepsilon^{m/2},$$

which, along with the definition of V^ε in (3.70), (3.3) ,the Sobolev embedding inequality and Lemma 3.1-Lemma 3.6 and Lemma 3.10, leads to

$$\begin{aligned}
|G_3^{\varepsilon}(x,t)| \leq & T^{1/2}\varepsilon\left(\|v_x^{I,0}\|_{L^2(0,T;L^\infty)} + \varepsilon^{-1/2}\|v_z^{B,0}\|_{L^2(0,T;L_z^\infty)}\right) \\
& + T^{1/2}\varepsilon\left(\varepsilon^{-1/2}\|v_\xi^{b,0}\|_{L^2(0,T;L_\xi^\infty)} + \varepsilon^{1/2}\|v_x^{I,1}\|_{L^2(0,T;L^\infty)}\right) \\
& + T^{1/2}\varepsilon\left(\|v_z^{B,1}\|_{L^2(0,T;L_z^\infty)} + \|v_\xi^{b,1}\|_{L^2(0,T;L_\xi^\infty)}\right) \\
& + T^{1/2}\varepsilon\left(\|b_{2x}^{\varepsilon}\|_{L^2(0,T;L^\infty)} + \varepsilon^{1/2}\|V_x^{\varepsilon}\|_{L^2(0,T;L^\infty)}\right) \\
\leq & CT^{1/2}\varepsilon\left(1 + \varepsilon^{-1/2} + \varepsilon^{1/2} + \varepsilon^{m/2}\right) \\
& + C_0 T^{1/2}\varepsilon^{3/2}\left(\|V_x^{\varepsilon}\|_{L^2(0,T;L^2)} + \|V_{xx}^{\varepsilon}\|_{L^2(0,T;L^2)}\right) \\
\leq & CT^{1/2}\left(\varepsilon + \varepsilon^{1/2} + \varepsilon^{3/2} + \varepsilon^{(m+2)/2} + \varepsilon^{5/4} + \varepsilon^{3/4}\right).
\end{aligned} \tag{3.100}$$

Collecting (3.98)-(3.100) and noticing that $0 < \varepsilon < 1$, we end up with

$$\left|G_1^{\varepsilon}(x,t) + G_2^{\varepsilon}(x,t) + G_3^{\varepsilon}(x,t)\right| \leq C\varepsilon^{1/2},$$

for some positive constant C independent of ε (but dependent on T). Thus it follows from the Taylor expansion and $0 < \varepsilon < 1$ that

$$\begin{aligned}
& \left|e^{G_1^{\varepsilon}(x,t)+G_2^{\varepsilon}(x,t)+G_3^{\varepsilon}(x,t)} - 1\right| \\
\leq & \sum_{k=1}^{\infty}\frac{1}{k!}\left|G_1^{\varepsilon}(x,t) + G_2^{\varepsilon}(x,t) + G_3^{\varepsilon}(x,t)\right|^k \leq C\varepsilon^{1/2}.
\end{aligned} \tag{3.101}$$

We proceed by setting $\varepsilon = 0$ in the second equation of (3.15) and find that

$$0 < c^0(x,t) = c_0(x)e^{-\int_0^t u^0(x,\tau)\,d\tau} \leq c_0(x) \leq C_0, \tag{3.102}$$

subject to the fact $u^0(x,t) \geq 0$ for $(x,t) \in [0,1] \times [0,T]$. The combination of (3.97), (3.101) and (3.102) yields (3.16).

To prove (3.17), we use the transformation $v^\varepsilon = -\frac{c_x^\varepsilon}{c^\varepsilon}$, Theorem 3.1 and (3.16) and get

$$\begin{aligned}
c_x^{\varepsilon} - c_x^0 = & -\left[v^{\varepsilon}c^{\varepsilon} - v^0 c^0\right] \\
= & -\left[(v^{\varepsilon} - v^0)c^{\varepsilon} + v^0(c^{\varepsilon} - c^0)\right] \\
= & -\left[(v^{B,0} + v^{b,0} + O(\varepsilon^{1/2}))(c^0 + O(\varepsilon^{1/2})) + v^0\,O(\varepsilon^{1/2})\right] \\
= & -c^0\left(v^{B,0} + v^{b,0}\right) + O(\varepsilon^{1/2}),
\end{aligned}$$

which implies (3.17) and completes the proof of Theorem 3.2. □

3.6 Formal Derivation of Outer/Inner Layer Profiles

In this section, we shall detail the derivation of (3.4)-(3.13) by the method of matched asymptotic expansions, which has been used in appendix of [29] to determine the thickness of boundary layers. Here we carry out further procedures to derive equations (3.4)-(3.13) for clarity and completeness though part of them are analogous with those in [29].

Step 1. Initial-boundary conditions. Upon the substitution of (3.2) into the initial conditions in (2.1), one has

$$\begin{aligned} u_0(x) &= \sum_{j\ge 0} \varepsilon^{j/2}\big(u^{I,j}(x,0)+u^{B,j}(z,0)+u^{b,j}(\xi,0)\big),\\ v_0(x) &= \sum_{j\ge 0} \varepsilon^{j/2}\big(v^{I,j}(x,0)+v^{B,j}(z,0)+v^{b,j}(\xi,0)\big). \end{aligned} \tag{3.103}$$

Observing that the initial data (u_0, v_0) is independent of ε, it follows from (3.103) and the hypothesis (H) that

$$\begin{aligned} &u^{I,0}(x,0) = u_0(x), \quad u^{B,0}(z,0) = u^{b,0}(\xi,0) = 0,\\ &v^{I,0}(x,0) = v_0(x), \quad v^{B,0}(z,0) = v^{b,0}(\xi,0) = 0 \end{aligned} \tag{3.104}$$

and for $j \ge 1$

$$\begin{aligned} &u^{I,j}(x,0) = u^{B,j}(z,0) = u^{b,j}(\xi,0) = 0,\\ &v^{I,j}(x,0) = v^{B,j}(z,0) = v^{b,j}(\xi,0) = 0. \end{aligned} \tag{3.105}$$

To derive the boundary conditions, one feeds (2.2) on (3.2) and use (3.3) to get

$$\begin{aligned} \bar u &= \sum_{j\ge 0} \varepsilon^{j/2}\big(u^{I,j}(0,t)+u^{B,j}(0,t)\big),\\ \bar u &= \sum_{j\ge 0} \varepsilon^{j/2}\big(u^{I,j}(1,t)+u^{b,j}(0,t)\big),\\ \bar v &= \sum_{j\ge 0} \varepsilon^{j/2}\big(v^{I,j}(0,t)+v^{B,j}(0,t)\big),\\ \bar v &= \sum_{j\ge 0} \varepsilon^{j/2}\big(v^{I,j}(1,t)+v^{b,j}(0,t)\big), \end{aligned} \tag{3.106}$$

where in the first expression of (3.106), term $u^{b,j}(-\frac{1}{\varepsilon^{1/2}},t)$ has been neglected due to the assumption (H) (see section 2) that $u^{b,j}(-\frac{1}{\varepsilon^{1/2}},t)$ decay to zero exponentially as $-\frac{1}{\varepsilon^{1/2}} \to -\infty$ (i.e. as $\varepsilon \to 0$). For the same reason, we have neglected $u^{B,j}(\frac{1}{\varepsilon^{1/2}},t)$, $v^{b,j}(-\frac{1}{\varepsilon^{1/2}},t)$ and $v^{B,j}(\frac{1}{\varepsilon^{1/2}},t)$ in the second, third and fourth expressions in (3.106), respectively. Since expressions in (3.106) hold for any $\varepsilon > 0$, we derive

$$\begin{aligned} &\bar u = u^{I,0}(0,t)+u^{B,0}(0,t), \qquad \bar u = u^{I,0}(1,t)+u^{b,0}(0,t),\\ &\bar v = v^{I,0}(0,t)+v^{B,0}(0,t), \qquad \bar v = v^{I,0}(1,t)+v^{b,0}(0,t), \end{aligned} \tag{3.107}$$

and we obtain for $j \geq 1$ that

$$
\begin{aligned}
u^{I,j}(0,t) + u^{B,j}(0,t) &= 0, \\
u^{I,j}(1,t) + u^{b,j}(0,t) &= 0, \\
v^{I,j}(0,t) + v^{B,j}(0,t) &= 0, \\
v^{I,j}(1,t) + v^{b,j}(0,t) &= 0.
\end{aligned}
\tag{3.108}
$$

Step 2. Equations for $u^{I,j}$, $u^{B,j}$ and $u^{b,j}$. For profiles of the outer solution $u^{I,j}$, we substitute (3.2) without the boundary layer solutions $u^{B,j}$, $u^{b,j}$, $v^{B,j}$ and $v^{b,j}$ into the first equation of (2.1) and immediately get:

$$
u_t^{I,j} - \sum_{k=0}^{j} \left(u^{I,k} v^{I,j-k} \right)_x = u_{xx}^{I,j}, \quad \text{for } j \geq 0. \tag{3.109}
$$

To find the profiles for left boundary-layer solutions $u^{B,j}$, we first neglect the right boundary-layer solutions $u^{b,j}$ and $v^{b,j}$ in (3.2) and substitute the remaining terms of (3.2) into the first equation of (2.1). By using (3.109), after some calculations, we end up with

$$
\sum_{j \geq -2} \varepsilon^{j/2} G_j(x,z,t) = 0, \tag{3.110}
$$

where

$$
\begin{cases}
G_{-2} = -u_{zz}^{B,0}, \\
G_{-1} = -u^{I,0} v_z^{B,0} - v^{I,0} u_z^{B,0} - \left(u^{B,0} v^{B,0} \right)_z - u_{zz}^{B,1}, \\
\quad G_j = u_t^{B,j} - \sum_{k=0}^{j} u^{B,k} v_x^{I,j-k} - \sum_{k=0}^{j+1} \left(u^{I,k} + u^{B,k} \right) v_z^{B,j+1-k} - \sum_{k=0}^{j} u_x^{I,k} v^{B,j-k} \\
\qquad - \sum_{k=0}^{j+1} u_z^{B,k} \left(v^{I,j+1-k} + v^{B,j+1-k} \right) - u_{zz}^{B,j+2}, \quad \text{for } j \geq 0.
\end{cases}
$$

Then recalling that $x = \varepsilon^{1/2} z$ and expanding $G_j(x,z,t)$ in ε by the Taylor expansion to have

$$
\begin{aligned}
G_j(x,z,t) &= G_j(\varepsilon^{1/2} z, z, t) \\
&= G_j(0,z,t) + \sum_{k=1}^{\infty} \frac{1}{k!} \left(\varepsilon^{1/2} z \right)^k \partial_x^k G_j(0,z,t),
\end{aligned}
\tag{3.111}
$$

for $j \geq -2$.

Next feeding (3.110) on (3.111), we get

$$
\sum_{j \geq -2} \varepsilon^{j/2} \tilde{G}_j(z,t) = 0, \tag{3.112}
$$

where

$$
\begin{cases}
\tilde{G}_{-2} = -\, u_{zz}^{B,0}, \\
\tilde{G}_{-1} = -\, u^{I,0}(0,t) v_z^{B,0} - v^{I,0}(0,t) u_z^{B,0} - \left(u^{B,0} v^{B,0}\right)_z - u_{zz}^{B,1}, \\
\quad \tilde{G}_0 = u_t^{B,0} - u^{B,0} v_x^{I,0}(0,t) - \left(u^{I,0}(0,t) + u^{B,0}\right) v_z^{B,1} - \left(u^{I,1}(0,t) + u^{B,1}\right) v_z^{B,0} \\
\qquad\quad - u_x^{I,0}(0,t) v^{B,0} - u_z^{B,0}\left(v^{I,1}(0,t) + v^{B,1}\right) - u_z^{B,1}\left(v^{I,0}(0,t) + v^{B,0}\right) - u_{zz}^{B,2} \\
\qquad\quad - z u_x^{I,0}(0,t) v_z^{B,0} - z v_x^{I,0}(0,t) u_z^{B,0}, \\
\qquad \cdots\cdots
\end{cases}
$$

and terms $\tilde{G}_j$ for $j \geq 1$ have been omitted for brevity. To make (3.112) hold for any $\varepsilon > 0$, it is required that $\tilde{G}_j = 0$ for $j \geq -2$, where in particular $\tilde{G}_{-2} = 0$ implies $u_{zz}^{B,0} = 0$. This, upon the integration with respect to z over (z, ∞) along with the assumption (H), gives $u_z^{B,0} = 0$. Integrating this over (z, ∞) entails that

$$
u^{B,0}(z,t) = 0, \quad \text{for } (z,t) \in [0,\infty) \times [0,T], \tag{3.113}
$$

which, applied to $\tilde{G}_{-1} = 0$ yields

$$
u_{zz}^{B,1} = -u^{I,0}(0,t) v_z^{B,0}. \tag{3.114}
$$

Then we integrate (3.114) over (z, ∞) and use the assumption (H) again to have

$$
u_z^{B,1} = -u^{I,0}(0,t) v^{B,0} = -\bar{u} v^{B,0}, \tag{3.115}
$$

where we also have used (3.113) and the first identity in (3.107).
Finally, by a similar procedure as deriving (3.115), that is, first inserting (3.113) into $\tilde{G}_0 = 0$ to get

$$
\begin{aligned}
u_{zz}^{B,2} = &- u^{I,0}(0,t) v_z^{B,1} - \left(u^{I,1}(0,t) + u^{B,1}\right) v_z^{B,0} \\
&- u_x^{I,0}(0,t) v^{B,0} - u_z^{B,1}\left(v^{I,0}(0,t) + v^{B,0}\right) - z u_x^{I,0}(0,t) v_z^{B,0},
\end{aligned} \tag{3.116}
$$

then integrating the above equation with respect to z twice, one finds that

$$
u^{B,2} = \bar{u} \int_z^\infty v^{B,1}(s,t)\, ds - \int_z^\infty \int_s^\infty \Phi(\zeta,t)\, d\zeta ds, \tag{3.117}
$$

with $\Phi(z,t) := (u^{I,1}(0,t) + u^{B,1}) v_z^{B,0} + u_x^{I,0}(0,t) v^{B,0} + u_z^{B,1}(v^{I,0}(0,t) + v^{B,0}) + z u_x^{I,0}(0,t) v_z^{B,0}$.

We next turn to the derivation for the right boundary-layer solutions. Indeed, we modify the above approach by neglecting the left boundary-layer solutions $u^{B,j}$ and $v^{B,j}$ in (3.2) and substitute the remaining terms into the first equation of (2.1), then using (3.109) and noting that in definition (3.3) the boundary layers have the same thickness $O(\varepsilon^{1/2})$ at both endpoints $x=0$ and $x=1$, we derive an expression similar to (3.110):

$$\sum_{j\ge -2} \varepsilon^{j/2} F_j(x,\xi,t) = 0, \tag{3.118}$$

where F_j is the same as G_j in (3.110), if we replace terms $u^{B,i}$, $v^{B,i}$ and z in the expression of G_j by $u^{b,i}$, $v^{b,i}$ and ξ, respectively. Then using the Taylor expansion at $x=1$:

$$F_j(x,\xi,t) = F_j(\varepsilon^{1/2}\xi+1,\xi,t) = F_j(1,\xi,t) + \sum_{k=1}^{\infty} \frac{1}{k!}\left(\varepsilon^{1/2}\xi\right)^k \partial_x^k F_j(1,\xi,t),\ j\ge -2,$$

we convert (3.118) into

$$\sum_{j\ge -2} \varepsilon^{j/2} \tilde{F}_j(\xi,t) = 0,$$

where $\tilde{F}_j$ is defined as $\tilde{G}_j$ in (3.112) by replacing $(u^{B,k},v^{B,k})$ with $(u^{b,k},v^{b,k})$, $(u^{I,k},v^{I,k})(0,t)$ with $(u^{I,k},v^{I,k})(1,t)$ and z with ξ, for $k\in\mathbb{N}$. Moreover, we deduce from $\tilde{F}_{-2}=0$, $\tilde{F}_{-1}=0$ and $\tilde{F}_0=0$ that

$$u^{b,0}(\xi,t) = 0, \quad \text{for } (\xi,t)\in(-\infty,0]\times[0,T], \tag{3.119}$$

$$u^{b,1}_{\xi} = -\bar{u}v^{b,0}, \tag{3.120}$$

and

$$\begin{aligned} u^{b,2}_{\xi\xi} =& -u^{I,0}(1,t)v^{b,1}_{\xi} - \left(u^{I,1}(1,t)+u^{b,1}\right)v^{b,0}_{\xi} \\ &- u^{I,0}_x(1,t)v^{b,0} - u^{b,1}_{\xi}\left(v^{I,0}(1,t)+v^{b,0}\right) - \xi u^{I,0}_x(1,t)v^{b,0}_{\xi}. \end{aligned} \tag{3.121}$$

Thus

$$u^{b,2} = \bar{u}\int_{\xi}^{-\infty} v^{b,1}(s,t)\,ds - \int_{\xi}^{-\infty}\int_{s}^{-\infty} \Psi(\zeta,t)\,d\zeta ds, \tag{3.122}$$

with $\Psi(\xi,t) := (u^{I,1}(1,t)+u^{b,1})v^{b,0}_{\xi} + u^{I,0}_x(1,t)v^{b,0} + u^{b,1}_{\xi}(v^{I,0}(1,t)+v^{b,0}) + \xi u^{I,0}_x(1,t)v^{b,0}_{\xi}$. Here the equations for $u^{b,0}$, $u^{b,1}$ and $u^{b,2}$ are similar to the ones for $u^{B,0}$, $u^{B,1}$ and $u^{B,2}$, respectively.

Step 3. Equations for $v^{I,j}$, $v^{B,j}$ and $v^{b,j}$. This part will be focused on deducing equations for each profile in the expansion of v^{ε} by substituting (3.2) into the second equation of (2.1) with analogous arguments as Step 2. As before, we first neglect the boundary-layer solutions in (3.2) and insert the remaining terms into the second equation of (2.1) to have

$$\begin{cases} v^{I,0}_t - u^{I,0}_x = 0, \\ v^{I,1}_t - u^{I,1}_x = 0, \\ v^{I,j}_t + 2\sum_{k=0}^{j-2} v^{I,k}v^{I,j-2-k}_x - u^{I,j}_x - v^{I,j-2}_{xx} = 0, \quad \text{for } j\ge 2. \end{cases} \tag{3.123}$$

Then neglecting right boundary-layer solutions $u^{b,j}$ and $v^{b,j}$ in (3.2) and plugging the remaining terms into the second equation of (2.1) and using (3.123), we find

$$\sum_{j\geq -1} \varepsilon^{\frac{j}{2}} R_j(x,z,t) = 0, \tag{3.124}$$

where

$$\begin{cases} R_{-1} = - u_z^{B,0}, \\ R_0 = v_t^{B,0} - u_z^{B,1} - v_{zz}^{B,0}, \\ R_1 = v_t^{B,1} + 2\left(v^{I,0} + v^{B,0}\right) v_z^{B,0} - u_z^{B,2} - v_{zz}^{B,1}, \\ \quad \cdots\cdots \end{cases} \tag{3.125}$$

Furthermore, using $x = \varepsilon^{1/2} z$ and expanding $R_j(x,z,t)$ formally in ε, one has

$$R_j(x,z,t) = R_j(\varepsilon^{1/2} z, z, t) = R_j(0,z,t) + \sum_{k=1}^{\infty} \frac{1}{k!} \left(\varepsilon^{1/2} z\right)^k \partial_x^k R_j(0,z,t), \quad j \geq -1.$$

Thus, inserting this into (3.125) yields

$$\sum_{j\geq -1} \varepsilon^{1/2} \tilde{R}_j(z,t) = 0, \tag{3.126}$$

where

$$\begin{cases} \tilde{R}_{-1} = -u_z^{B,0}, \\ \tilde{R}_0 = v_t^{B,0} - u_z^{B,1} - v_{zz}^{B,0}, \\ \tilde{R}_1 = v_t^{B,1} + 2\left(v^{I,0}(0,t) + v^{B,0}\right) v_z^{B,0} - u_z^{B,2} - v_{zz}^{B,1}, \\ \quad \cdots\cdots \end{cases}$$

On the other hand, one can get the following equations for right boundary-layer solutions by an analogous procedure as above:

$$\begin{aligned} v_t^{b,0} - u_\xi^{b,1} - v_{\xi\xi}^{b,0} &= 0, \\ v_t^{b,1} + 2\left(v^{I,0}(1,t) + v^{b,0}\right) v_\xi^{b,0} - u_\xi^{b,2} - v_{\xi\xi}^{b,1} &= 0. \end{aligned} \tag{3.127}$$

Finally, we collect the results obtained in Step 1 to Step 3 to derive the initial boundary value problems (3.4)-(3.13) given in section 2. First, from (3.109) with $j = 0$, (3.123), (3.104), (3.107), (3.113) and (3.119), we get (3.4). Combining (3.115), (3.126) with $j = 0$, (3.104) and (3.107), one gets (3.5)-(3.6). Similarly equations (3.120), (3.127), (3.104) and (3.107) lead to (3.7)-(3.8). Moreover, (3.109) with $j = 1$, (3.123), (3.105) and (3.108) give rise to (3.9), and equations (3.10)-(3.11) come from (3.117), (3.126) with $j = 1$, (3.105) and (3.108). Finally (3.12)-(3.13) follow from (3.122), (3.127), (3.105) and (3.108).

Chapter 4

Existence of Radial Boundary Layers

In Chapter 2 and Chapter 3, we have proved the existence and stability of boundary layers for one-dimensional system (1.4) with $\Omega = (0,1)$, but the boundary layer problem in multi-dimensional space is left open. Indeed, compared to the one dimensional case, the boundary layer problem in multi-dimensions turns out to be much more involved since an additional intrinsic curl-free condition for $\vec{v}$:

$$\nabla \times \vec{v} = 0 \tag{4.1}$$

is required so that the results on (1.4) can be passed to the original model (1.2) via transformation (1.3). We shall further discuss the differences between one and multi-dimensional boundary layer problems for system (1.4) at the beginning of Chapter 5. As the first step towards a thorough understanding of boundary layer theory of (1.4) in multi-dimensions, this chapter will be concentrated to study a special case, the radial solutions (with radial spatial domains) in order to gain some basic insights.

To get rid of the singularity of radial solutions at the origin $r = 0$ and to focus on the boundary layer effect, we set $\Omega = \{\vec{x} = (x_1, x_2, \cdots, x_d) \in \mathbb{R}^d \mid 0 < a < |\vec{x}| < b\}$. Assume that the solution component u of (1.4) is radially symmetric, depending only on the radial variable $r = |\vec{x}|$ and time variable t (i.e. $u(\vec{x},t) = u(r,t)$). To fulfill the requirement (4.1), instead of imposing a radial form on $\vec{v}$ we assume that the solution component $\vec{v}(\vec{x},t) = v(r,t)(\frac{x_1}{r}, \frac{x_2}{r}, \cdots, \frac{x_d}{r})$. Substituting these radial expressions for $(u, \vec{v})$ into (1.4), it follows that

$$\begin{cases} u_t = \frac{1}{r^{d-1}}(r^{d-1}u_r)_r + \frac{1}{r^{d-1}}(r^{d-1}uv)_r, \quad (r,t) \in (a,b) \times (0,\infty) \\ v_t = \varepsilon \frac{1}{r^{d-1}}(r^{d-1}v_r)_r - \varepsilon \frac{d-1}{r^2} v - \varepsilon (v^2)_r + u_r, \\ (u,v)(r,0) = (u_0, v_0)(r). \end{cases} \tag{4.2}$$

where $D = \chi = \mu = 1$ have been assumed for simplicity, but the results of this thesis hold for general values of $D > 0$, $\chi > 0$ and $\mu > 0$. With a similar discussion as for (2.2), the Dirichlet boundary conditions for (4.2) ought to be

$$\begin{cases} u|_{r=a,b} = \bar{u},\ v|_{r=a} = \bar{v}_1,\ v|_{r=b} = \bar{v}_2, & \text{if } \varepsilon > 0, \\ u|_{r=a,b} = \bar{u}, & \text{if } \varepsilon = 0. \end{cases} \tag{4.3}$$

In this chapter, we shall investigate the asymptotic behavior of solutions as $\varepsilon \to 0$ to (4.2)-(4.3) with $d \geq 2$ (if $d = 1$, it coincides with the one-dimensional model (2.1)-(2.2), which has been studied in Chapter 2 and Chapter 3). In particular, the solution component v will be proved to possess boundary layer singularity due to the mismatch of its boundary values between $\varepsilon > 0$ and $\varepsilon = 0$ (see Theorem 4.2). We emphasize that the main result in Theorem 4.2 derived below for system (4.2)-(4.3) is not a trivial generalization of the one-dimensional result in Theorem 2.2. Actually, the strategy used in proving Theorem 2.2 is not applicable to system (2.1)-(2.2) and we postpone the main ideas employed and additional difficulties encountered (compared to the one-dimensional case) in proving Theorem 4.2 to the discussions below Proposition 4.1 (see Section 4.1). The main results are stated in Section 4.1 and their proofs will be given in Section 4.2 and Section 4.3.

4.1 Results on Existence of Boundary Layers

To study the boundary layer effect, we first present in Theorem 4.1 the global well-posedness and regularity estimates for solutions of (4.2)-(4.3) with $\varepsilon = 0$. By these estimates, the convergence for u and boundary layer singularity for v are justified in Theorem 4.2, which is the main result. Finally, we convert the results of Theorem 4.2 to the original model (1.2).

The fist result is concerned with the global well-posedness of (4.2)-(4.3) with $\varepsilon = 0$.

Theorem 4.1. *Assume that* $(u_0, v_0) \in H^2 \times H^2$ *with* $u_0 \geq 0$ *satisfy the compatible conditions* $u_0(a) = u_0(b) = \bar{u}$. *Then the initial-boundary value problem* (4.2)-(4.3) *with* $\varepsilon = 0$ *has a unique solution* $(u^0, v^0) \in C([0,\infty); H^2 \times H^2)$, *such that*
(i) If $\bar{u} > 0$, *there is a constant* C_0 *independent of* t *such that*

$$\|(u^0 - \bar{u})(t)\|_{H^2}^2 + \|v^0(t)\|_{H^2}^2 + \int_0^t \left(\|(u^0 - \bar{u})(\tau)\|_{H^3}^2 + \bar{u}\|(r^{d-1}v^0)_r(\tau)\|_{H^1}^2 \right) d\tau \leq C_0. \quad (4.4)$$

Moreover,

$$\lim_{t\to\infty} \|(u^0 - \bar{u})(t)\|_{L^\infty} = 0. \quad (4.5)$$

(ii) If $\bar{u} = 0$, *for any* $0 < T < \infty$ *there exists a constant* C *depending on* T *such that*

$$\|u^0\|_{L^\infty(0,T;H^2)} + \|v^0\|_{L^\infty(0,T;H^2)} + \|u^0\|_{L^2(0,T;H^3)} \leq C. \quad (4.6)$$

The main result is as follows.

Theorem 4.2. *Suppose that* $(u_0, v_0) \in H^2 \times H^2$ *with* $u_0 \geq 0$ *satisfy the compatible conditions* $u_0(a) = u_0(b) = \bar{u}$ *and* $v_0(a) = \bar{v}_1, v_0(b) = \bar{v}_2$. *Let* (u^0, v^0) *be the solution obtained in Theorem 4.1. For* $0 < T < \infty$, *we denote*

$$\varepsilon_T = \min\left\{ \left(8C_0 \int_0^T F(t)\,dt\right)^{-2},\ \left(16C_0^2 T e^{C_0 \int_0^T F(t)\,dt} \int_0^T F(t)\,dt\right)^{-2} \right\},$$

with function $F(t)$ defined in (4.50) *and the constant C_0 depending only on a, b and n. Then* (4.2)-(4.3) *with $\varepsilon \in (0, \varepsilon_T]$ admits a unique solution $(u^\varepsilon, v^\varepsilon) \in C([0,T]; H^2 \times H^2)$. Moreover, for any function $\delta = \delta(\varepsilon)$ satisfying*

$$\delta(\varepsilon) \to 0 \text{ and } \varepsilon^{1/2}/\delta(\varepsilon) \to 0, \text{ as } \varepsilon \to 0 \tag{4.7}$$

it holds that

$$\|u^\varepsilon - u^0\|_{L^\infty(0,T;C[a,b])} \le C\varepsilon^{1/4}, \tag{4.8}$$

$$\|v^\varepsilon - v^0\|_{L^\infty(0,T;C[a+\delta,b-\delta])} \le C\varepsilon^{1/4}\delta^{-1/2} \tag{4.9}$$

and

$$\liminf_{\varepsilon \to 0} \|v^\varepsilon - v^0\|_{L^\infty(0,T;C[a,b])} > 0 \tag{4.10}$$

if and only if

$$\int_0^t u_r^0(a,\tau)\,d\tau \neq 0 \qquad \text{or} \qquad \int_0^t u_r^0(b,\tau)\,d\tau \neq 0, \quad \text{for some } t \in [0,T]. \tag{4.11}$$

Remark 4.1. *As stated in Remark 2.2, the BL-thickness $\delta(\varepsilon)$ (the BL-thickness) is not uniquely determined.*

By employing transformation (1.3), we next convert the results in Theorem 4.2 to the pre-transformed model (1.2). In particular, by inserting the expression $\vec{v}(\vec{x},t) = v(r,t)\left(\frac{x_1}{r}, \frac{x_2}{r}, \cdots, \frac{x_d}{r}\right)$ into (1.3) we derive

$$v(r,t)\left(\frac{x_1}{r}, \frac{x_2}{r}, \cdots, \frac{x_d}{r}\right) = -\nabla \ln c = -(\ln c)_r \left(\frac{x_1}{r}, \frac{x_2}{r}, \cdots, \frac{x_d}{r}\right),$$

which indicates that the solution component c is radially symmetric, that is $c(\vec{x},t) = c(r,t)$. Moreover,

$$v(r,t) = -(\ln c)_r = -\frac{c_r}{c}. \tag{4.12}$$

Noting that u and c are radially symmetric, the counterpart of the original model (1.2) corresponding to the transformed system (4.2)-(4.3) reads as follows:

$$\begin{cases} u_t = u_{rr} + \frac{d-1}{r}u_r - \left(u\frac{c_r}{c}\right)_r - \frac{d-1}{r}\left(u\frac{c_r}{c}\right), \\ c_t = \varepsilon c_{rr} + \varepsilon\frac{d-1}{r}c_r - uc, \\ u(0,r) = u_0(r), \ c(0,r) = c_0(r), \\ u|_{r=a,b} = \bar{u}, \ -\frac{c_r}{c}\Big|_{r=a} = \bar{v}_1, \ -\frac{c_r}{c}\Big|_{r=b} = \bar{v}_2, & \text{if } \varepsilon > 0; \\ u|_{r=a,b} = \bar{u}, & \text{if } \varepsilon = 0. \end{cases} \tag{4.13}$$

where (4.12) has been used to determine the boundary conditions for c.

Proposition 4.1. *Assume $c_0 \ge 0$ and $(u_0, \ln c_0) \in H^2 \times H^3$. Suppose that the assumptions in Theorem 4.2 hold with $v_0 = -(\ln c_0)_r$. Let $0 < T < \infty$. Then* (4.13) *with $\varepsilon \in [0, \varepsilon_T]$ admits a* *unique solution* $(u^\varepsilon, c^\varepsilon) \in C([0,T]; H^2 \times H^3)$ *such that*

$$\begin{aligned} &\|u^\varepsilon - u^0\|_{L^\infty(0,T;C[a,b])} \le C\varepsilon^{1/4}, \\ &\|c^\varepsilon - c^0\|_{L^\infty(0,T;C[a,b])} \le C\varepsilon^{1/4}. \end{aligned} \tag{4.14}$$

Moreover, for the function $\delta(\varepsilon)$ *defined* (4.7) *it holds that*

$$\|c_r^\varepsilon - c_r^0\|_{L^\infty(0,T;C[a+\delta,b-\delta])} \le C\varepsilon^{1/4}\delta^{-1/2} \tag{4.15}$$

and the following estimate is equivalent to (4.11)

$$\liminf_{\varepsilon\to 0} \|c_r^\varepsilon - c_r^0\|_{L^\infty(0,T;C[a,b])} > 0. \tag{4.16}$$

At the end of this section, we briefly introduce the main idea and arrangement of the following sections. We emphasize that although the system (4.2)-(4.3) with $d \ge 2$ is in a similar form to its counterpart with $d = 1$ for which the boundary layer problem has been studied in Chapter 2 based on a uniform-in-ε estimates of solutions $(u^\varepsilon, v^\varepsilon)$ with $\varepsilon > 0$, the methods used there can not be applied to study the present problem since when $d \ge 2$ the system (4.2)-(4.3) with $\varepsilon > 0$ lacks an energy-like structure to provide a preliminary estimate (or a Lyapunov function) of ε-independence. The challenge in our analysis will thus consist in deriving the $\varepsilon^{1/4}$-convergence rate in (4.8) and (4.9) without any uniform-in-ε priori estimates on solutions $(u^\varepsilon, v^\varepsilon)$. Inspired by the works [9, 85], this will be achieved in Section 4.3 by regarding $(u^\varepsilon, v^\varepsilon)$ with small $\varepsilon > 0$ as perturbations of (u^0, v^0) and then estimating their difference $(u^\varepsilon - u^0, v^\varepsilon - v^0)$ by employing the standard energy methods and a preliminary lemma (see Lemma 4.4) on ODEs. The proof of Theorem 4.1 is quite standard and will be given in next section.

4.2 Proof of Theorem 4.1

This section is to prove Theorem 4.1 based on the following lemmas where *a priori* estimates of solution (u^0, v^0) are derived by employing the standard energy methods. We set off by rewriting (4.2)-(4.3) with $\varepsilon = 0$ as follows:

$$\begin{cases} u_t^0 = \frac{1}{r^{d-1}}\left(r^{d-1}u_r^0\right)_r + \frac{1}{r^{d-1}}\left(r^{d-1}u^0v^0\right)_r, \\ v_t^0 = u_r^0, \\ (u,v)(r,0) = (u_0,v_0)(r), \\ u(a,t) = u(b,t) = \bar{u}. \end{cases} \tag{4.17}$$

Lemma 4.1. *Suppose the assumptions in Theorem 4.1 hold and $\bar{u} > 0$. Then there exists a positive constant C_0 independent of t such that*

$$\begin{aligned}
&\int_a^b r^{d-1}[(u^0 \ln u^0 - u^0)(t) - (\bar{u}\ln\bar{u} - \bar{u}) - \ln\bar{u}(u^0(t) - \bar{u})]dr \\
&+\frac{1}{2}\int_a^b r^{d-1}(v^0)^2(t)dr + \int_0^t\int_a^b r^{d-1}\frac{(u_r^0)^2}{u^0}drd\tau \le C_0
\end{aligned} \tag{4.18}$$

and

$$\|r^{(d-1)/2}(u^0-\bar{u})(t)\|_{L^2}^2 + \int_0^t \|r^{(d-1)/2}u_r^0(\tau)\|_{L^2}^2 d\tau \le C_0. \tag{4.19}$$

Proof. Taking the L^2 inner products of the first and second equation of (4.17) with $r^{d-1}(\ln u^0 - \ln\bar{u})$ and $r^{d-1}v^0$ respectively, we then add the results and use integration by parts to get

$$\begin{aligned}
&\frac{d}{dt}\int_a^b r^{d-1}[(u^0\ln u^0 - u^0) - (\bar{u}\ln\bar{u} - \bar{u}) - \ln\bar{u}(u^0 - \bar{u})]dr \\
&+\frac{1}{2}\frac{d}{dt}\int_a^b r^{d-1}(v^0)^2 dr + \int_a^b r^{d-1}\frac{(u_r^0)^2}{u^0}dr = 0,
\end{aligned}$$

which gives rise to (4.18) upon integration over $(0,t)$. We proceed to prove (4.19) by denoting $\tilde{u} = u^0 - \bar{u}$. Then from (4.17) we deduce that $\tilde{u}$ satisfies

$$\tilde{u}_t = \frac{1}{r^{d-1}}(r^{d-1}\tilde{u}_r)_r + \frac{1}{r^{d-1}}(r^{d-1}\tilde{u}v)_r + \frac{\bar{u}}{r^{d-1}}(r^{d-1}v)_r. \tag{4.20}$$

Multiplying (4.20) by $r^{d-1}\tilde{u}$ and the second equation of (4.17) by $\bar{u}r^{d-1}v$, respectively, and adding the results gives

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|r^{(d-1)/2}\tilde{u}\|_{L^2}^2 + \bar{u}\|r^{(d-1)/2}v^0\|_{L^2}^2\right) + \|r^{(d-1)/2}\tilde{u}_r\|_{L^2}^2 \\
&= -\int_a^b r^{d-1}\tilde{u}v^0\tilde{u}_r dr \\
&\le \frac{1}{2}\|r^{(d-1)/2}\tilde{u}_r\|_{L^2}^2 + \frac{1}{2}\|\tilde{u}\|_{L^\infty}^2\|r^{(d-1)/2}v^0\|_{L^2}^2,
\end{aligned} \tag{4.21}$$

with $\|\tilde{u}\|_{L^\infty}$ estimated as follows

$$|\tilde{u}(r,t)| = |u^0(r,t) - \bar{u}| = \left|\int_a^r u_r^0 dr\right| \le \left(\int_a^b u^0 dr\right)^{1/2}\left(\int_a^b \frac{(u_r^0)^2}{u^0}dr\right)^{1/2}.$$

Substituting the above estimate into (4.21) and then integrating the result over $(0,t)$ we have

$$\begin{aligned}
&\frac{1}{2}\|r^{(d-1)/2}\tilde{u}(t)\|_{L^2}^2 + \frac{1}{2}\bar{u}\|r^{(d-1)/2}v^0(t)\|_{L^2}^2 + \frac{1}{2}\int_0^t \|r^{(d-1)/2}\tilde{u}_r\|_{L^2}^2 d\tau \\
&\le \frac{1}{2}\int_0^t\int_a^b \frac{(u_r^0)^2}{u^0}drd\tau \cdot \|u^0\|_{L^\infty(0,t;L^\infty)}\|r^{(d-1)/2}v^0\|_{L^\infty(0,t;L^2)}^2,
\end{aligned}$$

which combined with the fact that

$$\|u^0\|_{L^\infty(0,t;L^1)} \le C_0 \sup_{\tau\in[0,t]} \int_a^b r^{d-1}[(u^0\ln u^0 - u^0)(\tau) - (\bar{u}\ln\bar{u} - \bar{u}) - \ln\bar{u}(u^0(\tau) - \bar{u})]dr$$

and (4.18) implies (4.19). The proof is completed.

We proceed to derive higher regularity properties for the solution.

Lemma 4.2. *Let the assumptions in Theorem 4.1 hold and $\bar{u} > 0$. Then there are constants C_0 independent of t such that*

$$\begin{aligned} &\|(r^{d-1}v^0)_r(t)\|_{L^2}^2 + \|r^{(d-1)/2}\tilde{u}_r(t)\|_{L^2}^2 \\ &+ \int_0^t \big(\bar{u}\|(r^{d-1}v^0)_r\|_{L^2}^2 + \|r^{(d-1)/2}\tilde{u}_t\|_{L^2}^2\big)d\tau \le C_0. \end{aligned} \tag{4.22}$$

Proof. We multiply the second equation of (4.17) with r^{d-1} and differentiate the resulting equation with respect to r, then from (4.20) we obtain

$$(r^{d-1}v^0)_{rt} = (r^{d-1}\tilde{u}_r)_r = r^{d-1}\tilde{u}_t - (r^{d-1}\tilde{u}v^0)_r - \bar{u}(r^{d-1}v^0)_r. \tag{4.23}$$

Taking the L^2 inner product of (4.23) with $2(r^{d-1}v^0)_r$ to get

$$\begin{aligned} &\frac{d}{dt}\|(r^{d-1}v^0)_r\|_{L^2}^2 + 2\bar{u}\|(r^{d-1}v^0)_r\|_{L^2}^2 \\ =&2\int_a^b r^{d-1}\tilde{u}_t(r^{d-1}v^0)_r dr - 2\int_a^b (r^{d-1}\tilde{u}v^0)_r(r^{d-1}v^0)_r dr \\ :=&I_1 + I_2. \end{aligned} \tag{4.24}$$

We rewrite I_1 as follows:

$$I_1 = 2\frac{d}{dt}\int_a^b (r^{d-1}\tilde{u})(r^{d-1}v^0)_r dr - 2\int_a^b (r^{d-1}\tilde{u})(r^{d-1}v^0)_{rt} dr := M_1 + M_2,$$

with

$$M_1 = \frac{d}{dt}\big(\|(r^{d-1}v^0)_r\|_{L^2}^2/2 + 2\|r^{d-1}\tilde{u}\|_{L^2}^2 - \|(r^{d-1}v^0)_r/\sqrt{2} - \sqrt{2}r^{d-1}\tilde{u}\|_{L^2}^2\big)$$

and

$$M_2 = -2\int_a^b (r^{d-1}\tilde{u})(r^{d-1}\tilde{u}_r)_r dr \quad = 2\int_a^b (r^{d-1}\tilde{u})_r(r^{d-1}\tilde{u}_r)dr \le C_0\|r^{(d-1)/2}\tilde{u}_r\|_{L^2}^2,$$

where (4.23), the Poincaré inequality and (4.39) have been used. The estimate for I_2 follows from (4.52), Sobolev embedding inequality and (4.39):

$$\begin{aligned} I_2 \le& \frac{\bar{u}}{2}\|(r^{d-1}v^0)_r\|_{L^2}^2 + \frac{4}{\bar{u}}\|\tilde{u}\|_{L^\infty}^2\|(r^{d-1}v^0)_r\|_{L^2}^2 + \frac{4}{\bar{u}}\|\tilde{u}_r\|_{L^2}^2\|(r^{d-1}v^0)\|_{L^\infty} \\ \le& \frac{\bar{u}}{2}\|(r^{d-1}v^0)_r\|_{L^2}^2 + C_0\|r^{(d-1)/2}\tilde{u}_r\|_{L^2}^2\big(\|r^{(d-1)/2}v^0\|_{L^2}^2 + \|(r^{d-1}v^0)_r\|_{L^2}^2\big). \end{aligned} \tag{4.25}$$

Substituting the above estimates for I_1 and I_2 into (4.24) then applying Gronwall's inequality to the results and using Lemma 4.1 to conclude that

$$\|(r^{d-1}v^0)_r(t)\|_{L^2}^2+\bar{u}\int_0^t\|(r^{d-1}v^0)_r\|_{L^2}^2d\tau\le C_0. \tag{4.26}$$

We proceed to estimate $\|r^{(d-1)/2}\tilde{u}_r(t)\|_{L^2}$ by multiplying (4.20) with $2r^{d-1}\tilde{u}_t$ in L^2 and derive

$$\begin{aligned}&\frac{d}{dt}\|r^{(d-1)/2}\tilde{u}_r\|_{L^2}^2+2\|r^{(d-1)/2}\tilde{u}_t\|_{L^2}^2\\ =&2\int_a^b(r^{d-1}\tilde{u}v^0)_r\tilde{u}_tdr+2\bar{u}\int_a^b(r^{d-1}v^0)_r\tilde{u}_tdr\\ :=&I_3+I_4.\end{aligned} \tag{4.27}$$

By similar arguments as deriving (4.25), we deduce that

$$I_3\le\frac{1}{2}\|r^{(d-1)/2}\tilde{u}_t\|_{L^2}^2+C_0\|r^{(d-1)/2}\tilde{u}_r\|_{L^2}^2(\|r^{(d-1)/2}v^0\|_{L^2}^2+\|(r^{d-1}v^0)_r\|_{L^2}^2).$$

Cauchy-Schwarz inequality leads to

$$I_4\le\frac{1}{2}\|r^{(d-1)/2}\tilde{u}_t\|_{L^2}^2+C_0\|(r^{d-1}v^0)_r\|_{L^2}^2.$$

Inserting the above estimates for I_3 and I_4 into (4.27), we derive

$$\begin{aligned}&\frac{d}{dt}\|r^{(d-1)/2}\tilde{u}_r\|_{L^2}^2+\|r^{(d-1)/2}\tilde{u}_t\|_{L^2}^2\\ \le&C_0\|r^{(d-1)/2}\tilde{u}_r\|_{L^2}^2(\|r^{(d-1)/2}v^0\|_{L^2}^2+\|(r^{d-1}v^0)_r\|_{L^2}^2)\\ &+C_0\|(r^{d-1}v^0)_r\|_{L^2}^2.\end{aligned} \tag{4.28}$$

Then applying Gronwall's inequality and (4.26) to (4.28), one arrives at

$$\|r^{(d-1)/2}\tilde{u}_r(t)\|_{L^2}^2+\int_0^t\|r^{(d-1)/2}\tilde{u}_t\|_{L^2}^2d\tau\le C_0,$$

which, in conjunction with (4.26) completes the proof.

□

Lemma 4.3. *Suppose that the assumptions in Theorem 4.1 hold and $\bar{u}>0$. Then there constants C_0 independent of t such that*

$$\begin{aligned}&\|r^{(d-1)/2}\tilde{u}_t(t)\|_{L^2}^2+\|(r^{d-1}v^0)_{rr}(t)\|_{L^2}^2\\ &+\int_0^t\left(\|r^{(d-1)/2}\tilde{u}_{rt}\|_{L^2}^2+\bar{u}\|(r^{d-1}v^0)_{rr}\|_{L^2}^2\right)d\tau\le C_0\end{aligned} \tag{4.29}$$

and

$$\|(r^{d-1}\tilde{u}_r)_r(t)\|_{L^2}^2+\int_0^t\left(\|(r^{d-1}\tilde{u}_r)_r\|_{L^2}^2+\|(r^{d-1}\tilde{u}_r)_{rr}\|_{L^2}^2\right)d\tau\le C_0. \tag{4.30}$$

Proof. Differentiating (4.20) with respect to t then multiplying the results with $2r^{d-1}\tilde{u}_t$, we derive upon integration by parts

$$\begin{aligned}\frac{d}{dt}\|r^{(d-1)/2}\tilde{u}_t\|_{L^2}^2+2\|r^{(d-1)/2}\tilde{u}_{rt}\|_{L^2}^2 &= -2\int_a^b (r^{d-1}\tilde{u}v^0)_t\tilde{u}_{rt}dr-2\bar{u}\int_a^b (r^{d-1}v^0)_t\tilde{u}_{rt} \\ &:= I_5+I_6.\end{aligned} \tag{4.31}$$

Applying (4.52) to $\tilde{u}_t$ and $\tilde{u}$ gives

$$\begin{aligned}I_5 \leq& C_0\|\tilde{u}_t\|_{L^\infty}\|v^0\|_{L^2}\|r^{(d-1)/2}\tilde{u}_{rt}\|_{L^2}+\|\tilde{u}\|_{L^\infty}\|v_t^0\|_{L^2}\|r^{(d-1)/2}\tilde{u}_{rt}\|_{L^2} \\ \leq& C_0\|r^{(d-1)/2}\tilde{u}_t\|_{L^2}^{1/2}\|r^{(d-1)/2}\tilde{u}_{rt}\|_{L^2}^{3/2}\|r^{(d-1)/2}v^0\|_{L^2} \\ &+\|r^{(d-1)/2}\tilde{u}_r\|_{L^2}\|\tilde{u}_r\|_{L^2}\|r^{(d-1)/2}\tilde{u}_{rt}\|_{L^2} \\ \leq& \frac{1}{2}\|r^{(d-1)/2}\tilde{u}_{rt}\|_{L^2}^2+C_0\|r^{(d-1)/2}\tilde{u}_t\|_{L^2}^2\|r^{(d-1)/2}v^0\|_{L^2}^4+\|r^{(d-1)/2}\tilde{u}_r\|_{L^2}^4,\end{aligned}$$

where the second equation of (4.17) has been used. We use again the second equation of (4.17) and Cauchy-Schwarz inequality to get

$$I_6 \leq \frac{1}{2}\|r^{(d-1)/2}\tilde{u}_{rt}\|_{L^2}^2+2\bar{u}^2\|r^{(d-1)/2}\tilde{u}_r\|_{L^2}^2.$$

Substituting the above estimates for I_5-I_6 into (4.31), then we integrate the results over $(0,t)$ and use Lemma 4.1 along with Lemma 4.2 to conclude that

$$\|r^{(d-1)/2}\tilde{u}_t(t)\|_{L^2}^2+\int_0^t \|r^{(d-1)/2}\tilde{u}_{rt}\|_{L^2}^2 d\tau \leq C_0. \tag{4.32}$$

Differentiating (4.23) with respect to r and multiplying the resulting equation with $2(r^{d-1}v)_{rr}$ we get

$$\begin{aligned}&\frac{d}{dt}\|(r^{d-1}v^0)_{rr}\|_{L^2}^2+2\bar{u}\|(r^{d-1}v^0)_{rr}\|_{L^2}^2 \\ =&2\int_a^b (r^{d-1}\tilde{u}_t)_r(r^{d-1}v^0)_{rr}dr-2\int_a^b (r^{d-1}\tilde{u}v^0)_{rr}(r^{d-1}v^0)_{rr} \\ :=&I_7+I_8.\end{aligned} \tag{4.33}$$

The estimate for I_7 follows from Cauchy-Schwarz inequality and (4.39)

$$I_7 \leq \frac{\bar{u}}{2}\|(r^{d-1}v^0)_{rr}\|_{L^2}^2+C_0(\|r^{(d-1)/2}\tilde{u}_t\|_{L^2}^2+\|r^{(d-1)/2}\tilde{u}_{rt}\|_{L^2}^2).$$

To bound I_8 we first estimate $\int_0^t \|(r^{d-1}\tilde{u}_r)_r\|_{L^2}^2 d\tau$ by the first equation of (4.17) as follows:

$$\begin{aligned}\int_0^t \|(r^{d-1}\tilde{u}_r)_r\|_{L^2}^2 d\tau \leq& \int_0^t \|r^{d-1}\tilde{u}_t\|_{L^2}^2 d\tau+C_0\int_0^t \|\tilde{u}_r\|_{L^2}^2 d\tau\cdot\|(r^{d-1}v^0)\|_{L^\infty(0,t;H^1)}^2 \\ &+C_0\bar{u}^2\int_0^t \|(r^{d-1}v^0)_r\|_{L^2}^2 d\tau \leq C_0,\end{aligned} \tag{4.34}$$

where Lemma 4.1 and Lemma 4.2 have been used. (4.34) along with (4.39) and (4.19) implies that

$$\int_0^t \|\tilde{u}_{rr}\|_{L^2}^2 d\tau \le C_0 \int_0^t (\|(r^{d-1}\tilde{u}_r)_r\|_{L^2}^2 + \|r^{(d-1)/2}\tilde{u}_r\|_{L^2}^2) d\tau \le C_0, \tag{4.35}$$

where constants C_0 depend on a, b. Then noting that $(r^{d-1}\tilde{u}v^0)_{rr} = (r^{d-1}v^0)_{rr}\tilde{u} + 2(r^{d-1}v^0)_r\tilde{u}_r + (r^{d-1}v^0)\tilde{u}_{rr}$ one deduces that

$$\begin{aligned} I_8 \le& \frac{\bar{u}}{2}\|(r^{d-1}v^0)_{rr}\|_{L^2}^2 + \frac{2}{\bar{u}}\|(r^{d-1}\tilde{u}v^0)_{rr}\|_{L^2}^2 \\ \le& \frac{\bar{u}}{2}\|(r^{d-1}v^0)_{rr}\|_{L^2}^2 + C_0\big(\|\tilde{u}_r\|_{L^2}^2\|(r^{d-1}v^0)_{rr}\|_{L^2}^2 \\ &+ \|\tilde{u}_r\|_{L^2}^2\|(r^{d-1}v^0)_r\|_{L^2}^2 + \|\tilde{u}_{rr}\|_{L^2}^2\|(r^{d-1}v^0)_r\|_{L^2}^2 + \|\tilde{u}_{rr}\|_{L^2}^2\|v^0\|_{L^2}^2\big). \end{aligned}$$

We feed (4.33) on the above estimates for I_7-I_8 then apply Gronwall's inequality, Lemma 4.1 - Lemma 4.2 and (4.32) and (4.35) to the result to find

$$\|(r^{d-1}v^0)_{rr}(t)\|_{L^2}^2 + \bar{u}\int_0^t \|(r^{d-1}v^0)_{rr}\|_{L^2}^2 d\tau \le C_0. \tag{4.36}$$

By similar arguments as deriving (4.34) one gets

$$\begin{aligned} \|(r^{d-1}\tilde{u}_r)_r(t)\|_{L^2}^2 \le& \|r^{d-1}\tilde{u}_t(t)\|_{L^2}^2 + C_0\|\tilde{u}_r(t)\|_{L^2}^2\|(r^{d-1}v^0)(t)\|_{H^1}^2 \\ &+ C_0\bar{u}^2\|(r^{d-1}v^0)_r(t)\|_{L^2}^2 \le C_0, \end{aligned} \tag{4.37}$$

where Lemma 4.1 - Lemma 4.2 have been used. We differentiate (4.23) with respect to r and conclude that

$$\begin{aligned} &\int_0^t \|(r^{d-1}\tilde{u}_r)_{rr}\|_{L^2}^2 d\tau \\ \le& \int_0^t \|r^{d-1}\tilde{u}_{rt}\|_{L^2}^2 d\tau + C_0\int_0^t \|r^{(d-1)/2}\tilde{u}_t\|_{L^2}^2 d\tau + \bar{u}^2\int_0^t \|(r^{d-1}v^0)_{rr}\|_{L^2}^2 d\tau \\ &+ C_0\int_0^t \big(\|\tilde{u}_r\|_{L^2}^2 + \|\tilde{u}_{rr}\|_{L^2}^2\big) d\tau \cdot \|(r^{d-1}v^0)\|_{L^\infty(0,t;H^2)}^2 \\ \le& C_0, \end{aligned} \tag{4.38}$$

where we have used Lemma 4.1 - Lemma 4.2 and (4.35)-(4.36). Finally collecting (4.32), (4.34), (4.36)-(4.38) we derive the desired estimates and complete the proof.

□

We are now in the position to prove Theorem 4.1 by the above Lemma 4.1 - Lemma 4.3.

Proof of Theorem 4.1. The first part of Theorem 4.1 follows from Lemma 4.1- Lemma 4.3. Indeed, we note for $g(r,t) \in L^2(a,b)$ with fixed $t > 0$, it follows that

$$b^{-(d-1)}\|r^{(d-1)/2}g(t)\|_{L^2}^2 \le \|g(t)\|_{L^2}^2 \le a^{-(d-1)}\|r^{(d-1)/2}g(t)\|_{L^2}^2. \tag{4.39}$$

Thus by Lemma 4.1 and (4.39), one derives

$$\|v^0(t)\|_{L^2}^2 \le C_0\|r^{(d-1)/2}v^0(t)\|_{L^2}^2 \le C_0, \qquad \|\tilde{u}(t)\|_{L^2}^2 + \int_0^t \|\tilde{u}\|_{H^1}^2 d\tau \le C_0, \tag{4.40}$$

where the constants C_0 depend on a,b and n and we have used the Poincaré inequality $\|\tilde{u}\|_{L^2}^2 \le C_0\|\tilde{u}_r\|_{L^2}^2$. Similarly, it follows from (4.40), Lemma 4.2 and (4.39) that

$$\|v^0(t)\|_{H^1}^2 + \|\tilde{u}(t)\|_{H^1}^2 + \int_0^t \big(\bar{u}\|(r^{d-1}v^0)_r\|_{L^2}^2 + \|\tilde{u}_t\|_{L^2}^2\big)d\tau \le C_0, \tag{4.41}$$

where the constant C_0 depends on a,b,n and we have also used the following fact:

$$\begin{aligned}\|v_r^0\|_{L^2}^2 =&\|r^{-(d-1)}[(r^{d-1}v^0)_r - (d-1)r^{d-2}v^0]\|_{L^2}^2\\ \le& a^{-2(d-2)}\|(r^{d-1}v^0)_r\|_{L^2}^2 + a^{-2(d-1)}(d-1)b^{2(d-2)}\|v^0\|_{L^2}^2.\end{aligned}$$

Thus, by applying (4.39) to (4.30) then using (4.41), we deduce the desired *a priori* estimate (4.4), which along with the fixed point theorem implies the existence of solution (u^0,v^0) in $C([0,\infty);H^2\times H^2)$. We next prove (4.5). Integrating (4.28) over $(0,\infty)$ with respect to t, then using (4.18), (4.19) and (4.22) we have

$$\begin{aligned}&\int_0^\infty \frac{d}{dt}\|r^{(d-1)/2}\tilde{u}_r\|_{L^2}^2 dt\\ \le& C_0\|r^{(d-1)/2}\tilde{u}_r\|_{L^2(0,\infty;L^2)}^2\Big(\|r^{(d-1)/2}v^0\|_{L^\infty(0,\infty;L^2)}^2 + \|(r^{d-1}v^0)_r\|_{L^\infty(0,\infty;L^2)}^2\Big)\\ &+ C_0\|(r^{d-1}v^0)_r\|_{L^2(0,\infty;L^2)}^2 \le C_0,\end{aligned}$$

which, along with (4.19) implies that $\|r^{(d-1)/2}\tilde{u}_r\|_{L^2}^2 \in W^{1,1}(0,\infty)$. Hence, it follows that

$$\lim_{t\to\infty}\|\tilde{u}_r\|_{L^2} = 0,$$

which, along with the Gagliardo-Nirenberg inequality $\|(u^0-\bar{u})(t)\|_{L^\infty}^2 \le C_0\|(u^0-\bar{u})(t)\|_{L^2}\|(u^0-\bar{u})_r(t)\|_{L^2}$ and (4.19) gives (4.5). The Part (i) of Theorem 4.1 is thus proved.

We proceed to prove the Part (ii). When $\bar{u}=0$, for $0<T<\infty$ one can easily deduce the *a priori* estimates (4.6) by the standard energy method that bootstraps the regularity of the solution (u^0,v^0) from L^2 to H^2. We omit this procedure for simplicity and refer readers to [44] for detail. Then the existence of solution (u^0,v^0) follows from (4.6) and the fixed point theorem. The proof is finished.

□

4.3 Proof of Theorem 4.2 and Proposition 4.1

Let $(u^\varepsilon,v^\varepsilon)$ and (u^0,v^0) be the solutions of (4.2)-(4.3) corresponding to $\varepsilon>0$ and $\varepsilon=0$ respectively. Then the initial-boundary problem for their differences $h := u^\varepsilon - u^0$, $w :=$

$v^\varepsilon - v^0$ satisfy:

$$\begin{cases} h_t = \frac{1}{r^{d-1}}(r^{d-1}h_r)_r + \frac{1}{r^{d-1}}(r^{d-1}hw)_r \\ \qquad + \frac{1}{r^{d-1}}(r^{d-1}u^0 w)_r + \frac{1}{r^{d-1}}(r^{d-1}hv^0)_r, \\ w_t = \varepsilon \frac{1}{r^{d-1}}(r^{d-1}w_r)_r - 2\varepsilon w w_r + h_r \\ \qquad + \varepsilon \frac{1}{r^{d-1}}(r^{d-1}v^0_r)_r - 2\varepsilon(w v^0_r + v^0 w_r + v^0 v^0_r) \\ \qquad - \varepsilon \frac{d-1}{r^2}(w+v^0), \qquad (r,t) \in (a,b) \times (0,\infty) \\ (h,w)(r,0) = (0,0), \\ h|_{r=a,b} = 0, \; w|_{r=a} = \bar{v}_1 - v^0(a,t), \; w|_{r=b} = \bar{v}_2 - v^0(b,t). \end{cases} \tag{4.42}$$

To prove Theorem 4.2 we shall invoke an elementary result (see Lemma 4.4) on an ordinary differential equation (ODE) and a series of lemmas on *a priori* estimates for the solutions of (4.42). In particular, the L^2-estimate for solution (h,w) and higher regularity estimate for the solution component h will be established in Lemma 4.5 - Lemma 4.8, and Lemma 4.9 will give a weighted L^2-estimate for the derivative of w.

We proceed with the following Lemma 4.4, which gives an upper bound for solution of an ODE involving a small parameter γ. It extends a result in [9, 85] for $k=2$ to any integer $k \geq 2$. To this end, we modify the proof of [9, 85].

Lemma 4.4. *Let $k \geq 2$ be an integer and $0 < T < \infty$. Let C_0 be a positive constant independent of T and $f_1(t)$, $f_2(t) \geq 0$ be two continuous functions on $[0,T]$. Consider the ODE*

$$\begin{cases} \frac{d}{dt}y(t) \leq \gamma f_1(t) + f_2(t)y(t) + C_0[y^2(t) + \cdots + y^k(t)], \\ y(0) = 0. \end{cases} \tag{4.43}$$

If we set

$$\gamma_0 = \min\left\{[4(k-1)]^{-1}\left(\int_0^T f_1(t)dt\right)^{-1}, [8TG(k-1)^2]^{-1}\left(\int_0^T f_1(t)dt\right)^{-1}\right\}, \tag{4.44}$$

with $G := C_0\left(e^{\int_0^T f_2(t)\,dt}\right)^{k-1}$. Then for every $\gamma \in (0,\gamma_0]$, the solution of (4.43) *satisfies for $t \in [0,t]$ that*

$$y(t) \leq e^{\int_0^T f_2(t)\,dt} \cdot \min\left\{3, \frac{3}{2T(k-1)G}, 12(k-1)\gamma \int_0^T f_1(t)\,dt\right\}. \tag{4.45}$$

Proof. Let $U(t) = y(t)e^{-\int_0^t f_2(\tau)\,d\tau}$. Then (4.43) can be rewritten as

$$\frac{d}{dt}U(t) \leq \gamma f_1(t)e^{-\int_0^t f_2(\tau)\,d\tau} + C_0\left(e^{\int_0^t f_2(\tau)\,d\tau}\right)U^2 + \cdots + C_0\left(e^{\int_0^t f_2(\tau)\,d\tau}\right)^{k-1}U^k,$$

from which, we deduce that

$$\begin{cases} \dfrac{d}{dt}U(t) \le \gamma f_1(t) + GU^2(t)(1+U(t))^{k-2}, \\ U(0) = 0. \end{cases} \tag{4.46}$$

For later use, we define

$$\sigma = \min\left\{ G,\ \frac{1}{4T^2(k-1)^2 G},\ 16(k-1)^2\gamma^2 G\left(\int_0^T f_1(t)\,dt\right)^2 \right\}. \tag{4.47}$$

Now dividing both sides of (4.46) by $\left(1+\sqrt{\frac{G}{\sigma}}\right)^k$, it follows that

$$\frac{\frac{d}{dt}U(t)}{\left(1+\sqrt{\frac{G}{\sigma}}U(t)\right)^k} \le \gamma f_1(t) + \frac{GU^2(t)}{\left(1+\sqrt{\frac{G}{\sigma}}U(t)\right)^2}\cdot\frac{(1+U(t))^{k-2}}{\left(1+\sqrt{\frac{G}{\sigma}}U(t)\right)^{k-2}}.$$

Then noting $\sigma \le G$ due to definition (4.47), we deduce from the above inequality that

$$\frac{\frac{d}{dt}U(t)}{\left(1+\sqrt{\frac{G}{\sigma}}U(t)\right)^k} \le \quad \gamma f_1(t) + \sigma,$$

which integrated over $(0,t)$ with $t \in (0,T]$ yields,

$$\begin{aligned} \frac{\sqrt{\frac{\sigma}{G}}}{k-1}\cdot\frac{1}{\left(1+\sqrt{\frac{G}{\sigma}}U(t)\right)^{k-1}} \ge & \frac{\sqrt{\frac{\sigma}{G}}}{k-1} - \sigma T - \gamma\int_0^T f_1(t)\,dt \\ \ge & \frac{\sqrt{\frac{\sigma}{G}}}{2(k-1)} - \gamma\int_0^T f_1(t)\,dt, \end{aligned} \tag{4.48}$$

where we have used the fact $\sigma T \le \frac{\sqrt{\frac{\sigma}{G}}}{2(k-1)}$, thanks to the definition of σ. We shall prove that

$$\gamma\int_0^T f_1(t)\,dt \le \frac{1}{4}\cdot\frac{\sqrt{\frac{\sigma}{G}}}{k-1}, \tag{4.49}$$

of which the proof is split into three cases by the values of σ.
Case 1, when $\sigma = G$, it follows from the definition of γ_0 in (4.44) that

$$\begin{aligned} \gamma\int_0^T f_1(t)\,dt \le & \gamma_0\int_0^T f_1(t)\,dt \\ \le & \frac{1}{4}\cdot\frac{\sqrt{\frac{\sigma}{G}}}{k-1}. \end{aligned}$$

Case 2, when $\sigma = \frac{1}{4T^2(k-1)^2G}$, we have by using (4.44) again

$$\gamma \int_0^T f_1(t)\,dt \le \gamma_0 \int_0^T f_1(t)\,dt \le \frac{1}{8TG(k-1)^2} = \frac{1}{4}\cdot\frac{\sqrt{\frac{\sigma}{G}}}{k-1}.$$

Case 3, when $\sigma = 16(k-1)^2\gamma^2 G\left(\int_0^T f_1(t)\,dt\right)^2$, one immediately get

$$\gamma \int_0^T f_1(t)\,dt = \frac{1}{4}\cdot\frac{\sqrt{\frac{\sigma}{G}}}{k-1}.$$

Hence combining the above Case 1 - Case 3, we conclude that (4.49) holds true and it follows from (4.49) and (4.48) that

$$\left(1+\sqrt{\frac{G}{\sigma}}U(t)\right)^{k-1} \le 4, \quad t\in[0,T]$$

thus

$$U(t) \le 3\sqrt{\frac{\sigma}{G}}, \quad t\in[0,T]$$

which, along with (4.47) and the definition of $U(t)$, yields the desired estimate (4.45). The proof is finished. □

In the sequel, for convenience we denote

$$\begin{aligned} E(t) &= \|r^{(d-1)/2}h(t)\|_{L^2}^2 + \|r^{(n-1)/2}w(t)\|_{L^2}^2 + \varepsilon\|r^{(d-1)/2}w_r(t)\|_{L^2}^2,\\ F(t) &= \|u^0(t)\|_{H^2}^2 + \|v^0(t)\|_{H^2} + \|v^0(t)\|_{H^2}^2 + \|v^0(t)\|_{H^2}^4 + |\bar v_1|^2 + |\bar v_2|^2 + 1. \end{aligned} \tag{4.50}$$

The following lemma gives the L^2-estimate for the solution (h,w) of problem (4.42).

Lemma 4.5. *Let $0<t<\infty$. Then there exists a constant C_0 independent of ε and t, such that*

$$\begin{aligned} &\frac{d}{dt}(\|r^{(d-1)/2}h(t)\|_{L^2}^2 + \|r^{(d-1)/2}w(t)\|_{L^2}^2) + \frac{3}{2}\|r^{(d-1)/2}h_r(t)\|_{L^2}^2\\ &\quad + 2\varepsilon\|r^{(d-1)/2}w_r(t)\|_{L^2}^2 + 2(d-1)\varepsilon\|r^{(d-3)/2}w(t)\|_{L^2}^2\\ &\le C_0\varepsilon^2F(t) + C_0F(t)E(t) + C_0E^2(t) + C_0E^3(t) + 2\varepsilon[r^{d-1}w_rw]|_a^b. \end{aligned} \tag{4.51}$$

Proof. Testing the first equation of (4.42) with $2r^{d-1}h$ in L^2 and using integration by parts, we get

$$\begin{aligned} \frac{d}{dt}\|r^{(d-1)/2}h\|_{L^2}^2 + 2\|r^{(d-1)/2}h_r\|_{L^2}^2 &= -2\int_a^b r^{d-1}hwh_r\,dr - 2\int_a^b r^{d-1}(u^0w+hv^0)h_r\,dr\\ &:= J_1 + J_2. \end{aligned}$$

To estimate J_1, we first note that for fixed $t>0$ if $f(r,t)\in H^1$ satisfying $f(r,t)|_{r=a,b}=0$ it follows that $f(r,t)^2=2\int_a^r ff_r dr\le 2\|f(t)\|_{L^2}\|f_r(t)\|_{L^2}$, which leads to

$$\|f(t)\|_{L^\infty}\le\sqrt{2}\|f(t)\|_{L^2}^{1/2}\|f_r(t)\|_{L^2}^{1/2}\quad\text{and}\quad\|f(t)\|_{L^\infty}\le C_0\|f_r(t)\|_{L^2},\tag{4.52}$$

thanks to the Poincaré inequality $\|f(t)\|_{L^2}\le C_0\|f_r(t)\|_{L^2}$. The estimate of J_1 then follows from (4.52) and (4.39):

$$\begin{aligned}J_1\le&C_0\|r^{(d-1)/2}h_r\|_{L^2}^{\frac{3}{2}}\|r^{(d-1)/2}h\|_{L^2}^{\frac{1}{2}}\|r^{(d-1)/2}w\|_{L^2}\\ \le&\frac{1}{8}\|r^{(d-1)/2}h_r\|_{L^2}^2+C_0\|r^{(d-1)/2}h\|_{L^2}^2\|r^{(d-1)/2}w\|_{L^2}^4.\end{aligned}$$

On the other hand, Sobolev embedding inequality and Cauchy-Schwarz inequality entail that

$$J_2\le\frac{1}{8}\|r^{(d-1)/2}h_r\|_{L^2}^2+C_0\|u^0\|_{H^1}^2\|r^{(d-1)/2}w\|_{L^2}^2+C_0\|v^0\|_{H^1}^2\|r^{(d-1)/2}h\|_{L^2}^2.$$

Finally collecting the above estimates for J_1 and J_2, we conclude that

$$\frac{d}{dt}\|r^{(d-1)/2}h(t)\|_{L^2}^2+\frac{7}{4}\|r^{(d-1)/2}h_r(t)\|_{L^2}^2\le C_0F(t)E(t)+C_0E^3(t).\tag{4.53}$$

We proceed by taking the L^2 inner product of the second equation of (4.42) with $2r^{d-1}w$ to get

$$\begin{aligned}&\frac{d}{dt}\|r^{(d-1)/2}w\|_{L^2}^2+2\varepsilon\|r^{(d-1)/2}w_r\|_{L^2}^2+2(d-1)\varepsilon\|r^{(d-3)/2}w\|_{L^2}^2\\ =&2\varepsilon[r^{d-1}w_rw]|_a^b-4\varepsilon\int_a^b r^{d-1}(ww_r+wv_r^0)wdr\\ &+2\int_a^b\left(r^{d-1}h_r+\varepsilon(r^{d-1}v_r^0)_r\right)wdr\\ &-2\varepsilon\int_a^b\left(2r^{d-1}v^0w_r+2r^{d-1}v^0v_r^0+(d-1)r^{d-3}v^0\right)wdr\\ :=&2\varepsilon[r^{d-1}w_rw]|_a^b+J_3+J_4+J_5.\end{aligned}$$

First Sobolev embedding inequality and (4.39) yield

$$\begin{aligned}J_3\le&4\varepsilon\|w\|_{L^\infty}\|r^{(d-1)/2}w_r\|_{L^2}\|r^{(d-1)/2}w\|_{L^2}\\ &+4\varepsilon\|v_r^0\|_{L^\infty}\|r^{(d-1)/2}w\|_{L^2}^2\\ \le&C_0\varepsilon\left(\|r^{(d-1)/2}w_r\|_{L^2}+\|r^{(d-1)/2}w\|_{L^2}\right)\|r^{(d-1)/2}w_r\|_{L^2}\|r^{(d-1)/2}w\|_{L^2}\\ &+C_0\varepsilon\|v^0\|_{H^2}\|r^{(d-1)/2}w\|_{L^2}^2.\end{aligned}$$

It follows from Cauchy-Schwarz inequality and (4.39) that

$$J_4\le\frac{1}{8}\|r^{(d-1)/2}h_r\|_{L^2}^2+C_0\|r^{(d-1)/2}w\|_{L^2}^2+C_0\varepsilon^2\|v^0\|_{H^2}^2.$$

Moreover Sobolev embedding inequality, Cauchy-Schwarz inequality and (4.39) lead to

$$\begin{aligned}J_5 \le& \varepsilon\|r^{(d-1)/2}w_r\|_{L^2}^2 + C_0\varepsilon\|v^0\|_{H^1}^2\|r^{(d-1)/2}w\|_{L^2}^2\\&+2\|r^{(d-1)/2}w\|_{L^2}^2 + C_0\varepsilon^2\|v^0\|_{H^2}^4 + C_0\varepsilon^2\|v^0\|_{L^2}^2.\end{aligned}$$

Collecting the above estimates for J_3-J_5 and recalling that $0<\varepsilon<1$, we end up with

$$\begin{aligned}&\frac{d}{dt}\|r^{(d-1)/2}w(t)\|_{L^2}^2 + 2\varepsilon\|r^{(d-1)/2}w_r(t)\|_{L^2}^2 + 2(d-1)\varepsilon\|r^{(d-3)/2}w(t)\|_{L^2}^2\\ \le& 2\varepsilon[r^{d-1}w_rw]|_a^b + \frac{1}{8}\|r^{(d-1)/2}h_r(t)\|_{L^2}^2 + C_0\varepsilon^2F(t) + C_0F(t)E(t) + C_0E^2(t),\end{aligned}$$

which, adding to (4.53) gives the desired estimate and finishes the proof.

□

We turn to estimate the derivative of w and the boundary term in (4.51).

Lemma 4.6. *Let $0<t<\infty$. Then there exist constants C_0 independent of ε and t, such that*

$$\begin{aligned}&\frac{d}{dt}\left(\varepsilon\|r^{(d-1)/2}w_r(t)\|_{L^2}^2\right) + \frac{1}{2}\|r^{(d-1)/2}w_t(t)\|_{L^2}^2\\ \le& C_0\varepsilon^2F(t) + C_0F(t)E(t) + C_0E^2(t)\\&+\|r^{(d-1)/2}h_r(t)\|_{L^2}^2 + 2\varepsilon[r^{d-1}w_rw_t]|_a^b\end{aligned} \tag{4.54}$$

and

$$\begin{aligned}&2\varepsilon[r^{d-1}w_rw]|_a^b + 2\varepsilon[r^{d-1}w_rw_t]|_a^b\\ \le& C_0\varepsilon^{1/2}F(t) + C_0F(t)E(t) + C_0E^2(t)\\&+\frac{1}{8}\|r^{(d-1)/2}w_t(t)\|_{L^2}^2 + \frac{1}{8}\|r^{(d-1)/2}h_r(t)\|_{L^2}^2.\end{aligned} \tag{4.55}$$

Proof. Taking the L^2 inner product of the second equation of (4.42) with $2r^{d-1}w_t$, then using integration by parts we have

$$\begin{aligned}&\frac{d}{dt}\varepsilon\|r^{(d-1)/2}w_r(t)\|_{L^2}^2 + 2\|r^{(d-1)/2}w_t(t)\|_{L^2}^2\\ =& 2\varepsilon[r^{d-1}w_rw_t]|_a^b - 4\varepsilon\int_a^b r^{d-1}ww_rw_t\,dr - 4\varepsilon\int_a^b r^{d-1}(wv_r^0 + v^0w_r + v^0v_r^0)w_t\,dr\\&+2\int_a^b\left[r^{d-1}h_r + \varepsilon(r^{d-1}v_r^0)_r - \varepsilon(d-1)r^{d-3}w - \varepsilon(d-1)r^{d-3}v_0\right]w_t\,dr\\ :=& 2\varepsilon[r^{d-1}w_rw_t]|_a^b + J_6 + J_7 + J_8.\end{aligned}$$

We first apply Sobolev embedding inequality and (4.39) to deduce that

$$\begin{aligned}J_6 \le& C_0\varepsilon\|w\|_{H^1}\|r^{(d-1)/2}w_r\|_{L^2}\|r^{(d-1)/2}w_t\|_{L^2}\\ \le& \frac{1}{8}\|r^{(d-1)/2}w_t\|_{L^2}^2 + C_0\varepsilon^2\left(\|r^{(d-1)/2}w_r\|_{L^2}^2 + \|r^{(d-1)/2}w\|_{L^2}^2\right)\|r^{(d-1)/2}w_r\|_{L^2}^2\end{aligned}$$

and that

$$J_7 \le \frac{1}{8}\|r^{(d-1)/2}w_t\|_{L^2}^2 + C_0\varepsilon^2\big(\|v^0\|_{H^2}^2\|r^{(d-1)/2}w\|_{L^2}^2 + \|v^0\|_{H^1}^2\|r^{(d-1)/2}w_r\|_{L^2}^2 + \|v^0\|_{H^1}^4\big).$$

Moreover Cauchy-Schwarz inequality entails that

$$J_8 \le \frac{9}{8}\|r^{(d-1)/2}w_t\|_{L^2}^2 + \|r^{(d-1)/2}h_r\|_{L^2}^2 + C_0\varepsilon^2(\|v^0\|_{H^2}^2 + \|r^{(d-1)/2}w\|_{L^2}^2).$$

Then (4.54) follows from the above estimates on J_6-J_8. It remains to prove (4.55). By the definition of w and Gagliardo-Nirenberg interpolation inequality, one deduces that

$$\begin{aligned}
2\varepsilon[r^{d-1}w_r w]|_a^b \le & C_0\varepsilon\|w_r\|_{L^\infty}(|\bar v_1| + |\bar v_2| + \|v^0\|_{L^\infty}) \\
\le & C_0\varepsilon(\|w_r\|_{L^2}^{\frac12}\|w_{rr}\|_{L^2}^{\frac12} + \|w_r\|_{L^2})(|\bar v_1| + |\bar v_2| + \|v^0\|_{H^1}) \\
\le & \zeta\varepsilon^2\|w_{rr}\|_{L^2}^2 + C_0(1+1/\zeta)\varepsilon\|w_r\|_{L^2}^2 \\
& + C_0(\varepsilon^{1/2}+\varepsilon)(|\bar v_1| + |\bar v_2| + \|v^0\|_{H^1})^2,
\end{aligned} \tag{4.56}$$

where ζ is a small constant to be determined. By a similar argument as deriving (4.56) and the second equation of (4.2) with $\varepsilon = 0$, we further get

$$\begin{aligned}
2\varepsilon[r^{d-1}w_r w_t]|_a^b \le & \zeta\varepsilon^2\|w_{rr}\|_{L^2}^2 + C_0(1+1/\zeta)\varepsilon\|w_r\|_{L^2}^2 + C_0(\varepsilon^{1/2}+\varepsilon)\|v_t^0\|_{H^1}^2 \\
\le & \zeta\varepsilon^2\|w_{rr}\|_{L^2}^2 + C_0(1+1/\zeta)\varepsilon\|w_r\|_{L^2}^2 + C_0(\varepsilon^{1/2}+\varepsilon)\|u^0\|_{H^2}^2.
\end{aligned} \tag{4.57}$$

To bound the term $\|w_{rr}\|_{L^2}^2$ in the above two inequalities, we use the second equation of (4.42) and Sobolev embedding inequality and derive

$$\begin{aligned}
\varepsilon^2\|w_{rr}\|_{L^2}^2 \le C_1\big(&\varepsilon^2\|r^{(d-1)/2}w\|_{L^2}^2\|v^0\|_{H^2}^2 + \varepsilon^2\|r^{(d-1)/2}w\|_{L^2}^2\|r^{(d-1)/2}w_r\|_{L^2}^2 \\
&+ \|r^{(d-1)/2}w_t\|_{L^2}^2 + \varepsilon^2\|r^{(d-1)/2}w_r\|_{L^2}^2 + \|r^{(d-1)/2}h_r\|_{L^2}^2 \\
&+ \varepsilon^2\|r^{(d-1)/2}w\|_{L^2}^2 + \varepsilon^2\|r^{(d-1)/2}w_r\|_{L^2}^2\|v^0\|_{H^1}^2 \\
&+ \varepsilon^2\|r^{(d-1)/2}w_r\|_{L^2}^4 + \varepsilon^2\|v^0\|_{H^1}^4 + \varepsilon^2\|v^0\|_{H^2}^2\big),
\end{aligned} \tag{4.58}$$

where we have used the notation C_1 to distinguish it from the constant C_0 in (4.56)-(4.57). Finally feeding (4.56) and (4.57) on (4.58) then adding the results, we obtain (4.55) by taking small ζ so that $C_1\zeta < \frac{1}{16}$ and by using $0 < \varepsilon < 1$. The proof is completed.

□

We next apply Lemma 4.4 to the combination of Lemma 4.5 and Lemma 4.6 to obtain the following result.

Lemma 4.7. *Let $0 < T < \infty$ and $\varepsilon \in (0, \varepsilon_T]$ with ε_T defined in Theorem 4.2. Then there exists a constant C independent of ε, depending on T such that*

$$\begin{aligned}
&\|h\|_{L^\infty(0,T;L^2)}^2 + \|w\|_{L^\infty(0,T;L^2)}^2 \\
&\quad + \varepsilon\|w_r\|_{L^\infty(0,T;L^2)}^2 + \|h_r\|_{L^2(0,T;L^2)}^2 + \|w_t\|_{L^2(0,T;L^2)}^2 \le C\varepsilon^{\frac12}.
\end{aligned} \tag{4.59}$$

Proof. We first add (4.54) and (4.55) to (4.51) and find

$$\begin{aligned}&\frac{d}{dt}E(t)+\frac{1}{4}\|r^{(d-1)/2}h_r(t)\|_{L^2}^2+2(n-1)\varepsilon\|r^{(d-3)/2}w(t)\|_{L^2}^2+\frac{1}{4}\|r^{(d-1)/2}w_t(t)\|_{L^2}^2\\&\le C_0\varepsilon^{\frac{1}{2}}F(t)+C_0F(t)E(t)+C_0E^2(t)+C_0E^3(t),\end{aligned}\tag{4.60}$$

where $0<\varepsilon<1$ has been used. Then we apply lemma 4.4 to (4.60) by taking $k=3$, $\gamma=\varepsilon^{1/2}$ and $f_1(t)=f_2(t)=C_0F(t)$ to conclude for $\varepsilon\in(0,\varepsilon_T]$ that

$$\begin{aligned}&\|h\|_{L^\infty(0,T;L^2)}^2+\|w\|_{L^\infty(0,T;L^2)}^2+\varepsilon\|w_r\|_{L^\infty(0,T;L^2)}^2\\&\le\left(C_0e^{C_0\int_0^T F(t)\,dt}\int_0^T F(t)dt\right)\varepsilon^{\frac{1}{2}},\end{aligned}\tag{4.61}$$

where (4.39) has been used. Then we integrate (4.60) over (0,T) and applying (4.61) to the result to deduce that

$$\|h\|_{L^\infty(0,T;L^2)}^2+\|w\|_{L^\infty(0,T;L^2)}^2+\varepsilon\|w_r\|_{L^\infty(0,T;L^2)}^2+\|h_r\|_{L^2(0,T;L^2)}^2+\|w_t\|_{L^2(0,T;L^2)}^2\le C\varepsilon^{\frac{1}{2}},$$

where the constant C which depends on T and $\int_0^T F(t)dt$ is finite thanks to Theorem 4.1. The proof is completed. □

Higher regularity estimate for the solution component h is given in the following lemma.

Lemma 4.8. *Suppose $0<T<\infty$ and $\varepsilon\in(0,\varepsilon_T]$. Then there exists a constant C independent of ε,depending on T such that*

$$\|h_r\|_{L^\infty(0,T;L^2)}^2+\|h_t\|_{L^\infty(0,T;L^2)}^2+\|h_{rt}\|_{L^2(0,T;L^2)}^2\le C\varepsilon^{1/2}.\tag{4.62}$$

Proof. We first take the L^2 inner product of the first equation of (4.42) with $2r^{d-1}h_t$ and use integration by parts to get

$$\begin{aligned}&\frac{d}{dt}\|r^{(d-1)/2}h_r\|_{L^2}^2+2\|r^{(d-1)/2}h_t\|_{L^2}^2\\&=-2\int_a^b r^{d-1}(hw+u^0w+hv^0)h_{rt}dr\\&\le\frac{1}{2}\|r^{(d-1)/2}h_{rt}\|_{L^2}^2+C_0(\|h_r\|_{L^2}^2\|w\|_{L^2}^2+\|u^0\|_{H^1}^2\|w\|_{L^2}^2+\|h_r\|_{L^2}^2\|v^0\|_{L^2}^2).\end{aligned}\tag{4.63}$$

Then differentiating the first equation of (4.42) with respect to t and multiplying the resulting equation with $2r^{d-1}h_t$ in L^2, we derive

$$\begin{aligned}&\frac{d}{dt}\|r^{(d-1)/2}h_t\|_{L^2}^2+2\|r^{(d-1)/2}h_{rt}\|_{L^2}^2\\&=-2\int_a^b r^{d-1}h_twh_{rt}dr-2\int_a^b r^{d-1}\left(hw_t+u_t^0w+u^0w_t+h_tv^0+hv_t^0\right)h_{rt}dr\\&:=K_1+K_2.\end{aligned}$$

The estimate for K_1 follows from (4.52) and (4.39):

$$
\begin{aligned}
K_1 \leq & C_0\|r^{(d-1)/2}h_{rt}\|_{L^2}^{3/2}\|r^{(d-1)/2}h_t\|_{L^2}^{1/2}\|w\|_{L^2} \\
\leq & \frac{1}{4}\|r^{(d-1)/2}h_{rt}\|_{L^2}^2 + C_0\|r^{(d-1)/2}h_t\|_{L^2}^2\|w\|_{L^2}^4.
\end{aligned}
$$

Moreover, Sobolev embedding inequality and (4.39) entail that

$$
\begin{aligned}
K_2 \leq & \frac{1}{4}\|r^{(d-1)/2}h_{rt}\|_{L^2}^2 + C_0\Big(\|r^{(d-1)/2}h_r\|_{L^2}^2\|w_t\|_{L^2}^2 + \|u_t^0\|_{H^1}^2\|w\|_{L^2}^2\Big) \\
& + C_0\Big(\|u^0\|_{H^1}^2\|w_t\|_{L^2}^2 + \|v^0\|_{H^1}^2\|r^{(d-1)/2}h_t\|_{L^2}^2 + \|v_t^0\|_{H^1}^2\|h\|_{L^2}^2\Big)
\end{aligned}
$$

Then collecting the above estimates for K_1-K_2, one derives

$$
\begin{aligned}
& \frac{d}{dt}\|r^{(d-1)/2}h_t\|_{L^2}^2 + \frac{3}{2}\|r^{(d-1)/2}h_{rt}\|_{L^2}^2 \\
\leq & C_0(\|w\|_{L^2}^4 + \|v^0\|_{H^1}^2)\|r^{(d-1)/2}h_t\|_{L^2}^2 + C_0\|w_t\|_{L^2}^2\|r^{(d-1)/2}h_r\|_{L^2}^2 \\
& + C_0(\|u^0\|_{H^1}^2\|w_t\|_{L^2}^2 + \|u^0\|_{H^2}^2\|h\|_{L^2}^2) + C_0(\|u^0\|_{H^3}^2 + \|u^0\|_{H^2}^2\|v^0\|_{H^2}^2)\|w\|_{L^2}^2,
\end{aligned}
\tag{4.64}
$$

where we have used inequalities $\|u_t^0\|_{H^1}^2 \leq C_0(\|u^0\|_{H^3}^2 + \|u^0\|_{H^2}^2\|v^0\|_{H^2}^2)$ and $\|v_t^0\|_{H^1}^2 \leq C_0\|u^0\|_{H^2}^2$, thanks to the first and second equations of (4.42) with $\varepsilon = 0$. Finally by adding (4.64) to (4.63) and applying Gronwall's inequality to the result, then using (4.59) we obtain (4.62). The proof is completed. □

We turn to establish a weighted L^2-estimate (enlightened by [30]) on the derivative of solution component w.

Lemma 4.9. *For $0 < T < \infty$ and $\varepsilon \in (0, \varepsilon_T]$, there exists a constant C independent of ε, depending on T such that*

$$
\|(r-a)(r-b)w_r\|_{L^\infty(0,T;L^2)}^2 + \varepsilon\|(r-a)(r-b)w_{rr}\|_{L^2(0,T;L^2)}^2 \leq C\varepsilon^{1/2}.
$$

Proof. Taking the L^2 inner product of the second equation of (4.42) with $-2(r-a)^2(r-b)^2 w_{rr}$ and using integration by parts to get

$$
\begin{aligned}
& \frac{d}{dt}\|(r-a)(r-b)w_r\|_{L^2}^2 + 2\varepsilon\|(r-a)(r-b)w_{rr}\|_{L^2}^2 \\
= & -2\varepsilon\int_a^b (r-a)^2(r-b)^2 w_{rr}\Big[\frac{1}{r^{d-1}}(r^{d-1}v_r^0)_r - \frac{d-1}{r^2}(w+v_0)\Big]dr \\
& + 4\varepsilon\int_a^b (r-a)^2(r-b)^2 w_{rr}(wv_r^0 + v^0 w_r + v^0 v_r^0)dr \\
& - 4\int_a^b (2r-a-b)(r-a)(r-b)w_r w_t dr - 2\int_a^b (r-a)^2(r-b)^2 w_{rr}h_r dr \\
& - 2\varepsilon\int_a^b (r-a)^2(r-b)^2 w_{rr}\Big(\frac{d-1}{r}w_r - 2ww_r\Big)dr \\
:= & \sum_{i=3}^{7} K_i.
\end{aligned}
\tag{4.65}
$$

We next estimate K_3-K_7. Indeed Cauchy-Schwarz inequality, Sobolev embedding inequality and (4.39) yield

$$\begin{aligned} K_3+K_4 \le & \frac{1}{8}\varepsilon\|(r-a)(r-b)w_{rr}\|_{L^2}^2 \\ & + C_0\varepsilon(\|v^0\|_{H^2}^2+\|w\|_{L^2}^2+\|v^0\|_{H^2}^2\|w\|_{L^2}^2+\|v^0\|_{H^1}^2\|w_r\|_{L^2}^2+\|v^0\|_{H^1}^4) \end{aligned}$$

and

$$\begin{aligned} K_5+K_7 \le & \frac{1}{8}\varepsilon\|(r-a)(r-b)w_{rr}\|_{L^2}^2+C_0\|w_t\|_{L^2}^2 \\ & + C_0(1+\varepsilon+\varepsilon\|w\|_{L^2}^2+\varepsilon\|w_r\|_{L^2}^2)\|(r-a)(r-b)w_r\|_{L^2}^2. \end{aligned}$$

For the term K_6, we use integration by parts and the first equation of (4.42) to get

$$\begin{aligned} K_6 =& 4\int_a^b (2r-a-b)(r-a)(r-b)w_r h_r dr + 2\int_a^b (r-a)^2(r-b)^2 w_r h_{rr} dr \\ =& 4\int_a^b (2r-a-b)(r-a)(r-b)w_r h_r dr \\ & + 2\int_a^b (r-a)^2(r-b)^2 w_r\Big(h_t-\frac{d-1}{r}h_r\Big)dr \\ & - 2\int_a^b (r-a)^2(r-b)^2 w_r(hw+u^0w)_r dr \\ & - 2\int_a^b (r-a)^2(r-b)^2 w_r\Big[(hv^0)_r+\frac{d-1}{r}(hw+u^0w+hv^0)\Big]dr \\ :=& R_1+R_2+R_3+R_4. \end{aligned}$$

We proceed to estimate R_1-R_4. First it follows from Cauchy-Schwarz inequality that

$$R_1+R_2 \le 2\|(r-a)(r-b)w_r\|_{L^2}^2+C_0(\|h_r\|_{L^2}^2+\|h_t\|_{L^2}^2).$$

Moveover, we use Cauchy-Schwarz inequality and apply (4.52) to h and $(r-a)(r-b)w$ to derive

$$\begin{aligned} R_3 \le & 2(\|h\|_{L^\infty}+\|u^0\|_{L^\infty})\|(r-a)(r-b)w_r\|_{L^2}^2 \\ & + 2(\|h_r\|_{L^2}+\|u_r^0\|_{L^2})\|(r-a)(r-b)w_r\|_{L^2}\|(r-a)(r-b)w\|_{L^\infty} \\ \le & C_0(\|h_r\|_{L^2}+\|u^0\|_{H^1})\|(r-a)(r-b)w_r\|_{L^2}^2 \\ & + C_0(\|h_r\|_{L^2}+\|u_r^0\|_{L^2})\|(r-a)(r-b)w_r\|_{L^2}\|[(r-a)(r-b)w]_r\|_{L^2} \\ \le & C_0(\|h_r\|_{L^2}+\|u^0\|_{H^1}+1)\|(r-a)(r-b)w_r\|_{L^2}^2+C_0(\|h_r\|_{L^2}^2+\|u^0\|_{H^1}^2)\|w\|_{L^2}^2. \end{aligned}$$

The estimate for R_4 follows from the Sobolev embedding inequality, (4.52) and Cauchy-Schwarz inequality:

$$\begin{aligned} R_4 \le & C_0\|(r-a)(r-b)w_r\|_{L^2}(\|hw\|_{L^2}+\|u^0w\|_{L^2}+\|hv^0\|_{H^1}) \\ \le & \|(r-a)(r-b)w_r\|_{L^2}^2+C_0(\|h_r\|_{L^2}^2\|w\|_{L^2}^2+\|u^0\|_{H^1}^2\|w\|_{L^2}^2+\|h_r\|_{L^2}^2\|v^0\|_{H^1}^2). \end{aligned}$$

We thus conclude from the above estimates for R_1-R_4 that

$$\begin{aligned}K_6 \leq & C_0(\|h_r\|_{L^2}+\|u^0\|_{H^1}+1)\|(r-a)(r-b)w_r\|_{L^2}^2\\ &+C_0(\|h_r\|_{L^2}^2+\|h_t\|_{L^2}^2+\|h_r\|_{L^2}^2\|w\|_{L^2}^2+\|u^0\|_{H^1}^2\|w\|_{L^2}^2+\|h_r\|_{L^2}^2\|v^0\|_{H^1}^2).\end{aligned}$$

Substituting the above estimates for K_3-K_7 into (4.65), then applying Gronwall's inequality, Lemma 4.7- Lemma 4.8 and Theorem 4.1 to the result, we obtain the desired estimate and complete the proof.

□

By a similar argument as the proof of Theorem 2.2, one can prove Theorem 4.2 based on the results derived in Lemma 4.7 - Lemma 4.9. For the sake of self-containedness and reader's convenience we briefly sketch the proof.

Proof of Theorem 4.2. By Lemma 4.7 - Lemma 4.8 and Sobolev embedding inequality, we deduce that

$$\|u^\varepsilon-u^0\|_{L^\infty(0,T;C[a,b])} \leq C_0\|u^\varepsilon-u^0\|_{L^\infty(0,T;H^1)} \leq C\varepsilon^{1/4},$$

which proves (4.8). Clearly $\delta^2 \leq \frac{4}{(b-a)^2}(r-a)^2(r-b)^2$ holds for $\delta < \frac{b+a}{2}$ and $r \in (a,b)$, thus it follows from Lemma 4.9 that

$$\delta^2\int_{a+\delta}^{b-\delta} w_r^2(r,t)\,dr \leq \frac{4}{(b-a)^2}\int_{a+\delta}^{b-\delta}(r-a)^2(r-b)^2w_r^2(r,t)\,dr \leq C\varepsilon^{1/2}, \quad t\in[0,T]$$

which, along with Lemma 4.7 and Gagliardo-Nirenberg inequality entails that

$$\begin{aligned}\|v^\varepsilon-v^0\|_{L^\infty(0,T;C[a+\delta,b-\delta])} \leq & C_0\big(\|w\|_{L^\infty(0,T;L^2(a+\delta,b-\delta))}\\ &+\|w\|^{1/2}_{L^\infty(0,T;L^2(a+\delta,b-\delta))}\|w_r\|^{1/2}_{L^\infty(0,T;L^2(a+\delta,b-\delta))}\big)\\ \leq & C\big(\varepsilon^{1/4}+\varepsilon^{1/8}\cdot\varepsilon^{1/8}\delta^{-1/2}\big)\\ \leq & C\varepsilon^{1/4}\delta^{-1/2},\end{aligned}$$

provided $\delta < 1$.

Hence we derive (4.9) and we next prove the equivalence between (4.10) and (4.11). Assume $\int_0^{t_0} u_r^0(a,\tau)\,d\tau \neq 0$ for some $t_0 \in [0,T]$. Then integrating the second equation (4.2) over $(0,t_0)$ along with compatible condition $\bar{v}_1 = v_0(a)$ gives

$$v^0(a,t_0) = \bar{v}_1 + \int_0^{t_0} u_r^0(a,\tau)\,d\tau. \tag{4.66}$$

We thus have

$$\liminf_{\varepsilon\to 0}\|v^\varepsilon-v^0\|_{L^\infty(0,T;C[a,b])} \geq \liminf_{\varepsilon\to 0}|\bar{v}_1-v^0(a,t_0)| > 0.$$

Similar arguments lead to (4.10) when assuming that $\int_0^{t_0} u_r^0(b,\tau)\,d\tau \neq 0$ for some $t_0 \in [0,T]$. We thus proved that (4.11) implies (4.10). The proof of (4.10) $\Rightarrow$ (4.11) will follow from the argument of contradiction. Indeed, if we assume that (4.10) holds and the opposite of (4.11) holds, that is

$$\int_0^t u_r^0(a,\tau)\,d\tau = 0 \quad \text{and} \quad \int_0^t u_r^0(b,\tau)\,d\tau = 0, \quad \text{for all } t \in [0,T],$$

which, along with (4.66) leads to $w|_{r=a,b} = (v^\varepsilon - v^0)|_{r=a,b} = 0$ and $w_t|_{r=a,b} = [(v^\varepsilon - v^0)|_{r=a,b}]_t = 0$. Thus the terms $2\varepsilon[r^{d-1}w_r w]|_a^b$, $2\varepsilon[r^{d-1}w_r w_t]|_a^b$ in (4.51) and (4.54) vanish and by similar arguments as deriving (4.61), we conclude that

$$\|h\|^2_{L^\infty(0,T;L^2)} + \|w\|^2_{L^\infty(0,T;L^2)} + \varepsilon\|w_r\|^2_{L^\infty(0,T;L^2)} \le C\varepsilon^2.$$

Applying Sobolev embedding inequality gives rise to

$$\lim_{\varepsilon\to 0}\|v^\varepsilon - v^0\|_{L^\infty(0,T;C[a,b])} \le C_0 \lim_{\varepsilon\to 0}(\|w\|_{L^\infty(0,T;L^2)} + \|w_r\|_{L^\infty(0,T;L^2)}) = 0,$$

which contradicts with (4.10), thus (4.10) implies (4.11) and they are equivalent. The proof is completed. $\square$

The proof for Proposition 4.1 is partially similar to the proof of [29, Proposition 2.8.], we shall mainly detail the proof (4.15) and (4.16) (which are newly derived estimates compared to [29]), and omit the similar part for brevity.

Proof of Proposition 4.1. By the argument used in the proof of [29, Proposition 2.8.], we derive from the second equation of (4.13) that

$$\begin{aligned} c^\varepsilon(r,t) &= c_0(r)\exp\left\{\int_0^t \left[-u^\varepsilon + \varepsilon(v^\varepsilon)^2 - v_x^\varepsilon - \varepsilon\frac{d-1}{r}v^\varepsilon\right]d\tau\right\}, \\ c^0(r,t) &= c_0(r)\exp\left\{-\int_0^t u^0 d\tau\right\}, \end{aligned} \tag{4.67}$$

with $v^\varepsilon = -\frac{c_r^\varepsilon}{c^\varepsilon}$. By a similar argument used in proof of [29, Proposition 2.8.], one derives (4.14) from (4.67). We proceed to prove (4.15). First, by the assumption $v_0 = -(\ln c_0)_r$ of Proposition 4.1 we deduce that

$$c_0(r) = c_0(a)e^{-\int_a^r v_0(s)ds} = c_0(b)e^{-\int_b^r v_0(s)ds}, \tag{4.68}$$

which implies that $c_0(a) \neq 0$, $c_0(b) \neq 0$ since otherwise it follows from (4.68) that $c_0(r) \equiv 0$ for $r \in [a,b]$, which contradicts with the assumption $\ln c_0 \in H^3$. Thus it follows from (4.68)

$$c_0(r) > 0, \qquad r \in [a,b].$$

From (4.68), (4.67) and (4.14) we conclude there exists $\gamma > 0$ depending on T such that

$$c^0(r,t) > \gamma,\ c^\varepsilon(r,t) > \gamma \quad \text{for } (r,t) \in [a,b] \times [0,T] \tag{4.69}$$

and for ε small enough. On the other hand, by the transformation $v = -\frac{c_r}{c}$ we derive

$$c_r^\varepsilon - c_r^0 = (v^\varepsilon - v^0)c^\varepsilon + v^0(c^\varepsilon - c^0), \tag{4.70}$$

which, in conjunction with (4.9) and (4.14) that

$$\begin{aligned}\|c_r^\varepsilon - c_r^0\|_{L^\infty(0,T;C[a+\delta,b-\delta])} \leq & \|v^\varepsilon - v^0\|_{L^\infty(0,T;C[a+\delta,b-\delta])}(\|c^0\|_{L^\infty(0,T;[a,b])} + C\varepsilon^{1/4}) \\ & + C\varepsilon^{1/4}\|c^0\|_{L^\infty(0,T;[a,b])} \\ \leq & C\varepsilon^{1/4}\delta^{-1/2},\end{aligned}$$

where $\delta < 1$ have been used thanks to $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. We thus derived (4.15). It remains to prove that (4.16) is equivalent to (4.11). First, it follows from (4.14) and (4.70) that

$$\begin{aligned}\liminf_{\varepsilon\to 0}\|c_r^\varepsilon - c_r^0\|_{L^\infty(0,T;C[a,b])} \geq & \|c^0\|_{L^\infty(0,T;C[a,b])} \liminf_{\varepsilon\to 0}\|v^\varepsilon - v^0\|_{L^\infty(0,T;C[a,b])} \\ \geq & C \liminf_{\varepsilon\to 0}\|v^\varepsilon - v^0\|_{L^\infty(0,T;C[a,b])}.\end{aligned} \tag{4.71}$$

Dividing (4.70) by c^ε then applying a similar argument as in (4.71), along with (4.69) and (4.14), one deduces that

$$\liminf_{\varepsilon\to 0}\|v^\varepsilon - v^0\|_{L^\infty(0,T;C[a,b])} \geq \liminf_{\varepsilon\to 0}\|c_r^\varepsilon - c_r^0\|_{L^\infty(0,T;C[a,b])},$$

which, in conjunction with (4.71) indicates the equivalence between (4.16) and (4.10). Then we conclude that (4.16) is equivalent to (4.11) by using Theorem 4.2. The proof is completed.

□

Chapter 5

Stability of Boundary Layers in a Half Plane

In the previous chapter, we have studied the multi-dimensional boundary layer problem of system (1.4) on its radial solutions, which is intrinsically still a one-dimensional problem though some additional challenges in analysis have been encountered compared to the one-dimensional case. Nevertheless, the result on radial solutions derived in Chapter 4 indicates that the boundary layer singularity indeed exists for system (1.4) in multi-dimensions. This chapter will proceed to investigate the boundary layer problem of (1.4) in two dimensions, which pertains to more realistic situations (cf. [78]).

Due to the special structure of (1.4), there are several essential differences between one and multi-dimensions as to be mentioned below. From the Cole-Hopf transformation (1.3), the curl for $\vec{v}$ must be intrinsically free:

$$\nabla \times \vec{v} = 0 \tag{5.1}$$

which implies that $\nabla|\vec{v}|^2 = 2\vec{v} \cdot \nabla\vec{v}$. Then the second equation of (1.4) becomes $\vec{v}_t + 2\varepsilon\vec{v} \cdot \nabla\vec{v} - \nabla u = \varepsilon\Delta\vec{v}$, which is surprisingly analogous to the incompressible Navier-Stokes (INS) equations (1.5) (see Chapter 1) by putting $\vec{w} = \vec{v}$ and $p = -u$. As aforementioned in Section 1.2 (see Chapter 1), it is well-known that the inviscid limit of the INS equations will generate boundary layers if the physical boundary conditions (1.6) are prescribed. However, the convergence of solutions of the INS equations to its limiting Euler equations (namely (1.5) with $\varepsilon = 0$), in two or higher dimensions as ε vanishes still remains unjustified due to the appearance of (degenerate) Prandtl's boundary layer equations (see [71]) whose well-posedness in Sobolev spaces is open except for analytic or monotonic data [1, 15, 20, 53, 65]. As such, due to the analogy between (1.4) and the INS equations, a natural concern is whether the KS system (1.4) with Dirichlet boundary conditions in multi-dimensions will generate similar Prandtl's boundary layers making the vanishing limit problem as $\varepsilon \to 0$ unverifiable? This question does not exist in one dimension but must be first elucidated in higher dimensions (see more details in the end of this section) before taking the next step. Moreover the system (1.4) is invariant under the scaling for any $\lambda > 0$:

$u_\lambda(\vec{x},t) = \lambda^2 u(\lambda\vec{x},\lambda^2 t)$, $\vec{v}_\lambda(\vec{x},t) = \lambda\vec{v}(\lambda\vec{x},\lambda^2 t)$ which indicates that the critical space dimension of (1.4) in the framework of Sobolev spaces is $N = 2$, and $N = 3$ is supercritical while $N = 1$ is subcritical, same as the Navier-Stokes equations (see [13]). But analysis of (1.4) is somewhat more difficult than the INS equations due to the lack of the divergence-free condition which is critical for the existence of large solutions to the INS equations in two dimensions (e.g. see [17, 58]). Indeed, although large-data solutions of (1.4) in one dimension have been obtained, none of the large-data solutions has been obtained in the multi-dimensions so far even for the critical space dimension $N = 2$ (cf.[69, 82]). This is the second difference from the one-dimensional case. Thirdly, in order to preserve the curl-free condition (5.1) so that the results of (1.4) can be transferred to the original Keller-Segel system (1.2), the condition (5.1) has to be taken into account when prescribing boundary conditions. However no such concern is needed in one dimension.

Bearing these structural differences between one and multi-dimensions, in this chapter, we shall exploit the inviscid limit and boundary layers for the system (1.4) with Dirichlet boundary conditions in two dimensions. For simplicity, we consider the problem in the half plane $\Omega = \{\vec{x} = (x,y) \in \mathbb{R}^2 \mid y > 0\}$ and hence $\partial\Omega = \{(x,y) \in \mathbb{R}^2 \mid y = 0\}$. For illustration, we rewrite (1.4) as follows

$$\begin{cases} u_t - \nabla\cdot(u\vec{v}) = \Delta u, \quad (x,y,t) \in \mathbb{R}\times\mathbb{R}_+\times(0,\infty) \\ \vec{v}_t + \nabla(\varepsilon|\vec{v}|^2 - u) = \varepsilon\Delta\vec{v}, \\ (u,\vec{v})(x,y,0) = (u_0,\vec{v}_0)(x,y). \end{cases} \tag{5.2}$$

In the sequel, let $\vec{v} = (v_1,v_2)$ and hence $\nabla\times\vec{v} = \partial_x v_2 - \partial_y v_1$. Taking the curl on both sides of the second equation of (5.2), one can get $\partial_t(\nabla\times\vec{v}) = \varepsilon\Delta(\nabla\times\vec{v})$. This indicates that to preserve the curl-free condition $\nabla\times\vec{v} = 0$ which is an intrinsic requirement from (1.3), except that the initial condition is required to satisfy $\nabla\times\vec{v}_0 = 0$, we also need the condition $\nabla\times\vec{v}|_{\partial\Omega} = (\partial_x v_2 - \partial_y v_1)|_{\partial\Omega} = 0$ for $\varepsilon > 0$. Thanks to this curl free condition, the physical boundary conditions of solution components v_1 and v_2 are dependent and can be prescribed as:

$$\begin{cases} u|_{y=0} = \bar{u}(x,t), \ (\nabla\times\vec{v})|_{y=0} = 0, \ v_2|_{y=0} = \bar{v}(x,t), & \text{if } \varepsilon > 0, \\ u|_{y=0} = \bar{u}(x,t), & \text{if } \varepsilon = 0. \end{cases} \tag{5.3}$$

where $\bar{u}(x,t)$ and $\bar{v}(x,t)$ are functions of x and t representing the boundary conditions.

We shall study the stability of boundary layers of system (5.2)-(5.3) in the present chapter. By the BL theory [71, 75], we anticipate that the solution $(u^\varepsilon,\vec{v}^\varepsilon)$ (where $\vec{v}^\varepsilon = (v_1^\varepsilon, v_2^\varepsilon)$) of (5.2), (5.3) with small $\varepsilon > 0$ consists of two parts: the outer layer profile and inner (boundary) layer profile. Furthermore the inner layer profile for u^ε will be absent since the boundary conditions for u between $\varepsilon > 0$ and $\varepsilon = 0$ are consistent (i.e. u^ε converges uniformly in ε). The approximation for $(u^\varepsilon, \vec{v}^\varepsilon)$ ought to be:

$$
\begin{aligned}
&u^\varepsilon(x,y,t) = u^0(x,y,t) + O(\varepsilon^{1/2}),\\
&\vec{v}^\varepsilon(x,y,t) = \vec{v}^0(x,y,t) + \Big(v_1^{B,0}\big(x, \frac{y}{\sqrt{\varepsilon}}, t\big), v_2^{B,0}\big(x, \frac{y}{\sqrt{\varepsilon}}, t\big)\Big) + O(\varepsilon^{1/2}),
\end{aligned}
$$

where the outer layer profile $(u^0, \vec{v}^0) = (u^0, v_1^0, v_2^0)$ is the solution of (5.2), (5.3) with $\varepsilon = 0$; the inner layer profile $(v_1^{B,0}, v_2^{B,0})$ rapidly adjusts from a value away from the boundary layer to another value on the boundary. Actually the inner layer profile $v_1^{B,0} \equiv 0$ as well, since formally it follows from the above expansion and the curl-free condition (5.1) that

$$
\begin{aligned}
|v_1^\varepsilon(x,y,t) - v_1^0(x,y,t)| &= \Big|\int_y^\infty \partial_s(v_1^\varepsilon(x,s,t) - v_1^0(x,s,t))\,ds\Big|\\
&= \Big|\int_y^\infty \partial_x(v_2^\varepsilon(x,s,t) - v_2^0(x,s,t))\,ds\Big|\\
&\le \varepsilon^{1/2}\int_0^\infty |\partial_x v_2^{B,0}(x,\eta,t)|\,d\eta + O(\varepsilon^{1/2}),
\end{aligned}
$$

where $\int_0^\infty |\partial_x v_2^{B,0}(x,\eta,t)|\,d\eta$ is uniformly bounded in ε. Therefore, the profile $(u^\varepsilon, \vec{v}^\varepsilon)$ is expected to be

$$
\begin{aligned}
&u^\varepsilon(x,y,t) = u^0(x,y,t) + O(\varepsilon^{1/2}),\\
&\vec{v}^\varepsilon(x,y,t) = \vec{v}^0(x,y,t) + \Big(0, v_2^{B,0}\big(x, \frac{y}{\sqrt{\varepsilon}}, t\big)\Big) + O(\varepsilon^{1/2}).
\end{aligned}
\tag{5.4}
$$

Due to the similarity between the second equation of (5.2) and the INS equations, justifying (5.4) seems to be a great challenge at first glance since we suspect if $v_2^{B,0}$ satisfies a degenerate Prandtl type equation (as INS equations do) whose well-posedness with general initial data in Sobolev space still remains as a grand open question in spite of numerous attempts (cf. [21, 27, 73, 74, 84, 86]), where the Prandtl equation lacks diffusive dissipation in x-direction to control the nonlinear convection term in x-direction. However, thanks to the special structure of (5.2), the nonlinear trouble convection term $\varepsilon\nabla|\vec{v}|^2$ in (5.2) vanishes as $\varepsilon \to 0$ and the resulting limit equation $\vec{v}_t + \nabla u = 0$ is fundamentally different from the Euler equation - limit equation of INS. Indeed a formal analysis will show that the boundary layer equation of (5.2) for $v_2^{B,0}$ is not of Prandtl's type in two dimensions (see (5.10) in section 5.2). This key observation promises us a possibility to justifying (5.4). For brevity, instead of (5.4) we shall prove in Theorem 5.2 a similar result with convergence rate for $\vec{v}$ replaced by $O(\varepsilon^{1/4})$, followed with a remark stating the sufficient conditions on initial and boundary data to rigorously prove (5.4).

The organization of this chapter is as follows. In Section 5.1, the initial-boundary problems for outer and inner layer profiles will be exhibited. We shall give the main results on the boundary layer problem (5.2)-(5.3) and the ideas for their proofs in Section 5.2. Section 5.3 and Section 5.4 are devoted to the regularity estimates for the outer/inner profiles and the remainders, respectively. With these regularity estimates in hand, we shall prove the main results, Theorem 5.1 and Theorem 5.3 in Section 5.5. Finally, Section 5.6 is a formal derivation for the outer and inner layer profiles of Section 5.1.

5.1 Equations for Outer/Inner Layer Profiles

In this section, we are devoted to deriving the equations for outer and inner layer profiles by applying formal asymptotic analysis to solutions $(u^\varepsilon, \vec{v}^\varepsilon)$ of (5.2), (5.3) with small $\varepsilon > 0$. Note that the thickness of boundary layer has been formally justified as $O(\varepsilon^{1/2})$ in appendix of [29] and thus for (5.2), (5.3) by similar arguments used there. Hence based on the WK-B theory (see e.g. [29], [26, Chapter 4], [22, 72]), solutions $(u^\varepsilon, \vec{v}^\varepsilon)$ have the following expansions in ε for $j \in \mathbb{N}$:

$$
\begin{aligned}
u^\varepsilon(x,y,t) &= \sum_{j=0}^{\infty} \varepsilon^{j/2}\left(u^{I,j}(x,y,t) + u^{B,j}(x,\eta,t)\right), \\
\vec{v}^\varepsilon(x,y,t) &= \sum_{j=0}^{\infty} \varepsilon^{j/2}\left(\vec{v}^{I,j}(x,y,t) + \vec{v}^{B,j}(x,\eta,t)\right),
\end{aligned}
\tag{5.5}
$$

where the boundary layer coordinate is defined as:

$$
\eta = \frac{y}{\varepsilon^{1/2}}, \quad y \in (0,\infty). \tag{5.6}
$$

Each term in (5.5) is assumed to be smooth and the boundary-layer profiles $(u^{B,j}, \vec{v}^{B,j})$ enjoy the following basic hypothesis (see also [26, Chapter 4], [22], [72]):

(H*) $u^{B,j}$ and $\vec{v}^{B,j}$ decay to zero exponentially as $\eta \to \infty$.

In order to obtain the equations for outer and inner layer profiles in (5.5), the analysis will be split into three steps. First the initial and boundary values follow from the substitution of (5.5) into the third equality of (5.2) and (5.3). Then we deduce the equations for layer profiles by inserting (5.5) into the first and second equations of (5.2) successively. Applying these procedures and using the asymptotic matching method (detail is given in Section 4.6) we deduce that the leading-order outer layer profile $(u^{I,0}, \vec{v}^{I,0})(x,y,t)$ satisfies the following initial-boundary value problem:

$$
\begin{cases}
u_t^{I,0} = \nabla\cdot(u^{I,0}\vec{v}^{I,0}) + \Delta u^{I,0}, & (x,y,t) \in \mathbb{R}\times\mathbb{R}_+\times(0,T) \\
\vec{v}_t^{I,0} = \nabla u^{I,0}, \\
(u^{I,0}, \vec{v}^{I,0})(x,y,0) = (u_0, \vec{v}_0)(x,y). \\
u^{I,0}(x,0,t) = \bar{u}(x,t).
\end{cases}
\tag{5.7}
$$

Note that (5.7) is exactly the system (5.2), (5.3) with $\varepsilon = 0$, whose solution is denoted as $(u^0, \vec{v}^0)(x,y,t)$. Then we conclude that

$$
(u^{I,0}, \vec{v}^{I,0})(x,y,t) = (u^0, \vec{v}^0)(x,y,t), \quad (x,y,t) \in \mathbb{R}\times\mathbb{R}_+\times(0,T) \tag{5.8}
$$

thanks to the uniqueness of solutions. The leading-order inner layer profile $u^{B,0}(x,\eta,t)$ satisfies

$$u^{B,0}(x,\eta,t) \equiv 0,$$

and $v_1^{B,0}(x,\eta,t)$, the first component of $\vec{v}^{B,0}(x,\eta,t)$, solves

$$\begin{cases} \partial_t v_1^{B,0} = \partial_\eta^2 v_1^{B,0}, & (x,\eta,t) \in \mathbb{R} \times \mathbb{R}_+ \times (0,T) \\ v_1^{B,0}(x,\eta,0) = 0, \\ \partial_\eta v_1^{B,0}(x,0,t) = 0, \end{cases} \tag{5.9}$$

which gives rise to $v_1^{B,0}(x,\eta,t) \equiv 0$, by the uniqueness of solutions.
The second component of $\vec{v}^{B,0}(x,\eta,t)$ fulfills

$$\begin{cases} \partial_t v_2^{B,0} + \bar{u}(x,t) v_2^{B,0} = \partial_\eta^2 v_2^{B,0}, & (x,\eta,t) \in \mathbb{R} \times \mathbb{R}_+ \times (0,T) \\ v_2^{B,0}(x,\eta,0) = 0, \\ v_2^{B,0}(x,0,t) = \bar{v}(x,t) - v_2^{I,0}(x,0,t) \end{cases} \tag{5.10}$$

and the first-order inner layer profile $u^{B,1}(x,\eta,t)$ is determined by $v_2^{B,0}(x,\eta,t)$ via

$$u^{B,1}(x,\eta,t) = \bar{u}(x,t) \int_\eta^\infty v_2^{B,0}(x,\zeta,t)\, d\zeta. \tag{5.11}$$

Moreover, the first-order outer layer profile $(u^{I,1}, \vec{v}^{I,1})(x,y,t)$ is the solution of

$$\begin{cases} u_t^{I,1} = \nabla \cdot (u^{I,0} \vec{v}^{I,1}) + \nabla \cdot (u^{I,1} \vec{v}^{I,0}) + \Delta u^{I,1}, & (x,y,t) \in \mathbb{R} \times \mathbb{R}_+ \times (0,T), \\ \vec{v}_t^{I,1} = \nabla u^{I,1}, \\ (u^{I,1}, \vec{v}^{I,1})(x,y,0) = (0,0), \\ u^{I,1}(x,0,t) = -\bar{u}(x,t) \displaystyle\int_0^\infty v_2^{B,0}(x,\eta,t)\, d\eta. \end{cases} \tag{5.12}$$

For the first-order inner layer profile $\vec{v}^{B,1}(x,\eta,t)$, its first component $v_1^{B,1}(x,\eta,t)$ satisfies

$$\begin{cases} \partial_t v_1^{B,1} - \partial_x u^{B,1} = \partial_\eta^2 v_1^{B,1}, & (x,\eta,t) \in \mathbb{R} \times \mathbb{R}_+ \times (0,T) \\ v_1^{B,1}(x,\eta,0) = 0, \\ \partial_\eta v_1^{B,1}(x,0,t) = \partial_x \bar{v}(x,t) - \partial_y v_1^{I,0}(x,0,t), \end{cases} \tag{5.13}$$

and its second component $v_2^{B,1}(x,\eta,t)$ solves

$$\begin{cases} \partial_t v_2^{B,1} + \bar{u}(x,t) v_2^{B,1} = \partial_\eta^2 v_2^{B,1} - 2(v_2^{I,0}(x,0,t) + v_2^{B,0}) \partial_\eta v_2^{B,0} + \displaystyle\int_\eta^\infty \Phi(x,\zeta,t)\, d\zeta, \\ v_2^{B,1}(x,\eta,0) = 0, \qquad (x,\eta) \in \mathbb{R} \times \mathbb{R}_+ \\ v_2^{B,1}(x,0,t) = -v_2^{I,1}(x,0,t). \end{cases} \tag{5.14}$$

The second-order inner layer profile $u^{B,2}(x,\eta,t)$ is given as

$$u^{B,2}(x,\eta,t)=\bar{u}(x,t)\int_{\eta}^{\infty}v_2^{B,1}(x,\zeta,t)\,d\zeta-\int_{\eta}^{\infty}\int_{\zeta}^{\infty}\Phi(x,s,t)\,dsd\zeta, \tag{5.15}$$

where

$$\begin{aligned}\Phi(x,\eta,t):=&(u^{I,1}(x,0,t)+u^{B,1})\partial_\eta v_2^{B,0}+\partial_y u^{I,0}(x,0,t)v_2^{B,0}\\&+\partial_\eta u^{B,1}(v_2^{I,0}(x,0,t)+v_2^{B,0})+\eta\partial_y u^{I,0}(x,0,t)\partial_\eta v_2^{B,0}.\end{aligned} \tag{5.16}$$

Finally, $v_1^{B,2}(x,\eta,t)$ the first component of $\vec{v}^{B,2}(x,\eta,t)$ solves the following problem:

$$\begin{cases}\partial_t v_1^{B,2}=-\partial_x[2v_2^{I,0}(x,0,t)v_2^{B,0}+v_2^{B,0}v_2^{B,0}]+\partial_x u^{B,2}+\partial_\eta^2 v_1^{B,2},\\ v_1^{B,2}(x,\eta,0)=0, \qquad (x,\eta)\in\mathbb{R}\times\mathbb{R}_+\\ \partial_\eta v_1^{B,2}(x,0,t)=-\partial_y v_1^{I,1}(x,0,t).\end{cases} \tag{5.17}$$

The derivation of (5.7)-(5.17) will be detailed in Section 4.6 and their well-posedness will be readily discussed below. One can go further to deduce the initial boundary value problems for $(u^{I,j},v^{I,j})$, $(u^{B,j+1},v_1^{B,j+1},v_2^{B,j})$ with $j\geq 2$ by the asymptotic analysis, however the higher-order terms (5.9) - (5.17) are sufficient to conclude our results.

5.2 Results on Stability of Boundary Layers

For later use, we first introduce the following compatibility conditions:

$$(A^*)\begin{cases}\bar{u}(x,0)=&u_0(x,0),\\ \partial_t\bar{u}(x,0)=&[\nabla\cdot(u_0\vec{v}_0)+\Delta u_0](x,0),\\ \partial_t^2\bar{u}(x,0)=&\nabla\cdot[\partial_t\bar{u}(x,0)\vec{v}_0(x,0)]+\nabla\cdot[u_0\nabla u_0]+\Delta\partial_t\bar{u}(x,0),\\ \partial_t^3\bar{u}(x,0)=&\nabla\cdot[\partial_t^2\bar{u}(x,0)\vec{v}_0(x,0)]+2\nabla\cdot[\partial_t\bar{u}(x,0)\nabla u_0]\\ &+\nabla\cdot[u_0\nabla\partial_t\bar{u}(x,0)]+\Delta\partial_t^2\bar{u}(x,0),\\ \partial_t^4\bar{u}(x,0)=&\nabla\cdot[\partial_t^3\bar{u}(x,0)\vec{v}_0(x,0)]+3\nabla\cdot[\partial_t^2\bar{u}(x,0)\nabla u_0(x,0)]\\ &+3\nabla\cdot[\partial_t\bar{u}(x,0)\nabla\partial_t\bar{u}(x,0)]+\nabla\cdot[u_0\nabla\partial_t^2\bar{u}(x,0)\vec{v}_0],\\ \bar{v}(x,0)=&v_{02}(x,0),\\ \partial_t\bar{v}(x,0)=&\partial_y u_0(x,0),\\ \partial_t^2\bar{v}(x,0)=&\partial_y[\nabla\cdot(u_0\vec{v}_0)+\Delta u_0](x,0),\\ \partial_t^3\bar{v}(x,0)=&\partial_y[\nabla\cdot(\nabla\cdot(u_0\vec{v}_0)+\Delta u_0)\vec{v}_0](x,0)+\nabla\cdot(u_0\vec{v}_0)(x,0)\\ &+\Delta[\nabla\cdot(u_0\vec{v}_0)+\Delta u_0](x,0),\end{cases}$$

where the first five and the last four equalities are respectively the compatibility conditions of problem (5.2), (5.3) with $\varepsilon=0$ (up to order 4) and problem (5.10) (up to order 3). The definition of the compatibility conditions (for initial-boundary problem (3.18)) has been stated below (3.18) in section 3.3 and the little lengthy compatibility conditions (A^*) here are to guarantee the well-posedness of out/inner layer solutions (see (5.7)-(5.17)) in the required Sobolev space (see Lemma 5.1- Lemma 5.5 in section 5.3) and ultimately to prove the main result, Theorem 5.2.

To prove the stability of boundary layers, we need the following regularity on solutions of (5.2)-(5.3) with $\varepsilon = 0$.

Theorem 5.1. *Assume that the initial and boundary data satisfy*

$$u_0, \vec{v}_0 \in H^9,\ u_0 \geq 0, \nabla \times \vec{v}_0 = 0;\ \ \partial_t^k \bar{u}, \partial_t^k \bar{v} \in L^2_{loc}(0,\infty; H_x^{10-2k}),\ 0 \leq k \leq 5$$

and the first five equalities of (A^*) *hold. Then there exists* $0 < T < \infty$ *such that the problem* (5.2), (5.3) *has a unique solution* $(u^0, \vec{v}^0)(x,y,t)$ *with* $\varepsilon = 0$ *on* $[0,T]$ *satisfying* $\nabla \times \vec{v}^0(x,y,t) \equiv 0$ *and*

$$\begin{aligned}
&\partial_t^k u^0 \in L^2(0,T; H^{10-2k}), \quad k = 0,1,2,3,4,5;\\
&\partial_t^k \vec{v}^0 \in L^2(0,T; H^{11-2k}), \quad k = 1,2,3,4,5;\\
&\vec{v}^0 \in L^\infty(0,T; H^9).
\end{aligned}$$

The proof of Theorem 5.1 is quite standard and we omit it for brevity and refer the reader to [42, Theorem 1.1] for details, where the local well-posedness of (5.2) with $\Omega = \mathbb{R}^d$ $(d \geq 2)$ is proved.

We are now in a position to state the main results of this chapter.

Theorem 5.2. *Suppose that the initial and boundary data*

$$u_0, \vec{v}_0 \in H^9,\ u_0 \geq 0, \nabla \times \vec{v}_0 = 0;\ \ \partial_t^k \bar{u}, \partial_t^k \bar{v} \in L^2_{loc}(0,\infty; H_x^{10-2k}),\ 0 \leq k \leq 5$$

satisfy the compatibility conditions (A^*). *Let* $(u^0, \vec{v}^0)(x,y,t)$ *be the solution of* (5.2), (5.3) *with* $\varepsilon = 0$ *and let* $T > 0$ *be less than the maximal existence time of* $(u^0, \vec{v}^0)$. *Let* $\varepsilon_T > 0$ *be the constant defined in Lemma 5.8, which is decreasing in* T. *Then for any* $\varepsilon \in (0, \varepsilon_T]$, *problem* (5.2), (5.3) *admits a unique solution* $(u^\varepsilon, \vec{v}^\varepsilon) \in C([0,T]; H^2 \times H^2)$ *on* $[0,T]$ *satisfying* $\nabla \times \vec{v}^\varepsilon(x,y,t) \equiv 0$ *and*

$$\begin{aligned}
&\|u^\varepsilon(x,y,t) - u^0(x,y,t)\|_{L^\infty(0,T;L^\infty)} \leq C\varepsilon^{1/2},\\
&\|\vec{v}^\varepsilon(x,y,t) - \vec{v}^0(x,y,t) - \big(0, v_2^{B,0}\big)\big(x, \frac{y}{\sqrt{\varepsilon}}, t\big)\|_{L^\infty(0,T;L^\infty)} \leq C\varepsilon^{1/4},
\end{aligned} \tag{5.18}$$

where the constant C *is independent of* ε *and*

$$v_2^{B,0}(x,\eta,t) := \int_0^t \int_{-\infty}^0 \frac{1}{\sqrt{\pi(t-s)}} e^{-\left(\frac{(\eta-\zeta)^2}{4(t-s)} + \bar{u}(t-s)\right)} [\bar{u}(\bar{v} - v_2^0(x,0,s) - \partial_s v_2^0(x,0,s))] d\zeta ds. \tag{5.19}$$

Remark 5.1. *The convergence rate for $\vec{v}$ in* (5.18) *can be enhanced to $O(\varepsilon^{1/2})$ by first including the higher-order profiles $(u^{I,2},\vec{v}^{I,2})$, $(u^{B,3},v_1^{B,3},v_2^{B,2})$ in the approximation $(U^a,\vec{V}^a)$ (see Section 5.5), then applying the similar procedures as proving* (5.18) *based on a stronger assumption on initial-boundary data: $u^0,\vec{v}^0 \in H^{11}$, $\partial_t^k \bar{u}$, $\partial_t^k \bar{v} \in L^2_{loc}(0,\infty;H_x^{12-2k})$.*

Remark 5.2. *The regularity of $(u^\varepsilon,\vec{v}^\varepsilon) \in C([0,T];H^2\times H^2)$ in Theorem 5.2 is much lower than that of the given initial data $(u_0,\vec{v}_0)\in H^9$, since the conditions (A^*) only provide the* 0-*th order compatibility condition for problem* (5.2), (5.3) *with $\varepsilon>0$. By assuming further that the initial-boundary data satisfy the compatibility conditions of* (5.2), (5.3) *(with $\varepsilon>0$) up to order* 4, *the regularity space of $(u^\varepsilon,\vec{v}^\varepsilon)$ can be improved to $C([0,T];H^9\times H^9)$, however the regularity derived in Theorem 5.2 is sufficient to conclude the main result* (5.18).

The corresponding initial-boundary value problem of the original chemotaxis model (1.2) reads as

$$\begin{cases} u_t = \nabla\cdot[\nabla u - u\nabla(\ln c)], \qquad (x,y,t)\in\mathbb{R}\times\mathbb{R}_+\times(0,T), \\ c_t = \varepsilon\Delta c - uc, \\ (u,c)(x,y,0) = (u_0,c_0)(x,y), \\ u|_{y=0} = \bar{u}(x,t), \quad [\nabla c\cdot\vec{n} + \bar{v}(x,t)c]|_{y=0} = 0, & \text{if } \varepsilon>0, \\ u|_{y=0} = \bar{u}(x,t), & \text{if } \varepsilon = 0, \end{cases} \tag{5.20}$$

where $\vec{n}$ is the unit outward normal of $\mathbb{R}^2_+ = \{(x,y)\in\mathbb{R}^2\,|\,y>0\}$. Then as a consequence of Theorem 5.2, we get the following results for the problem (5.20).

Theorem 5.3. *Suppose $(u_0,\ln c_0)\in H^9\times H^{10}$ with u_0, $c_0\geq 0$. Let the assumptions in Theorem 5.2 hold with $\vec{v}_0 = -\frac{\nabla c_0}{c_0}$. Then* (5.20) *admits a unique solution $(u^\varepsilon,c^\varepsilon)\in C([0,T];H^2\times H^3)$ for $\varepsilon\in(0,\varepsilon_T]$ and $(u^0,c^0)\in C([0,T];H^9\times H^{10})$ for $\varepsilon=0$ such that*

$$\begin{aligned} &\|u^\varepsilon(x,y,t)-u^0(x,y,t)\|_{L^\infty(0,T;L^\infty)} \leq C\varepsilon^{1/2}, \\ &\|c^\varepsilon(x,y,t)-c^0(x,y,t)\|_{L^\infty(0,T;L^\infty)} \leq C\varepsilon^{1/4} \end{aligned} \tag{5.21}$$

and

$$\|\nabla c^\varepsilon(x,y,t) - \nabla c^0(x,y,t) + c^0(x,y,t)\big(0,\, v_2^{B,0}\big(x,\frac{y}{\sqrt{\varepsilon}},t\big)\big)\|_{L^\infty(0,T;L^\infty)} \leq C\varepsilon^{1/4}, \tag{5.22}$$

where the constant C is independent of ε.

5.3 Regularity of Outer/Inner Layer Profiles

To assert the well-posedness on solutions of (5.10)-(5.17), we first introduce some preliminary results. In particular, to solve (5.10) and (5.14) we introduce the following system

$$\begin{cases} \theta_t(x,\eta,t)+\bar{u}(x,t)\theta(x,\eta,t)=\partial_\eta^2\theta(x,\eta,t)+\rho(x,\eta,t), & (x,\eta,t)\in\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}_+ \\ \theta(x,\eta,0)=0, \\ \theta|_{\eta=0}=0. \end{cases} \tag{5.23}$$

The corresponding regularity result on the solution of (5.23) is as follows.

Proposition 5.1. *Let $0<T<\infty$ and $m\in\mathbb{N}_+$. Suppose ρ satisfies for all $l\in\mathbb{N}$ that*

$$\langle\eta\rangle^l\partial_t^k\rho\in L^2(0,T;H_x^{2m-2k}L_\eta^2),\quad k=0,1,\cdots,m$$

and $\bar{u}(x,t)$ satisfies

$$\partial_t^k\bar{u}\in L^2(0,T;H_x^{2m+1-2k}),\quad k=0,1,\cdots,m.$$

Assume further that ρ and $\bar{u}$ satisfy the compatibility conditions up to order $(m-1)$ for the problem (5.23)*. Then* (5.23) *admits a unique solution $\theta(x,\eta,t)$ on $[0,T]$ such that for any $l\in\mathbb{N}$*

$$\langle\eta\rangle^l\partial_t^k\theta\in L^\infty(0,T;H_x^{2m-2k}H_\eta^1)\cap L^2(0,T;H_x^{2m-2k}H_\eta^2),\quad k=0,1,\cdots,m.$$

We omit the proof of Proposition 5.1 since it is standard and refer the reader to [16, page 380-388] for detail.

To study (5.12) we shall employ the following initial-boundary problem

$$\begin{cases} h_t=\Delta h+\nabla\cdot(\vec{f}_1h)+\nabla\cdot(f_2\vec{w})+f, & (x,y,t)\in\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}_+ \\ \vec{w}_t=\nabla h+\vec{g}, \\ (h,\vec{w})(x,y,0)=(h_0,\vec{w}_0)(x,y), \\ h|_{y=0}=0, \end{cases} \tag{5.24}$$

whose well-posedness is as follows.

Proposition 5.2. *Let $0<T<\infty$ and $m\in\mathbb{N}_+$. Suppose that $(h_0,\vec{w}_0)\in H^{2m-1}\times H^{2m-1}$ and*

$$\begin{aligned} &\partial_t^kf\in L^2(0,T;H^{2m-2-2k}),\quad \partial_t^k\vec{g}\in L^2(0,T;H^{2m-1-2k})\quad \text{for } k=0,1,\cdots,m-1; \\ &\partial_t^k\vec{f}_1\in L^\infty(0,T;H^{2m-1-2k}),\quad \partial_t^kf_2\in L^2(0,T;H^{2m-2k})\quad \text{for } k=0,1,\cdots,m-1. \end{aligned}$$

Assume further that $(h_0,\vec{w}_0)$ and f, $\vec{g}$, $\vec{f}_1$, f_2 satisfy the compatibility conditions up to order $(m-1)$ for problem (5.24)*. Then* (5.24) *admits a unique solution $(h,\vec{w})(x,y,t)$ on $[0,T]$ such* *that*

$$
\begin{aligned}
&\partial_t^k h \in L^2(0,T;H^{2m-2k}), \quad \text{for } k=0,1,\cdots,m;\\
&\partial_t^k \vec{w} \in L^2(0,T;H^{2m+1-2k}), \quad \text{for } k=1,\cdots,m;\\
&\vec{w} \in L^\infty(0,T;H^{2m-1}).
\end{aligned}
$$

The proof of Proposition 5.2 is similar to that of Proposition 3.1, thus we omit it for brevity.

Finally, for the regularity on solutions of (5.13) and (5.17), we introduce the following system

$$
\begin{cases}
\psi_t(x,\eta,t) = \partial_\eta^2 \psi(x,\eta,t) + r(x,\eta,t), \quad (x,\eta,t) \in \mathbb{R}\times\mathbb{R}_+\times\mathbb{R}_+\\
\psi(x,\eta,0) = 0,\\
\partial_\eta \psi(x,0,t) = s(x,t).
\end{cases}
\tag{5.25}
$$

For system (5.25), we have the following result.

Proposition 5.3. *Let* $0 < T < \infty$ *and take the integer* $m \geq 3$. *Assume that* $r(x,\eta,t)$ *fulfills for all* $l \in \mathbb{N}$ *that*

$$
\langle\eta\rangle^l r,\ \langle\eta\rangle^l \partial_t r \in L^2(0,T;H_x^m L_\eta^2); \qquad \langle\eta\rangle^l \partial_t^2 r \in L^2(0,T;H_x^{m-2} L_\eta^2);
$$

and $s(x,t)$ *satisfies*

$$
s,\ \partial_t s \in L^2(0,T;H_x^m); \qquad \partial_t^2 s \in L^2(0,T;H_x^{m-2}).
$$

Assume further that r *and* s *satisfy the compatibility conditions up to order* 1 *for the initial-boundary problem* (5.25). *Then there exists a unique solution* $\psi(x,\eta,t)$ *of* (5.25) *on* $[0,T]$ *such that for any* $l \in \mathbb{N}$,

$$
\begin{aligned}
&\langle\eta\rangle^l \psi,\ \langle\eta\rangle^l \partial_\eta \psi,\ \langle\eta\rangle^l \partial_t \psi \in L^\infty(0,T;H_x^m L_\eta^2) \cap L^2(0,T;H_x^m H_\eta^1);\\
&\langle\eta\rangle^l \partial_\eta \partial_t \psi,\ \langle\eta\rangle^l \partial_t^2 \psi \in L^\infty(0,T;H_x^{m-2} L_\eta^2) \cap L^2(0,T;H_x^{m-2} H_\eta^1).
\end{aligned}
$$

Proof. With $0 \leq j \leq m$ and $l \in \mathbb{N}$, we first apply ∂_x^j (j-th order differentiation) to (5.25), then multiply the resulting equation with $2\langle\eta\rangle^{2l}\partial_x^j \psi$ in $L_{x\eta}^2$ and use integration by parts to derive

$$
\begin{aligned}
&\frac{d}{dt}\|\langle\eta\rangle^l \partial_x^j \psi\|_{L_{x\eta}^2}^2 + 2\|\langle\eta\rangle^l \partial_x^j \partial_\eta \psi\|_{L_{x\eta}^2}^2\\
=&-4l\int_0^\infty\int_{-\infty}^\infty \langle\eta\rangle^{2l-2}\eta(\partial_\eta\partial_x^j\psi)(\partial_x^j\psi)dxd\eta + 2\int_0^\infty\int_{-\infty}^\infty \langle\eta\rangle^{2l}(\partial_x^j r)(\partial_x^j\psi)dxd\eta\\
&+2\int_{-\infty}^\infty (\partial_x^j\partial_\eta\psi(x,0,t))(\partial_x^j\psi(x,0,t))dx\\
\leq&\frac{1}{2}\|\langle\eta\rangle^l\partial_x^j\partial_\eta\psi\|_{L_{x\eta}^2}^2 + C_0(l^2+1)\|\langle\eta\rangle^l\partial_x^j\psi\|_{L_{x\eta}^2}^2 + \|\langle\eta\rangle^l\partial_x^j r\|_{L_{x\eta}^2}^2\\
&+2\int_{-\infty}^\infty (\partial_x^j s(x,t))(\partial_x^j\psi(x,0,t))dx,
\end{aligned}
\tag{5.26}
$$

with

$$
\begin{aligned}
2\int_{-\infty}^{\infty}(\partial_x^j s(x,t))(\partial_x^j \psi(x,0,t))dx \le & 2\int_{-\infty}^{\infty}|\partial_x^j s(x,t)|\,\|\partial_x^j \psi(x,\eta,t)\|_{L^\infty_\eta}dx \\
\le & C_0\int_{-\infty}^{\infty}|\partial_x^j s(x,t)|\,\|\partial_x^j \psi(x,\eta,t)\|_{H^1_\eta}dx \\
\le & \frac{1}{2}\|\langle\eta\rangle^l\partial_x^j\partial_\eta\psi\|^2_{L^2_{x\eta}}+\frac{1}{2}\|\langle\eta\rangle^l\partial_x^j\psi\|^2_{L^2_{x\eta}}+C_0\|\partial_x^j s\|^2_{L^2_x}.
\end{aligned}
$$

where the Sobolev embedding inequality has been used. Summing (5.26) from $j=0$ to $j=m$ and applying Gronwall's inequality, one deduces that

$$
\|\langle\eta\rangle^l\psi\|^2_{L^\infty_T H^m_x L^2_\eta}+\|\langle\eta\rangle^l\partial_\eta\psi\|^2_{L^2_T H^m_x L^2_\eta}\le C. \tag{5.27}
$$

We proceed to derive higher regularity estimates for ψ. Similar to the above procedure in deriving (5.26), we apply ∂_x^j to (5.25) and multiply the resulting equation with $2\langle\eta\rangle^{2l}\partial_x^j\partial_t\psi$ in $L^2_{x\eta}$ to have

$$
\begin{aligned}
&\frac{d}{dt}\|\langle\eta\rangle^l\partial_x^j\partial_\eta\psi\|^2_{L^2_{x\eta}}+2\|\langle\eta\rangle^l\partial_x^j\partial_t\psi\|^2_{L^2_{x\eta}} \\
\le&\frac{1}{2}\|\langle\eta\rangle^l\partial_x^j\partial_t\psi\|^2_{L^2_{x\eta}}+C_0(l^2+1)\|\langle\eta\rangle^l\partial_x^j\partial_\eta\psi\|^2_{L^2_{x\eta}}+C_0\|\langle\eta\rangle^l\partial_x^j r\|^2_{L^2_{x\eta}} \\
&+2\int_{-\infty}^{\infty}(\partial_x^j s(x,t))(\partial_x^j\partial_t\psi(x,0,t))dx,
\end{aligned}
\tag{5.28}
$$

with

$$
\begin{aligned}
2\int_{-\infty}^{\infty}(\partial_x^j s(x,t))(\partial_x^j\partial_t\psi(x,0,t))dx \le & \frac{1}{2}\|\langle\eta\rangle^l\partial_x^j\partial_\eta\partial_t\psi\|^2_{L^2_{x\eta}} \\
&+\frac{1}{2}\|\langle\eta\rangle^l\partial_x^j\partial_t\psi\|^2_{L^2_{x\eta}}+C_0\|\partial_x^j s\|^2_{L^2_x}.
\end{aligned}
$$

On the other hand, after applying ∂_t to (5.25) one finds that $\partial_t\psi$ solves a similar system as (5.25) with $r(x,\eta,t)$, $s(x,t)$ and the initial condition replaced by $\partial_t r(x,\eta,t)$, $\partial_t s(x,t)$ and $\partial_t\psi(x,\eta,0)=r(x,\eta,0)$, respectively. Thus it follows from (5.26) that

$$
\begin{aligned}
&\frac{d}{dt}\|\langle\eta\rangle^l\partial_x^j\partial_t\psi\|^2_{L^2_{x\eta}}+2\|\langle\eta\rangle^l\partial_x^j\partial_\eta\partial_t\psi\|^2_{L^2_{x\eta}} \\
\le&\|\langle\eta\rangle^l\partial_x^j\partial_\eta\partial_t\psi\|^2_{L^2_{x\eta}}+C_0(l^2+1)\|\langle\eta\rangle^l\partial_x^j\partial_t\psi\|^2_{L^2_{x\eta}} \\
&+\|\langle\eta\rangle^l\partial_x^j\partial_t r\|^2_{L^2_{x\eta}}+C_0\|\partial_x^j\partial_t s\|^2_{L^2_x}.
\end{aligned}
\tag{5.29}
$$

We add (5.29) to (5.28) and then sum the results from $j=0$ to $j=m$ to get

$$
\begin{aligned}
&\frac{d}{dt}(\|\langle\eta\rangle^l\partial_\eta\psi\|^2_{H^m_x L^2_\eta}+\|\langle\eta\rangle^l\partial_t\psi\|^2_{H^m_x L^2_\eta})+\|\langle\eta\rangle^l\partial_t\psi\|^2_{H^m_x L^2_\eta}+\|\langle\eta\rangle^l\partial_\eta\partial_t\psi\|^2_{H^m_x L^2_\eta} \\
\le&C_0(\|\langle\eta\rangle^l\partial_\eta\psi\|^2_{H^m_x L^2_\eta}+\|\langle\eta\rangle^l\partial_t\psi\|^2_{H^m_x L^2_\eta}) \\
&+C_0(\|\langle\eta\rangle^l r\|^2_{H^m_x L^2_\eta}+\|\langle\eta\rangle^l\partial_t r\|^2_{H^m_x L^2_\eta}+\|s\|^2_{H^m_x}+\|\partial_t s\|^2_{H^m_x}),
\end{aligned}
$$

which, along with Gronwall's inequality, leads to

$$\begin{aligned}&\|\langle\eta\rangle^l\partial_\eta\psi\|^2_{L^\infty_T H^m_x L^2_\eta}+\|\langle\eta\rangle^l\partial_t\psi\|^2_{L^\infty_T H^m_x L^2_\eta}\\&\quad+\|\langle\eta\rangle^l\partial_t\psi\|^2_{L^2_T H^m_x L^2_\eta}+\|\langle\eta\rangle^l\partial_\eta\partial_t\psi\|^2_{L^2_T H^m_x L^2_\eta}\le C.\end{aligned}\tag{5.30}$$

By an analogous argument as deriving (5.30) one can deduce for all $l\in\mathbb{N}$ that

$$\begin{aligned}&\|\langle\eta\rangle^l\partial_\eta\partial_t\psi\|^2_{L^\infty_T H^{m-2}_x L^2_\eta}+\|\langle\eta\rangle^l\partial_t^2\psi\|^2_{L^\infty_T H^{m-2}_x L^2_\eta}\\&\quad+\|\langle\eta\rangle^l\partial_t^2\psi\|^2_{L^2_T H^{m-2}_x L^2_\eta}+\|\langle\eta\rangle^l\partial_\eta\partial_t^2\psi\|^2_{L^2_T H^{m-2}_x L^2_\eta}\le C.\end{aligned}\tag{5.31}$$

Combining (5.27), (5.30) and (5.31), we get the desired estimates and complete the proof. □

With the above results in hand, we establish the well-posedness of (5.10)-(5.17).

Lemma 5.1. *Suppose the assumptions in Theorem 5.2 hold. Let* $(u^0,\vec{v}^0)(x,y,t)$ *be the solution obtained in Theorem 5.1 and let* $T>0$ *be less than the maximal existence time of* $(u^0,\vec{v}^0)$. *Then*

$$v_2^{B,0}(x,\eta,t):=\int_0^t\int_{-\infty}^0\frac{1}{\sqrt{\pi(t-s)}}e^{-\left(\frac{(\eta-\zeta)^2}{4(t-s)}+\bar{u}(t-s)\right)}[\bar{u}(\bar{v}-v_2^0(x,0,s)-\partial_s v_2^0(x,0,s))]d\zeta ds \tag{5.32}$$

is the unique solution of (5.10) *on* $[0,T]$ *satisfying for all* $l\in\mathbb{N}$ *that*

$$\langle\eta\rangle^l\partial_t^k v_2^{B,0}\in L^\infty(0,T;H_x^{8-2k}H_\eta^1)\cap L^2(0,T;H_x^{8-2k}H_\eta^2),\quad k=0,1,2,3,4.\tag{5.33}$$

Furthermore, it follows from the equations (5.10) *and* (5.11) *that*

$$\langle\eta\rangle^l v_2^{B,0}\in L^\infty(0,T;H_x^6H_\eta^3),\quad \langle\eta\rangle^l\partial_t v_2^{B,0}\in L^\infty(0,T;H_x^4H_\eta^3)\tag{5.34}$$

and that

$$\langle\eta\rangle^l\partial_t^k u^{B,1}\in L^\infty(0,T;H_x^{8-2k}H_\eta^2)\cap L^2(0,T;H_x^{8-2k}H_\eta^3),\quad k=0,1,2,3,4.$$

Proof. Observing that for fixed $x\in\mathbb{R}$, (5.10) can be converted to the one dimensional heat equation with independent variables $(t,\eta)\in(0,T)\times\mathbb{R}_+$, which has been solved explicitly by a formula similar to (5.32) using the reflection method with odd extension in Lemma 3.2. Thus we omit the derivation of (5.32) for brevity and refer the reader to Lemma 3.2 for details. We proceed to prove (5.33). Let $\varphi(\eta)$ be a smooth function defined on $[0,\infty)$ satisfying

$$\varphi(0)=1,\qquad \varphi(\eta)=0 \text{ for } \eta>1.\tag{5.35}$$

Denote

$$\tilde{v}_2^{B,0}(x,\eta,t)=v_2^{B,0}(x,\eta,t)-\big(\bar{v}(x,t)-v_2^0(x,0,t)\big)\varphi(\eta).$$

Then one deduces from (5.10) and (5.8) that

$$\begin{cases} \partial_t \tilde{v}_2^{B,0} + \bar{u}(x,t)\tilde{v}_2^{B,0} = \partial_\eta^2 \tilde{v}_2^{B,0} + \rho, \quad (x,\eta,t) \in \mathbb{R}\times\mathbb{R}_+\times(0,T) \\ \tilde{v}_2^{B,0}(x,\eta,0) = 0, \\ \tilde{v}_2^{B,0}(x,0,t) = 0, \end{cases} \tag{5.36}$$

where $\rho(x,\eta,t) = \big(\bar{v}(x,t) - v_2^0(x,0,t)\big)\partial_\eta^2\varphi(\eta) - \partial_t\big(\bar{v}(x,t) - v_2^0(x,0,t)\big)\varphi(\eta) - \bar{u}(x,t)\big(\bar{v}(x,t) - v_2^0(x,0,t)\big)\varphi(\eta)$. The compatibility condition $\bar{v}(x,0) = v_{02}(x,0)$ has been used to determine the initial data of $\tilde{v}_2^{B,0}$ in (5.36). We next prove that ρ satisfies the assumptions in Proposition 5.1 with $m = 4$. First note that for $f(x,y,t) \in H^{k+1}$ with fixed $t > 0$ and $k \geq 0$ the following holds

$$\begin{aligned} \|f(x,0,t)\|_{H_x^k}^2 \leq & \sum_{j=0}^k \int_{-\infty}^{\infty} |\partial_x^j f(x,0,t)|^2\,dx \\ \leq & \sum_{j=0}^k \int_{-\infty}^{\infty} \|\partial_x^j f(x,y,t)\|_{L_y^\infty}^2 dx \\ \leq & C_0 \sum_{j=0}^k \int_{-\infty}^{\infty} \|\partial_x^j f(x,y,t)\|_{H_y^1}^2 dx \\ \leq & C_0 \|f(x,y,t)\|_{H^{k+1}}^2, \end{aligned} \tag{5.37}$$

where the Sobolev embedding inequality has been used. Then it follows from Theorem 5.1 and (5.37) that

$$\|\partial_t^k v_2^0(x,0,t)\|_{L_T^2 H_x^{10-2k}} \leq \|\partial_t^k v_2^0\|_{L_T^2 H^{11-2k}} \leq C, \quad k = 1,2,3,4,5 \tag{5.38}$$

and that

$$\|v_2^0(x,0,t)\|_{L_T^2 H_x^8} \leq \|v_2^0\|_{L_T^2 H^9} \leq C.$$

Hence from the above estimates we deduce for $l \in \mathbb{N}$ and $k = 0,1,2,3,4$ that

$$\begin{aligned} & \|\langle\eta\rangle^l \partial_t^k \rho\|_{L_T^2 H_x^{8-2k} L_\eta^2} \\ \leq & \big(\|\partial_t^k \bar{v}\|_{L_T^2 H_x^{8-2k}} + \|\partial_t^k v_2^0(x,0,t)\|_{L_T^2 H_x^{8-2k}}\big)\|\langle\eta\rangle^l \partial_\eta^2\varphi\|_{L_\eta^2} \\ & + \big(\|\partial_t^{k+1}\bar{v}\|_{L_T^2 H_x^{10-2(k+1)}} + \|\partial_t^{k+1} v_2^0(x,0,t)\|_{L_T^2 H_x^{10-2(k+1)}}\big)\|\langle\eta\rangle^l\varphi\|_{L_\eta^2} \\ & + \sum_{j=0}^k \big(\|\partial_t^j \bar{v}\|_{L_T^2 H_x^{8-2j}} + \|\partial_t^j v_2^0(x,0,t)\|_{L_T^2 H_x^{8-2j}}\big)\|\partial_t^{k-j}\bar{u}\|_{L_T^\infty H_x^{9-2(k-j)}}\|\langle\eta\rangle^l\varphi\|_{L_\eta^2} \\ \leq & C, \end{aligned} \tag{5.39}$$

where $\|\partial_t^{k-j}\bar{u}\|_{L_T^\infty H_x^{9-2(k-j)}} \leq C$ has been used thanks to the assumptions on $\partial_t^k \bar{u}$ in Theorem 5.2 and [77, Lemma 1.2]. Moreover, it is easy to verify that ρ and $\bar{u}$ satisfy the compatibility conditions up to order 3 for problem (5.36) under assumption (A^*). We apply Proposition 5.1 with $m=4$ to (5.36) to conclude that

$$\langle\eta\rangle^l \partial_t^k \tilde{v}_2^{B,0} \in L^\infty(0,T;H_x^{8-2k}H_\eta^1)\cap L^2(0,T;H_x^{8-2k}H_\eta^2), \quad k=0,1,2,3,4$$

which along with the definition of $\tilde{v}_2^{B,0}$ and (5.38) gives rise to (5.33). The estimate for $u^{B,1}$ follows directly from (5.11), (5.33) and the assumptions on $\partial_t^k \bar{u}(x,t)$. It remains to prove (5.34). Indeed, by (5.10) and (5.33) we deduce for all $l\in\mathbb{N}$ that

$$\begin{aligned}\|\langle\eta\rangle^l v_2^{B,0}\|_{L_T^\infty H_x^6 H_\eta^3} \le & C_0(\|\bar{u}\|_{L_T^\infty H_x^6}\|\langle\eta\rangle^l v_2^{B,0}\|_{L_T^\infty H_x^6 H_\eta^1}+\|\langle\eta\rangle^l\partial_t v_2^{B,0}\|_{L_T^\infty H_x^6 H_\eta^1})\\ \le & C.\end{aligned} \tag{5.40}$$

Similar argument leads to $\|\langle\eta\rangle^l\partial_t v_2^{B,0}\|_{L_T^\infty H_x^4 H_\eta^3}\le C$. The proof is completed.

$\square$

Lemma 5.2. *Let the assumptions in Theorem 5.2 hold. Let $(u^0,\vec{v}^0)(x,y,t)$ and $v_2^{B,0}(x,\eta,t)$ be as obtained in Theorem 5.1 and Lemma 5.1 respectively. Then* (5.12) *admits a unique solutions $(u^{I,1},\vec{v}^{I,1})(x,y,t)$ on $[0,T]$ such that*

$$\begin{aligned}&\partial_t^k u^{I,1}\in L^2(0,T;H^{8-2k}), \quad k=0,1,2,3,4;\\ &\partial_t^k \vec{v}^{I,1}\in L^2(0,T;H^{9-2k}), \quad k=1,2,3,4;\\ &\vec{v}^{I,1}\in L^\infty(0,T;H^7).\end{aligned} \tag{5.41}$$

Proof. Let φ be as defined in (5.35). We denote

$$\tilde{u}^{I,1}(x,y,t)=u^{I,1}(x,y,t)+\varphi(y)\bar{u}(x,t)\int_0^\infty v_2^{B,0}(x,\eta,t)d\eta.$$

Then it follows from (5.12) that

$$\begin{cases}\partial_t\tilde{u}^{I,1}=\nabla\cdot(\vec{v}^0\tilde{u}^{I,1})+\nabla\cdot(u^0\vec{v}^{I,1})+\Delta\tilde{u}^{I,1}+f,\\ \vec{v}_t^{I,1}=\nabla\tilde{u}^{I,1}+\vec{g},\\ (\tilde{u}^{I,1},\vec{v}^{I,1})(x,y,0)=(0,0),\\ \tilde{u}^{I,1}(x,0,t)=0,\end{cases} \tag{5.42}$$

where

$$\vec{g}(x,y,t)=-\nabla\big[\varphi(y)\bar{u}(x,t)\int_0^\infty v_2^{B,0}(x,\eta,t)d\eta\big]$$

and

$$\begin{aligned}f(x,y,t)=&\varphi(y)\partial_t\big[\bar{u}(x,t)\int_0^\infty v_2^{B,0}(x,\eta,t)d\eta\big]-\Delta\big[\varphi(y)\bar{u}(x,t)\int_0^\infty v_2^{B,0}(x,\eta,t)d\eta\big]\\ &-\nabla\cdot\big[\varphi(y)\bar{u}(x,t)\vec{v}^0(x,y,t)\int_0^\infty v_2^{B,0}(x,\eta,t)d\eta\big].\end{aligned}$$

To apply Proposition 5.2 with $m=4$ to (5.42) we next verify that $\vec{v}^0$, u^0, f and $\vec{g}$ satisfy the assumptions. By the Cauchy-Schwarz inequality and Lemma 5.1 we deduce for $j=0,1,2,3,4$ that

$$\Big\|\int_0^\infty \partial_t^j v_2^{B,0} d\eta\Big\|_{L_T^\infty H_x^{8-2j}} \le \Big(\int_0^\infty \langle\eta\rangle^{-2} d\eta\Big)^{1/2} \|\langle\eta\rangle \partial_t^j v_2^{B,0}\|_{L_T^\infty H_x^{8-2j} L_\eta^2} \le C. \tag{5.43}$$

Thus it follows for $k=0,1,2,3$ that

$$\begin{aligned}
&\|\partial_t^k f\|_{L_T^2 H^{6-2k}}\\
\le& C_0 \sum_{j=0}^{k+1} \|\partial_t^{k+1-j}\bar{u}\|_{L_T^2 H_x^{9-2(k+1-j)}} \Big\|\int_0^\infty \partial_t^j v_2^{B,0} d\eta\Big\|_{L_T^\infty H_x^{8-2j}} \|\varphi\|_{H_y^6}\\
&+C_0 \sum_{i+j=0}^{k} \|\partial_t^{k-(i+j)}\bar{u}\|_{L_T^2 H_x^{7-2(k-i-j)}} \|\partial_t^i \vec{v}^0\|_{L_T^\infty H^{7-2i}} \Big\|\int_0^\infty \partial_t^j v_2^{B,0} d\eta\Big\|_{L_T^\infty H_x^{7-2j}} \|\varphi\|_{H_y^7}\\
&+C_0 \sum_{j=0}^{k} \|\partial_t^{k-j}\bar{u}\|_{L_T^2 H_x^{8-2(k-j)}} \Big\|\int_0^\infty \partial_t^j v_2^{B,0} d\eta\Big\|_{L_T^\infty H_x^{8-2j}} \|\varphi\|_{H_y^8} \le C.
\end{aligned}$$

Similarly, for $k=0,1,2,3$, one gets

$$\|\partial_t^k \vec{g}\|_{L_T^2 H^{7-2k}} \le C.$$

It is easy to verify that f, $\vec{g}$, u^0 and $\vec{v}^0$ satisfy the compatibility conditions up to order 3 for problem (5.42) under assumption (A^*). By the above estimates for $\vec{g}$, f and Theorem 5.1, we apply Proposition 5.2 with $m=4$ to (5.42) to conclude that

$$\begin{aligned}
&\partial_t^k \tilde{u}^{I,1} \in L^2(0,T;H^{8-2k}), \quad k=0,1,2,3,4;\\
&\partial_t^k \vec{v}^{I,1} \in L^2(0,T;H^{9-2k}), \quad k=1,2,3,4; \qquad \vec{v}^{I,1} \in L^\infty(0,T;H^7),
\end{aligned}$$

which, along with the definition of $\tilde{u}^{I,1}$ and (5.43), leads to (5.41) and completes the proof. □

Lemma 5.3. *Suppose the assumptions in Theorem 5.2 hold true. Let $(u^0,\vec{v}^0)(x,y,t)$ and $u^{B,1}(x,\eta,t)$ be as derived in Theorem 5.1 and Lemma 5.1 respectively. Then there exists a unique solution $v_1^{B,1}(x,\eta,t)$ of* (5.13) *on $[0,T]$ such that for any $l\in\mathbb{N}$*

$$\begin{aligned}
&\langle\eta\rangle^l v_1^{B,1},\ \langle\eta\rangle^l \partial_\eta v_1^{B,1},\ \langle\eta\rangle^l \partial_t v_1^{B,1} \in L^\infty(0,T;H_x^5 L_\eta^2)\cap L^2(0,T;H_x^5 H_\eta^1);\\
&\langle\eta\rangle^l \partial_\eta\partial_t v_1^{B,1},\ \langle\eta\rangle^l \partial_t^2 v_1^{B,1} \in L^\infty(0,T;H_x^3 L_\eta^2)\cap L^2(0,T;H_x^3 H_\eta^1).
\end{aligned} \tag{5.44}$$

Furthermore, it follows from (5.13) *that*

$$\langle\eta\rangle^l v_1^{B,1} \in L^\infty(0,T;H_x^5 H_\eta^2), \quad \langle\eta\rangle^l \partial_t v_1^{B,1} \in L^\infty(0,T;H_x^3 H_\eta^2). \tag{5.45}$$

Proof. Let $r(x,\eta,t)=\partial_x u^{B,1}(x,\eta,t)$ and $s(x,t)=\partial_x\bar{v}(x,t)-\partial_y v_1^0(x,0,t)$. We next verify that $r(x,\eta,t)$ and $s(x,t)$ satisfy the assumptions in Proposition 5.3 with $m=5$. In fact, for

$l \in \mathbb{N}$ one deduces from Lemma 5.1 that

$$
\begin{aligned}
&\|\langle\eta\rangle^l r\|_{L^2_T H^5_x L^2_\eta} + \|\langle\eta\rangle^l \partial_t r\|_{L^2_T H^5_x L^2_\eta} + \|\langle\eta\rangle^l \partial_t^2 r\|_{L^2_T H^3_x L^2_\eta} \\
\le &\|\langle\eta\rangle^l u^{B,1}\|_{L^2_T H^6_x L^2_\eta} + \|\langle\eta\rangle^l \partial_t u^{B,1}\|_{L^2_T H^6_x L^2_\eta} + \|\langle\eta\rangle^l \partial_t^2 u^{B,1}\|_{L^2_T H^4_x L^2_\eta} \\
\le &C.
\end{aligned}
$$

Moreover, (5.37) and Theorem 5.1 entail that

$$
\begin{aligned}
&\|s\|_{L^2_T H^5_x} + \|\partial_t s\|_{L^2_T H^5_x} + \|\partial_t^2 s\|_{L^2_T H^3_x} \\
\le &\|\bar{v}\|_{L^2_T H^6_x} + \|v_1^0\|_{L^2_T H^7} + \|\partial_t \bar{v}\|_{L^2_T H^6_x} + \|\partial_t v_1^0\|_{L^2_T H^7} \\
&+ \|\partial_t^2 \bar{v}\|_{L^2_T H^4_x} + \|\partial_t^2 v_1^0\|_{L^2_T H^5} \le C.
\end{aligned}
$$

It is easy to verify that the compatibility conditions up to order 1 for problem (5.13) is fulfilled by r and s under assumption (A^*). By the above estimates on $r(x,\eta,t)$ and $s(x,t)$, we can apply Proposition 5.3 to (5.13) and derive (5.44). Moreover, (5.45) follows from (5.13) and (5.44) by a similar argument as deriving (5.40). The proof is completed.

□

Lemma 5.4. *Suppose the assumptions in Theorem 5.2 hold. Let* $(u^0,\vec{v}^0)(x,y,t)$*,* $(v_2^{B,0},u^{B,1})(x,\eta,t)$ *and* $(u^{I,1},\vec{v}^{I,1})(x,y,t)$ *be as derived in Theorem 5.1, Lemma 5.1 and Lemma 5.2 respectively. Then* (5.14) *admits a unique solution* $v_2^{B,1}(x,\eta,t)$ *on* $[0,T]$ *satisfying for all* $l \in \mathbb{N}$ *that*

$$
\langle\eta\rangle^l \partial_t^k v_2^{B,1} \in L^\infty(0,T;H_x^{6-2k}H^1_\eta) \cap L^2(0,T;H_x^{6-2k}H^2_\eta), \quad k=0,1,2,3. \tag{5.46}
$$

Moreover, it follows from (5.14) *and* (5.15) *that*

$$
\langle\eta\rangle^l v_2^{B,1} \in L^\infty(0,T;H^4_x H^3_\eta), \quad \langle\eta\rangle^l \partial_t v_2^{B,1} \in L^\infty(0,T;H^2_x H^3_\eta) \tag{5.47}
$$

and that

$$
\langle\eta\rangle^l \partial_t^k u^{B,2} \in L^\infty(0,T;H_x^{6-2k}H^2_\eta) \cap L^2(0,T;H_x^{6-2k}H^3_\eta), \quad k=0,1,2,3. \tag{5.48}
$$

Proof. Let φ be as defined in (5.35). Denote

$$
\tilde{v}_2^{B,1}(x,\eta,t) = v_2^{B,1}(x,\eta,t) + \varphi(\eta) v_2^{I,1}(x,0,t).
$$

From (5.14) one deduces that

$$
\begin{cases}
\partial_t \tilde{v}_2^{B,1} + \bar{u}(x,t)\tilde{v}_2^{B,1} = \partial_\eta^2 \tilde{v}_2^{B,1} + \rho, \\
\tilde{v}_2^{B,1}(x,\eta,0) = 0, \\
\tilde{v}_2^{B,1}(x,0,t) = 0
\end{cases} \tag{5.49}
$$

where

$$\begin{aligned}\rho(x,\eta,t)=&\partial_t v_2^{I,1}(x,0,t)\varphi(\eta)+\bar{u}(x,t)v_2^{I,1}(x,0,t)\varphi(\eta)-v_2^{I,1}(x,0,t)\partial_\eta^2\varphi(\eta)\\&-2(v_2^0(x,0,t)+v_2^{B,0})\partial_\eta v_2^{B,0}+\int_\eta^\infty \Phi(x,\zeta,t)\,d\zeta\end{aligned}$$

with $\Phi(x,\eta,t)$ defined in (5.16). For $k=0,1,2,3$ and $l\in\mathbb{N}$ one has

$$\begin{aligned}\langle\eta\rangle^l\partial_t^k\rho=&[\langle\eta\rangle^l\varphi(\eta)\partial_t^{k+1}v_2^{I,1}(x,0,t)+\langle\eta\rangle^l\varphi(\eta)\partial_t^k(\bar{u}(x,t)v_2^{I,1}(x,0,t))\\&-\langle\eta\rangle^l\partial_\eta^2\varphi(\eta)\partial_t^k v_2^{I,1}(x,0,t)]\\&-2\langle\eta\rangle^l\partial_t^k[(v_2^0(x,0,t)+v_2^{B,0})\partial_z v_2^{B,0}]+[\langle\eta\rangle^l\int_z^\infty \partial_t^k\Phi(x,\eta,t)\,d\eta]\\:=&R_1-R_2+R_3.\end{aligned}$$

We proceed to estimate R_1, R_2 and R_3. First it follows from (5.37) and Lemma 5.2 that

$$\|\partial_t^k v_2^{I,1}(x,0,t)\|_{L_T^2H_x^{8-2k}}\le\|\partial_t^k v_2^{I,1}\|_{L_T^2H^{9-2k}}\le C,\quad k=1,2,3,4 \tag{5.50}$$

and that

$$\|v_2^{I,1}(x,0,t)\|_{L_T^2H_x^6}\le\|v_2^{I,1}\|_{L_T^2H^7}\le C.$$

Thus by (5.50) and a similar argument as deriving (5.39) one gets $\|R_1\|_{L_T^2H_x^{6-2k}L_\eta^2}\le C$. Moreover, it follows from the Sobolev embedding inequality that

$$\begin{aligned}\|R_2\|_{L_T^2H_x^{6-2k}L_\eta^2}\le&\sum_{j=0}^k(\|\partial_t^j v_2^0\|_{L_T^2H^{8-2j}}\|\langle\eta\rangle^l\partial_t^{k-j}\partial_\eta v_2^{B,0}\|_{L_T^\infty H_x^{6-2(k-j)}L_\eta^2}\\&+\|\partial_t^j v_2^{B,0}\|_{L_T^2H_x^{8-2j}H_\eta^2}\|\langle\eta\rangle^l\partial_t^{k-j}\partial_\eta v_2^{B,0}\|_{L_T^\infty H_x^{6-2(k-j)}L_\eta^2})\\\le&C,\end{aligned}$$

where we have used the following inequality

$$\begin{aligned}\|f(x,\eta,t)g(x,\eta,t)\|_{H_x^lL_\eta^2}\le&C_0\sum_{i=0}^l\|\partial_x^i f\|_{L_{x\eta}^\infty}\sum_{j=0}^l\|\partial_x^j g\|_{L_{x\eta}^2}\\\le&C_0\sum_{i=0}^l\|\partial_x^i f\|_{H_{x\eta}^2}\sum_{j=0}^l\|\partial_x^i g\|_{L_{x\eta}^2}\\\le&C_0\|f\|_{H_x^{l+2}H_\eta^2}\|g\|_{H_x^lL_\eta^2}\end{aligned}\tag{5.51}$$

for fixed $t>0$. By (5.37), Theorem 5.1, Lemma 5.1 and a similar argument as estimating $\|R_2\|_{L_T^2H_x^{6-2k}L_\eta^2}$ one derives for all $l\in\mathbb{N}$ and $k=0,1,2,3$ that

$$\|\langle\eta\rangle^{l+2}\partial_t^k\Phi\|_{L_T^2H_x^{6-2k}L_\eta^2}\le C. \tag{5.52}$$

On the other hand, the Cauchy-Schwarz inequality entails for fixed $t \in [0,T]$ that

$$
\begin{aligned}
\|R_3\|^2_{H_x^{6-2k}L^2_\eta} \le & \int_0^\infty \left(\langle\eta\rangle^l \int_\eta^\infty \|\partial_t^k \Phi(x,\zeta,t)\|_{H_x^{6-2k}} d\zeta \right)^2 d\eta \\
\le & \int_0^\infty \langle\eta\rangle^{-2} d\eta \cdot \left(\int_0^\infty \|\langle\zeta\rangle^{l+1}\partial_t^k \Phi\|_{H_x^{6-2k}} d\zeta \right)^2 \\
\le & \int_0^\infty \langle\eta\rangle^{-2} d\eta \cdot \int_0^\infty \langle\zeta\rangle^{-2} d\zeta \cdot \int_0^\infty \|\langle\zeta\rangle^{l+2}\partial_t^k \Phi\|^2_{H_x^{6-2k}} d\zeta \\
\le & C_0 \|\langle\eta\rangle^{l+2}\partial_t^k \Phi\|^2_{H_x^{6-2k}L^2_\eta},
\end{aligned}
$$

which along with (5.52) gives rise to

$$
\|R_3\|_{L^2_T H_x^{6-2k} L^2_\eta} \le C.
$$

Collecting the above estimates for R_1, R_2 and R_3 we deduce for all $l \in \mathbb{N}$ and $k = 0,1,2,3$ that $\|\langle\eta\rangle^l \partial_t^k \rho\|_{L^2_T H_x^{6-2k} L^2_\eta} \le C$. It is easy to verify that ρ and $\bar{u}$ fulfill the compatibility conditions up to order 2 for problem (5.49) under assumption (A^*). Then we apply Proposition 5.1 with $m = 3$ to (5.49) to conclude that

$$
\langle\eta\rangle^l \partial_t^k \tilde{v}_2^{B,1} \in L^\infty(0,T;H_x^{6-2k}H_\eta^1) \cap L^2(0,T;H_x^{6-2k}H_\eta^2), \quad k = 0,1,2,3
$$

which, in conjunction with the definition of $\tilde{v}_2^{B,1}$ and (5.50), implies (5.46). Then (5.48) follows directly from (5.15) and (5.52). Finally, by a similar argument used in deriving (5.40), one deduces (5.47) from (5.46), (5.14) and (5.52). The proof is finished.

□

Lemma 5.5. *Suppose the assumptions in Theorem 5.2 hold true. Let $(u^0,\vec{v}^0)(x,y,t)$, $v_2^{B,0}(x,\eta,t)$, $(u^{I,1},\vec{v}^{I,1})(x,y,t)$ and $u^{B,2}(x,\eta,t)$ be as derived in Theorem 5.1, Lemma 5.1 - Lemma 5.4 respectively. Then* (5.17) *admits a unique solution $v_1^{B,2}(x,\eta,t)$ on $[0,T]$ such that for any $l \in \mathbb{N}$,*

$$
\begin{aligned}
&\langle\eta\rangle^l v_1^{B,2},\ \langle\eta\rangle^l \partial_\eta v_1^{B,2},\ \langle\eta\rangle^l \partial_t v_1^{B,2} \in L^\infty(0,T;H_x^3 L^2_\eta) \cap L^2(0,T;H_x^3 H_\eta^1); \\
&\langle\eta\rangle^l \partial_\eta \partial_t v_1^{B,2},\ \langle\eta\rangle^l \partial_t^2 v_1^{B,2} \in L^\infty(0,T;H_x^1 L^2_\eta) \cap L^2(0,T;H_x^1 H_\eta^1).
\end{aligned}
\tag{5.53}
$$

Moreover, it follows from (5.17) *that*

$$
\langle\eta\rangle^l v_1^{B,2} \in L^\infty(0,T;H_x^3 H_\eta^2), \quad \langle\eta\rangle^l \partial_t v_1^{B,2} \in L^\infty(0,T;H^2_{x\eta}). \tag{5.54}
$$

Proof. Let $r(x,\eta,t) = -\partial_x[2v_2^0(x,0,t)v_2^{B,0} + v_2^{B,0}v_2^{B,0}] + \partial_x u^{B,2}$ and $s(x,t) = -\partial_y v_1^{I,1}(x,0,t)$. To apply Proposition 5.3 to (5.17) we shall prove that r and s satisfy the assumptions of Proposition 5.3 with $m = 3$. First, it is easy to verify that r and s fulfill the compatibility conditions up to order 1 for problem (5.17) under assumption (A^*). Moreover, for all $l \in \mathbb{N}$

we deduce from (5.38) and (5.51) that

$$\begin{aligned}\|\langle\eta\rangle^l\partial_t r\|_{L^2_TH^3_xL^2_\eta} \le& C_0(\|v_2^0\|_{L^2_TH^5}\|\langle\eta\rangle^l\partial_t v_2^{B,0}\|_{L^\infty_TH^4_xL^2_\eta}+\|\partial_t v_2^0\|_{L^2_TH^5}\|\langle\eta\rangle^l v_2^{B,0}\|_{L^\infty_TH^4_xL^2_\eta}\\ &+\|v_2^{B,0}\|_{L^\infty_TH^6_xH^2_\eta}\|\langle\eta\rangle^l\partial_t v_2^{B,0}\|_{L^2_TH^4_xL^2_\eta}+\|\langle\eta\rangle^l\partial_t u^{B,2}\|_{L^2_TH^4_xL^2_\eta})\\ \le& C.\end{aligned}$$

Similarly, one derives

$$\|\langle\eta\rangle^l r\|_{L^2_TH^3_xL^2_\eta}+\|\langle\eta\rangle^l\partial_t^2 r\|_{L^2_TH^1_xL^2_\eta}\le C.$$

On the other hand, it follows from (5.37) and Lemma 5.2 that

$$\begin{aligned}\|s\|_{L^2_TH^3_x}+\|\partial_t s\|_{L^2_TH^3_x}+\|\partial_t^2 s\|_{L^2_TH^1_x}\le&\|v_1^{I,1}\|_{L^2_TH^5}+\|\partial_t v_1^{I,1}\|_{L^2_TH^5}+\|\partial_t^2 v_2^{I,1}\|_{L^2_TH^3}\\ \le& C.\end{aligned}$$

Combining the above estimates for $r(x,\eta,t)$ and $s(x,t)$ we then apply Proposition 5.3 with $m=3$ to (5.17) and derive (5.53). By a similar argument as deriving (5.40), we get (5.54) from (5.17) and the proof is completed.

□

5.4 Regularity Estimates on Remainders

To show the convergence results in (5.18), we first approximate solutions $(u^\varepsilon,\vec{v}^\varepsilon)$ of (5.2), (5.3) with $\varepsilon>0$ by a combination of outer and boundary layer profiles derived in the previous section, then estimate the remainders by the standard energy method and *bootstrap principle*. In particular the approximation $(U^a,\vec{V}^a)(x,y,t)$ is defined as follows:

$$\begin{aligned}U^a(x,y,t)=&u^0(x,y,t)+\varepsilon^{1/2}u^{I,1}(x,y,t)+\varepsilon^{1/2}u^{B,1}\Big(x,\frac{y}{\sqrt{\varepsilon}},t\Big)\\ &+\varepsilon u^{B,2}\Big(x,\frac{y}{\sqrt{\varepsilon}},t\Big)-\varepsilon\varphi(y)u^{B,2}(x,0,t),\end{aligned}$$

$$\begin{aligned}\vec{V}^a(x,y,t)=&\vec{v}^0(x,y,t)+\Big(0,\ v_2^{B,0}\Big(x,\frac{y}{\sqrt{\varepsilon}},t\Big)\Big)+\varepsilon^{1/2}\vec{v}^{I,1}(x,y,t)\\ &+\varepsilon^{1/2}\vec{v}^{B,1}\Big(x,\frac{y}{\sqrt{\varepsilon}},t\Big)+\varepsilon\Big(v_1^{B,2}\Big(x,\frac{y}{\sqrt{\varepsilon}},t\Big),\ 0\Big)\end{aligned}$$

and the remainder $(U^\varepsilon,\vec{V}^\varepsilon)(x,y,t)$ is as follows

$$U^\varepsilon(x,y,t):=\varepsilon^{-1/2}(u^\varepsilon-U^a)(x,y,t),\qquad \vec{V}^\varepsilon(x,y,t):=\varepsilon^{-1/2}(\vec{v}^\varepsilon-\vec{V}^a)(x,y,t),$$

where φ is defined in (5.35) and $\varepsilon\varphi(y)u^{B,2}(x,0,t)$, $\varepsilon v_1^{B,2}\Big(x,\frac{y}{\sqrt{\varepsilon}},t\Big)$ in the definition of U^a, $\vec{V}^a$ are to homogenize the boundary values of U^ε and $\vec{V}^\varepsilon$. The initial-boundary problem for the remainder follows directly from (5.2), (5.3) and initial and boundary conditions in (5.9)-(5.17), and read as

$$\begin{cases} U_t^\varepsilon = \varepsilon^{1/2}\nabla\cdot(U^\varepsilon\vec{V}^\varepsilon) + \nabla\cdot(U^\varepsilon\vec{V}^a) + \nabla\cdot(\vec{V}^\varepsilon U^a) + \Delta U^\varepsilon + \varepsilon^{-1/2}f^\varepsilon, \\ \vec{V}_t^\varepsilon = -\varepsilon^{3/2}\nabla(|\vec{V}^\varepsilon|^2) - 2\varepsilon\nabla(\vec{V}^\varepsilon\cdot\vec{V}^a) + \nabla U^\varepsilon + \varepsilon\Delta\vec{V}^\varepsilon + \varepsilon^{-1/2}\vec{g}^\varepsilon, \\ (U^\varepsilon,\vec{V}^\varepsilon)(x,0) = (0,0), \\ (U^\varepsilon, V_2^\varepsilon)(x,0,t) = (0,0), \quad \partial_y V_1^\varepsilon(x,0,t) = 0, \end{cases} \tag{5.55}$$

where

$$f^\varepsilon = \Delta U^a + \nabla\cdot(U^a\vec{V}^a) - U_t^a, \quad \vec{g}^\varepsilon = \varepsilon\Delta\vec{V}^a + \nabla U^a - \varepsilon\nabla(|\vec{V}^a|^2) - \vec{V}_t^a. \tag{5.56}$$

For the initial-boundary problem (5.55), we derive the following result.

Proposition 5.4. *Suppose that the assumptions in Theorem 5.2 hold and that $T > 0$ is less than the maximal existence time of $(u^0, \vec{v}^0)$. Let ε_T be as defined in Lemma 5.8. Then for any $\varepsilon \in (0, \varepsilon_T]$, problem* (5.55) *admits a unique solution $(U^\varepsilon, \vec{V}^\varepsilon) \in C([0,T]; H^2 \times H^2)$ on $[0,T]$ satisfying*

$$\|U^\varepsilon\|^2_{L_T^\infty L^2} + \|\vec{V}^\varepsilon\|^2_{L_T^\infty L^2} + \|\nabla U^\varepsilon\|^2_{L_T^\infty L^2} + \varepsilon\|\nabla\vec{V}^\varepsilon\|^2_{L_T^\infty L^2} \le C\varepsilon^{1/2} \tag{5.57}$$

and

$$\varepsilon\|U^\varepsilon\|^2_{L_T^\infty H^2} + \varepsilon^2\|\vec{V}^\varepsilon\|^2_{L_T^\infty H^2} + \varepsilon^3\|\vec{V}^\varepsilon\|^2_{L_T^2 H^3} \le C\varepsilon^{1/2}, \tag{5.58}$$

where the constant C is independent of ε, depending on T.

We emphasize that the estimates (5.57) and (5.58) are crucial to prove the main result, Theorem 5.2. Before proceeding, we introduce the additional difficulties encountered (compared with one-dimensional case) and main ideas used in proving Proposition 5.4. When estimating the remainders $(U^\varepsilon, \vec{V}^\varepsilon)$ (see section 3.4), an L^2 uniform-in-ε estimates of $(u^\varepsilon, \vec{v}^\varepsilon)$ is used in the one dimensional case (see Lemma 2.1 of Chapter 2), while system (5.2)-(5.3) in multi-dimensions lacks an energy-like structure to provide such L^2-estimates of ε-independence. The challenge in our analysis thus consists in deriving the estimates (5.57) and (5.58) for $(U^\varepsilon, \vec{V}^\varepsilon)$ without any uniform-in-ε priori estimates of solutions $(u^\varepsilon, \vec{v}^\varepsilon)$. This will be achieved by regarding $(u^\varepsilon, \vec{v}^\varepsilon)$ as small perturbations of $(U^a, \vec{V}^a)$ and employing the bootstrap method by choosing ε small enough.

We next introduce some preliminaries for later use. For $G_1(x,\eta,t) \in H_x^k H_\eta^m$ with $k, m \in \mathbb{N}$ and fixed $t > 0$, we have from the change of variables that

$$\left\|\partial_y^m G_1\left(x, \frac{y}{\sqrt{\varepsilon}}, t\right)\right\|_{H_x^k L_y^2} = \varepsilon^{\frac{1}{4}-\frac{m}{2}}\|\partial_z^m G_1(x,\eta,t)\|_{H_x^k L_\eta^2}. \tag{5.59}$$

Similar arguments in deriving (5.37) entail that

$$\begin{aligned}\|G_2(x,0,t)\|_{H_x^k}^2 \leq &\sum_{j=0}^{k}\int_{-\infty}^{\infty}|\partial_x^j G_1(x,0,t)|^2dx\\ \leq &C_0\sum_{j=0}^{k}\int_{-\infty}^{\infty}\|\partial_x^j G_2(x,\eta,t)\|_{H_\eta^1}^2dx\\ = &C_0\|G_2(x,\eta,t)\|_{H_x^kH_\eta^1}^2,\end{aligned} \tag{5.60}$$

provided $G_2(x,\eta,t) \in H_x^kH_\eta^1$ for fixed $t>0$. Furthermore, if $G_3(x,\eta,t) \in H_x^3H_\eta^2$ one has

$$\begin{aligned}&\|G_3(x,0,t)\|_{L_x^\infty} \leq C_0\|G_3(x,\eta,t)\|_{L_{x\eta}^\infty} \leq C_0\|G_3(x,\eta,t)\|_{L_T^\infty H_{x\eta}^2},\\ &\|\partial_x G_3(x,0,t)\|_{L_x^\infty} \leq C_0\|G_3(x,\eta,t)\|_{H_x^3H_\eta^2}.\end{aligned} \tag{5.61}$$

For $G_4(x,\eta,t) \in H_{x\eta}^2$ one deduces by the Sobolev embedding inequality that

$$\left\|G_4\left(x,\frac{y}{\sqrt{\varepsilon}},t\right)\right\|_{L_{xy}^\infty} = \|G_4(x,\eta,t)\|_{L_{x\eta}^\infty} \leq C_0\|G_4(x,\eta,t)\|_{H_{x\eta}^2}. \tag{5.62}$$

For $h_1(x,y,t) \in H^1$ with fixed $t>0$, it follows from the Gagliardo-Nirenberg interpolation inequality that

$$\|h_1\|_{L^4} \leq C_0(\|h_1\|_{L^2}^{1/2}\|\nabla h_1\|_{L^2}^{1/2} + \|h_1\|_{L^2}) \tag{5.63}$$

and

$$\|h_1\|_{L^4} \leq C_0\|h_1\|_{L^2}^{1/2}\|\nabla h_1\|_{L^2}^{1/2}, \tag{5.64}$$

provided further $h_1|_{y=0}=0$. For $h_2(x,y,t) \in H^2$ one gets

$$\|h_2\|_{L^\infty} \leq C_0(\|h_2\|_{L^2}^{1/2}\|\nabla^2 h_2\|_{L^2}^{1/2} + \|h_2\|_{L^2}) \tag{5.65}$$

and

$$\|h_2\|_{L^\infty} \leq C_0\|h_2\|_{L^2}^{1/2}\|\nabla^2 h_2\|_{L^2}^{1/2}, \tag{5.66}$$

provided $h_2|_{y=0}=0$.

The assumption $0<\varepsilon<1$ and the results of Theorem 5.1, Lemma 5.1- Lemma 5.5 will be frequently used in the proof of Lemma 5.6- Lemma 5.9 without further clarification.

We shall prove Proposition 5.4 by the following series of lemmas where a priori estimates on the solutions $(U^\varepsilon,\vec{V}^\varepsilon)$ is derived based on the L^2 regularity on external force $f^\varepsilon(x,y,t)$ and $\vec{g}^\varepsilon(x,y,t)$. The estimates on f^ε and $\vec{g}^\varepsilon$ are as follows.

Lemma 5.6. *Suppose that the assumptions in Theorem 5.2 hold. Let $T>0$ be less than the maximal existence time of $(u^0,\vec{v}^0)$. Then there exists a constant C independent of ε, such that*

$$\|f^\varepsilon\|_{L_T^\infty L_{xy}^2} \leq C\varepsilon^{3/4}; \qquad \|\partial_t f^\varepsilon\|_{L_T^\infty L_{xy}^2} \leq C\varepsilon^{3/4}.$$

Proof. First it follows from the definition of U^a, $\vec{V}^a$, f^ε, (5.7) and (5.12) that

$$
\begin{aligned}
f^\varepsilon =& \varepsilon^{1/2}\partial_x^2 u^{B,1} + \varepsilon^{1/2}\partial_y^2 u^{B,1} + \varepsilon\partial_x^2 u^{B,2} + \varepsilon\partial_y^2 u^{B,2} \\
&- \varepsilon\varphi(y)\partial_x^2 u^{B,2}(x,0,t) - \varepsilon u^{B,2}(x,0,t)\partial_y^2\varphi(y) \\
&+ \partial_x\big[-\varepsilon\varphi(y)u^{B,2}(x,0,t)\big(v_1^{I,0} + \varepsilon^{1/2}v_1^{I,1} + \varepsilon^{1/2}v_1^{B,1} + \varepsilon v_1^{B,2}\big)\big] \\
&+ \partial_y\big[-\varepsilon\varphi(y)u^{B,2}(x,0,t)\big(v_2^{I,0} + v_2^{B,0} + \varepsilon^{1/2}v_2^{I,1} + \varepsilon^{1/2}v_2^{B,1}\big)\big] \\
&+ \partial_x\big[\big(u^{I,0} + \varepsilon^{1/2}u^{I,1}\big)\big(\varepsilon^{1/2}v_1^{B,1} + \varepsilon v_1^{B,2}\big)\big] + \varepsilon\partial_x(u^{I,1}v_1^{I,1}) \\
&+ \partial_x\big[\big(\varepsilon^{1/2}u^{B,1} + \varepsilon u^{B,2}\big)\big(v_1^{I,0} + \varepsilon^{1/2}v_1^{I,1} + \varepsilon^{1/2}v_1^{B,1} + \varepsilon v_1^{B,2}\big)\big] \\
&+ \partial_y\big[\big(u^{I,0} + \varepsilon^{1/2}u^{I,1}\big)\big(v_2^{B,0} + \varepsilon^{1/2}v_2^{B,1}\big)\big] + \varepsilon\partial_y(u^{I,1}v_2^{I,1}) \\
&+ \partial_y\big[\big(\varepsilon^{1/2}u^{B,1} + \varepsilon u^{B,2}\big)\big(v_2^{I,0} + v_2^{B,0} + \varepsilon^{1/2}v_2^{I,1} + \varepsilon^{1/2}v_2^{B,1}\big)\big] \\
&- \varepsilon^{1/2}\partial_t u^{B,1} - \varepsilon\partial_t u^{B,2} + \varepsilon\varphi(y)\partial_t u^{B,2}(x,0,t).
\end{aligned}
$$

Moreover, from the transformation $\eta = \frac{y}{\sqrt{\varepsilon}}$ and (3.114), (3.116) we deduce that

$$
\begin{aligned}
\varepsilon^{1/2}\partial_y^2 u^{B,1} =& \varepsilon^{-1/2}\partial_\eta^2 u^{B,1} = -\varepsilon^{-1/2}u^{I,0}(x,0,t)\partial_\eta v_2^{B,0} = -u^{I,0}(x,0,t)\partial_y v_2^{B,0} \\
\varepsilon\partial_y^2 u^{B,2} =& -\varepsilon^{1/2}u^{I,0}(x,0,t)\partial_y v_2^{B,1} - \varepsilon^{1/2}(u^{I,1}(x,0,t) + u^{B,1})\partial_y v_2^{B,0} \\
&- \varepsilon^{1/2}\partial_y u^{B,1}(v_2^{I,0}(x,0,t) + v_2^{B,0}) - y\partial_y u^{I,0}(x,0,t)\partial_y v_2^{B,0} \\
&- \partial_y u^{I,0}(x,0,t)v_2^{B,0},
\end{aligned}
$$

which, substituted into the above expression for f^ε gives rise to

$$
\begin{aligned}
f^\varepsilon =& \varepsilon^{1/2}\partial_x^2 u^{B,1} + \varepsilon\partial_x^2 u^{B,2} - \varepsilon\varphi(y)\partial_x^2 u^{B,2}(x,0,t) - \varepsilon\partial_y^2\varphi(y)u^{B,2}(x,0,t) \\
&+ \partial_x\big[-\varepsilon\varphi(y)u^{B,2}(x,0,t)\big(v_1^{I,0} + \varepsilon^{1/2}v_1^{I,1} + \varepsilon^{1/2}v_1^{B,1} + \varepsilon v_1^{B,2}\big)\big] \\
&+ \partial_y\big[-\varepsilon\varphi(y)u^{B,2}(x,0,t)\big(v_2^{I,0} + v_2^{B,0} + \varepsilon^{1/2}v_2^{I,1} + \varepsilon^{1/2}v_2^{B,1}\big)\big] \\
&+ \partial_x\big[\big(u^{I,0} + \varepsilon^{1/2}u^{I,1}\big)\big(\varepsilon^{1/2}v_1^{B,1} + \varepsilon v_1^{B,2}\big)\big] + \varepsilon\partial_x(u^{I,1}v_1^{I,1}) + \varepsilon\partial_y(u^{I,1}v_2^{I,1}) \\
&+ \partial_x\big[\big(\varepsilon^{1/2}u^{B,1} + \varepsilon u^{B,2}\big)\big(v_1^{I,0} + \varepsilon^{1/2}v_1^{I,1} + \varepsilon^{1/2}v_1^{B,1} + \varepsilon v_1^{B,2}\big)\big] \\
&+ \big(u^{I,0}(x,y,t) - u^{I,0}(x,0,t) - y\partial_y u^{I,0}(x,0,t)\big)\partial_y v_2^{B,0} \\
&+ \big(\partial_y u^{I,0}(x,y,t) - \partial_y u^{I,0}(x,0,t)\big)v_2^{B,0} + \varepsilon^{1/2}\big(u^{I,0}(x,y,t) - u^{I,0}(x,0,t)\big)\partial_y v_2^{B,1} \\
&+ \varepsilon^{1/2}\big(u^{I,1}(x,y,t) - u^{I,1}(x,0,t)\big)\partial_y v_2^{B,0} + \varepsilon^{1/2}\big(v_2^{I,0}(x,y,t) - v_2^{I,0}(x,0,t)\big)\partial_y u^{B,1} \\
&+ \varepsilon^{1/2}\big[\partial_y u^{I,0}v_2^{B,1} + \partial_y u^{I,1}v_2^{B,0} + \partial_y v_2^{I,0}u^{B,1}\big] \\
&+ \varepsilon\partial_y\big[u^{I,1}v_2^{B,1} + u^{B,1}\big(v_2^{I,1} + v_2^{B,1}\big) + u^{B,2}\big(v_2^{I,0} + v_2^{B,0} + \varepsilon^{1/2}v_2^{I,1} + \varepsilon^{1/2}v_2^{B,1}\big)\big] \\
&- \varepsilon^{1/2}\partial_t u^{B,1} - \varepsilon\partial_t u^{B,2} + \varepsilon\varphi(y)\partial_t u^{B,2}(x,0,t) \\
:=& \sum_{i=1}^{11} K_i,
\end{aligned} \tag{5.67}
$$

where K_i represents the entirety of the i-th line in the above expression. We first prove $\|f^\varepsilon\|_{L_T^\infty L_{xy}^2} \le C\varepsilon^{3/4}$ by estimating each K_i $(1 \le i \le 10)$. Indeed, (5.59), (5.60), (5.61) and

(5.62) lead to

$$
\begin{aligned}
\|K_3\|_{L_T^\infty L_{xy}^2} \leq & \varepsilon\|\phi\|_{L_y^\infty}\|u^{B,2}(x,0,t)\|_{L_T^\infty L_x^\infty}\big(\|\partial_y v_2^{I,0}\|_{L_T^\infty L_{xy}^2}+\|\partial_y v_2^{B,0}\|_{L_T^\infty L_{xy}^2} \\
& \qquad\qquad +\|\partial_y v_2^{I,1}\|_{L_T^\infty L_{xy}^2}+\|\partial_y v_2^{B,1}\|_{L_T^\infty L_{xy}^2}\big) \\
& +\varepsilon\|\partial_y\phi\|_{L_y^2}\|u^{B,2}(x,0,t)\|_{L_T^\infty L_x^2}\big(\|v_2^{I,0}\|_{L_T^\infty L_{xy}^\infty}+\|v_2^{B,0}\|_{L_T^\infty L_{xy}^\infty} \\
& \qquad\qquad +\|v_2^{I,1}\|_{L_T^\infty L_{xy}^\infty}+\|v_2^{B,1}\|_{L_T^\infty L_{xy}^\infty}\big) \\
\leq & C\varepsilon^{3/4}\|u^{B,2}\|_{L_T^\infty H_{x\eta}^2}\big(\|v_2^{I,0}\|_{L_T^\infty H^3}+\|v_2^{B,0}\|_{L_T^\infty H_{x\eta}^2}+\|v_2^{I,1}\|_{L_T^\infty H^3}+\|v_2^{B,1}\|_{L_T^\infty H_{x\eta}^2}\big) \\
\leq & C\varepsilon^{3/4},
\end{aligned}
$$

where $0<\varepsilon<1$ has been used. Similar argument further gives the estimates for K_2, K_1 and K_{11} as follows:

$$
\begin{aligned}
\|K_2\|_{L_T^\infty L_{xy}^2} \leq & C\varepsilon^{3/4}\|u^{B,2}\|_{L_T^\infty H_{x\eta}^2}\left(\|v_1^{I,0}\|_{L_T^\infty H^3}+\|v_1^{B,1}\|_{L_T^\infty H_{x\eta}^2}\right) \\
& +C\varepsilon^{3/4}\|u^{B,2}\|_{L_T^\infty H_{x\eta}^2}\left(\|v_1^{I,1}\|_{L_T^\infty H^3}+\|v_1^{B,2}\|_{L_T^\infty H_{x\eta}^2}\right) \\
\leq & C\varepsilon^{3/4}
\end{aligned}
$$

and

$$
\begin{aligned}
\|K_1\|_{L_T^\infty L_{xy}^2} \leq & \varepsilon^{3/4}\|u^{B,1}\|_{L_T^\infty H_x^2 L_\eta^2}+\varepsilon^{5/4}\|u^{B,2}\|_{L_T^\infty H_x^2 L_\eta^2} \\
& +C_0\varepsilon\left(\|u^{B,2}\|_{L_T^\infty H_x^2 H_\eta^1}+\|u^{B,2}\|_{L_T^\infty L_x^2 H_\eta^1}\right) \\
\leq & C\varepsilon^{3/4}
\end{aligned}
$$

and

$$
\begin{aligned}
\|K_{11}\|_{L_T^\infty L_{xy}^2} \leq & \varepsilon^{3/4}\|\partial_t u^{B,1}\|_{L_T^\infty L_{x\eta}^2}+\varepsilon^{5/4}\|\partial_t u^{B,2}\|_{L_T^\infty L_{x\eta}^2} \\
& +C_0\varepsilon\|\varphi(y)\|_{L_y^2}\|\partial_t u^{B,2}\|_{L_T^\infty L_x^2 H_\eta^1} \\
\leq & C\varepsilon^{3/4}.
\end{aligned}
$$

By the Sobolev embedding inequality and (5.59) we have that

$$
\begin{aligned}
& \|K_4\|_{L_T^\infty L_{xy}^2} \\
\leq & \left(\|\partial_x u^{I,0}\|_{L_T^\infty L_{xy}^\infty}+\varepsilon^{1/2}\|\partial_x u^{I,1}\|_{L_T^\infty L_{xy}^\infty}\right)\left(\varepsilon^{1/2}\|v_1^{B,1}\|_{L_T^\infty L_{xy}^2}+\varepsilon\|v_1^{B,2}\|_{L_T^\infty L_{xy}^2}\right) \\
& +\left(\|u^{I,0}\|_{L_T^\infty L_{xy}^\infty}+\varepsilon^{1/2}\|u^{I,1}\|_{L_T^\infty L_{xy}^\infty}\right)\left(\varepsilon^{1/2}\|\partial_x v_1^{B,1}\|_{L_T^\infty L_{xy}^2}+\varepsilon\|\partial_x v_1^{B,2}\|_{L_T^\infty L_{xy}^2}\right) \\
& +\varepsilon\|\nabla u^{I,1}\|_{L_T^\infty L_{xy}^\infty}\|\vec{v}^{I,1}\|_{L_T^\infty L_{xy}^2}+\varepsilon\|u^{I,1}\|_{L_T^\infty L_{xy}^\infty}\|\nabla\vec{v}^{I,1}\|_{L_T^\infty L_{xy}^2} \\
\leq & C_0\big(\|u^{I,0}\|_{L_T^\infty H^3}+\varepsilon^{1/2}\|u^{I,1}\|_{L_T^\infty H^3}\big)\left(\varepsilon^{3/4}\|v_1^{B,1}\|_{L_T^\infty H_x^1 L_\eta^2}+\varepsilon^{5/4}\|v_1^{B,2}\|_{L_T^\infty H_x^1 L_\eta^2}\right) \\
& +C_0\varepsilon\|u^{I,1}\|_{L_T^\infty H^3}\|\vec{v}^{I,1}\|_{L_T^\infty H^1} \\
\leq & C\varepsilon^{3/4}.
\end{aligned}
$$

To bound K_5, K_9 and K_{10}, we use (5.59), (5.62) the similar argument in estimating K_4 and derive

$$\begin{aligned}\|K_5\|_{L_T^\infty L_{xy}^2} \le & C_0\varepsilon^{3/4}\left(\|\vec{v}^{I,0}\|_{L_T^\infty H^3}+\|\vec{v}^{I,1}\|_{L_T^\infty H^3}+\|v_1^{B,1}\|_{L_T^\infty H_x^3 H_\eta^2}\right)\\ & \times\left(\|u^{B,1}\|_{L_T^\infty H_x^1 L_\eta^2}+\|u^{B,2}\|_{L_T^\infty H_x^1 L_\eta^2}\right)\\ \le & C\varepsilon^{3/4}\end{aligned}$$

and

$$\begin{aligned}\|K_9\|_{L_T^\infty L_{xy}^2} \le & C_0\varepsilon^{3/4}\left(\|u^{I,0}\|_{L_T^\infty H^3}\|v_2^{B,1}\|_{L_T^\infty L_{x\eta}^2}+\|u^{I,1}\|_{L_T^\infty H^3}\|v_2^{B,0}\|_{L_T^\infty L_{x\eta}^2}\right)\\ & +C_0\varepsilon^{3/4}\|\vec{v}^{I,0}\|_{L_T^\infty H^3}\|u^{B,1}\|_{L_T^\infty L_{x\eta}^2}\\ \le & C\varepsilon^{3/4}\end{aligned}$$

and

$$\begin{aligned}&\|K_{10}\|_{L_T^\infty L_{xy}^2}\\ \le & C_0\varepsilon^{3/4}\left[\|u^{I,1}\|_{L_T^\infty H^3}\|v_2^{B,1}\|_{L_T^\infty L_x^2 H_\eta^1}+\left(\|v_2^{I,1}\|_{L_T^\infty H^3}+\|v_2^{B,1}\|_{L_T^\infty H_x^2 H_\eta^3}\right)\|u^{B,1}\|_{L_T^\infty L_x^2 H_\eta^1}\right]\\ & +C_0\varepsilon^{3/4}\left(\|v_2^{I,0}\|_{L_T^\infty H^3}+\|v_2^{B,0}\|_{L_T^\infty H_x^2 H_\eta^3}+\|v_2^{I,1}\|_{L_T^\infty H^3}+\|v_2^{B,1}\|_{L_T^\infty H_x^2 H_\eta^3}\right)\|u^{B,2}\|_{L_T^\infty L_x^2 H_\eta^1}\\ \le & C\varepsilon^{3/4}.\end{aligned}$$

We come to estimate K_6 by applying the change of variables $y=\varepsilon^{1/2}z$, Taylor's formula, (5.59), Theorem 5.1 and Lemma 5.1 to arrive at

$$\begin{aligned}\|K_6\|_{L_T^\infty L_{xy}^2} = & \varepsilon\left\|\frac{u^{I,0}(x,y,t)-u^{I,0}(x,0,t)-y\partial_y u^{I,0}(x,0,t)}{y^2}\cdot\eta^2\partial_y v_2^{B,0}\right\|_{L_T^\infty L_{xy}^2}\\ \le & \varepsilon\|\partial_y^2 u^{I,0}\|_{L_T^\infty L_{xy}^\infty}\|\eta^2\partial_y v_2^{B,0}\|_{L_T^\infty L_{xy}^2}\\ \le & C_0\varepsilon^{3/4}\|u^{I,0}\|_{L_T^\infty H^4}\|\langle\eta\rangle^2\partial_\eta v_2^{B,0}\|_{L_T^\infty L_{x\eta}^2}\\ \le & C\varepsilon^{3/4}.\end{aligned}$$

A similar arguments as estimating K_5 also leads to

$$\begin{aligned}\|K_7\|_{L_T^\infty L_{xy}^2} \le & C_0\varepsilon^{3/4}\left(\|u^{I,0}\|_{L_T^\infty H^4}\|\langle\eta\rangle v_2^{B,0}\|_{L_T^\infty L_{x\eta}^2}+\|u^{I,0}\|_{L_T^\infty H^3}\|\langle\eta\rangle\partial_\eta v_2^{B,1}\|_{L_T^\infty L_{x\eta}^2}\right)\\ \le & C\varepsilon^{3/4}\end{aligned}$$

and

$$\begin{aligned}\|K_8\|_{L_T^\infty L_{xy}^2} \le & C_0\varepsilon^{3/4}\left(\|u^{I,1}\|_{L_T^\infty H^3}\|\langle\eta\rangle\partial_\eta v_2^{B,0}\|_{L_T^\infty L_{x\eta}^2}+\|\vec{v}^{I,0}\|_{L_T^\infty H^3}\|\langle\eta\rangle\partial_\eta u^{B,1}\|_{L_T^\infty L_{x\eta}^2}\right)\\ \le & C\varepsilon^{3/4}.\end{aligned}$$

Substituting the above estimates for K_1 to K_{11} into (5.67) we conclude that

$$\|f^{\varepsilon}\|_{L_T^\infty L_{xy}^2} \le C\varepsilon^{3/4}.$$

It remains to prove $\|\partial_t f^{\varepsilon}\|_{L_T^\infty L_{xy}^2} \le C\varepsilon^{3/4}$. To this end, we first note that with Banach spaces X,Y,Z if $\|fg\|_Z \le C_0\|f\|_X\|g\|_Y$ holds for all $f\in X$, $g\in Y$, then it follows that

$$\|\partial_t(fg)\|_Z \le \|\partial_t f\|_X\|g\|_Y + \|f\|_X\|\partial_t g\|_Y, \tag{5.68}$$

provided that $\partial_t f\in X$ and $\partial_t g\in Y$. Thus from the estimates on K_3, (5.68) and Lemma 5.6-Lemma 5.9, one deduces that

$$\begin{aligned}
&\|\partial_t K_3\|_{L_T^\infty L_{xy}^2}\\
\le & C\varepsilon^{3/4}\|u^{B,2}\|_{L_T^\infty H_{x\eta}^2}\big(\|\partial_t v_2^{I,0}\|_{L_T^\infty H^3}+\|\partial_t v_2^{B,0}\|_{L_T^\infty H_{x\eta}^2}+\|\partial_t v_2^{I,1}\|_{L_T^\infty H^3}+\|\partial_t v_2^{B,1}\|_{L_T^\infty H_{x\eta}^2}\big)\\
&+C\varepsilon^{3/4}\|\partial_t u^{B,2}\|_{L_T^\infty H_{x\eta}^2}\big(\|v_2^{I,0}\|_{L_T^\infty H^3}+\|v_2^{B,0}\|_{L_T^\infty H_{x\eta}^2}+\|v_2^{I,1}\|_{L_T^\infty H^3}+\|v_2^{B,1}\|_{L_T^\infty H_{x\eta}^2}\big)\\
\le & C\varepsilon^{3/4}.
\end{aligned}$$

Similarly it follows from (5.68) and the above estimates on K_1, K_2 and K_3 to K_{11} that

$$\|\partial_t K_i\|_{L_T^\infty L_{xy}^2} \le C\varepsilon^{3/4}, \qquad i=1,2,4,5,\cdots,11.$$

Combing the above estimates for $\partial_t K_1$ to $\partial_t K_{11}$ with (5.67) we end up with $\|\partial_t f^{\varepsilon}\|_{L_T^\infty L_{xy}^2} \le C\varepsilon^{3/4}$ and the proof is completed.

□

Lemma 5.7. *Suppose the assumptions in Theorem 5.2 hold. Let* $0<T<\infty$ *be less than the maximal existence time of* $(u^0,\vec v^0)$. *Then there exists a positive constant* C *independent of* ε, *depending on* T *such that*

$$\|\vec g^{\varepsilon}\|_{L_T^\infty L_{xy}^2} \le C\varepsilon; \qquad \|\partial_t \vec g^{\varepsilon}\|_{L_T^\infty L_{xy}^2} \le C\varepsilon.$$

Proof. By the definition of $\vec g^{\varepsilon}$ in (5.56) we write its first component g_1^{ε} as follows:

$$\begin{aligned}
g_1^{\varepsilon} =&\big[\varepsilon\Delta v_1^{I,0}+\varepsilon^{3/2}\Delta v_1^{I,1}+\varepsilon^{3/2}\partial_x^2 v_1^{B,1}+\varepsilon^2\partial_x^2 v_1^{B,2}+\varepsilon^2\partial_y^2 v_1^{B,2}+\varepsilon\partial_x u^{B,2}-\varepsilon\varphi(y)\partial_x u^{B,2}(x,0,t)\big]\\
&-\big[2\varepsilon\vec V^a\cdot\partial_x\vec V^a+\varepsilon\partial_t v_1^{B,2}\big]\\
:=&M_1-M_2,
\end{aligned}$$

where the second equation of (5.7), (5.12) and the first equation of (5.13) have been used. We proceed to estimate M_1 and M_2. First (5.59) and (5.60) lead to

$$\begin{aligned}
\|M_1\|_{L_T^\infty L_{xy}^2} \le& C_0\big(\varepsilon\|\vec v^{I,0}\|_{L_T^\infty H^2}+\varepsilon^{3/2}\|\partial_t\vec v^{I,1}\|_{L_T^\infty H^2}+\varepsilon^{7/4}\|v_1^{B,1}\|_{L_T^\infty H_x^2 L_\eta^2}\\
&+\varepsilon^{9/4}\|v_1^{B,2}\|_{L_T^\infty H_x^2 L_\eta^2}+\varepsilon^{5/4}\|v_1^{B,2}\|_{L_T^\infty L_x^2 H_\eta^2}+\varepsilon^{5/4}\|u^{B,2}\|_{L_T^\infty H_x^1 H_\eta^1}\big)\\
\le& C\varepsilon.
\end{aligned}$$

To bound M_2 we first estimate $\|\vec{V}^a\|_{L^\infty_T L^\infty_{xy}}$ by the Sobolev embedding inequality, (5.62) and $0<\varepsilon<1$ as follows

$$\begin{aligned}\|\vec{V}^a\|_{L^\infty_T L^\infty_{xy}} \leq & C_0\big(\|\vec{v}^{I,0}\|_{L^\infty_T H^2}+\|v_2^{B,0}\|_{L^\infty_T H^2_{x\eta}}+\varepsilon^{1/2}\|\vec{v}^{I,1}\|_{L^\infty_T H^2} \\ & +\varepsilon^{1/2}\|v_1^{B,1}\|_{L^\infty_T H^2_{x\eta}}+\varepsilon^{1/2}\|v_2^{B,1}\|_{L^\infty_T H^2_{x\eta}}+\varepsilon\|v_1^{B,2}\|_{L^\infty_T H^2_{x\eta}}\big) \\ \leq & C.\end{aligned} \tag{5.69}$$

Similar arguments further yield

$$\|\partial_t\vec{V}^a\|_{L^\infty_T L^2_{xy}},\ \|\partial_x\vec{V}^a\|_{L^\infty_T L^2_{xy}},\ \|\partial_x\partial_t\vec{V}^a\|_{L^\infty_T L^2_{xy}} \leq C. \tag{5.70}$$

Thus by (5.69), (5.70) and (5.62) we obtain

$$\|M_2\|_{L^\infty_T L^2_{xy}} \leq C_0\varepsilon(\|\vec{V}^a\|_{L^\infty_T L^\infty_{xy}}\|\partial_x\vec{V}^a\|_{L^\infty_T L^2_{xy}}+\|\partial_t v_1^{B,2}\|_{L^\infty_T L^2_{xy}}) \leq C\varepsilon.$$

Hence from the above estimates for M_1, M_2 one derives $\|g_1^\varepsilon\|_{L^\infty_T L^2_{xy}} \leq C\varepsilon$. By (5.68), the above estimates for M_1, M_2 and (5.70), we deduce that $\|\partial_t g_1^\varepsilon\|_{L^\infty_T L^2_{xy}} \leq C\varepsilon$. It remains to estimate g_2^ε and $\partial_t g_2^\varepsilon$. Indeed from the definition of $\vec{g}^\varepsilon$ it follows that

$$\begin{aligned} g_2^\varepsilon = & \big[\varepsilon\Delta v_2^{I,0}+\varepsilon^{3/2}\Delta v_2^{I,1}+\varepsilon\partial_x^2 v_2^{B,0}+\varepsilon^{3/2}\partial_x^2 v_2^{B,1}-\varepsilon\partial_y\varphi(y)u^{B,2}(x,0,t)\big] \\ & +\big[2\varepsilon(v_2^{I,0}(x,0,t)-v_2^{I,0}(x,y,t))\partial_y v_2^{B,0}-2\varepsilon\partial_y v_2^{B,0}(\varepsilon^{1/2}v_2^{I,1}+\varepsilon^{1/2}v_2^{B,1})\big] \\ & -2\varepsilon(v_1^{I,0}+\varepsilon^{1/2}v_1^{I,1}+\varepsilon^{1/2}v_1^{B,1}+\varepsilon v_1^{B,2})(\partial_y v_1^{I,0}+\varepsilon^{1/2}\partial_y v_1^{I,1}+\varepsilon^{1/2}\partial_y v_1^{B,1}+\varepsilon\partial_y v_1^{B,2}) \\ & -2\varepsilon(v_2^{I,0}+v_2^{B,0}+\varepsilon^{1/2}v_2^{I,1}+\varepsilon^{1/2}v_2^{B,1})(\partial_y v_2^{I,0}+\varepsilon^{1/2}\partial_y v_2^{I,1}+\varepsilon^{1/2}\partial_y v_2^{B,1}) \\ := & M_3+M_4-M_5-M_6,\end{aligned}$$

where the second equation of (5.7), (5.12) and $F_2^0=F_2^1=0$ in (3.126) have been used. First by (5.59) and (5.60) we get

$$\begin{aligned}\|M_3\|_{L^\infty_T L^2_{xy}} \leq & C_0\varepsilon(\|\vec{v}^{I,0}\|_{L^\infty_T H^2}+\|\vec{v}^{I,1}\|_{L^\infty_T H^2}+\|v_2^{B,0}\|_{L^\infty_T H^2_x L^2_\eta} \\ & +\|v_2^{B,1}\|_{L^\infty_T H^2_x L^2_\eta}+\|u^{B,2}\|_{L^\infty_T L^2_x H^1_\eta}) \leq C\varepsilon.\end{aligned}$$

The boundness of M_4 follows from an analogous argument as estimating K_6 and (5.62):

$$\begin{aligned}\|M_4\|_{L^\infty_T L^2_{xy}} \leq & C_0\varepsilon^{5/4}\|v_2^{B,0}\|_{L^\infty_T L^2_x H^1_\eta}(\|\vec{v}^{I,1}\|_{L^\infty_T H^2}+\|v_2^{B,1}\|_{L^\infty_T H^2_{x\eta}}) \\ & +C_0\varepsilon^{5/4}\|v_2^{I,0}\|_{L^\infty_T H^3}\|\langle\eta\rangle v_2^{B,0}\|_{L^\infty_T L^2_x H^1_\eta} \leq C\varepsilon^{5/4}.\end{aligned}$$

Similarly as estimating K_4 we use the Cauchy-Schwarz inequality, (5.59) and (5.62) to derive

$$\begin{aligned}\|M_5\|_{L^\infty_T L^2_{xy}} \leq & C_0\varepsilon(\|\vec{v}^{I,0}\|_{L^\infty_T H^2}+\|\vec{v}^{I,1}\|_{L^\infty_T H^2}+\|v_1^{B,1}\|_{L^\infty_T H^2_{x\eta}}+\|v_1^{B,2}\|_{L^\infty_T H^2_{x\eta}}) \\ & \times(\|\partial_y\vec{v}^{I,0}\|_{L^\infty_T L^2_{xy}}+\|\partial_y\vec{v}^{I,1}\|_{L^\infty_T L^2_{xy}}+\|\partial_\eta v_1^{B,1}\|_{L^\infty_T L^2_{x\eta}}+\|\partial_\eta v_1^{B,2}\|_{L^\infty_T L^2_{x\eta}}) \\ \leq & C\varepsilon.\end{aligned}$$

Moreover, $\|M_6\|_{L_T^\infty L_{xy}^2} \le C\varepsilon$ follows from a similar argument. Now collecting the above estimates from M_3 to M_6, we conclude that $\|g_2^\varepsilon\|_{L_T^\infty L_{xy}^2} \le C\varepsilon$. Finally, by (5.68), the above estimates from M_3 to M_6 we deduce that $\|\partial_t g_2^\varepsilon\|_{L_T^\infty L_{xy}^2} \le C\varepsilon$. The proof is completed.

□

We next exhibit the L^2 estimates for U^ε and $\vec{V}^\varepsilon$.

Lemma 5.8. *Suppose that the assumptions in Proposition 5.4 hold. Denote $C_1 = \max\{C_0, \tilde{C}_0\}$ with C_0 and $\tilde{C}_0$ derived in* (5.76) *and* (5.77)*, respectively. Denote $C_2 = C_3 T e^{(3C_1+C_3)T}$ with C_3 derived in* (5.75)*. Set $\varepsilon_T = \min\{(2C_2)^{-2}, (12C_1)^{-2}, 1\}$. Assume further that the solution $(U^\varepsilon, \vec{V}^\varepsilon)(x,y,t)$ of* (5.55) *on $[0,T]$ satisfies*

$$\|U^\varepsilon\|_{L_T^\infty L^2}^2 + \|\vec{V}^\varepsilon\|_{L_T^\infty L^2}^2 < 1. \tag{5.71}$$

Then for any $\varepsilon \in (0, \varepsilon_T]$ the following holds true:

$$\|U^\varepsilon\|_{L_T^\infty L^2}^2 + \|\vec{V}^\varepsilon\|_{L_T^\infty L^2}^2 \le C_2 \varepsilon^{1/2} < \frac{1}{2}. \tag{5.72}$$

Moreover, there exists a constant C independent of ε such that

$$\|\nabla U^\varepsilon\|_{L_T^2 L^2}^2 + \varepsilon \|\nabla \vec{V}^\varepsilon\|_{L_T^2 L^2}^2 \le C\varepsilon^{1/2}. \tag{5.73}$$

Proof. First, it follows from a similar argument as deriving (5.69) that

$$\|U^a\|_{L_T^\infty L_{xy}^\infty} \le C, \quad \|\partial_t U^a\|_{L_T^\infty L_{xy}^\infty} \le C, \quad \|\partial_t \vec{V}^a\|_{L_T^\infty L_{xy}^\infty} \le C. \tag{5.74}$$

Thus we conclude from (5.74), (5.69), Lemma 5.6 and Lemma 5.7 that there exists a constant C_3 independent of ε, depending on T satisfying:

$$8\left(\|U^a\|_{L_T^\infty L_{xy}^\infty}^2 + \|\vec{V}^a\|_{L_T^\infty L_{xy}^\infty}^2\right) \le C_3; \qquad \left(\|f^\varepsilon\|_{L_T^\infty L_{xy}^2}^2 + \|\vec{g}^\varepsilon\|_{L_T^\infty L_{xy}^2}^2\right) \le C_3 \varepsilon^{3/2}. \tag{5.75}$$

We proceed by taking the L^2 inner products of the first and second equations of (5.55) with $2U^\varepsilon$ and $2\vec{V}^\varepsilon$ respectively, then adding the results to obtain

$$\begin{aligned}
&\frac{d}{dt}\left(\|U^\varepsilon(t)\|_{L^2}^2 + \|\vec{V}^\varepsilon(t)\|_{L^2}^2\right) + 2\|\nabla U^\varepsilon(t)\|_{L^2}^2 + 2\varepsilon\|\nabla \vec{V}^\varepsilon(t)\|_{L^2}^2 \\
&\quad = 2\int_0^\infty \int_{-\infty}^\infty \left(-\varepsilon^{1/2} U^\varepsilon \vec{V}^\varepsilon \cdot \nabla U^\varepsilon + \varepsilon^{3/2} |\vec{V}^\varepsilon|^2 \nabla \cdot \vec{V}^\varepsilon\right) dxdy \\
&\quad + 2\int_0^\infty \int_{-\infty}^\infty \left(-U^\varepsilon \vec{V}^a \cdot \nabla U^\varepsilon - U^a \vec{V}^\varepsilon \cdot \nabla U^\varepsilon + 2\varepsilon\left(\vec{V}^a \cdot \vec{V}^\varepsilon\right) \nabla \cdot \vec{V}^\varepsilon\right) dxdy \\
&\quad + 2\int_0^\infty \int_{-\infty}^\infty \left(\varepsilon^{-1/2} f^\varepsilon U^\varepsilon + \nabla U^\varepsilon \cdot \vec{V}^\varepsilon + \varepsilon^{-1/2} \vec{g}^\varepsilon \cdot \vec{V}^\varepsilon\right) dxdy \\
&:= I_1 + I_2 + I_3.
\end{aligned}$$

The estimate for I_1 follows from (5.63), (5.64) and the Cauchy-Schwarz inequality as:

$$
\begin{aligned}
I_1 \leq & 2\varepsilon^{1/2}\|U^\varepsilon\|_{L^4}\|\vec{V}^\varepsilon\|_{L^4}\|\nabla U^\varepsilon\|_{L^2}+2\varepsilon^{3/2}\|\vec{V}^\varepsilon\|_{L^4}^2\|\nabla\vec{V}^\varepsilon\|_{L^2} \\
\leq & C_0\varepsilon^{1/2}\|U^\varepsilon\|_{L^2}^{1/2}\|\nabla U^\varepsilon\|_{L^2}^{3/2}(\|\vec{V}^\varepsilon\|_{L^2}^{1/2}\|\nabla\vec{V}^\varepsilon\|_{L^2}^{1/2}+\|\vec{V}^\varepsilon\|_{L^2}) \\
& +C_0\varepsilon^{3/2}(\|\vec{V}^\varepsilon\|_{L^2}\|\nabla\vec{V}^\varepsilon\|_{L^2}+\|\vec{V}^\varepsilon\|_{L^2}^2)\|\nabla\vec{V}^\varepsilon\|_{L^2} \\
\leq & \frac{1}{2}\|\nabla U^\varepsilon\|_{L^2}^2+\frac{1}{4}\varepsilon\|\nabla\vec{V}^\varepsilon\|_{L^2}^2+C_0(\varepsilon^2\|U^\varepsilon\|_{L^2}^2\|\vec{V}^\varepsilon\|_{L^2}^2+\varepsilon)\|\vec{V}^\varepsilon\|_{L^2}^2 \\
& +C_0(\varepsilon^2\|U^\varepsilon\|_{L^2}^2\|\vec{V}^\varepsilon\|_{L^2}^2+\varepsilon^{3/2}\|\vec{V}^\varepsilon\|_{L^2}+\varepsilon^2\|\vec{V}^\varepsilon\|_{L^2}^2)\|\nabla\vec{V}^\varepsilon\|_{L^2}^2 \\
\leq & \frac{1}{2}\|\nabla U^\varepsilon\|_{L^2}^2+\frac{1}{2}\varepsilon\|\nabla\vec{V}^\varepsilon\|_{L^2}^2+2C_0\|\vec{V}^\varepsilon\|_{L^2}^2,
\end{aligned}
\tag{5.76}
$$

where we have used the estimates $C_0(\varepsilon^2\|U^\varepsilon\|_{L^2}^2\|\vec{V}^\varepsilon\|_{L^2}^2+\varepsilon^{3/2}\|\vec{V}^\varepsilon\|_{L^2}+\varepsilon^2\|\vec{V}^\varepsilon\|_{L^2}^2)<3C_0\varepsilon^{3/2}<\frac{1}{4}\varepsilon$ and $(\varepsilon^2\|U^\varepsilon\|_{L^2}^2\|\vec{V}^\varepsilon\|_{L^2}^2+\varepsilon)<2$ thanks to (5.71) and the assumption $\varepsilon\in(0,\varepsilon_T]$. Moreover, by the Cauchy-Schwarz inequality and (5.75), we deduce that

$$
\begin{aligned}
I_2 \leq & \frac{1}{4}\|\nabla U^\varepsilon\|_{L^2}^2+\frac{1}{2}\varepsilon\|\nabla\vec{V}^\varepsilon\|_{L^2}^2+8\|\vec{V}^a\|_{L^\infty}^2\|U^\varepsilon\|_{L^2}^2 \\
& +8\|U^a\|_{L^\infty}^2\|\vec{V}^\varepsilon\|_{L^2}^2+8\varepsilon\|\vec{V}^a\|_{L^\infty}^2\|\vec{V}^\varepsilon\|_{L^2}^2 \\
\leq & \frac{1}{4}\|\nabla U^\varepsilon\|_{L^2}^2+\frac{1}{2}\varepsilon\|\nabla\vec{V}^\varepsilon\|_{L^2}^2+C_3(\|U^\varepsilon\|_{L^2}^2+\|\vec{V}^\varepsilon\|_{L^2}^2).
\end{aligned}
$$

It follows from Lemma 5.6 and Lemma 5.7 that

$$
\begin{aligned}
I_3 \leq & \frac{1}{4}\|\nabla U^\varepsilon\|_{L^2}^2+\tilde{C}_0(\|U^\varepsilon\|_{L^2}^2+\|\vec{V}^\varepsilon\|_{L^2}^2)+\varepsilon^{-1}(\|f^\varepsilon\|_{L^2}^2+\|\vec{g}^\varepsilon\|_{L^2}^2) \\
\leq & \frac{1}{4}\|\nabla U^\varepsilon\|_{L^2}^2+\tilde{C}_0(\|U^\varepsilon\|_{L^2}^2+\|\vec{V}^\varepsilon\|_{L^2}^2)+C_3\varepsilon^{1/2}.
\end{aligned}
\tag{5.77}
$$

Now collecting the above estimates for I_1- I_3 one gets that

$$
\begin{aligned}
& \frac{d}{dt}(\|U^\varepsilon(t)\|_{L^2}^2+\|\vec{V}^\varepsilon(t)\|_{L^2}^2)+\|\nabla U^\varepsilon(t)\|_{L^2}^2+\varepsilon\|\nabla\vec{V}^\varepsilon(t)\|_{L^2}^2 \\
\leq & (2C_0+\tilde{C}_0+C_3)(\|U^\varepsilon(t)\|_{L^2}^2+\|\vec{V}^\varepsilon(t)\|_{L^2}^2)+C_3\varepsilon^{1/2},
\end{aligned}
\tag{5.78}
$$

which, along with Gronwall's inequality and $\varepsilon\in(0,\varepsilon_T]$ yields (5.72). Finally integrating (5.78) over $[0,T]$ and using (5.72) we derive (5.73). The proof is completed.

□

The H^2 regularity estimate on U^ε and $\vec{V}^\varepsilon$ is given in the following lemma.

Lemma 5.9. *Let the assumptions in Lemma 5.8 hold true. Then there exists a constant C independent of ε such that*

$$
\begin{aligned}
& \|\nabla U^\varepsilon\|_{L_T^\infty L^2}^2+\varepsilon\|\nabla\vec{V}^\varepsilon\|_{L_T^\infty L^2}^2+\|\partial_t U^\varepsilon\|_{L_T^\infty L^2}^2 \\
& +\|\partial_t\vec{V}^\varepsilon\|_{L_T^\infty L^2}^2+\|\nabla\partial_t U^\varepsilon\|_{L_T^2 L^2}^2+\varepsilon\|\nabla\partial_t\vec{V}^\varepsilon\|_{L_T^2 L^2}^2\leq C\varepsilon^{1/2}.
\end{aligned}
\tag{5.79}
$$

Consequently, it follows from (5.55) *that*

$$\varepsilon\|U^\varepsilon\|^2_{L^\infty_T H^2}+\varepsilon^2\|\vec{V}^\varepsilon\|^2_{L^\infty_T H^2}+\varepsilon^3\|\vec{V}^\varepsilon\|^2_{L^2_T H^3}\le C\varepsilon^{1/2}. \tag{5.80}$$

Proof. Taking the L^2 inner products of the first and second equation of (5.55) with $2\partial_t U^\varepsilon$ and $2\partial_t\vec{V}^\varepsilon$ respectively and using integration by parts, one derives after adding the results

$$\begin{aligned}
&\frac{d}{dt}(\|\nabla U^\varepsilon(t)\|^2_{L^2}+\varepsilon\|\nabla\vec{V}^\varepsilon(t)\|^2_{L^2})+2\|\partial_t U^\varepsilon(t)\|^2_{L^2}+2\|\partial_t\vec{V}^\varepsilon(t)\|^2_{L^2}\\
=&2\int_0^\infty\int_{-\infty}^\infty(-\varepsilon^{1/2}U^\varepsilon\vec{V}^\varepsilon\cdot\nabla\partial_t U^\varepsilon+\varepsilon^{3/2}|\vec{V}^\varepsilon|^2\nabla\cdot\partial_t\vec{V}^\varepsilon)dxdy\\
&+2\int_0^\infty\int_{-\infty}^\infty(-U^\varepsilon\vec{V}^a\cdot\nabla\partial_t U^\varepsilon-U^a\vec{V}^\varepsilon\cdot\nabla\partial_t U^\varepsilon+2\varepsilon(\vec{V}^a\cdot\vec{V}^\varepsilon)\nabla\cdot\partial_t\vec{V}^\varepsilon)dxdy\\
&+2\int_0^\infty\int_{-\infty}^\infty(\varepsilon^{-1/2}f^\varepsilon\partial_t U^\varepsilon+\nabla U^\varepsilon\cdot\partial_t\vec{V}^\varepsilon+\varepsilon^{-1/2}\vec{g}^\varepsilon\cdot\partial_t\vec{V}^\varepsilon)dxdy\\
:=&I_4+I_5+I_6.
\end{aligned}$$

By (5.63), (5.64) and Cauchy-Schwarz inequality we have

$$\begin{aligned}
I_4\le&2\varepsilon^{1/2}\|U^\varepsilon\|_{L^4}\|\vec{V}^\varepsilon\|_{L^4}\|\nabla\partial_t U^\varepsilon\|_{L^2}+2\varepsilon^{3/2}\|\vec{V}^\varepsilon\|^2_{L^4}\|\nabla\partial_t\vec{V}^\varepsilon\|_{L^2}\\
\le&C_0\varepsilon^{1/2}\|U^\varepsilon\|^{1/2}_{L^2}\|\nabla U^\varepsilon\|^{1/2}_{L^2}\left(\|\vec{V}^\varepsilon\|^{1/2}_{L^2}\|\nabla\vec{V}^\varepsilon\|^{1/2}_{L^2}+\|\vec{V}^\varepsilon\|_{L^2}\right)\|\nabla\partial_t U^\varepsilon\|_{L^2}\\
&+C_0\varepsilon^{3/2}\left(\|\vec{V}^\varepsilon\|_{L^2}\|\nabla\vec{V}^\varepsilon\|_{L^2}+\|\vec{V}^\varepsilon\|^2_{L^2}\right)\|\nabla\partial_t\vec{V}^\varepsilon\|_{L^2}\\
\le&\frac{1}{4}\|\nabla\partial_t U^\varepsilon\|^2_{L^2}+\frac{1}{4}\varepsilon\|\nabla\partial_t\vec{V}^\varepsilon\|^2_{L^2}\\
&+C_0\left(\|U^\varepsilon\|^2_{L^2}\|\nabla U^\varepsilon\|^2_{L^2}+\varepsilon^2\|\vec{V}^\varepsilon\|^2_{L^2}\|\nabla\vec{V}^\varepsilon\|^2_{L^2}+\varepsilon^2\|\vec{V}^\varepsilon\|^4_{L^2}\right).
\end{aligned}$$

Moreover, a similar argument as estimating I_2 and I_3 yields:

$$\begin{aligned}
I_5\le&\frac{1}{4}\|\nabla\partial_t U^\varepsilon\|^2_{L^2}+\frac{1}{2}\varepsilon\|\nabla\partial_t\vec{V}^\varepsilon\|^2_{L^2}+8\|U^\varepsilon\|^2_{L^2}\|\vec{V}^a\|^2_{L^\infty_{xy}}\\
&+8\|\vec{V}^\varepsilon\|^2_{L^2}\left(\|U^a\|^2_{L^\infty_{xy}}\|+\vec{V}^a\|^2_{L^\infty_{xy}}\right)\\
\le&\frac{1}{4}\|\nabla\partial_t U^\varepsilon\|^2_{L^2}+\frac{1}{2}\varepsilon\|\nabla\partial_t\vec{V}^\varepsilon\|^2_{L^2}\\
&+C_3\left(\|U^\varepsilon\|^2_{L^2}+\|\vec{V}^\varepsilon\|^2_{L^2}\right)
\end{aligned}$$

and

$$\begin{aligned}
I_6\le&14\|\nabla U^\varepsilon\|^2_{L^2}+\tilde{C}_0\left(\|\partial_t U^\varepsilon\|^2_{L^2}+\|\partial_t\vec{V}^\varepsilon\|^2_{L^2}\right)\\
&+\varepsilon^{-1}\left(\|f^\varepsilon\|^2_{L^\infty_T L^2_{xy}}+\|\vec{g}^\varepsilon\|^2_{L^\infty_T L^2_{xy}}\right)\\
\le&14\|\nabla U^\varepsilon\|^2_{L^2}+\tilde{C}_0\left(\|\partial_t U^\varepsilon\|^2_{L^2}+\|\partial_t\vec{V}^\varepsilon\|^2_{L^2}\right)+C_3\varepsilon^{1/2}.
\end{aligned}$$

We proceed by differentiating the first equation of (5.55) with respect to t, then multiplying the resulting equation with $2\partial_t U^\varepsilon$ in L^2 and using integration by parts to derive

$$\begin{aligned}
&\frac{d}{dt}\|\partial_t U^\varepsilon(t)\|_{L^2}^2+2\|\nabla\partial_t U^\varepsilon(t)\|_{L^2}^2\\
&=-2\varepsilon^{1/2}\int_0^\infty\int_{-\infty}^\infty(\partial_t U^\varepsilon\vec{V}^\varepsilon+U^\varepsilon\partial_t\vec{V}^\varepsilon)\cdot\nabla\partial_t U^\varepsilon dxdy\\
&\quad-2\int_0^\infty\int_{-\infty}^\infty\big(\partial_t(U^\varepsilon\vec{V}^a)+\partial_t(U^a\vec{V}^\varepsilon)\big)\cdot\nabla\partial_t U^\varepsilon dxdy\\
&\quad+2\varepsilon^{-1/2}\int_0^\infty\int_{-\infty}^\infty\partial_t f^\varepsilon\partial_t U^\varepsilon dxdy\\
&:=I_7+I_8+I_9.
\end{aligned}$$

The estimate for I_7 follows from (5.63), (5.64) and Cauchy-Schwarz inequality

$$\begin{aligned}
I_7\le&C_0\varepsilon^{1/2}\|\nabla\partial_t U^\varepsilon\|_{L^2}^{3/2}\|\partial_t U^\varepsilon\|_{L^2}^{1/2}(\|\vec{V}^\varepsilon\|_{L^2}^{1/2}\|\nabla\vec{V}^\varepsilon\|_{L^2}^{1/2}+\|\vec{V}^\varepsilon\|_{L^2})\\
&+C_0\varepsilon^{1/2}\|\nabla\partial_t U^\varepsilon\|_{L^2}\|\nabla U^\varepsilon\|_{L^2}^{1/2}\|U^\varepsilon\|_{L^2}^{1/2}(\|\partial_t\vec{V}^\varepsilon\|_{L^2}^{1/2}\|\nabla\partial_t\vec{V}^\varepsilon\|_{L^2}^{1/2}+\|\partial_t\vec{V}^\varepsilon\|_{L^2})\\
\le&\frac{1}{8}\|\nabla\partial_t U^\varepsilon\|_{L^2}^2+\frac{1}{8}\varepsilon\|\nabla\partial_t\vec{V}^\varepsilon\|_{L^2}^2+C_0\varepsilon^2(\|\vec{V}^\varepsilon\|_{L^2}^2\|\nabla\vec{V}^\varepsilon\|_{L^2}^2+\|\vec{V}^\varepsilon\|_{L^2}^4)\|\partial_t U^\varepsilon\|_{L^2}^2\\
&+C_0\varepsilon(\|U^\varepsilon\|_{L^2}^2\|\nabla U^\varepsilon\|_{L^2}^2+\|U^\varepsilon\|_{L^2}\|\nabla U^\varepsilon\|_{L^2})\|\partial_t\vec{V}^\varepsilon\|_{L^2}^2.
\end{aligned}$$

By (5.69), (5.74) and Cauchy-Schwarz inequality one derives

$$I_8\le\frac{1}{8}\|\nabla\partial_t U^\varepsilon\|_{L^2}^2+C(\|\partial_t U^\varepsilon\|_{L^2}^2+\|\partial_t\vec{V}^\varepsilon\|_{L^2}^2)+C(\|U^\varepsilon\|_{L^2}^2+\|\vec{V}^\varepsilon\|_{L^2}^2).$$

The Cauchy-Schwarz inequality leads to $I_9\le\|\partial_t U^\varepsilon\|_{L^2}^2+\varepsilon^{-1}\|\partial_t f^\varepsilon\|_{L^2}^2$. We next differentiate the second equation of (5.55) with respect to t, then take the L^2 inner product of $2\partial_t\vec{V}^\varepsilon$ with the resulting equation and use integration by parts to have

$$\begin{aligned}
\frac{d}{dt}\|\partial_t\vec{V}^\varepsilon(t)\|_{L^2}^2+2\varepsilon\|\nabla\partial_t\vec{V}^\varepsilon(t)\|_{L^2}^2=&4\varepsilon^{3/2}\int_0^\infty\int_{-\infty}^\infty\vec{V}^\varepsilon\cdot\partial_t\vec{V}^\varepsilon(\nabla\cdot\partial_t\vec{V}^\varepsilon)dxdy\\
&+2\varepsilon\int_0^\infty\int_{-\infty}^\infty\partial_t(\vec{V}^\varepsilon\cdot\vec{V}^a)(\nabla\cdot\partial_t\vec{V}^\varepsilon)dxdy\\
&+2\int_0^\infty\int_{-\infty}^\infty(\nabla\partial_t U^\varepsilon\partial_t\vec{V}^\varepsilon+\varepsilon^{-1/2}\partial_t\vec{g}^\varepsilon\cdot\partial_t\vec{V}^\varepsilon)dxdy\\
:=&I_{10}+I_{11}+I_{12}.
\end{aligned}$$

First, (5.63) and the Cauchy-Schwarz inequality entail that

$$\begin{aligned}
I_{10}\le&C_0\varepsilon^{3/2}(\|\vec{V}^\varepsilon\|_{L^2}^{1/2}\|\nabla\vec{V}^\varepsilon\|_{L^2}^{1/2}+\|\vec{V}^\varepsilon\|_{L^2})(\|\partial_t\vec{V}^\varepsilon\|_{L^2}^{1/2}\|\nabla\partial_t\vec{V}^\varepsilon\|_{L^2}^{1/2}+\|\partial_t\vec{V}^\varepsilon\|_{L^2})\|\nabla\partial_t\vec{V}^\varepsilon\|_{L^2}\\
\le&\frac{1}{8}\varepsilon\|\nabla\partial_t\vec{V}^\varepsilon\|_{L^2}^2+C_0\varepsilon^2(\|\vec{V}^\varepsilon\|_{L^2}^2\|\nabla\vec{V}^\varepsilon\|_{L^2}^2+\|\vec{V}^\varepsilon\|_{L^2}\|\nabla\vec{V}^\varepsilon\|_{L^2}+\|\vec{V}^\varepsilon\|_{L^2}^4+\|\vec{V}^\varepsilon\|_{L^2}^2)\|\partial_t\vec{V}^\varepsilon\|_{L^2}^2,
\end{aligned}$$

where $0<\varepsilon<1$ has been used. Moreover, from (5.69) and (5.74) one gets

$$I_{11}\le\frac{1}{8}\varepsilon\|\nabla\partial_t\vec{V}^\varepsilon\|_{L^2}^2+C_0(\|\vec{V}^\varepsilon\|_{L^2}^2+\|\partial_t\vec{V}^\varepsilon\|_{L^2}^2).$$

Finally, it follows from the Cauchy-Schwarz inequality that $I_{12} \le \frac{1}{8}\|\nabla\partial_t U^\varepsilon\|_{L^2}^2 + \varepsilon^{-1}\|\partial_t \vec{g}^\varepsilon\|_{L^2}^2 + C^0\|\partial_t \vec{V}^\varepsilon\|_{L^2}^2$. Collecting the above estimates for I_4-I_{12} we arrive at

$$
\begin{aligned}
&\frac{d}{dt}(\|\nabla U^\varepsilon\|_{L^2}^2 + \varepsilon\|\nabla\vec{V}^\varepsilon\|_{L^2}^2 + \|\partial_t U^\varepsilon\|_{L^2}^2 + \|\partial_t\vec{V}^\varepsilon\|_{L^2}^2) \\
&\quad + \|\partial_t U^\varepsilon\|_{L^2}^2 + \|\partial_t\vec{V}^\varepsilon\|_{L^2}^2 + \|\nabla\partial_t U^\varepsilon(t)\|_{L^2}^2 + \varepsilon\|\nabla\partial_t\vec{V}^\varepsilon(t)\|_{L^2}^2 \\
&\le C(\varepsilon\|\vec{V}^\varepsilon\|_{L^2}^2\|\nabla\vec{V}^\varepsilon\|_{L^2}^2 + \|U^\varepsilon\|_{L^2}^2\|\nabla U^\varepsilon\|_{L^2}^2 \\
&\qquad + \|U^\varepsilon\|_{L^2}^2 + \|\vec{V}^\varepsilon\|_{L^2}^4 + 1) \times (\|\partial_t U^\varepsilon\|_{L^2}^2 + \|\partial_t\vec{V}^\varepsilon\|_{L^2}^2 + 1) \\
&\qquad + C\varepsilon^{1/2} + \varepsilon^{-1}(\|\partial_t f^\varepsilon\|_{L^2}^2 + \|\partial_t\vec{g}^\varepsilon\|_{L^2}^2).
\end{aligned}
\tag{5.81}
$$

On the other hand, from (5.55), Lemma 5.6 and Lemma 5.7, we have

$$
\|\partial_t U^\varepsilon(x,y,0)\|_{L^2}^2 = \varepsilon^{-1}\|f^\varepsilon(x,y,0)\|_{L^2}^2 \le \varepsilon^{-1}\|f^\varepsilon\|_{L_T^\infty L^2}^2 \le C\varepsilon^{1/2}
$$

and similarly $\|\partial_t\vec{V}^\varepsilon(x,y,0)\|_{L^2}^2 = \varepsilon^{-1}\|\vec{g}^\varepsilon(x,y,0)\|_{L^2}^2 \le C\varepsilon$. Thus we can apply Gronwall's inequality, Lemma 5.6- Lemma 5.8 to (5.81) and derive (5.79). The estimate (5.80) follows immediately from the system (5.55) and (5.79). Indeed, by the second equation of (5.55) and (5.65) one deduces for fixed $t \in [0,T]$ that

$$
\begin{aligned}
\varepsilon^2\|\vec{V}^\varepsilon\|_{H^2}^2 \le & C_0(\varepsilon^3\|\nabla\vec{V}^\varepsilon\|_{L^2}^2\|\vec{V}^\varepsilon\|_{L^\infty}^2 + \varepsilon^2\|\nabla\vec{V}^\varepsilon\|_{L^2}^2\|\vec{V}^a\|_{L^\infty}^2 + \varepsilon^2\|\vec{V}^\varepsilon\|_{L^\infty}^2\|\nabla\vec{V}^a\|_{L^2}^2 \\
& + \|U^\varepsilon\|_{H^1}^2 + \|\partial_t\vec{V}^\varepsilon\|_{L^2}^2 + \varepsilon^{-1}\|\vec{g}^\varepsilon\|_{L^2}^2) \\
\le & C_0(\varepsilon^3\|\nabla\vec{V}^\varepsilon\|_{L^2}^2\|\vec{V}^\varepsilon\|_{L^2}\|\vec{V}^\varepsilon\|_{H^2} + \varepsilon^2\|\nabla\vec{V}^\varepsilon\|_{L^2}^2\|\vec{V}^a\|_{L^\infty}^2 \\
& + \varepsilon^2\|\vec{V}^\varepsilon\|_{L^2}\|\vec{V}^\varepsilon\|_{H^2}\|\nabla\vec{V}^a\|_{L^2}^2 + \|U^\varepsilon\|_{H^1}^2 + \|\partial_t\vec{V}^\varepsilon\|_{L^2}^2 + \varepsilon^{-1}\|\vec{g}^\varepsilon\|_{L^2}^2) \\
\le & \frac{1}{2}\varepsilon^2\|\vec{V}^\varepsilon\|_{H^2}^2 + C_0(\varepsilon^4\|\nabla\vec{V}^\varepsilon\|_{L^2}^4\|\vec{V}^\varepsilon\|_{L^2}^2 + \varepsilon^2\|\nabla\vec{V}^\varepsilon\|_{L^2}^2\|\vec{V}^a\|_{L^\infty}^2 \\
& + \varepsilon^2\|\vec{V}^\varepsilon\|_{L^2}^2\|\nabla\vec{V}^a\|_{L^2}^4 + \|U^\varepsilon\|_{H^1}^2 + \|\partial_t\vec{V}^\varepsilon\|_{L^2}^2 + \varepsilon^{-1}\|\vec{g}^\varepsilon\|_{L^2}^2).
\end{aligned}
$$

Subtracting $\frac{1}{2}\varepsilon^2\|\vec{V}^\varepsilon\|_{H^2}^2$ from both side of the above inequality, then using (5.79), (5.72), (5.69) and Lemma 5.7 one gets

$$
\varepsilon^2\|\vec{V}^\varepsilon\|_{L_T^\infty H^2}^2 \le C\varepsilon^{1/2},
$$

where we have also used $\|\nabla\vec{V}^a\|_{L_T^\infty L^2}^2 \le C\varepsilon^{-1/2}$, which follows from (5.59) and a similar argument in deriving (5.69). Moreover, one derives $\varepsilon\|U^\varepsilon\|_{L_T^\infty H^2}^2 + \varepsilon^3\|\vec{V}^\varepsilon\|_{L_T^2 H^3}^2 \le C\varepsilon^{1/2}$ by a similar argument as estimating $\varepsilon^2\|\vec{V}^\varepsilon\|_{L_T^\infty H^2}^2$. The proof is completed.

We come to prove Proposition 5.4 by the results of Lemma 5.8 and Lemma 5.9.

Proof of Proposition 5.4. First, by the *bootstrap principle*, Lemma 5.8 and Lemma 5.9 we deduce (5.57) and (5.58). Thus $(U^\varepsilon, \vec{V}^\varepsilon) \in C([0,T]; H^2 \times H^2)$. The uniqueness can be proved by the method used in [83], we omit the details for brevity.

□

5.5 Proof of Theorem 5.2 and Theorem 5.3

We next prove Theorem 5.2 and Theorem 5.3 by the results of Proposition 5.4.

Proof of Theorem 5.2. First, by the fact that $(U^\varepsilon, \vec{V}^\varepsilon)$ uniquely solves problem (5.55) one deduces that $(u^\varepsilon, \vec{v}^\varepsilon)$ with $u^\varepsilon = \varepsilon^{1/2}U^\varepsilon + U^a$, $\vec{v}^\varepsilon = \varepsilon^{1/2}\vec{V}^\varepsilon + \vec{V}^a$ is the unique solution of (5.2), (5.3) with $\varepsilon \in (0, \varepsilon_T]$. The regularity $(u^\varepsilon, \vec{v}^\varepsilon) \in C([0,T]; H^2 \times H^2)$ follows from $(U^\varepsilon, \vec{V}^\varepsilon)$, $(U^a, \vec{V}^a) \in C([0,T]; H^2 \times H^2)$. We next prove the curl-free property of $\vec{v}^\varepsilon$ by applying operator "$\nabla\times$" to the second equation of (5.2) with $\varepsilon > 0$ to find

$$\begin{cases} (\nabla \times \vec{v}^\varepsilon)_t = \varepsilon\Delta(\nabla \times \vec{v}^\varepsilon), \\ (\nabla \times \vec{v}^\varepsilon)(x,y,0) = 0 \\ \nabla \times \vec{v}^\varepsilon|_{y=0} = 0, \end{cases} \tag{5.82}$$

where the assumption $\nabla \times \vec{v}_0 = 0$ and the boundary conditions (5.3) have been used. Consequently, the uniqueness on solution of (5.82) entails that $\nabla \times \vec{v}^\varepsilon = 0$. Moreover, (5.19) follows from Lemma 5.1.

It remains to prove (5.18). By the Gagliardo-Nirenberg interpolation inequality, (5.57) and (5.58) we get

$$\begin{aligned} \|\vec{V}^\varepsilon\|_{L^\infty_T L^\infty} \le& C_0\big(\|\nabla^2\vec{V}^\varepsilon\|^{1/2}_{L^\infty_T L^2}\|\vec{V}^\varepsilon\|^{1/2}_{L^\infty_T L^2} + \|\vec{V}^\varepsilon\|_{L^\infty_T L^2}\big) \\ \le& C(\varepsilon^{-\frac{3}{8}} \cdot \varepsilon^{\frac{1}{8}} + \varepsilon^{\frac{1}{4}}) \le C\varepsilon^{-1/4}. \end{aligned} \tag{5.83}$$

Similarly it follows that

$$\|U^\varepsilon\|_{L^\infty_T L^\infty} \le C_0\|\nabla^2 U^\varepsilon\|^{1/2}_{L^\infty_T L^2}\|U^\varepsilon\|^{1/2}_{L^\infty_T L^2} \le C\varepsilon^{-1/8} \cdot \varepsilon^{1/8} \le C. \tag{5.84}$$

Then the definition of U^ε, $\vec{V}^\varepsilon$, Sobolev embedding inequality, (5.62) and (5.83) lead to

$$\begin{aligned} &\|\vec{v}^\varepsilon(x,y,t) - \vec{v}^0(x,y,t) - \big(0, v_2^{B,0}\big)\big(x, \frac{y}{\sqrt{\varepsilon}}, t\big)\|_{L^\infty_T L^\infty} \\ \le& C_0\big(\varepsilon^{1/2}\|\vec{v}^{I,1}\|_{L^\infty_T H^2} + \varepsilon^{1/2}\|v_1^{B,1}\|_{L^\infty_T H^2_{x\eta}} + \varepsilon^{1/2}\|v_2^{B,1}\|_{L^\infty_T H^2_{x\eta}} \\ &\quad + \varepsilon\|v_1^{B,2}\|_{L^\infty_T H^2_{x\eta}} + \varepsilon^{1/2}\|\vec{V}^\varepsilon\|_{L^\infty_T L^\infty}\big) \\ \le& C\varepsilon^{1/4}. \end{aligned} \tag{5.85}$$

Similarly, by (5.84) and the definition of U^ε we have

$$\|u^\varepsilon(x,y,t) - u^0(x,y,t)\|_{L^\infty_T L^\infty} \le C\varepsilon^{1/2}. \tag{5.86}$$

The combination of (5.85) and (5.86) gives (5.18) and completes the proof. □

Proof of Theorem 5.3. Denote $(u^\varepsilon, \vec{v}^\varepsilon)$ and $(u^0, \vec{v}^0)$ the solutions of problem (5.2), (5.3) obtained in Theorem 5.2 and Theorem 5.1, respectively. Let

$$
\begin{aligned}
c^{\varepsilon}(x,y,t) &= c_0(x,y)\exp\Big\{\int_0^t [-\varepsilon\nabla\cdot\vec{v}^{\varepsilon}+\varepsilon|\vec{v}^{\varepsilon}|^2-u^{\varepsilon}](x,y,\tau)\,d\tau\Big\},\\
c^{0}(x,y,t) &= c_0(x,y)\exp\Big\{-\int_0^t u^{0}(x,y,\tau)\,d\tau\Big\}.
\end{aligned}
\tag{5.87}
$$

It is easy to verify that $(u^{\varepsilon},c^{\varepsilon})(x,y,t)$ and $(u^0,c^0)(x,y,t)$ solve (5.20) with $\varepsilon\in(0,\varepsilon_T]$ and $\varepsilon=0$ respectively by directly substituting (5.87) into the second equation of (5.20) along with some elementary calculations and using the curl-free property $\nabla\times\vec{v}^{\varepsilon}(x,y,t)=0$, $\nabla\times\vec{v}^{0}(x,y,t)=0$. We further deduce that $(u^{\varepsilon},c^{\varepsilon})\in C([0,T];H^2\times H^3)$ and $(u^0,c^0)\in C([0,T];H^9\times H^{10})$ by the regularity estimates of $(u^{\varepsilon},\vec{v}^{\varepsilon})$, $(u^0,\vec{v}^0)$ in Theorem 5.2 and Theorem 5.1. The uniqueness follows from the standard method used in [83]. Finally, one derives (5.21) and (5.22) by (5.87), (5.18), (5.58) and following the arguments used in proving Theorem 3.2 of Chapter 3. We omit it for brevity.

□

5.6 Formal Derivation of Outer/Inner Layer Profiles

This section is devoted to the derivation of equations (5.7)-(5.17), by employing the asymptotic analysis, which has been used in Section 3.6 to derive layer profiles of one dimensional case and in [29, Appendix] to determine thickness of the boundary layer. For brevity we just sketch the procedure and refer the reader to Section 3.6 and [29, Appendix] for details.

Step 1. Initial and boundary conditions. Substituting (5.5) into the initial conditions in (5.2) and following the arguments used in [29, Appendix], we have

$$
\begin{aligned}
&u^{I,0}(x,y,0)=u_0(x,y), \quad u^{B,0}(x,\eta,0)=0,\\
&\vec{v}^{I,0}(x,y,0)=\vec{v}_0(x,y), \quad \vec{v}^{B,0}(x,\eta,0)=0
\end{aligned}
\tag{5.88}
$$

and for $j\geq 1$

$$
\begin{aligned}
&u^{I,j}(x,y,0)=u^{B,j}(x,\eta,0)=0,\\
&\vec{v}^{I,j}(x,y,0)=\vec{v}^{B,j}(x,\eta,0)=0.
\end{aligned}
\tag{5.89}
$$

For the boundary conditions, we insert (5.5) into (5.3) and use (5.6) to get for $j\in\mathbb{N}$ that

$$
\begin{aligned}
\bar{u}(x,t) &= \sum_{j=0}^{\infty}\varepsilon^{j/2}\big[u^{I,j}(x,0,t)+u^{B,j}(x,0,t)\big],\\
\bar{v}(x,t) &= \sum_{j=0}^{\infty}\varepsilon^{j/2}\big[v_2^{I,j}(x,0,t)+v_2^{B,j}(x,0,t)\big],\\
0 &= \sum_{j=0}^{\infty}\varepsilon^{j/2}\big[\partial_y v_1^{I,j}(x,0,t)+\varepsilon^{-1/2}\partial_\eta v_1^{B,j}(x,0,t)\big]-\sum_{j=0}^{\infty}\varepsilon^{j/2}\partial_x\big[v_2^{I,j}(x,0,t)+v_2^{B,j}(x,0,t)\big].
\end{aligned}
$$

To fulfill the above boundary conditions for all small $\varepsilon > 0$, it is required that

$$\begin{aligned}
\bar{u}(x,t) &= u^{I,0}(x,0,t) + u^{B,0}(x,0,t),\\
\bar{v}(x,t) &= v_2^{I,0}(x,0,t) + v_2^{B,0}(x,0,t),\\
0 &= \partial_\eta v_1^{B,0}(x,0,t),\\
\partial_x \bar{v}(x,t) &= \partial_y v_1^{I,0}(x,0,t) + \partial_\eta v_1^{B,1}(x,0,t)
\end{aligned} \tag{5.90}$$

and for $j \geq 1$ that

$$\begin{aligned}
0 &= u^{I,j}(x,0,t) + u^{B,j}(x,0,t),\\
0 &= v_2^{I,j}(x,0,t) + v_2^{B,j}(x,0,t),\\
0 &= \partial_y v_1^{I,j}(x,0,t) + \partial_\eta v_1^{B,j+1}(x,0,t).
\end{aligned} \tag{5.91}$$

Step 2. Equations for $u^{I,j}$ and $u^{B,j}$. We first substitute (5.5) without the inner layer profiles $u^{B,j}$, $\vec{v}^{B,j}$ into the first equation of (5.2) to get the equations for outer layer profiles $u^{I,j}$:

$$u_t^{I,j} - \sum_{k=0}^{j} \nabla \cdot (u^{I,k} \vec{v}^{I,j-k}) = \Delta u^{I,j}, \quad \text{for } j \in \mathbb{N}. \tag{5.92}$$

To find the equations for inner layer profiles $u^{B,j}$, by a similar argument used in Step 2 of subsection 2.2.6, that is inserting (5.5) into the first equation of (5.2) and subtracting (5.92) from the resulting equation then applying Taylor expansion to $u^{I,j}$, $\vec{v}^{I,j}$, we end up with

$$\sum_{j=-2}^{\infty} \varepsilon^{j/2} \tilde{G}^j(x,\eta,t) = 0, \tag{5.93}$$

where

$$\begin{cases}
\tilde{G}^{-2} = -\partial_\eta^2 u^{B,0},\\
\tilde{G}^{-1} = -u^{I,0}(x,0,t)\partial_\eta v_2^{B,0} - v_2^{I,0}(x,0,t)\partial_\eta u^{B,0} - \partial_\eta(u^{B,0} v_2^{B,0}) - \partial_\eta^2 u^{B,1},\\
\tilde{G}^0 = \partial_t u^{B,0} - \partial_x[(u^{I,0}(x,0,t) + u^{B,0})v_1^{B,0}] - \partial_x(u^{B,0} v_1^{I,0}(x,0,t)) - u^{B,0}\partial_y v_2^{I,0}(x,0,t)\\
\qquad - (u^{I,0}(x,0,t) + u^{B,0})\partial_\eta v_2^{B,1} - (u^{I,1}(x,0,t) + u^{B,1})\partial_\eta v_2^{B,0} - \partial_y u^{I,0}(x,0,t) v_2^{B,0}\\
\qquad - \partial_\eta u^{B,0}(v_2^{I,1}(x,0,t) + v_2^{B,1}) - \partial_\eta u^{B,1}(v_2^{I,0}(x,0,t) + v_2^{B,0})\\
\qquad - \partial_x^2 u^{B,0} - \partial_\eta^2 u^{B,2} - \eta\,\partial_y u^{I,0}(x,0,t)\partial_\eta v_2^{B,0} - \eta\,\partial_y v_2^{I,0}(x,0,t)\partial_\eta u^{B,0},\\
\quad \cdots\cdots
\end{cases}$$

where $\tilde{G}_j = 0$ for $j \geq -2$. From $\tilde{G}_{-2} = 0$ we get $\partial_\eta^2 u^{B,0} = 0$, which integrated twice with respect to η over (η,∞) along with the assumption (H*), yields

$$u^{B,0}(x,\eta,t) = 0, \quad \text{for } (x,\eta,t) \in \mathbb{R} \times \mathbb{R}_+ \times [0,T]. \tag{5.94}$$

Furthermore, it follows from (5.94), $\tilde{G}_{-1} = 0$ and the first identity of (5.90) that

$$\partial_\eta^2 u^{B,1} = -u^{I,0}(x,0,t)\partial_\eta v_2^{B,0} = -\bar{u}(x,t)\partial_\eta v_2^{B,0}, \tag{5.95}$$

which, upon integration over (η,∞) gives rise to

$$\partial_\eta u^{B,1} = -\bar{u}(x,t)v_2^{B,0}, \tag{5.96}$$

where assumption (H*) has been used. Applying a similar procedure as deriving (5.96) that is, first inserting (5.94) into $\tilde{G}_0 = 0$ to get

$$\begin{aligned}\partial_\eta^2 u^{B,2} =& -\partial_x(u^{I,0}(x,0,t)v_1^{B,0}) - u^{I,0}(x,0,t)\partial_\eta v_2^{B,1} - (u^{I,1}(x,0,t)+u^{B,1})\partial_\eta v_2^{B,0} \\ & -\partial_y u^{I,0}(x,0,t)v_2^{B,0} - \partial_\eta u^{B,1}(v_2^{I,0}(x,0,t)+v_2^{B,0}) - \eta\partial_y u^{I,0}(x,0,t)\partial_\eta v_2^{B,0},\end{aligned} \tag{5.97}$$

then integrating the above equation with respect to η twice, we have

$$u^{B,2} = \bar{u}(x,t)\int_\eta^\infty v_2^{B,1}(x,\zeta,t)\,d\zeta - \int_\eta^\infty\int_\zeta^\infty \Phi(x,s,t)\,ds d\zeta, \tag{5.98}$$

where

$$\begin{aligned}\Phi(x,\eta,t) :=& \partial_x(u^{I,0}(x,0,t)v_1^{B,0}) + (u^{I,1}(x,0,t)+u^{B,1})\partial_\eta v_2^{B,0} + \partial_y u^{I,0}(x,0,t)v_2^{B,0} \\ & + \partial_\eta u^{B,1}(v_2^{I,0}(x,0,t)+v_2^{B,0}) - \eta\partial_y u^{I,0}(x,0,t)\partial_\eta v_2^{B,0}.\end{aligned}$$

Step 3. Equations for $\vec{v}^{I,j}$ and $\vec{v}^{B,j}$. Applying an analogous argument as Step 2 to the second equation of (5.2), we derive

$$\begin{cases}\vec{v}_t^{I,0} - \nabla u^{I,0} = 0, \\ \vec{v}_t^{I,1} - \nabla u^{I,1} = 0, \\ \vec{v}_t^{I,j} + 2\displaystyle\sum_{k=0}^{j-2}\nabla(\vec{v}^{I,k}\cdot\vec{v}^{I,j-2-k}) - \nabla u^{I,j} - \Delta\vec{v}^{I,j-2} = 0, \quad \text{for } j \geq 2\end{cases} \tag{5.99}$$

and

$$\sum_{j\geq -1}\varepsilon^{\frac{j}{2}}\vec{F}^j(x,\eta,t) = 0, \tag{5.100}$$

where $\vec{F}^j(x,\eta,t) = (F_1^j, F_2^j)(x,\eta,t)$ with

$$\begin{cases}F_1^{-1} = 0, \\ F_1^0 = \partial_t v_1^{B,0} - \partial_x u^{B,0} - \partial_\eta^2 v_1^{B,0}, \\ F_1^1 = \partial_t v_1^{B,1} - \partial_x u^{B,1} - \partial_\eta^2 v_1^{B,1}, \\ F_1^2 = \partial_t v_1^{B,2} + \partial_x(2v_1^{I,0}(x,0,t)v_1^{B,0} + v_1^{B,0}v_1^{B,0} + 2v_2^{I,0}(x,0,t)v_2^{B,0} + v_2^{B,0}v_2^{B,0}) \\ \qquad - \partial_x u^{B,2} - \partial_x^2 v_1^{B,0} - \partial_\eta^2 v_1^{B,2}, \\ \qquad \cdots\cdots\end{cases}$$

and

$$
\begin{cases}
F_2^{-1} = -\partial_\eta u^{B,0}, \\
F_2^0 = \partial_t v_2^{B,0} - \partial_\eta u^{B,1} - \partial_\eta^2 v_2^{B,0}, \\
F_2^1 = \partial_t v_2^{B,1} + 2(v_1^{I,0}(x,0,t) + v_1^{B,0})\partial_\eta v_1^{B,0} + 2(v_2^{I,0}(x,0,t) + v_2^{B,0})\partial_\eta v_2^{B,0} \\
\qquad - \partial_\eta u^{B,2} - \partial_\eta^2 v_2^{B,1}, \\
\quad \cdots\cdots
\end{cases}
$$

which leads to $F_1^j = 0$, $F_2^j = 0$ with $j \geq -1$ to guarantee that (5.100) holds true for all small $\varepsilon > 0$. Finally, the initial boundary value problems (5.7)-(5.17) follow directly from the results derived in Step 1- Step 3. Indeed, by (5.92) with $j = 0$, (5.99), (5.88) and (5.90), we derive (5.7). From (5.100) with $j = 0$, (5.94), (5.88) and (5.90) one deduces (5.9). Similarly, (5.10) is the combination of (5.96), (5.100) with $j = 0$, (5.88) and (5.90). (5.12) comes from (5.92) with $j = 1$, (5.99), (5.89) and (5.91). Moreover (5.100), (5.89) and (5.91) with $j = 1$ lead to (5.13). The combination of (5.97), (5.100) with $j = 1$, (5.89), (5.91) and $v_1^{B,0} = 0$ yields (5.14). Lastly, (5.17) follows from (5.100) with $j = 1$, (5.89) and (5.91).

Chapter 6

Conclusions and Future Works

6.1 Conclusions

The zero-diffusion limit of a viscous hyperbolic system (1.4) transformed from a chemotaxis model (1.2) is investigated in this thesis. The following conclusions are rigorously justified.

1. With spatial domain $\Omega = (0,1)$, the system (1.4) subject to Dirichlet boundary conditions possesses boundary layers at each endpoint $x = 0$ and $x = 1$, whose thickness are of order $O(\varepsilon^{1/2})$. Denote by $(u^\varepsilon, v^\varepsilon)$ and (u^0, v^0) the solutions with $\varepsilon > 0$ and $\varepsilon = 0$, respectively. Outside the boundary layers, the solution component v^ε converges to v^0 as the chemical diffusion rate ε goes to zero. However, the convergence does not hold inside the boundary layers. Indeed, v^ε approaches to the boundary layer profiles $v^{B,0}$ and $v^{b,0}$ inside the boundary layer at endpoint $x = 0$ and $x = 1$, respectively. Hence, v^ε converges uniformly in $(x,t) \in [0,1] \times [0,T]$ (for any $0 < T < \infty$) to v^0 (outer layer profile) plus the inner layer profiles $v^{B,0}$, $v^{b,0}$ as $\varepsilon \to 0$ based on the asymptotic matching theory.

2. For the multi-dimensional case, the radial solution of (1.4) with $\Omega = \{\vec{x} \in \mathbb{R}^d \mid 0 < a < |\vec{x}| < b\}$ also possesses boundary layers near $|\vec{x}| = a$ and $|\vec{x}| = b$, when it subjects to the Dirichlet boundary conditions.

3. With $\Omega = \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$, denote by $(u^\varepsilon, \vec{v}^\varepsilon) = (u^\varepsilon, v_1^\varepsilon, v_2^\varepsilon)$ and $(u^0, \vec{v}^0) = (u^0, v_1^0, v_2^0)$ the solutions of (1.4) with $\varepsilon > 0$ and $\varepsilon = 0$, respectively; where u^ε, v_2^ε and u^0 satisfy the Dirichlet boundary conditions and v_1^ε satisfies the Neumann boundary condition. Then for any $0 < T < T_{\max}$ (here $T_{\max}$ is the maximal existence time of $(u^0, \vec{v}^0)$), the solution component v_2^ε converges uniformly in $(x,y,t) \in \Omega \times [0,T]$ to v_2^0 plus the boundary layer profile $v_2^{B,0}$ as $\varepsilon \to 0$. Moreover, u^ε and v_1^ε approach to u^0 and v_1^0, respectively.

4. The above results on system (1.4) are converted back to the original chemotaxis model (1.2) and it is found that the chemical concentration (denoted by c in (1.2)) has no boundary layer but its gradient ∇c does, which indicates that although both cell

density u and chemical concentration c have no boundary layer as chemical diffusion ε goes to zero, the chemotactic flux, namely the term $u\nabla c = -u\vec{v}$, has a sharp transition near the boundary (i.e. the endothelial cells cross the blood vessel wall quickly). Hence our results imply that the diffusion of chemical signal (i.e. vascular endothelial growth factor) plays an essential role in the transition of cell mass from boundaries to the field away from boundaries during the initiation of tumor angiogenesis.

6.2 Future Works

As a newly discovered phenomenon in chemotaxis (cf. [44]), the study of boundary layers for chemotaxis models is still in its infant stage. Except the topics investigated in this thesis, there are many other interesting problems left open and we just list part of them below, which we plan to explore in the future.

1. In our result (see the above Conclusion 3) for $\Omega = \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$, the convergence merely holds for T with $T < T_{\max}$, where $T_{\max}$ denotes the maximal existence time of $(u^0, \vec{v}^0)$. Hence to guarantee this convergence holds for arbitrary $0 < T < \infty$, it is required that $T_{\max} = \infty$. However, for system (1.2) (with $\varepsilon \geq 0$) in multi-dimensions with large initial data, only the local well-posedness is justified and its global well-posedness is still an open problem in spite of numerous attempts (as mentioned in the literature review). How to resolve this challenging problem is of great interest and we shall exploit this open question in the future.

2. For two dimensional case, we have only investigated the boundary layer problem with spatial domain $\Omega = \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$, whose geometric structure is relatively simple. In the future, extending our result to more general spatial domains of $\mathbb{R}^2$ and even of $\mathbb{R}^3$ is worthwhile and challenging due to the complicated geometric structures that may involve.

3. Our results indicate that the solution component v, denoting the chemical concentration possesses boundary layers as the chemical diffusion rate $\varepsilon \to 0$. It is natural to ask whether the solution component u wound possess boundary layer if we pass the cell diffusion coefficient D to zero but fix $\varepsilon > 0$. Actually, the boundary layer phenomenon for cell density has already been found in [78], where an experiment was conducted by using the sessile drop technique in suspensions of Bacillus subtilis, and the generation of aggregations in a thin layer near the air-water contact lines (the boundary of the suspensions) was observed. Based on this experimental observations, to study the boundary layer problem for chemotaxis-fluid models (Chemotaxis models coupled with fluid evolution) in cell diffusion limit is particularly relevant and promising by applying the analytic framework of this thesis. Moreover, this analytic framework can also be applied to investigate boundary layer problem for other chemotaxis models as the diffusion of the chemical or cells vanishes.

4. When $\Omega = (0,1)$ and Dirichlet boundary conditions are prescribed, our results in Chapter 2 and Chapter 3 assert that the solution component v^{ε} of (2.1) does not approach to v^0 near the boundary as $\varepsilon \to 0$. However, if Neumann-Dirichlet boundary conditions are imposed (i.e. u and v subject to Neumann and Dirichlet boundary conditions respectively), Wang and Zhao previously proved in [83] that v^{ε} converges to v^0 uniformly on the entire interval $[0,1]$, as $\varepsilon \to 0$. Enlightened by this distinction in solution behaviors caused by preassigning different boundary conditions, it is promising to expect different outcomes by changing the boundary conditions when studying other problems. In particular, the steady state of (1.2) with the second equation replaced by $c_t = \varepsilon\Delta c - \alpha c + \beta u$ (where α, β are positive constants) has been studied in [51, 63] and it is proved that the stationary system with ε/α large enough, only admits constant stationary solutions that subject to Neumann boundary conditions (i.e. both u and c are imposed with Neumann boundary conditions). By replacing the Neumann boundary conditions with other boundary conditions (for instance, the no-flux boundary conditions $[D\nabla u - \chi\nabla \ln c] \cdot \vec{n}\,|_{\partial\Omega} = 0$), we shall explore in the future whether non-constant stationary solutions would exist for system (1.2).

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