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SPECTRAL HYPERGRAPH THEORY VIA TENSORS

CHEN OUYANG

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Department of Applied Mathematics

Spectral Hypergraph Theory via Tensors

Chen Ouyang

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Certificate of Originality

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______________________________(Signed)

OUYANG Chen (Name of student)

Abstract

The thesis is devoted to a few problems on spectral hypergraphs theory via the adjacency tensor, Laplacian tensor and signless Laplacian tensor of a hypergraph. These problems are analogy and generalization of problems usually concerned in spectral graph theory. Three topics are included:

1. Characterization of uniform hypergraphs with largest spectral radii under certain conditions.

2. The property of symmetric spectrum for uniform hypergraphs with applications.

3. Properties on the spectra of non-uniform and general hypergrpahs.

For the first topic, two types of connected hypergraphs with fixed vertex number and cyclomatic number called unicyclic and bicyclic hypergraphs are studied. By combining recent developed spectral techniques, the first five hypergraphs with largest spectral radii among all unicyclic hypergraphs and the first three over all bicyclic hypergraphs are determined, together with two orderings of the corresponding hypergraphs.

For topic 2, we investigate the newly introduced odd-colorable hypergraphs and employ their symmetric spectra to obtain conditions for a uniform hypergraph to have equal Laplacian spectrum and signless Laplacian spectrum.

For the last topic, some spectral bounds in terms of graph invariants are extended from uniform case to general hypergraphs, and a new way is found to bound the spectral radius from below for a special class of non-uniform hypergraphs. Moreover, the property of symmetric spectrum for general hypergraphs is investigated. Equivalent conditions are extened from uniform case to general case. Besides, the capability of a non-uniform hypergraph to have symmetric (H-)spectrum, equal Laplacian (H-)spectrum (spectral radius) and signless Lapalcian (H-)spectrum (spectral radius) is discussed.

This thesis forms from the following articles written by the author during her study period at the Department of Applied Mathematics, The Hong Kong Polytechnic University as a research graduate student:

- C. Ouyang, L. Qi, X. Yuan, The first few unicyclic and bicyclic hypergraphs with largest spectral radii, Linear Algebra and its Applications 527 (2017) 141–162.

- X. Yuan, L. Qi, J. Shao, C. Ouyang, Some properties and applications of odd colorable r-hypergraphs, Discrete Applied Mathematics 236 (2018) 446–452.

- C. Ouyang, L. Qi, Spectral bounds and properties of general hypergraphs, 2018, preprint.

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Chapter 1

Introduction

1.1 Overview

Spectral graph theory is a well studied branch of graph theory and is highly applicable in combinatorics, computer sciences, chemistry and the social sciences. It concentrates on the connection between properties of a graph and the eigenvalues or singular values of matrices representing that graph [7]. Its major topics and methods have been naturally extended to a generalization of graphs called hypergraphs [3, 4], a subject with numerous application in mathematical sciences [23] and other applied sciences [6, 25, 32, 52]. This trend gradually shapes a novel and active field named spectral hypergraph theory around the last twenty years.

As a generalization of spectral graph theory, spectral hypergraph theory inherits the intrinsic idea from spectra graph theory of using spectral methods to explore hypergraph properties from various aspects. In details, hypergraphs are studied via their representations of matrices/tensors (hypermatrices) by employing algebraic tools and matrix/tensor theory.

The early trails on defining hypergraph spectrum were through matrices, where some meaningful results had been achieved on the Laplacian matrices of hypergraphs [13, 20, 22]. However, the limitation of matrix representation for hypergraphs lead subsequent exploration to obstacles quickly. It was after the set up of tensor eigenvalues and some basic theoretical developments that the study of spectral hypergraph theory via tensors initiated.

In 2005, several generalizations of matrix eigenvalues were introduced for higher order tensors in the independent works of Qi [46] and Lim [37], meanwhile a proposal of using tensor eigenvalues to study hypergraphs were roughly mentioned [37]. In 2009, through utilizing the H-eigenvalues of tensors defined by Qi [46], Bulò and Pelillo [10] employed the spectral radius of a hypergraph to characterize the clique number, a well-known graphic invariant, which was acknowledged as the first paper on spectral hypergraph theory via tensors. Another pioneer work was carried out by Cooper and Dutle [16] in 2012, which is a systematic study on the adjacency tensor of uniform hypergraphs with an outcome of many analogs of elemental results in spectral graph theory. Since then, accompanying with a blossom in spectral theory of tensors [11, 21, 27, 47, 49, 60], a vast and rapid development occurred to the emerged spectral hypergraph theory via tensors [1, 19, 28, 39, 34, 38, 44, 48].

In this new branch of hypergraph theory, studies are mainly based upon three types of tensor representations, the Laplacian tensor, the signless Laplacian tensor, and the adjacency tensor mentioned above. During the lasts five years, fruitful results have been obtained on a wide range of topics including: spectral properties of the three tensors for uniform hypergraphs [18, 28, 65, 69], general hypergraphs [1, 9, 66], directed hypergraphs [12, 58, 2], multi-hypergraphs [44, 45, 59], and random hypergraphs [15]; the property of symmetric spectrum of uniform hypergraphs with applications [42, 53, 54, 63]; spectral parameters such as spectral radius [19, 39, 34, 38, 31, 64, 43] and analytic connectivity [24, 36] of uniform hypergraphs, as well as their connection with chromatic number [16], clique/independent number [57, 62], isoperimetric number [36], and other graph invariants; applications of hypergraph eigenvalue in image processing and multi-page rank.

Among various research directions on hypergraph spectra with respect to the three tensors, the spectral radius may be the index attracting the most scholarly interest [16, 19, 34, 38, 39, 64], which may be attributed to its pleasant properties reflected by the Perron-Frobenius theorem for nonnegative tensors [11, 21, 35, 60] and the historical popularity of its analogs in the literature of spectral graph theory [8, 14, 29, 41, 51]. On the other hand, the symmetric property of spectra for graphs is a classic question concerned by many scholars. In the case of hypergraphs, researches are constantly exploring possible conditions for hypergraphs to have symmetric spectra [17, 18, 42, 54].

This thesis is concerned with these two aspects of spectral hypergraph theory. We investigate the spectral radii of uniform hypergraphs (Chapter 4) and general hypergraphs (Chapter 6) and the spectral symmetry of a special class of uniform hypergraphs (Chapter 5) and general hypergraphs (Chapter 6).

1.2 Outline

In addition to the present introduction chapter, this thesis has five other parts. This section will briefly introduce the organization of the following Chapters 2-6.

These five chapters can be separate into two parts. Chapters 2 and 3 are the preliminary knowledge of tensor eigenvalues and hypergraph spectra relevant to our study, while our main work on spectral hypergraph theory is included in Chapters 4-6.

Chapter 2: This chapter is devoted to the basic knowledge of tensor and its eigenvalues with some elementary properties and theorems. Section 2.1 presents the defintions of tensor eigenvalues and H-eigenvalues, spectrum, H-spectrum and spectral radius of tensors, and some basic properties of tensors eigenvalues. Then in Section 2.2 is on nonnegative tensors which is a significant class of tensors and closely relevant to spectral hypergraph theory. Some important concepts include irreducible tensor, weakly irreducible tensor are introduced, following with the Perron-Frobenius theory and Collatz-Wielandt minimax theorm for nonnegative tensors, as well as other pleasant properties for (symmetric) nonnegative tensors.

Chapter 3: This chapter includes basic concepts of spectral hypergraph theory and some properties on hypergraph spectra. Section 3.1 presents the three major types of tensors associated with a (general) hypergraph with a brief literature review. In Section 3.2, some basic properties for the three tensors are displayed, including distribution of eigenvalues, the weak irreducibility of adjacency, Laplacian and signless Lapalcian tensors for connected hypergraphs together with properties on the spectral radius yields from the theorems in Section 2.2. In Section 3.3, spectral methods to deal with uniform and non-uniform hypergraphs are introduced, including moving edge operations, the eigenvalue equation of power hypergraphs and the weighted incidence matrix for hypergraphs.

For the next three chapters reporting our study on spectral hypergraph theory, we briefly explain the main results instead.

Chapter 4: The study in this Chapter belongs to the area of extremal spectral hypergraph theory, an intersection of extremal hypergraphs theory and spectral hypergraph theory, which is about finding extremal hypergraphs satisfying some spectral property under certain conditions/restrictions. There are two main results.

The first result characterizes five unicyclic hypergraphs with spectral radius larger than the remaining hypergraphs among all unicyclic hypergraphs. The second one determined three bicyclic hypergraphs with spectral radius greater than the remaining hypergraphs among all bicyclic hypergraphs.

Chapter 5: In this chapter, properties of symmetric spectra of an odd-colorable k-graph is investigated. By the Perron-Frobenius theorem on nonnegative weakly irreducible tensors together with the relation between the spectra of a hypergraph and that of its connected components, we obatained a result for the disconnected uniform hypergraphs to have equivalent Lapalcian and signless Laplacian spectra in Theorems 5.2 and 5.3. Based on the above outcomes, we obtain an affirmative answer to Question 5.1 propsed in [54] about the relations between HSpec($\mathcal{L}$),HSpec($\mathcal{Q}$) and Spec($\mathcal{L}$),Spec($\mathcal{Q}$) for the remaining unsolved case $k \not\equiv 0$ $(mod\ 4)$ in Theorem 5.4.

Chapter 6: In the last chapter, we obtained three types of upper bounds for the spectral radius and the signless Laplacian spectral radius of general hypergraphs in Theorems 6.1, 6.2 and 6.3 of Section 6.1. By the idea of characterizing the spectral radius of general hypergraphs by uniform hypergraphs, we find a lower bound for (non-uniform) generalized power hypergraph stated as in Theorem 6.4 of Section 6.2. Finally, equivalent conditions of symmetric spectrum (i.e. $Spec(\mathcal{A}) = Spec(-\mathcal{A})$ for the adjacency tensor $\mathcal{A}$) is given in Proposition 6.1. It is verified in Proposition 6.2 and Corollary 6.3 that the spectrum of a non-uniform hypergraph will not be symmetric and there will be a gap between the Laplacian spectral radius and signless Laplacian spectral radius for a connected non-uniform hypergraph.

1.3 Basic concepts and notations

Let $\mathbb{C}$ and $\mathbb{R}$ be the fields of complex and real numbers respectively.

A tensor (also known as hypermatrix) $\mathcal{T} = (T_{i_1 \cdots i_k})$ in a certain field $\mathbb{F}$ generally refers to a multi-array with entries $T_{i_1 i_2 \cdots i_k} \in \mathbb{F}$, where each index i_j runs from 1 to a natural number n_j for $j = 1, \ldots, n$. The above k and $(n_1, \ldots, n_k)$ are called the order and the dimension of tensor $\mathcal{T}$, respectively. Throughout this thesis, only real tensors are considered which means $\mathbb{F} = \mathbb{R}$. Moreover, we are focusing on square tensors with $n_1 = \cdots = n_k = n$ and $\mathcal{T}$ is called a k-order n-dimensional tensor in this case, meanwhile k, n are assumed to be integers no less than 2. The set of all k-order n-dimensional real tensors is denoted as $\mathbb{T}_{k,n}$.

For a tensor $\mathcal{T} \in \mathbb{T}_{k,n}$, the n entries $\mathcal{T}_{i\cdots i}$ are called its diagonal entries, and others are called off-diagonal entries. If $\mathcal{T}$ does not have non-zero off-diagonal entry, then it is called a diagonal tensor. The identity tensor, denoted by $\mathcal{I}$, is a diagonal tensor with all diagonal entries being 1. Let $a \in \mathbb{C}$, the notation $a\mathcal{T}$ represents the tensor of same order and dimension as $\mathcal{T}$ such that $(a\mathcal{T})_{i_1\cdots i_k} = a\mathcal{T}_{i_1\cdots i_k}$. Given $\mathcal{A}, \mathcal{B} \in \mathbb{T}_{k,n}$, $\mathcal{A}+\mathcal{B}$, $\mathcal{A}-\mathcal{B}$ are two tensors in $\mathbb{T}_{k,n}$ with $(\mathcal{A} \pm \mathcal{B})_{i_1\cdots i_k} = \mathcal{A}_{i_1\cdots i_k} \pm \mathcal{B}_{i_1\cdots i_k}$.

A tensor is called nonnegative if its entries are nonnegative. For a square tensor $\mathcal{T}$, if entries using indices permutated from a common number sequence are equal, then $\mathcal{T}$ is called a symmetric tensor.

An elementary and significant part of tensor analysis is the theory on tensor eigenvalues and eigenvectors. In the literature, various types of tensor eigenvalues were proposed according to distinct ways of generalization from matrix counterpart. In this thesis, we adopt the one introduced by Qi [46] as defined in Definition 2.1, and apply the theory developed based on that to study hypergraphs.

The absolute main role of our study is hypergraph, which is a natural generalization of graph in the Graph Theory. A hypergraph is in general expressed by $H := (V, E)$, where V is a nonempty set and E is a set consisting some nonempty subsets of V. The terms vertex and edge refer to elements of V and E, respectively. Denote $V = [n] := \{1, \cdots, n\}$ and $E = \{e_1, \cdots, e_m\}$. If every $e_i \in E$ is a simple set where elements therein are mutually distinct, and $e_p \subset e_q$ only if $p = q$, then H is called a simple hypergraph, otherwise it is a multi-hypergraph. A loop in H is an edge with a single vertex. In the scope of this thesis, unless otherwise stated, the mentioned hypergraphs are simple and without loops.

If all edges of H has common cardinality k, i.e. each edge contains exactly k vertices of H and $k \geq 2$, then H is called a k-uniform hypergraph (or simply a k-graph). Otherwise it is called non-uniform. To emphasis to possibility of being non-uniform, a hypergraph not necessarily being uniform is also referred to as a general hypergraph.

Apart from the above settings, a very basic concept in graph and hypergraph theory is connectivity. A hypergraph is connected if every two of its vertices are connected through an edge chain. To strictly introduce this concept, we may review the definition of paths.

A path is a graphic structure generally in terms of an alternative sequence of distinct vertices and edges, say $v_1e_1v_2\cdots v_se_sv_{s+1}$, such that two consecutive vertices v_i, v_{i+1} are contained by the edge e_i for each $i \in [s]$. In this case, we say that the vertices v_1 and v_{s+1} are connected by the path. A connected hypergraph is one with all pairs of its vertices being connected.

The connectivity of a graph is linked with the irreducibility of its matrix representations which acts as a fundamental property during the analysis of graph spectrum. In the scenario of hypergraphs and tensors, similar relationship has been obtained.

A classic way to study graphs and hypergraphs is to analyze the sub-structures, namely subgraphs and connected components. Given $H = (V, E)$, a subhypergraph of H means a hypergraph consisting of subsets of V and E. A subhypergraph with vertex set $V' \subset V$ possessing all edges in E formed from V' is called a vertex-induced subhypergraph, denoted by $H[V']$. If for a connected vertex-induced subhypergraph $H[V']$, $H[V'']$ is disconnected for any $V'' \supsetneqq V'$, then $H[V']$ is called a connected component of H. Let $H_1, \ldots, H_s$ be all connected components of H, then we say that H is the disjoint union of $H_1, \ldots, H_s$. We use the notation $H = H_1 \cup \cdots \cup H_s$.

In the majority of cases, properties and spectra of disconnected hypergraphs are investigated through their connected subhypergraphs/components, where advantages of connectivity can be well employed. This has made it reasonable to focus our attention on connected hypergraphs.

Another graphic structure relevant to our study is called cycle. A cycle is expressed in a cyclic and alternative sequence of distinct vertices and edges, say

$v_1e_1v_2\cdots v_se_sv_{s+1}$, with each $e_i \supseteq \{v_i, v_{i+1}\}$ and $v_1 = v_{s+1}$. A hypergraph that contains no cycle is called acyclic. If it is furthermore connected, we call it a supertree. A unicylic hypergraph refers to a connected hypergraph having exactly one cycle, i.e. all cycles in it are expressed by sequences with a common collection of vertices and edges. For k-uniform hypergraphs with at least two cycles, a further classification uses a parameter extended from graph scenario called cyclomatic number, which is defined in terms of vertex number n, edge number m and the number of connected components s as $m(k-1)-n+s$ [19]. A hypergraph with cyclomatic number l is simply called l-cyclic. A connected 2-cyclic hypergraph is also called bicyclic.

For uniform hypergraphs with fixed k and edge number, larger cyclomatic number implies less vertices and more intersection between edges which usually means more complex structures. However, for some certain problems, we can start with hypergraphs in small cyclomatic number and gradually find a framework suitable for more general cases. The first part of our main work provides such a framework toward ordering problems.

Chapter 2

Tensor and its eigenvalues

2.1 Eigenvalues and H-eigenvalues

In 2005, Qi introduced the notions of eigenvalues and H-eigenvalues for symmetric tensors in [46] which can also be used on all square real tensors. The same year, Lim [37] indepently proposed the definition of eigenvalues for real tensors in real field which is slightly different in the odd-order case. Here, we use the definition from [46].

Let $\mathcal{T} \in \mathbb{T}_{k,n}$. Define a n-dimensional vector $\mathcal{T}\mathbf{x}^{k-1}$ by

$$(\mathcal{T}\mathbf{x}^{k-1})_i = \sum_{i_2,\ldots,i_k=1}^{n} T_{ii_2\cdots i_k} x_{i_2} \cdots x_{i_k},\ i \in [n],$$

and denote $\mathcal{T}\mathbf{x}^k = x^\top(\mathcal{T}\mathbf{x}^{k-1})$.

Definition 2.1. *[46] Let $\mathcal{T} \in \mathbb{T}_{k,n}$. If there exist a number $\lambda \in \mathbb{C}$ and a nonzero vector $\mathbf{x} \in \mathbb{C}^n$ satisfying the following homogeneous polynomial equations:*

$$\lambda x_i^{k-1} = (\mathcal{T}\mathbf{x}^{k-1})_i,\ i = 1, \ldots, n,$$

then λ is called an eigenvalue of $\mathcal{T}$ and $\mathbf{x}$ is an eigenvector of $\mathcal{T}$ associated with (corresponding to) λ. The pair $(\lambda, \mathbf{x})$ is called an eigenpair of $\mathcal{T}$. If furthermore an eigenvalue λ has a real eigenvector $\mathbf{y} \in \mathbb{R}^n$ associated to it, then λ is called an H-eigenvalue and $\mathbf{y}$ an H-eigenvector.

The equations in the above that define eigenvalues and eigenvectors are also referred as eigenequations. We see that an H-eigenvalue itself is a real eigenvalue. An H-eigenvalue is further called H^+-eigenvalue ($H^{+}+$-eigenvalue) if it has an associated real non-negative (positive) eigenvector.

The spectrum of $\mathcal{T}$ is the set of all its eigenvalues (multiplicity counted), denoted by $\mathrm{Spec}(\mathcal{T})$. The H-spectrum of $\mathcal{T}$, denoted $\mathrm{HSpec}(\mathcal{T})$ is the collection of all H-eigenvalues of $\mathcal{T}$. The notation $\rho(\mathcal{T})$ represents the largest modulus (magnitude) among elements in $\mathrm{Spec}(\mathcal{T})$, which is called the spectral radius of $\mathcal{T}$.

Note that unlike matrix case, the sum of any two distinct eigenvectors corresponding to one eigenvalue λ of $\mathcal{T}$ may no longer be an eigenvector of $\mathcal{T}$ associated to λ. Fortunately, we have the following multiplicity and additivity properties for tensor eigenpairs.

Proposition 2.1. *Let $\mathcal{T} \in \mathbb{T}_{k,n}$ and $a, b \in \mathbb{R}$. If $(\lambda, \mathbf{x})$ is an eigenpair of $\mathcal{T}$, then*

(1) $(\lambda, a\mathbf{x})$ *is also an eigenpair of $\mathcal{T}$ for $a \neq 0$;*

(2) $(a\lambda + b, \mathbf{x})$ *is an eigenpair of $a\mathcal{T} + b\mathcal{I}$, where $\mathcal{I}$ is the identity tensor.*

In more cases, one may have more interests on H-eigenvalues and their associated real eigenvectors. Two major classes of tensors had been verified to possess at least one H-eigenvalue which are even-order symmetric tensors and nonnegative tensors. Besides, the above proposition indicates that tensors with off-diagonal entries either all nonnegative or all nonpositive have H-eigenvalues.

Additionally for each nonnegative tensor, the spectral radius is its largest H-eigenvalue with an associated nonnegative eigenvector. This result is a tensor version of the Perron-Frobenius theorem which will be stated in detail in the next section.

Apart from the existence of H-eigenvalues, another fundamental and highly concerned problem is the distribution of eigenvalues of a tensor in the complex plane. Following is an analogue of the well-known Gershgorin circle theorem in matrix counterpart.

Proposition 2.2. *[46] Let $\mathcal{T} \in \mathbb{T}_{k,n}$ and $\lambda \in$Spec($\mathcal{T}$). Then there exists $i \in [n]$ such that*

$$|\lambda - \mathcal{T}_{i\cdots i}| \leq \sum_{i_2,\ldots,i_k=1}^{n} |\mathcal{T}_{i_2\cdots i_k}| - |\mathcal{T}_{i\cdots i}|.$$

In other words, the eigenvalues of $\mathcal{T}$ lie in the union of n disks in $\mathbb{C}$ and each of these disks has a diagonal entry of $\mathcal{T}$ as its center and the sum of the absolute values of off-diagonal entries of $\mathcal{T}$ as the radius.

We call the sum $\sum_{i_2\cdots i_k=1}^{n} \mathcal{T}_{ii_2\cdots i_k}$ as the ith row sum of the tensor $\mathcal{T}$ and denote it by $r_i(\mathcal{T})$. Then an observation of nonnegative tensors can be obtained from Proposition 2.2.

Corollary 2.1. *Let $\mathcal{T} \in \mathbb{T}_{k,n}$ be a nonnegative tensor. Then*
(1) $\lambda \leq \max\{r_i(\mathcal{T}), 1 \leq i \leq n\}$*;*
(2) *if $\mathcal{T}$ has equal row sum, say $r_1 = \cdots = r_n = r$, then r is the spectral radius of $\mathcal{T}$ with associated eigenvector* $(1, \ldots, 1)^\top$*.*

This corollary yields a basic result on spectral radius and signless Laplacian spectral radius of regular hypergraphs where each vertex is contained in the same number of edges, which readers may refer to Chapter 3.

2.2 Nonnegative tensors

Similar to the status of nonnegative matrices in matrix theory, nonnegative tensors is one of the most significant classes of tensors which have been extensively studied and widely applied to relevant fields such as signal processing, higher order Markov chains and our most concerned spectral hypergraph theory. Among numerous properties of nonnegative tensors having been found in the literature of tensor analysis, the Perron-Frobenius theory, which characterizes the association properties of eigenvalues and eigenvectors, is evidently one of the most critical and fundamental theoretical pieces.

While the Perron-Frobenius theory for matrices is based on irreducible nonnegative matrices, for tensors, the theory has two main versions which are based upon irreducible nonnegative tensors [11, 60] and weakly irreducible nonnegative tensors [21], respectively. In this section, both two versions are introduced with an emphasis to the later one. We will also present the well-known Collatz-Wielandt minimax theorem in the tensor case and some other basic properties for nonnegative tensors.

It is known that the irreducibility of matrices has many equivalent definitions. Two popular definitions in the literature, one based on index partition and the other based on the associated directed graph, are naturally extended to higher order tensors as bellow.

Definition 2.2. *Let $\mathcal{T} \in \mathbb{T}_{k,n}$. If there exists a nonempty proper index subset $J \subset [n]$ satisfying $\mathcal{T}_{i_1 \cdots i_k} = 0$ for any $i_1 \in J$ and $i_2, \dots, i_k \notin J$, then $\mathcal{T}$ is called a reducible tensor, otherwise it is called irreducible.*

To introduce the second definition, we need to review concepts on strongly connected directed graphs. A directed graph (or simply digraph) $G = (V, A)$ consists of a finite vertex set V and an arc set A whose elements are ordered pairs of V. The digraph G is strongly connected if for each pair of vertices i, j, there are vertices $v_0 = i, v_1, \dots, v_s = j$ such that $(v_{l-1}, v_l) \in A$ for $l \in [s]$.

For every $\mathcal{T} \in \mathbb{T}_{k,n}$, an associated directed graph is constructed as $V = [n]$ and $(i, j) \in A$ if

$$\sum_{j \in \{i_2, \dots, i_k\}} |\mathcal{T}_{i i_2 \cdots i_k}| > 0.$$

Definition 2.3. *A tensor $\mathcal{T}$ is weakly irreducible if its associated directed graph is* *strongly connected, otherwise it is called weakly irreducible.*

It can be found that this two definitions are not equivalent for higher order tensors, while they are equivalent in the definition of weakly irreducible matrices. The weakly irreducibity tensors has a wider range than irreducible tensors. In the following, we show a comprehensive statement of the Perron-Frobenius theorem for nonnegative tensors.

Theorem 2.1 (The Perron-Frobenius theorem for tensors [11, 21, 60])**.** *Let $\mathcal{T} \in \mathbb{T}_{k,n}$ be a nonnegative tensor. Then*
(1) $\rho(\mathcal{T})$ *is an* H^+*-eigenvalue of* $\mathcal{T}$*;*
(2) *if* $\mathcal{T}$ *is weakly irreducible, then* $\rho(\mathcal{T}) > 0$ *is the unique* H^{++}*-eigenvalue of* $\mathcal{T}$ *with a unique positive eigenvector, up to a multiplicative constant;*
(3) *if furthermore* $\mathcal{T}$ *is irreducible, then* ρ *is the unique* H^+*-eigenvalue of* $\mathcal{T}$ *as well.*

Among various type of tensors, symmetric tensors is closely relevant to the study of hypergraph spectra. We collect some special properties of symmetric nonnegative tensors which have been extensively applied in spectral hypergraph theory.

Theorem 2.2. *[47] Let* $\mathcal{T} \in \mathbb{T}_{k,n}$ *be a symmetric nonnegative tensor, where* $k \geq 2, n \geq 1$*. Then*
(1) $\rho(\mathcal{T}) = \max\{\mathcal{T}\mathbf{x}^k : \mathbf{x} \in \mathbb{R}^n_+, \sum_{i=1}^n x_i^k = 1\}$*;*
(2) $\rho(\mathcal{T}) \geq \max\{\bar{r}(\mathcal{T}), d_{\max}(\mathcal{T})\}$*, where* $\bar{r}(\mathcal{T})$ *and* $d_{\max}(\mathcal{T})$ *are the average row sum and the maximum diagonal entries of* $\mathcal{T}$*, respectively.*

The well-know Collatz-Wielandt minimax theorem is another important result in matrix theory. In the scenario of tensors, it was first generalized to irreducible nonnegative tensors which helped the development of algorithms in the computation of spectral radius. Then the irreducibility condition was then relaxed to the weak irreducibility and further to the existence of a positive eigenvector [60]. In 2013, Zhou, Qi and Wu [67] obtained another version for symmetric nonnegative tensors as stated below.

Theorem 2.3 (The Collatz-Wielandt minimax theorem for symmetric nonnegative tensors)**.** *[67] Let $\mathcal{T} \in \mathbb{T}_{k,n}$ be a symmetric nonnegative tensor with $k \geq 2, n \geq 2$. Then*

$$\max_{\mathbf{x}\geq\mathbf{0},\mathbf{x}\neq\mathbf{0}} \min_{x_i\neq 0} \frac{(\mathcal{T}\mathbf{x}^{k-1})_i}{x_i^{k-1}} = \rho(\mathcal{T}) = \min_{\mathbf{x}>\mathbf{0}} \max_{i} \frac{(\mathcal{T}\mathbf{x}^{k-1})_i}{x_i^{k-1}}.$$

Symmetry and nonnegativity of a tensor also enables the existence of index partition satisfying the weak irreducibility of the yielded sub-tensors.

Given $\mathcal{T} \in \mathbb{T}_{k,n}$. Let J be a nonempty subset of $[n]$. We use the notation $\mathcal{T}_J$ to denote the tensor in $\mathbb{T}_{k,|J|}$ with entries being $\mathcal{T}_{i_1\cdots i_k}$ for $i_1, \ldots, i_k \in J$. This tensor $\mathcal{T}_J$ is called a principal sub-tensor of $\mathcal{T}$.

Proposition 2.3. *[21, 27] Let $\mathcal{T} \in \mathbb{T}_{k,n}$ be a symmetric nonnegative tensor. Then there is a partition $(J_1, \ldots, J_s)$ of $[n]$ such that $\mathcal{T}(J_r)$ is weakly irreducible for $r \in [s]$ and $\mathcal{T}_{i_1\cdots i_k} = 0$ for all $i \in J_r, i_2, \ldots, i_k \notin J_r, r \in [s]$. Furthermore,*

$$\rho(\mathcal{T}) = \max_{1\leq r\leq s}\{\rho(\mathcal{T}(J_r))\}.$$

From this proposition, we see that the spectral radii of a symmetric nonnegative tensor can be obtained from its weakly irreducible principal sub-tensors. When it turns to hypergraph spectra, Proposition 2.3 leads to a critical result on the spectral radius and signless Laplacian spectra radius of disconnected hypergraphs.

For any tensor in $\mathbb{T}_{k,n}$, denote $|\mathcal{T}|$ as a tensor with entries $(|\mathcal{T}|)_{i_1\cdots i_k} = |\mathcal{T}_{i_1\cdots i_k}|$ for $i_j \in [n]$ and $j \in [k]$. Apparently, $|\mathcal{T}|$ is a nonnegative tensor. The following proposition show that we can bound the spectral radius of an arbitrary tensor from above by computing the spectral radius of a nonnegative tensor.

Proposition 2.4. *[21] Let $\mathcal{A}, \mathcal{B} \in \mathbb{T}_{k,n}$ and $\mathcal{B}$ is a nonnegative tensor with $k \geq 2, n \geq 2$. If $|\mathcal{A}| \leq B$, i.e $|\mathcal{A}_{i_1 \cdots i_k}| \leq \mathcal{B}_{i_1 \cdots i_k}$, then $\rho(\mathcal{A}) \leq \rho(\mathcal{B})$.*

In the next chapter, the Laplacian tensor $\mathcal{L}$ and signless Laplacian tensor $\mathcal{Q}$ of a hypergraph are such two tensors and $|\mathcal{L}| = \mathcal{Q}$.

Chapter 3

Preliminaries of spectral hypergraph theory

The purpose of this chapter is to introduce the primary part of spectral hypergraph theory and some useful tools to analyse hypergraph spectrum. We begin with three major tensor representations of hypergraphs, the Adjacency tensor, the Laplacian tensor and the signless Laplacian tensor. Then we review some basic properties of hypergraph spectra derived from the theory of nonnegative tensors, symmetric tensors in Chapter 2. Finally, we introduce some major methods to study spectra of hypergraphs, including edge operations, power hypergraph etc., with a majority of which target on the spectra radius.

3.1 Tensors associated to a hypergraph

The idea of representing a hypergraph by matrix or tensor stems from spectral graph theory. Using an array to represent a graph diminishes storage cost and facilitates computation, enables researchers to deal with tricky combinatorial problems in graph theory by the powerful matrix theory and algebraic methods. Initially, hypergraphs were considered to be characterized also by matrices and various representations have been proposed [13, 20] in the literature. However, they ran into obstacle quickly as those matrices could just partially describe their corresponding hypergraphs but lost intrinsic information more or less as edges are specified by more than two vertices.

A more natural way is to generalize the representation from 2-array (matrix) to multi-array (hypermatrix or tensor). It was until the year of 2008, after the preliminary stage of developments in the theory of tensor eigenvalues, that the first tensor representation called the adjacency tensor of hypergraphs was proposed by Lim in his talk during a seminar in 2008. From then on, studying hypergraphs via tensors jumped into researchers' horizons and gradually attracted scholarly interests. In 2012, Cooper and Dutle [16] published the first paper that systematically studied the adjacency tensor of uniform hypergraphs. The definition therein is as follows.

Definition 3.1. *[16] Let $H = (V, E)$ be a k-graph with $V = [n]$. The adjacency tensor $\mathcal{A} = \mathcal{A}(H)$ of H is a k-order n-dimensional tensor with entries $\mathcal{A}_{i_1 \cdots i_k}$ such that*

$$\mathcal{A}_{i_1 \cdots i_k} = \begin{cases} \frac{1}{(k-1)!} & \text{if } \{i_1 \cdots i_k\} \in E, \\ 0 & \text{otherwise,} \end{cases}$$

It can be observed that $\mathcal{A}$ is a nonnegative and symmetric tensor.

The other two generally accepted representations are the Laplacian tensor and the signless Laplacian tensor. It should be mentioned that the Laplacian tensors defined in the literature were in relatively complicated forms based on the schemes of sum of powers [33, 56], which deviate the brief formalization as the Laplacian matrix has in graph counterpart. In 2014, Qi [48] introduced the following definition of Laplacian and signless Laplacian tensors, which are constructed from the adjacency tensor through adding the information of vertex degrees along its diagonal entries. Since then, studied on the two Laplacian tensors are utilizing this definition.

Let E_i denote the set of edges in H that contain a vertex i. The parameter $d_i = |E_i|$ is called the degree of i in H. A hypergraph with all vertices having a common degree r is called an r-regular hypergraph.

Definition 3.2. *[48] Let* $H = (V, E)$ *be a* k*-graph with* $V = [n]$ *and* d_i *be the degree of vertex* i*. The degree tensor* $\mathcal{D} = \mathcal{D}(H)$ *of* H *is a* k*-order* n*-dimensional tensor with entries* $\mathcal{D}_{i_1 \cdots i_k}$ *such that*

$$\mathcal{D}_{i_1 \cdots i_k} = \begin{cases} d_i & \text{if } i_1 = \cdots = i_k = i, \ i \in [n], \\ 0 & \text{otherwise,} \end{cases}$$

The Lapalcian tensor $\mathcal{L} = \mathcal{L}(H)$ *and signless Laplacian tenosr* $\mathcal{Q} = \mathcal{Q}(H)$ *are two* k*-order* n*-dimensional tensors with entries* $\mathcal{L}_{i_1 \cdots i_k} = \mathcal{D}_{i_1 \cdots i_k} - \mathcal{A}_{i_1 \cdots i_k}$ *and* $\mathcal{Q}_{i_1 \cdots i_k} = \mathcal{D}_{i_1 \cdots i_k} + \mathcal{A}_{i_1 \cdots i_k}$*, respectively. They are simply expressed as* $\mathcal{L} = \mathcal{D} - \mathcal{A}$ *and* $\mathcal{Q} = \mathcal{D} + \mathcal{A}$ *as well.*

After half a decade of rather intense study on uniform hypergraphs, some researchers turned their first steps to more general cases. Recently, two extended versions of the three tensors $\mathcal{A}, \mathcal{L}, \mathcal{Q}$ for general hypergraphs were proposed by Banerjee et al [1] and Bu et al [9] respectively. In this thesis, we adopt the first version as it keeps the property of symmetry and received wider acknowledgement from peers.

Definition 3.3. *[1] Let* $H = (V, E)$ *be a hypergraph with maximum edge cardinality* $k \geq 2$ *and* $V = [n]$*. The adjacency tensor of* H *is a* k*-order* n*-dimensional tensor* $\mathcal{A} = \mathcal{A}(H)$ *defined as*

$$\mathcal{A}_{i_1 \cdots i_k} = \frac{|e|}{\beta_e}, \text{ where } \beta_e = \sum_{\substack{\alpha_1, \ldots, \alpha_{|e|} \geq 1 \\ \alpha_1 + \cdots + \alpha_{|e|} = k}} \frac{k!}{\alpha_1! \cdots \alpha_{|e|}!}$$

if the set of all distinct elements in $\{i_1, \ldots, i_k\}$ *is an edge* $e \in E$*, and* $\mathcal{A}_{i_1 \cdots i_k} = 0$ *for other* $i_1, \ldots, i_k \in [n]$*. The Laplacian tensor* $\mathcal{L}$ *and signless Laplacian tensor* $\mathcal{Q}$ *of* H *are formulated as* $\mathcal{D} - \mathcal{A}$ *and* $\mathcal{D} + \mathcal{A}$ *respectively. Here,* $\mathcal{D}$ *is the degree tensor of* H *with order* k *and dimension* n *whose* i*th diagonal entry being* d_i *for* $i \in [n]$ *and non-diagonal entries being zero.*

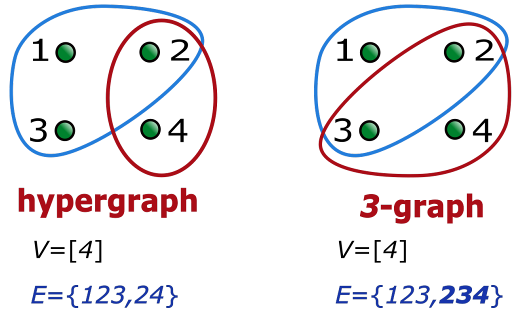


Figure 3.1: Two general hypergraphs

The maximum and minimum edge cardinality of a hypergraph will also be called rank and co-rank of it for abbreviation.

Following is an example to show difference between the adjacency tensor of a 3-graph and a non-uniform hypergraph.

Example 3.1. *Figure 3.1 depicts a 3-graph H and a non-uniform hypergraph F with rank 3 and co-rank 2. According to Definitions 3.1 and 3.3, the entries of their adjacency tensors $\mathcal{A}(H)$ and $\mathcal{A}(F)$ are listed as below.*

For the uniform hypergraph H we have $\mathcal{A}(H) = (a_{i_1 i_2 i_3})$:

$a_{123} = a_{132} = a_{213} = a_{231} = a_{312} = a_{321} = a_{234} = a_{243} = a_{324} = a_{342} = a_{423} = a_{432} = \frac{1}{2}$, *and other* $a_{i_1 i_2 i_3} = 0$.

For the non-uniform hypergraph F we have $\mathcal{A}(F) = (b_{i_1 i_2 i_3})$:

$b_{123} = b_{132} = b_{213} = b_{231} = b_{312} = b_{321} = \frac{1}{2}$,

$b_{224} = b_{242} = b_{422} = b_{244} = b_{424} = b_{442} = \frac{2}{2 \times \frac{3!}{2!1!}} = \frac{1}{3}$,

and other $a_{i_1 i_2 i_3} = 0$.

Recall from Definiton 2.1 that for each square tensor $\mathcal{T}$, eigenvalues and eigenvectors are defined through a homogeneous polynomial system and its spectrum $\operatorname{Spec}(\mathcal{T})$

is the collection of all eigenvalues with multiplicity. We call the eigenvalues (spectrum) of $\mathcal{A}$ as the eigenvalues (spectrum) of the hypergraph H, while eigenvalues (spectrum) of $\mathcal{L}$ and $\mathcal{Q}$ are called the Laplaican and signless Laplacian eigenvalues (spectrum) of H respectively. Besides, $\rho(\mathcal{A}), \rho(\mathcal{L}), \rho(\mathcal{Q})$ are called the spectra radius, the Laplacian spectral radius and the signless Laplacian spectral radius of H respectively.

It deserves remark that as the above tensor are symmetric, according to a proposition for symmetric tensors [46], permuting some of the indices does not influence the eigenvalues, which means the spectra of $\mathcal{A}$, $\mathcal{L}$, $\mathcal{Q}$ do not depend on the labeling of vertices.

Apart from symmertry, $\mathcal{A}, \mathcal{Q}$ are nonnegative which make the spectral theory of nonnegative tensors applicable in our study of hypergraphs spectra. Hence the elemental properties for symmetric and nonnegative tensors reviewed in the last chapter can be applied.

3.2 Properties of the adjacency, Laplacian and signless Laplacian tensors

Denote by Δ and d the maximum and the average vertex degree (the average number taken over all vertex degrees) of a hypergraph $H = (V, E)$. From the definition and structure of the three tensors, it is evident that the ith row sum of $\mathcal{A}$ and $\mathcal{Q}$ are the vertex degree d_i and $2d_i$ respectively, and the Laplacian tensor $\mathcal{L}$ has equal row sum zero.

According to Propositions 2.1, 2.2, Corollary 2.1 and Theorem 2.2 (2), we have the following basic properties on the three types of spectra of a hypergraph.

Theorem 3.1. *Let $H = (V, E)$ be a hypergraph with $n \geq 2$ vertices, $\mathcal{A}, \mathcal{L}$ and $\mathcal{Q}$ are the adjacency tensor, the Laplacian tensor and the signless Laplacian tensor of H*

respectively. Then

(1) $|\lambda| \le \Delta$ *for* $\lambda \in Spec(\mathcal{A})$, $|\mu - \Delta| \le \Delta$ *for* $\mu \in Spec(\mathcal{L}) \cup Spec(\mathcal{Q})$;

(2) $d \le \rho(\mathcal{A}) \le \Delta$ *and* $2d \le \rho(\mathcal{Q}) \le \rho(\mathcal{A}) + \Delta$;

(3) $\rho(L) \le \rho(\mathcal{Q}) \le 2\Delta$;

(4) 0 *is an* H*-eigenvalue of* $\mathcal{L}$ *with* H*-eigenvector* $(1, \dots, 1)^\top$;

(5) *if furthermore* H *is* r*-regular, then* $\rho(\mathcal{A}) = r$ *and* $\rho(\mathcal{Q}) = 2r$.

We now discuss the irreducibility and weak irreducibility of the ajacency tensor, the Lapalcian tensor and the signless Laplacian tensor of a hypergraph.

Suppose that $H = (V, E)$ is a hypergraph with $V = [n]$, rank k and co-rank at least 3. Then by setting $J = \{1, \dots, n-1\}$, we have that for any $i_1 \in J$ and $i_2, \dots, i_k \notin J$, $\mathcal{A}_{i_1 \cdots i_k} \equiv \mathcal{A}_{i_1 n \cdots n} = 0$. Thus the adjacency tensor $\mathcal{A}$ is irreducible. The reducibility of $\mathcal{L}, \mathcal{Q}$ can be similarly proved. However, the weak irreducibility of the adjacency tensor and the two Laplacian tensors were verified respectively in [44] and [48] for uniform hypergraphs. Recently, the results were extended to general hypergraphs as well [66].

Theorem 3.2. *Let* $H = (V, E)$ *be a hypergraph with* $n \ge 2$ *vertices,* $\mathcal{A}, \mathcal{L}$ *and* $\mathcal{Q}$ *are the adjacency tensor, the Laplacian tensor and the signless Laplacian tensor of* H *respectively. Then* $\mathcal{A}, \mathcal{L}$ *and* $\mathcal{Q}$ *are weakly irreducible if and only if* H *is connected.*

The above results establishes a link between the weak irreducibility of tensors and the connectivity of hypergraphs which enables us to employ the pleasant properties of weakly irreducible tensors in the study of hypergraphs. Then by Theorem 2.1, the property for $\rho(\mathcal{A}), \rho(\mathcal{Q})$ from the Perron-Frobenius theorem is stated as below.

Theorem 3.3. *Let* $H = (V, E)$ *be a hypergraph with* $n \ge 2$ *vertices. Then* $\rho(\mathcal{A}) > 0$ *is the unique* H^{++}*-eigenvalue of* $\mathcal{A}$ *with a unique positive eigenvector, up to a multiplicative constant. Similar conclusions hold for* $\mathcal{Q}$.

As the adjacency tensor and the signless Laplacian tensor are symmetric and nonnegative, we obtain the following from Theorems 2.2, 2.3 and Propostion 2.3.

Theorem 3.4. *Let $H = (V, E)$ be a hypergraph with $n \geq 2$ vertices. Then*
(1) $\rho(\mathcal{A}) = \max\{\mathcal{A}\mathbf{x}^k : \mathbf{x} \in \mathbb{R}^n_+, \sum_{i=1}^n x_i^k = 1\}$;
(2) $\max_{\mathbf{x}\geq\mathbf{0},\mathbf{x}\neq\mathbf{0}} \min_{x_i\neq 0} \frac{(\mathcal{A}\mathbf{x}^{k-1})_i}{x_i^{k-1}} = \rho(\mathcal{A}) = \min_{\mathbf{x}>\mathbf{0}} \max_i \frac{(\mathcal{A}\mathbf{x}^{k-1})_i}{x_i^{k-1}}$;
(3) *If H has s connected components $G_1, \ldots, G_s$ and $\mathcal{A}(G_r)$ is the adjacency tensor of G_r for $r \in [s]$, then $\rho(\mathcal{A}) = \max\{\rho(\mathcal{A}(G_r)), r \in [s]\}$.*

For simplicity, we write $\mathcal{A}_{i_1\cdots i_k}$ as $\mathcal{A}_e$ if the distinct elements of $\{i_1, \ldots, i_k\}$ consists an edge $e \in E$. Then the eigenequations in Definition 2.1 at vertex i for $\mathcal{A}, \mathcal{L}$ and $\mathcal{Q}$ of a hypergraph can be rewritten as

$$\lambda x_i^{k-1} = \sum_{e\in E_i} \mathcal{A}_e \mathbf{x}^{e\setminus i},$$

$$(d_i - \mu) y_i^{k-1} = \sum_{e\in E_i} \mathcal{A}_e \mathbf{y}^{e\setminus i},$$

$$(\nu - d_i) z_i^{k-1} = \sum_{e\in E_i} \mathcal{A}_e \mathbf{z}^{e\setminus i},$$

where $(\lambda, \mathbf{x})$, $(\mu, \mathbf{y})$ and $(\nu, \mathbf{z})$ are corresponding eigenpairs.

Denote by $\mathbf{1}_j$ the unit vector with a unique nonzero entry 1 as the ith entry. Note that for a k-graph, any off-diagonal entry of $\mathcal{A}, \mathcal{L}, \mathcal{Q}$ is zero if a number repeats in the indices. Due to this special structure, we have the following proposition.

Proposition 3.1. *Let $H = (V, E)$ be a k-graph with $n \geq 2$ vertices. Then the n unit vectors $\mathbf{1}_j$ for $i \in [n]$ are eigenvectors of $\mathcal{A}, \mathcal{L}, \mathcal{Q}$ associated with the eigenvalue 0 for $\mathcal{A}$ and eigenvalues d_j for $\mathcal{L}, \mathcal{Q}$, respectively.*

3.3 Some relevant spectral methods

This section may contain three main ingredients: edge operations, power hypergraphs and the weighted incidence matrices.

We begin with the edge operations introduced in [34] and [64] for k-graphs which are helpful in comparing spectral radius between different hypergraphs.

Two vertices appeared in one edge are called adjacent and said to be connected by this edge. If an edge contains a vertex v, then it is an incident edge of v. A vertex has a unique incident edge is called a pendent vertex, otherwise we call it non-pendent. In a k-graph, a pendent edge refers to an edge containing $k-1$ pendent vertices.

The first edge operation may be refered to as edge grafting as it grafts edge(s) from a bunch of vertices to aother vertex.

Definition 3.4. *[34] Let $H = (V, E)$ be a k-graph. Suppose there exists $u \in V$, $e_1, \cdots, e_r \in E$ for $r \geq 1$ such that $u \notin \cup_{i=1}^{r} e_i$. Let $v_i \in e_i$ and write $e_i' = (e_i \backslash \{v_i\}) \cup \{u\}$ for $i \in [r]$. Denote $H' = (V, E')$ as the hypergraph with $E' = (E \backslash \{e_i : i \in [r]\}) \cup \{e_i' : i \in [r]\}$. Then H' is said to be obtained from H by moving edges $(e_1, \cdots, e_r)$ from $(v_1, \cdots, v_r)$ to u.*

It was proved that this operation, if it generates no multiple edges (two edges containing exactly the same vertices), then the spectral radius increases as long as a requirement of the associated eigenvector (Perron vector)is satisfied.

Lemma 3.1. *[34] Let H be a connected k-graph. Let H' be the k-graph obtained from H by moving edges $(e_1, \cdots, e_r)$ from $(v_1, \cdots, v_r)$ to u where $r \geq 1$. Suppose that H' does not contain multiple edges. If $\mathbf{x}$ is a Perron vector of $\mathcal{A}(H)$ and $x_u \geq \max_{1 \leq i \leq r} x_{v_i}$, then $\rho(\mathcal{A}(H')) > \rho(\mathcal{A}(H))$.*

The second edge operation is a special case of edge grafting. Through adding restriction on the relevant vertices and edges, an increase of spectral radius is obtained without the condition on Perron vector.

Lemma 3.2. *[64] Let H be a connected k-graph with n vertices and $k \geq 3$. Suppose that H has two edges e and f satisfying $|e \cap f| = k - r$ $(2 \leq r \leq k-1)$. Let $V_1 = e \cap f$, $e \backslash V_1 = \{u_1, \cdots, u_r\}$ and $f \backslash V_1 = \{v_1, \cdots, v_r\}$, where $r \geq 2$, u_1, v_1 are non-pendent vertices while $u_2, \cdots, u_r$ and $v_2, \cdots, v_r$ are pendent vertices. Denote by $H_{e,f}$ the hypergraph obtained from H through moving all the edges incident with v_1 except f from v_1 to u_2. Then $\rho(\mathcal{A}(H_{e,f})) > \rho(\mathcal{A}(H))$.*

Besides, we may present another statement of Lemma 3.1 as below which fits our proof better in the next chapter.

Lemma 3.3. *Let H be a connected k-graph. Let $r \geq 2$ and $v_1, \cdots, v_r$ be some vertices of H. If H_i is a simple hypergraph obtained from H through moving at least one edge from vertices $\{v_j : j \in [r], j \neq i\}$ to v_i, then*

$$\max\{\rho(\mathcal{A}(H_i)) : i \in [r]\} > \rho(\mathcal{A}(H)).$$

From this restatement, we derive an corollary for a special case.

Corollary 3.1. *Let H be a connected k-graph and u_1, u_2 are two adjacent vertices in H. Denote by H' the k-graph obtained from H through moving all edges containing u_2 except edges containing both u_1, u_2 from u_2 to u_1. If $H' \ncong H$, then*

$$\rho(\mathcal{A}(H)) < \rho(\mathcal{A}(H')).$$

Proof. If u_1 or u_2 is only contained by their common edges, then $H \cong H'$. Thus the assumption $H' \ncong H$ means both u_1, u_2 have incident edges other than their common edges. Denote by H'' the hypergraph obtained from H through moving each incident edge of u_1 except common edges of u_1, u_2 from u_1 to u_2. Observe that H' and H'' have no multiple edges as each common edge of u_1, u_2 remains unchanged. In addition, $H'' \cong H'$. By Lemma 3.3, $\rho(\mathcal{A}(H)) < \max\{\rho(\mathcal{A}(H')), \rho(\mathcal{A}(H''))\} = \rho(\mathcal{A}(H'))$. $\square$

Before the introduction of power hypergraphs, we may review a spectral technique on graphs which will be used in the preliminary work of next chapter. Recall that for a multi-graph [5] with n vertices and no loops, its the adjacency matrix is an n by n matrix with the (ij)-entry being the number of parallel edges containing i and j if $i \neq j$ and zero elsewhere.

The characteristic polynomial of a graph G is denoted as $\phi_G(x) = \det(xI - A(G))$, where $A(G)$ denotes the adjacency matrix of G and I is the identity matrix. Suppose that the graph G can be obtained from two disjoint graphs K and F through amalgamating a vertex v of F and w of K, then the following relation are showed in [50, Remark 1.6]:

$$\phi_G(x) = \phi_K(x)\phi_{F-v}(x) + \phi_{K-w}(x)\phi_F(x) - x\phi_{F-v}(x)\phi_{K-w}(x), \tag{$*$}$$

where $F-v$ and $K-w$ are the graphs obtained from F and K through an elimination of v and w and all edges incident with them respectively.

Now we move on to power hyeprgraphs and generalized power hypergraphs.

The definition of power hypergraph was first proposed in [28] based on a simple graph. Actually, a power hypergraph can also be constructed with graphs without loops but having multiple edges, i.e. edges with the same two ends.

Let G be a graph containing no loops. For an integer $k \geq 3$, the kth power of G is a k-graph obtained from G through blowing up each edge to hyperedge by adding $k-2$ new vertices, and denoted as G^k. If a hypergraph is a power of some graph without loops, then we call it a power hypergraph [28]. The following figure depicts a star and its 3rd power.

Later, for the investigation of the non-odd-bipartiteness of even uniform hypergraphs, the generalized power hypergraph of a 2-graph was introduced and studied in [31, 30]. Given a 2-graph without loops, a generalized power hypergraph $G^{k,s}$ with $s \leq \frac{k}{2}$ is obtained from G through blowing up each vertex into an s-set, and every

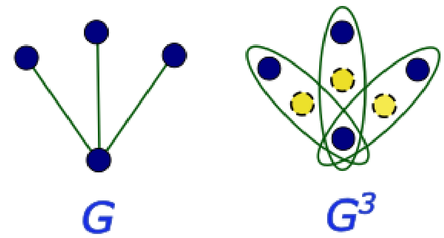


Figure 3.2: The 3rd power of a star

edge into a $(k2s)$-set. It is clear that when $s = 1$, $G^{k,s}$ is the power hypergraph of G. This concept was further generalized to the case when the generating hypergraph G is a uniform hypergraph in [30].

The definition of [30] is now extended such that the resulting hypergraph can be a non-uniform hypergraph.

Definition 3.5. *Let $G = (V, E)$ be a simple r-uniform hypergraph with $r \geq 2$. For some $k \geq r$ and $1 \leq s \leq \lfloor \frac{k}{r} \rfloor$, a generalized power $H = (V', E')$ of G is obtained by constructing new mutually disjoint vertex sets V_i (of size s) and V_e (of maximum size $k - rs$) for all $i \in V$, $e \in E$ such that*

(1) *$V' = (\cup_{i \in V} V_i) \cup (\cup_{e \in E} V_e)$;*

(2) *for each $e' \in E'$, there exists $e \in E$ satisfying $e' = (\cup_{i \in e} V_i) \cup V_e$.*

The set of all generalized power hypergraph obtained from G with rank k and fixed s is denoted as $\mathbb{G}^{k,s}$, and G is called a base of $H \in \mathbb{G}^{k,s}$.

One of the most useful result on this class of hypergraphs is the relation between spectral radius of power hypergraph and that of its base, which was first proposed in [69] and then extended to generalized power hypergraphs of an r-graph in [30]. We now state them in one theorem.

Theorem 3.5. *Let $H \in \mathbb{G}^{k,s}$ be a generalized power hypergraph as in Definition 3.5* *of an r-graph $G = (V, E)$, $k \geq \max\{r, 3\}$. Then*

$$\rho(\mathcal{A}^H) = [\rho(\mathcal{A}^G)]^{\frac{rs}{k}}.$$

This equation in the above theorem acts as one of the main tools in the next chapter and inspires us to consider the parallel situation for non-uniform case in Chapter 6.

Finally, we shall introduce another significant tool called the weighted incidence matrix for hypergraphs introduced by Lu and Man [39].

Definition 3.6. *[39] Let$H = (V, E)$ be a k-graph. A weighted incidence matrix B of H is a $|V| \times |E|$ matrix satisfies that any $v \in V$ and $e \in E$, if $v \in e$, then the entry $B(v, e) > 0$, and $B(v, e) = 0$ if $v \notin e$.*

Through a series of results derived in [39], we can characterize the spectral radius by a particular value α through a construction of consistent α-normal, α-subnormal or α-supernormal weighted incidence matrix for the corresponding hypergraph. The part on α-subnormal case are presented as below.

Definition 3.7. *[39] A hypergraph H is called α-subnormal if we can find a weighted incidence matrix B such that*

(1) $\sum_{e:v\in e} B(v, e) \leq 1$, *for each $v \in V(H)$;*

(2) $\prod_{v\in e} B(v, e) \geq \alpha$, *for each $e \in E(H)$.*

If no strict inequality appears in (1) and (2), then H is called α-normal. Otherwise, H is strictly α-subnormal. If furthermore,

$$\prod_{i=1}^{l} \frac{B(v_i, e_i)}{B(v_{i-1}, e_i)} = 1$$

for each cycle $v_0e_1v_1e_2 \cdots e_lv_0$ $(l \geq 1)$ of H, then B is consistent. In this case, we say that H is strictly and consistently α-subnormal.

Lemma 3.4. *[39] Let H be a k-graph. Then*

(1) $\rho(\mathcal{A}(H)) = \alpha^{-\frac{1}{k}}$ *if and only if H is consistently α-normal;*

(2) *if H is strictly and consistently α-subnormal, then*

$$\rho(\mathcal{A}(H)) < \alpha^{-\frac{1}{k}}.$$

It should be remarked here that the spectral radius of the original paper [39] , say $\rho^*(\mathcal{A}(H))$, is multiplied by $(k-1)!$, i.e. $\rho^*(\mathcal{A}(H)) = (k-1)!\rho(\mathcal{A}(H))$. The formula therein $\rho^*(\mathcal{A}(H)) < (k-1)!\alpha^{-\frac{1}{k}}$ has been adjusted to the above one in [43].

Chapter 4

Spectral radius of unicyclic and bicyclic hypergraphs

Recall from Section 1.2 that a connected k-graph with n vertices and m edges is called r-cyclic if $n = m(k-1) - r + 1$. Particularly when r takes the value 1 or 2, the k-graph is called unicyclic or bicyclic. This chapter is devoted to our investigation on k-graphs with largest spectral radii among the set of all simple connected unicyclic and bicyclic k-graphs.

With a initial work on the adjacency tensor by Cooper and Dutle [16], Li, Shao and Qi [34] took the first step on extremal problems on hypergraph spectra and determined the maximum spectral radius among all supertrees. In the following year, Yuan, Shao and Shan [64] proceeded with the study on uniform supertrees and obtained a detailed order of the supertrees attaining largest spectral radii by using edge operations generalized to hypergraphs and the spectral property of power hypergraphs [69].

The first study of extremal problems for uncyclic and bicyclic hypergraphs was carried out by Fan, Tan, Peng and Liu [19] three years ago. In this paper, they revealed a necessary condition for hypergraphs with maximum spectral radius among a certain group, based on which the unique hypergraph attaining the maximum spectral radii over all unicyclic k-graphs was determined. Besides, they found the linear hypergraph with maximum spectral radius over all linear unicyclic k-graphs and proposed several candidates for the bicyclic case. Later, based on that nomination, Kang et al. [30] determined the hypergraph maximizing the spectral radius among all linear bicyclic k-graphs.

However, the above preceding work do not cover non-linear bicyclic hypergraphs, which are actually the major part of bicyclic hypergraphs. On the other hand, there is also a blank about uncyclic hypergraphs with spectral radius between the maximum spectral radius among all unicyclic hypergraphs and that over all linear unicyclic hypergraphs.

With a motivation to make up the deficiency and do some contributions to the undeveloped extremal spectral hypergraph theory, we conduct a study on the entire groups of uncyclic and bicyclic hypergraphs. By modifying edge operations introduced in [34, 64], utilizing the spectral property of power hypergraphs, the weighted incidence matrix and a previous result on linear unicyclic hypergraphs, we find out the first five k-graphs attaining largest spectral radius over all unicyclic hypergraphs and the first three among all bicyclic hypergraphs.

4.1 Structures of unicyclic and bicyclic k-graphs

Let $\mathbb{U}^m$ and $\mathbb{B}^m$ be the set of all connected unicyclic and bicyclic hypergraphs of m edges respectively with $m \geq 2$. In this section, some basic structural properties of unicyclic and bicyclic hypergraphs will be presented.

Recall from Section 1.2 that a cycle in H can be formed from a path and an edge containing the two end vertices of that path. The length of a cycle refers to the edge number of this cycle. We call an edge appeared in a cycle as a cycle edge.

A k-graph of is called r-cyclic if $m(k-1) - n + l = r$, where n, m, l are the numbers of its vertices, edges and connected components [19], respectivly. Observe that $r \geq 0$, thus for every simple k-graph there is $n \leq m(k-1)+l$. Additionally, it was verified in the Proposition 4 of [3] that $r = 0$ if and only if the k-graph is acyclic, i.e. it does not have cycles.

In the following, we prove that an r-cyclic k-graph can not have an $(r+1)$-cyclic subgraph.

Lemma 4.1. *Let $H = (V, E)$ be a simple connected r-cyclic k-graph of n vertices and m edges. Suppose $H_1 = (V_1, E_1)$ is a connected subgraph of H. If H_1 is r_1-cyclic, then $r_1 \leq r$.*

Proof. Denote $E_2 = E \backslash E_1$ and $V_2 = \cup_{e \in E_2} e$. Then $H_2 = (V_2, E_2)$ is a k-uniform subgraph of H. Let $|V_i| = n_i$ and $|E_i| = m_i$ for $i = 1, 2$. Then we have

$$n_1 = m_1(k-1) - r_1 + 1,$$

as H_1 is connected and r-cyclic. Let l be the number of connected components of H_2. Then $n_2 \leq m_2(k-1) + l$. Due to the connectivity of H, each connected component of H_2 intersects with H_1 at some vertices. Thus $n_1 + n_2 \geq n + l$. Then we have

$$n \leq n_1 + n_2 - l \leq m_1(k-1) - r_1 + 1 + m_2(k-1) + l - l = m(k-1) - r_1 + 1.$$

Therefore $r_1 \leq m(k-1) - n + 1 = r$. □

According to the property of Lemma 4.1, we derive the following restrictions on edge intersection for unicyclic and bicyclic k-graphs.

Proposition 4.1. *Let $F \in \mathbb{U}^m$ and $K \in \mathbb{B}^m$. Then*

(1) *each pair of vertices in F have at most two common edges;*

(2) *each triple of vertices in F share at most one common edge;*

(3) *each pair of vertices in K have at most three common edges;*

(4) *each triple of vertices in K share at most two common edges;*

Proof. If there are a pair of vertices in F with three common edges, or there are a triple of vertices with two common edges, then there is a bicyclic subgraph in H containing those common edges, which is yields a contradiction to Lemma 4.1.

If K has two vertices with four common edges, then those common edges yields a 3-cyclic subgraph in K, which is a contradiction to Lemma 4.1. If there are a triple of vertices in K with three common edges, then the three edges yields a 4-cyclic subgraph, which contradicts Lemma 4.1. □

The Lemma 2 of [19] shows that if there is merely one cycle in H, then H is unicyclic ($r = 1$). Following is a proof of the inverse.

Lemma 4.2. *Let H be a simple connected k-graph. Then H is unicyclic if and only if it contains exactly one cycle.*

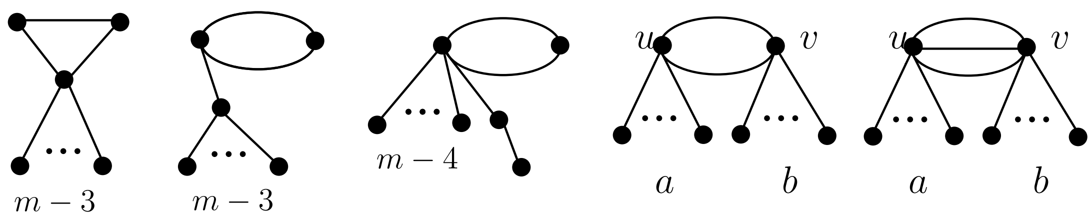
Proof. It is sufficent to prove the necessity. Suppose $H = (V, E)$ with $|V| = n$, $|E| = m$.

Let e_1 be an edge appeared in an cycle $C = v_1e_1v_2 \cdots v_se_sv_1$. Let u be a new vertex and $f = (e_1 \backslash \{v_1\}) \cup \{u\}$. Then the k-graph $H' = (V \cup \{u\}, (E \backslash \{e_1\}) \cup \{f\})$ is connected and with $n+1$ vertices and m edges.

Note that $n = m(k-1)$ as H is unicyclic. Therefore, we have $n+1 = m(k-1)+1$ implying H' is acyclic. Thus e_1 is contained by every cycle in H. Due to the arbitrariness of e_1, we can concluded that every cycle edge of H appears in each cycle of H, i.e., all cycles of H have a common edge set with a common length s.

Suppose that $s = 2$, then from Proposition 4.1 (ii), we know that the two cycle edges intersect at exactly two vertices. Therefore H has only one cycle.

If $s \geq 3$. Let K be the subgraph consisting of all cycle edges in H with n' vertices. Since we can arrange all edges of K in a cyclic sequence such that any two consecutive edges have at least one common vertices. If we can find two consecutive



G_1 G_2 G_3 $G(a,b)$ $M(a,b)$

Figure 4.1: Some unicyclic and bicyclic graphs

edges in K with an intersection of two vertices, then

$$n' \leq s + 1 + s(k-2) - 2.$$

Therefore $r' = s(k-1) - n' + 1$ is at least 2, which indicates that K is r'-cyclic and $r' \geq 2$. Now we obtain a contradiction to Lemma 4.1. Thus, each two consecutive edges in K intersects at merely one vertex, implying that H has exactly one cycle. □

A different statement to verify the above lemma can be found in the Propsition 4.57 of [49].

4.2 Comparison of spectral radius

Previous to the proof of main results, we list some comparison of spectral radius between graphs and hypergraphs that are necessary to the next two sections. Throughout this section, the spectral methods introduced in the last chapter are well empolyed.

We may begin with the comparison between graphs. Denote by $G(a,b)$ a multigraph obtained from a cycle of length 2 through attaching a and b pendent edges at its two vertices u and v respectively. Let $M(a,b)$ be the multi-graph obtained from $G(a,b)$ through adding an extra edge connecting the vertices u and v (See Figure 4.1).

Lemma 4.3. *Let G_1, G_2, G_3 and $G(a,b)$ be the graphs shown in Figure 4.1 with m edges. Then we have for $m \geq 8$ that*

$$\rho(A(G(m-2,0))) > \rho(A(G_3)) \geq \rho(a(G(m-4,2))) > \max\{\rho(A(G_1)), \rho(A(G_2))\},$$

where the second equality holds if and only if $m = 8$.

Proof. Note that G_1 can be obtained from a triangle C_3 and a star $K_{1,m-3}$ through amalgamating a vertex in C_3 and the unique non-pendent vertex in $K_{1,m-3}$. Utilizing the formula $(*)$ in Section 3.3, we have

$$\begin{aligned} \phi_{G_1}(x) &= x^{m-3} \cdot \phi_{C_3}(x) + \phi_{P_2}(x) \cdot \phi_{K_{1,m-3}}(x) - x \cdot x^{m-3} \cdot \phi_{P_2}(x) \\ &= x^{m-4}(x+1)[x^3 - x^2 - (m-1)x + m - 3], \end{aligned}$$

where P_2 denotes a path with one edge. Similarly, we obtain the characteristic polynomials for $G(a,b), G_2$ and G_3 as follows.

$$\begin{aligned} \phi_{G(a,b)}(x) &= x^{m-4}[x^4 - (m+2)x^2 + ab], \\ \phi_{G_2}(x) &= x^{m-4}[x^4 - (m+2)x^2 + 4(m-3)], \\ \phi_{G_3}(x) &= x^{m-4}[x^4 - (m+2)x^2 + m]. \end{aligned}$$

Thus

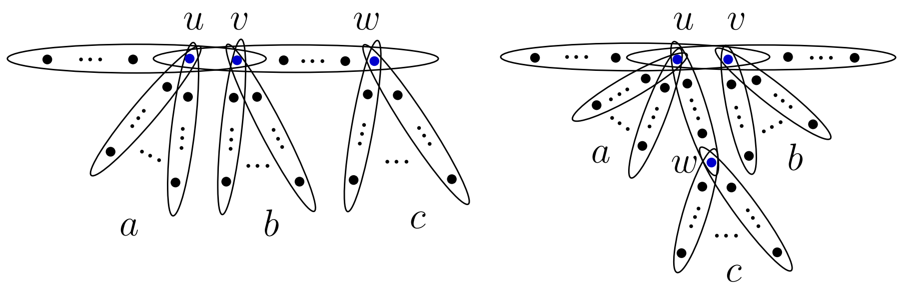
$$\rho(A(G(m-2,0)))^2 = m+2, \;\; \rho(A(G(m-4,2)))^2 = \frac{1}{2}\left(m+2+\sqrt{m^2-4m+36}\right),$$

$$\rho(A(G_2))^2 = \frac{1}{2}\left(m+2+\sqrt{m^2-12m+52}\right), \;\; \rho(A(G_3))^2 = \frac{1}{2}\left(m+2+\sqrt{m^2+4}\right).$$

If $m \geq 8$, then

$$\rho(\mathcal{A}(G(m-2,0)))^2 > \rho(A(G_3))^2 \geq \rho(A(G(m-4,2)))^2 > \rho(A(G_2))^2,$$

equality holds only if $m = 8$. Note that the above inequalities holds for the spectral radii as well.



$U_3^1(a, b; c)$ $U_3^2(a, b; c)$

Figure 4.2: Some unicyclic k-graphs

Now we compare $\rho(A(G(m-4,2)))$ and $\rho(A(G_1))$.

Let $\rho = \rho(A(G_1))$ for simplicity. By the characteristic polynomial $\phi_{G_1}(x)$, we have $\rho^3 = \rho^2 + (m-1)\rho - m + 3$. Define $h(x) = x^4 - (m+2)x^2 + 2(m-4)$. Then

$$\begin{aligned} h(\rho) &= \rho^4 - (m+2)\rho^2 + 2(m-4) \\ &= \rho[\rho^2 + (m-1)\rho - m + 3] - (m+2)\rho^2 + 2(m-4) \\ &= \rho^3 - 3\rho^2 - (m-3)\rho + 2(m-4) \\ &= \rho^2 + (m-1)\rho - m + 3 - 3\rho^2 - (m-3)\rho + 2(m-4) \\ &= -2\left(\rho - \frac{1}{2}\right)^2 + m - \frac{9}{2}. \end{aligned}$$

Since $\rho > \rho(A(K_{1,m-1})) = \sqrt{m-1}$, we obtain for $m \geq 6$ that

$$h(\rho) < m - \frac{9}{2} - 2\left(\sqrt{m-1} - \frac{1}{2}\right)^2 < 2\sqrt{m-1} - m < 0.$$

By the characteristic polynomial $\phi_{G(a,b)}(x)$, $\rho(A(G(m-4,2)))$ is the largest zero point of $h(x)$, hence it is strictly larger than $\rho = \rho(A(G_1))$, which completes the proof. $\square$

Now we present some spectral radii comparison between uncyclic hypergraphs using the weighted incidence matrices.

Let $U_2(a,b)$ be the kth power of $G(a,b)$. Let $U_3^1(a,b;c)$ be the k-graph obtained from $U_2(a,b)$ through attaching c pendent edges at an arbitrary pendent vertex w in a cycle edge. Denote by $U_3^2(a,b;c)$ the k-graph obtained from $U_2(a+1,b)$ through attaching c pendent edges at a pendent vertex w adjacent to u outside the cycle.

The k-graphs $U_3^1(a,b;c)$ and $U_3^2(a,b;c)$ are presented in Figure 4.2, with each edge representing by a closed curve and all non-pendent vertices in distinct color.

By setting α as an expression of the spectral radius of a certain hypergraph and constructing specific weighted incidence matrices, the following lemma builds a connection of spectral radii between two subclasses of unicyclic hypergraphs.

Lemma 4.4. *Let $m \geq 8$. Then for $a \leq 1$,*

$$\rho(\mathcal{A}(U_3^1(a,0;m-2-a))) < \rho(\mathcal{A}(U_2(m-4,2))).$$

Proof. Set $\alpha = \rho(G(m-4,2))^{-2}$. As the hypergraph $U_2(m-4,2)$ is the kth power of the graph $G(m-4,2)$, according to Lemma 2 we obtain $\alpha^{-\frac{1}{k}} = \rho(A(G(m-4,2)))^{\frac{2}{k}} = \rho(A(U_2(m-4,2)))$.

We claim that for $m \geq 8$ and $a \leq 1$, $U_3^1(a,0;m-2-a)$ is strictly and consistently α-subnormal.

We now construct a weighted incidence matrix B for $U_3^1(a,0;m-2-a)$. Set $B(p,e) = 1$ for every pendent vertex $p \in e$. For non-pendent vertex q contained in a pendent edge f, set $B(q,f) = \alpha$.

Denote by e_1 and e_2 the two non-pendent edges of $U_3^1(a,0;m-2-a)$ and suppose $w \in e_2$. Let $x_i = B(u,e_i)$, $y_i = B(v,e_i)$ for $i = 1,2$ and let $z = B(w,e_2)$. Set

$$x_1 + x_2 = 1 - a\alpha,\ y_1 + y_2 = 1,\ z = 1 - (m-2-a)\alpha,\ x_1 y_2 = x_2 y_1,\ x_1 y_1 = \alpha.$$

Since $x_1 y_2 = x_2 y_1$ for the unique cycle ue_1ve_2u, we know that B is consistent by Definition 3.7. It can be verified that all equalities hold for (1) and (2) of Definition 3.7 except on e_2.

Now we compare x_2y_2z with α. Let $A = \frac{x_2}{x_1} = \frac{y_2}{y_1} > 0$. Then

$$1 - a\alpha = (x_1 + x_2)(y_1 + y_2) = (1 + A)^2 x_1 y_1 = (1 + A)^2\alpha.$$

Thus $A = \sqrt{\frac{1}{\alpha} - a} - 1 \geq \sqrt{\frac{1}{\alpha} - 1} - 1$. Since

$$\frac{1}{\alpha} = \rho(A(G(m-4,2)))^2 = \frac{1}{2}\left(m + 2 + \sqrt{m^2 - 4m + 36}\right) > m,$$

when $m \geq 15$ we have that

$$\begin{aligned}
\frac{x_2y_2z}{\alpha} &= [1 - (m - 2 - a)\alpha]A^2 \\
&\geq \left[\frac{1}{\alpha} - (m-2)\right]\left(\sqrt{1-\alpha} - \sqrt{\alpha}\right)^2 \\
&> 2\left(\sqrt{1 - \frac{1}{m}} - \sqrt{\frac{1}{m}}\right)^2 \\
&\geq 2\left(\sqrt{\frac{14}{15}} - \sqrt{\frac{1}{15}}\right)^2 > 1.
\end{aligned}$$

It is shown by direct computation that the value of $\left[\frac{1}{\alpha} - (m-2)\right](\sqrt{1-\alpha} - \sqrt{\alpha})^2$ lies in the interval $(1.1, 1.4)$ in the case $8 \leq m \leq 14$. Thus $\prod_{t \in e_2} B(t, e_2) = x_2y_2z > \alpha$ for $m \geq 8$. Hence by Definition 3.7, $U_3^1(a, 0; m-2-a)$ is strictly α-subnormal. According to Lemma 3.4 (ii),

$$\rho(A(U_3^1(a, 0; m-2-a))) < \alpha^{-\frac{1}{k}} = \rho(A(U_2(m-4,2))).$$

□

Finally, we have some results for bicyclic case.

Denote by $B_2(a,b)$ the kth power of $M(a,b)$ as shown in Figure 4.1. Let $B_3^1(a,b,c)$ be the k-graph with exactly two non-pendent edges with intersection of three vertices

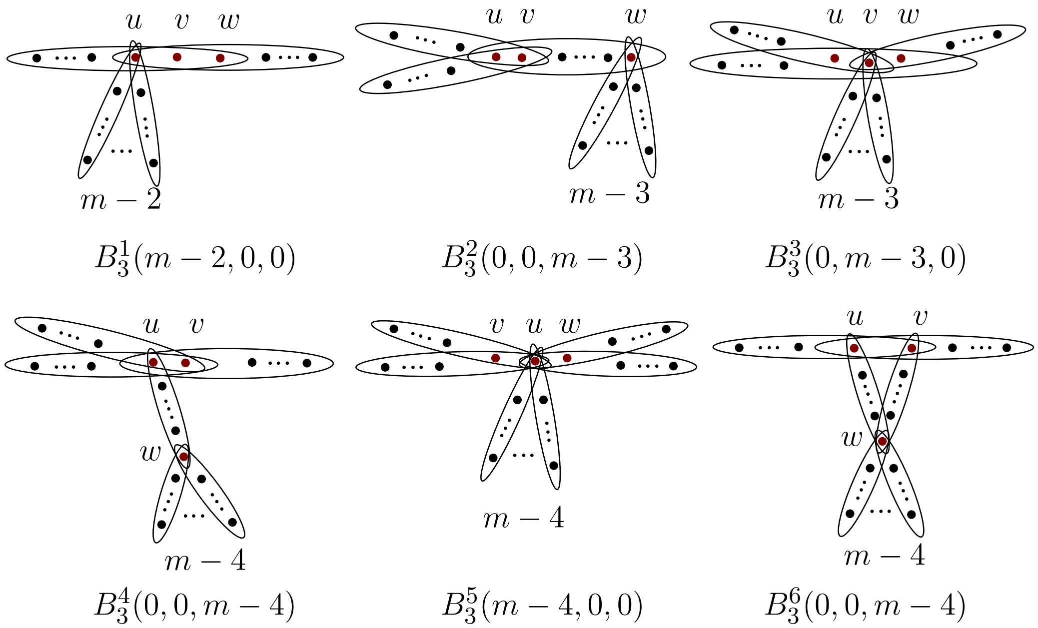


Figure 4.3: Some bicyclic k-graphs

u, v, w, and a, b, c be the number of pendent edges attached at u, v, w respectively. Denote by $B_3^2(a, b, c)$ ($B_3^3(a, b, c)$, resp.) the hypergraph obtained from $U_3^1(a, b; c)$ through adding a new edge containing u, v (v, w resp.) and $k-2$ new pendent vertices. Denote by $B_3^4(a, b, c)$ ($B_3^5(a, b, c)$ and $B_3^6(a, b, c)$ resp.) the hypergraph obtained from $U_3^2(a, b; c)$ through adding a new edge containing u, v (u, w and v, w resp.) and $k-2$ new pendent vertices. Denote by B_4 the bicyclic hypergraph obtained from $B_3^1(0, 0, 0)$ through attaching $m-2$ pendent edges at an arbitrary pendent vertex t in a cycle edge.

Lemma 4.5. *Let* $m \geq 5$. *Then*

(i) $\rho(\mathcal{A}(B_3^1(m-2, 0, 0))) = \rho(\mathcal{A}(B_2(m-3, 0)))$;

(ii) $\max\{\rho(\mathcal{A}(B_3^1(m-3, 1, 0))), \rho(\mathcal{A}(B_3^3(0, m-3, 0))), \rho(\mathcal{A}(B_4))\} < \rho(\mathcal{A}(B_2(m-4, 1)))$.

Proof. Employing the amalgamating operation introduced in Section 3.3 and the formula $(*)$, we obtain the characteristic polynomial of $M(a,b)$ as below:

$$\phi_{M(a,b)}(x) = x^{m-5}[x^4 - (m+6)x^2 + ab],$$

where $a+b+3 = m$. Hence

$$\rho(\mathcal{A}(M(m-3,0)))^2 = m+6,\ \ \rho(\mathcal{A}(M(m-4,1)))^2 = \frac{1}{2}\left(m+6+\sqrt{m^2+8m+52}\right).$$

Let

$$\alpha = \rho(\mathcal{A}(M(m-3,0)))^{-2} = \frac{1}{m+6},\quad \beta = \rho(\mathcal{A}(M(m-4,1)))^{-2}.$$

Claim 1. $B_3^1(m-2,0,0)$ is consistently α-normal.

We should construct a weighted incidence matrix B for $B_3^1(m-2,0,0)$. Set $B(p,e) = 1$ for each pendent vertex $p \in e$ and set $B(q,f) = \alpha$ for every non-pendent vertex q in a pendent edge f. Let e_1 and e_2 be the two edges intersecting at u, v, w. Set $B(u,e_i) = \frac{1-(m-2)\alpha}{2}$ and $B(v,e_i) = B(w,e_i) = \frac{1}{2}$ for $i = 1,2$.

Observe that $\sum_{e:t\in e} B(t,e) = 1$ for each vertex t and $\prod_{t\in e} B(t,e) = \alpha$ for each edge e in $B_3^1(m-2,0,0)$. Besides, B is consistent for both three cycles in $B_3^1(m-2,0,0)$. Hence by Definition 3.7, $B_3^1(m-2,0,0)$ is consistently α-normal.

Therefore by Lemma 2 and Lemma 3.4 (1),

$$\rho(\mathcal{A}(B_3^1(m-2,0,0))) = \alpha^{-\frac{1}{k}} = \rho(\mathcal{A}(M_2(m-3,0)))^{\frac{2}{k}} = \rho(\mathcal{A}(B_2(m-3,0))).$$

Claim 2. $B_3^1(m-3,1,0)$ is strictly and consistently β-subnormal.

We may construct a weighted incidence matrix B for $B_3^1(m-3,1,0)$ first. Set $B(p,e) = 1$ for every pendent vertex $p \in e$ and set $B(q,f) = \beta$ for every non-pendent vertex q contained in a pendent edge f. Denote by e_1 and e_2 the two non-pendent edges.

Set $x_i = B(u,e_i)$, $y_i = B(v,e_i)$ and $z_i = B(w,e_i)$ for $i = 1,2$. Let

$$x_1 + x_2 = 1-(m-3)\beta,\quad y_1 + y_2 = 1-\beta,\quad z_1 + z_2 = 1,\quad x_2y_2z_2 = \beta,$$

and let $A = \frac{x_1}{x_2} = \frac{y_1}{y_2} = \frac{z_1}{z_2} > 0$.

Since $x_1 y_2 = x_2 y_1$, $x_1 z_2 = x_2 z_1$ and $y_1 z_2 = y_2 z_1$ for both of the three cycles, B is consistent by Definition 3.7. Obeserve that all equalities hold for (1) and (2) of Definition 3.7 except on e_1.

Compare $x_1 y_1 z_1$ with β. Observe that

$$(1-\beta)[1-(m-3)\beta] = (x_1+x_2)(y_1+y_2)(z_1+z_2) = (1+A)^3 x_2 y_2 z_2 = (1+A)^3 \beta.$$

Since $\beta^{-\frac{1}{2}} = \rho(\mathcal{A}(M(m-4,1)))$ is the largest root of $x^4 - (m+6)x^2 + m - 4 = 0$, there is $\beta^{-2} - (m-2)\beta^{-1} + (m-3) = 8\beta^{-1} + 1$. Hence

$$\begin{aligned} (1+A)^3 &= (1-\beta)[1-(m-3)\beta]\beta^{-1} \\ &= [\beta^{-2} - (m-2)\beta^{-1} + (m-3)]\beta \\ &= (8\beta^{-1}+1)\beta > 8. \end{aligned}$$

Thus $A > 1$ and we have

$$\prod_{t \in e_2} B(t, e_2) = x_1 y_1 z_1 = A^3 x_2 y_2 z_2 = A^3 \beta > \beta.$$

Therefore, $B_3^1(m-3,1,0)$ is strictly and consistently β-subnormal.

Claim 3. $B_3^3(0,m-3,0)$ is strictly and consistently β-subnormal.

Now a weighted incidence matrix B may be constructed for $B_3^3(0,m-3,0)$. Let e_1, e_2, e_3 be the non-pendent edges with $\{u,v\} \subset e_2$, $\{v,w\} \subset e_3$, and all of u, v, w are contained in e_1.

Let $x_1 = B(u,e_1), x_2 = B(u,e_2), z_1 = B(w,e_1), z_3 = B(w,e_3)$ and $y_i = B(v,e_i)$ for $i = 1,2,3$. Let $A = \frac{x_1}{x_2} = \frac{y_1}{y_2} = \frac{y_1}{y_3} = \frac{z_1}{z_3} > 0$,

$$x_2 = z_3 = \frac{1}{A+1}, \quad y_2 = y_3 = \frac{1-(m-3)\beta}{A+2}, \quad x_2 y_2 = z_3 y_3 = \beta.$$

Assign 1 to $B(p,e)$ for each pendent vertex $p \in e$ and β to $B(q,f)$ for every non-pendent vertex q in a pendent edge f.

By the above settings, B is consistent for the three cycles ue_1ve_2u, ve_1we_3v and $ue_1we_3ve_2u$. Additionally, all equalities hold for (1) and (2) of Definition 3.7 except on e_1.

Since $\frac{1}{\beta} = \rho(\mathcal{A}(M(m-4,1)))^2 > m+5$, there is

$$(A+2)(A+1) = \frac{1-\beta(m-3)}{x_2y_2} = \frac{1-\beta(m-3)}{\beta} > 8.$$

Hence $A > \frac{\sqrt{33}-3}{2} > 1.37$. Thus,

$$\frac{x_1y_1z_1}{\beta} = \frac{A^3x_2y_2z_3}{\beta} = \frac{A^3}{A+1} = \frac{1}{A^{-2}+A^{-3}} > 1.$$

Since $\prod_{t\in e_1} B(t,e_1) = x_1y_1z_1 > \beta$, $B_3^3(0,m-3,0)$ is strictly and consistently β-subnormal.

Claim 4. B_4 is strictly and consistently β-subnormal.

A weighted incidence matrix B will be constructed for B_4. Denote by e_1, e_2 the two non-pendent edges in B_4 and $t \in e_2$. Assign 1 to $B(p,e)$ for each pendent vertex $p \in e$ and β to $B(q,f)$ for every non-pendent vertex q in a pendent edge f. Set

$$\frac{B(u,e_2)}{B(u,e_1)} = \frac{B(v,e_2)}{B(v,e_1)} = \frac{B(w,e_2)}{B(w,e_1)} = A, \quad B(t,e_2) = 1-\beta(m-2),$$

$$B(u,e_1) = B(v,e_1) = B(w,e_1) = \frac{1}{1+A} = \beta^{\frac{1}{3}}.$$

Observe that B is consistent and all equalities hold for (1) and (2) of Definition 3.7 except on e_2. As $A = \sqrt[3]{\frac{1}{\beta}} - 1 > 1$ and

$$\frac{1}{\beta} = \rho(\mathcal{A}(M(m-4,1)))^2 > m+5 \geq 10$$

in the case $m \geq 5$, it follows that

$$\begin{aligned}\frac{1}{\beta}\prod_{s\in e_2} B(s,e_2) &= A^3[1-(m-2)\beta] \\ &= (1-\sqrt[3]{\beta})^3\left(\frac{1}{\beta}-(m-2)\right) \\ &> 7\left(1-\sqrt[3]{\frac{1}{10}}\right)^3 > 1.\end{aligned}$$

Therefore, B_4 is strictly and consistently β-subnormal.

According to Claims $2, 3, 4$, Lemma 3.5 and Lemma 3.4 (ii), we have

$$\rho(\mathcal{A}(H)) < \beta^{-\frac{1}{k}} = \rho(\mathcal{A}(M(m-4,1)))^{\frac{2}{k}} = \rho(\mathcal{A}(B_2(m-4,1)))$$

if $H \in \{B_3^1(m-3,1,0), B_3^3(0,m-3,0), B_4\}$. $\square$

4.3 Unicyclic hypergraphs with largest spectral radii

To derive the desired results, we will discuss hypergraphs in $\mathbb{U}^m$ via the number of non-pendent vertices. Let $\mathbb{U}_i^m$ be the set of hypergraphs in $\mathbb{U}^m$ containing exactly i non-pendent vertices. As a cycle in a simple hypergraph is of length at least 2, we have $i \geq 2$.

Observe that the kth power of G_1 depicted in Figure 4.1 belongs to $\mathbb{U}_3^m$. In [19], G_1^k was proved to uniquely obtain the largest spectral radius among all linear k-graphs in $\mathbb{U}^m$. Thus our work is focused on non-linear k-graphs.

Suppose $H\mathbb{U}_i^m$ is a non-linear. Then there are a pair of vertices sharing two common edges, which forms the unique cycle of H according to Lemma 4.2. Denote by ue_1ve_2u the unique cycle in H.

If $H \in \mathbb{U}_2^m$, then by Proposition 4.1 (1), there are merely two non-pendent edges e_1, e_2 containing u and v. Thus $H \cong U_2(a, b)$ for some nonnegative a and b. We can say that k-graphs in $\mathbb{U}_2^m$ are in the form of $U_2(a, b)$ where $a, b \in \mathbb{N}$.

Lemma 4.6. *Let $a \geq b \geq 1$ and $a + b + 2 = m$. Then*

$$\rho(\mathcal{A}(U_2(a, b))) < \rho(\mathcal{A}(U_2(a + 1, b - 1))) \leq \rho(\mathcal{A}(U_2(m - 2, 0))).$$

Proof. Observe that $U_2(a + 1, b - 1)$ can be obtained from $U_2(a, b)$ through moving one pendent edge from v to u, or through moving $a - b + 1$ pendent edges from u to v, it follows from Lemma 3.3 that $\rho(\mathcal{A}(U_2(a, b))) < \rho(\mathcal{A}(U_2(a + 1, b - 1)))$. By induction, $\rho(\mathcal{A}(U_2(a + 1, b - 1))) \leq \rho(\mathcal{A}(U_2(m - 2, 0)))$, equality holds if and only if $b = 1$. $\square$

Now we discuss $H \in \mathbb{U}_3^m$. Let w be the third non-pendent vertex of H. If w is contained by a cycle edge e_1, then by Proposition 4.1 (2), $w \notin e_2$. Therefore $H \cong U_3^1(a, b; c)$ for some a, b and $c \geq 1$. If w does not appear on the cycle, then it is contained by an edge outside the cycle together with one of u, v, say u. Henve $H \cong U_3^2(a, b; c)$ for some a, b and $c \geq 1$. Therefore, we can express non-linear k-graphs in $\mathbb{U}_3^m$ either by $U_3^1(a, b; c)$, or by $U_3^2(a, b; c)$ with $c \geq 1$.

Lemma 4.7. *Suppose that H is a non-linear k-graph in $\mathbb{U}_3^m \backslash \{U_3^1(m-3, 0; 1), U_3^2(m-4, 0; 1)\}$. If $m \geq 8$, then*

$$\rho(\mathcal{A}(H)) < \rho(\mathcal{A}(U_2(m - 4, 2))) \leq \rho(\mathcal{A}(U_3^2(m - 4, 0; 1))) < \rho(\mathcal{A}(U_3^1(m - 3, 0; 1))),$$

where the second equality holds if and only if $m = 8$.

Proof. Two cases are considered.

Case 1. $H \cong U_3^1(a, b; c)$.

Assume that $a \geq b$. Since $H \ncong U_3^1(m - 3, 0; 1)$, we have $b \geq 1$ or $c \geq 2$.

If $a \geq 2$, then by Corollary 3.1, set $u_1 = v$ and $u_2 = w$, it follows

$$\rho(\mathcal{A}(H)) < \rho(\mathcal{A}(U_2(a, b+c))) \leq \rho(\mathcal{A}(U_2(m-4, 2))).$$

The second inequality follows from Lemma 4.6 for $a \geq 2$ and $b + c \geq 2$.

If $a = b = 1$, by Corollary 3.1 and set $u_1 = u, u_2 = v$, it follows $\rho(\mathcal{A}(H)) < \rho(\mathcal{A}(U_3^1(2, 0; c)))$ with $c = m - 4 > 2$, which is ascribed to the case $a \geq 2$.

If $a \leq 1, b = 0$, then by Lemma 4.4, $\rho(\mathcal{A}(H)) < \rho(\mathcal{A}(U_2(m-4, 2)))$.

Case 2. $H \cong U_3^2(a, b; c)$.

Since $H \ncong U_3^2(m-4, 0; 1)$, $b \geq 1$ or $c \geq 2$. We can obtain $U_3^1(a+1, b; c)$ from H through moving c pendent edges from w to an arbitrary pendent vertex in a cycle edge, which coincides with the operation of Lemma 3.2. Therefore for $b \geq 1$ or $c \geq 2$,

$$\rho(\mathcal{A}(H)) < \rho(\mathcal{A}(U_3^1(a+1, b; c))) < \rho(\mathcal{A}(U_2(m-4, 2))),$$

and the second inequality follows by Case 1.

Since $U_2(m-4, 2)$ and $U_3^2(m-4, 0; 1)$ are the kth powers of $G(m-4, 2)$ and G_3 respectively (See Figure 4.1), according to Lemma 3.5 and Lemma 4.3,

$$\rho(\mathcal{A}(U_2(m-4, 2))) = \rho(A(G(m-4, 2)))^{\frac{2}{k}} \leq \rho(A(G_3))^{\frac{2}{k}} = \rho(\mathcal{A}(U_3^2(m-4, 0; 1))),$$

equality holds only if $m = 8$.

Next we prove the last inequality. Note that $U_3^1(m-3, 0; 1)$ can be obtained from $U_3^2(m-4, 0; 1)$ through moving the pendent edge attached at w from w to an arbitrary pendent vertex in a cycle edge. By Lemma 3.2,

$$\rho(\mathcal{A}(U_3^2(m-4, 0; 1))) < \rho(\mathcal{A}(U_3^1(m-3, 0; 1))),$$

which completes the proof. □

Next is $H \in \mathbb{U}_4^m$.

Lemma 4.8. *Let H be a non-linear k-graph in $\mathbb{U}_4^m$ with $m \geq 8$. Then*

$$\rho(\mathcal{A}(H)) < \rho(\mathcal{A}(U_2(m-4,2))).$$

Proof. If H is non-linear, then the unique cycle is denoted by ue_1ve_2u with length 2. Let w and t be the third and fourth non-pendent vertices of H and a, b, c, d be the number of pendent edges attached at u, v, w, t, respectively.

We discuss with four cases by locations of w, t.

Case 1. Both w, t appear on the cycle. By Lemma 4.2, each of w, t is contained in exactly one non-pendent edge. Hence $c \geq 1, d \geq 1$. Through moving all pendent edges from w to t or from t to w, we can obtain $U_3^1(a, b; c+d)$ with $c + d \geq 2$. By Lemma 3.3 and Lemma 4.7, $\rho(\mathcal{A}(H)) < \rho(\mathcal{A}(U_3^1(a, b; c+d))) < \rho(\mathcal{A}(U_2(m-4,2)))$.

Case 2. Merely one of w, t appears on the cycle, say w. Then $d \geq 1$, otherwise either t is a pendent vertex, or H is not unicyclic.

Subcase 2.1. w and t are contained in an edge f. Set $u_1 = w$ and $u_2 = t$, by Corollary 3.1 and Lemma 4.7 it follows $\rho(\mathcal{A}(H)) < \rho(\mathcal{A}(U_3^1(a, b; c+d+1))) < \rho(\mathcal{A}(U_2(m-4,2)))$ for $c + d \geq 1$.

Subcase 2.2. w, t are not adjacent. Then t is adjacent to u or v. Let $w \in e_1 \backslash e_2$. Moving d pendent edges from t to an arbitrary pendent vertex in e_2, we obtain a hypergraph H' of Case 1 with larger spectral radius by Lemma 3.2. Thus $\rho(\mathcal{A}(H)) < \rho(\mathcal{A}(H')) < \rho(\mathcal{A}(U_2(m-4,2)))$.

Case 3. Both w and t are not on the cycle. Then at least one of w, t, say w, is adjacent to a vertex on the cycle, say u. Suppose that u, w are connected by an edge g not on the cycle. Moving all edges incident with w expect g from w to an arbitrary pendent vertex on the cycle, we obtain a hypergraph H'' of Case 2. According to Lemma 3.2 and the discussion in Case 2, $\rho(\mathcal{A}(H)) < \rho(\mathcal{A}(H'')) < \rho(\mathcal{A}(U_2(m-4,2)))$.

This completes the proof. $\square$

Lemma 4.9. *Let H be a k-graph in $\mathbb{U}_i^m$ for $i \geq 3$. Then*

$$\rho(\mathcal{A}(H)) < \max\{\rho(\mathcal{A}(F)) : F \textit{ is a } k\textit{-graph in } \mathbb{U}_{i-1}^m\}.$$

Proof. If all non-pendent vertices of H are contained by an edge f, then there is a non-pendent vertex w whose incident edges are pendent edges except f . Otherwise, every non-pendent vertex is incident with more than one non-pendent edge and there will be two distinct cycles of length 2 in the case $i \geq 3$, which contradicts Lemma 4.2. By moving all pendent edges from w to another non-pendent vertex t in f, we may obtain a k-graph in $\mathbb{U}_{i-1}^m$, denoted H'. By Corollary 3.1, set $u_1 = t$ and $u_2 = w$, it follows $\rho(\mathcal{A}(H)) < \rho(\mathcal{A}(H'))$.

Suppose H have two non-pendent vertices u, v without any common edge. Let $P = ue_1 \cdots e_s v$ be a shortest path connecting u and v in H, $s \geq 2$. Denote by H_1 the k-graph obtained from H through moving all edges incident with u except e_1 from u to v. Denote by H_2 the k-graph obtained from H through moving all edges incident with v except e_s from v to u. Observe that both H_1 and H_2 are in $\mathbb{U}_{i-1}^m$. By Lemma 3.3, $\rho(\mathcal{A}(H)) < \max\{\rho(\mathcal{A}(H_1)), \rho(\mathcal{A}(H_2))\}$.

In both cases, $\rho(\mathcal{A}(H))$ is bounded up by the spectral radius of a k-graph in $\mathbb{U}_{i-1}^m$, which completes the proof. □

According to Lemma 4.8 and Lemma 4.9, we have the following result.

Lemma 4.10. *Let H be a non-linear k-graph in $\mathbb{U}_i^m$ with $i \geq 4$ and $m \geq 8$. Then*

$$\rho(\mathcal{A}(H)) < \rho(\mathcal{A}(U_2(m-4, 2))).$$

Now the first main result of this chapter is ready to serve.

Theorem 4.1. *Let H be a k-graph in $\mathbb{U}^m$ with $m \geq 8$. Then*

$$\begin{aligned}(1)\ \rho(\mathcal{A}(U_2(m-4,2))) &\leq \rho(\mathcal{A}(U_3^2(m-4,0;1))) < \rho(\mathcal{A}(U_3^1(m-3,0;1))) \\ &< \rho(\mathcal{A}(U_2(m-3,1))) < \rho(\mathcal{A}(U_2(m-2,0))),\end{aligned}$$

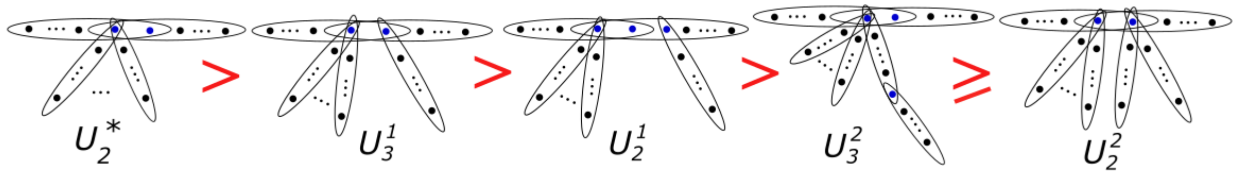


Figure 4.4: The first five unicyclic hypergraphs

where the first equality holds if and only if $m = 8$.

(2) *If* $H \notin \{U_2(m-2,0), U_2(m-3,1), U_2(m-4,2), U_3^1(m-3,0;1), U_3^2(m-4,0;1)\}$, *then*

$$\rho(\mathcal{A}(H)) < \rho(\mathcal{A}(U_2(m-4,2))).$$

Proof. Firstly we prove the relationships in (1).

The first two inequalities follow directly from Lemma 4.7 and the fourth inequality results from Lemma 4.6.

For the k-graph $U_3^1(m-3,0;1)$, by the Corollary 3.1 and set $u_1 = v, u_2 = w$, we have the third inequality $\rho(\mathcal{A}(U_3^1(m-3,0;1))) < \rho(\mathcal{A}(U_2(m-3,1)))$.

In the case of H being non-linear, the inequality of (2) can be obtained by Lemma 4.6, Lemma 4.7 and Lemma 4.10 through specifying the number of non-pendent vertices. If H is linear, then by Corollary 3.7 of [19], Lemma 2 and Lemma 4.3, it follows that

$$\rho(\mathcal{A}(H)) \leq \rho(\mathcal{A}(G_1^k)) < \rho(\mathcal{A}(G^k(m-4,2))) = \rho(\mathcal{A}(U_2(m-4,2))).$$

Now we complete the proof. $\square$

In Figure 4.4, we present the unicyclic hypergraphs in Theorem 4.1 (1), which are the first five with respect to a descending order of spectral radius over all unicyclic hypergraphs.

4.4 Bicyclic hypergraphs with largest spectral radii

Let $\mathbb{B}_i^m$ be the set of hypergraphs in $\mathbb{B}^m$ with exactly i non-pendent vertices with $i \geq 2$. We discuss the k-graph H in $\mathbb{B}_i^m$ with the three cases $i = 2, 3$ and $i \geq 4$.

First we suppose that $H \in \mathbb{B}_2^m$. Denote by u, v the non-pendent vertices of H. As H is bicyclic, u, v have more than two common edges, otherwise H is acyclic or unicyclic. By Proposition 4.1 (4), exactly three edges contain both of u, v and the remaining edges are pendent edges. Therefore, $H \cong B_2(a, b)$ for some a, b and each k-graph in $\mathbb{B}_2^m$ can be expressed as $B_2(a, b)$ with $a, b \in \mathbb{N}$.

Lemma 4.11. *Let $a \geq b \geq 1$ and $a + b + 3 = m$. Then*

$$\rho(\mathcal{A}(B_2(a, b))) < \rho(\mathcal{A}(B_2(a + 1, b - 1))) \leq \rho(\mathcal{A}(B_2(m - 3, 0))).$$

Proof. Observe that $B_2(a + 1, b - 1)$ can be obtained from $B_2(a, b)$ through moving one pendent edge from v to u, or moving $a - b + 1$ pendent edges from u to v. According to Lemma 3.3, $\rho(\mathcal{A}(B_2(a, b))) < \rho(\mathcal{A}(B_2(a + 1, b - 1)))$. By induction, $\rho(\mathcal{A}(B_2(a + 1, b - 1))) \leq \rho(\mathcal{A}(B_2(m - 3, 0)))$, equality holds if and only if $b = 1$. $\square$

Now suppose $H \in \mathbb{B}_3^m$. Denote by u, v, w the three non-pendent vertices of H. We may discuss by the number of edges containing all of u, v, w. According to Proposition 4.1 (3), u, v, w have no more than two common edges.

If u, v, w have two common edges, then any two of them can not share another edge, otherwise a 3-cyclic subgraph formed by the three non-pendent edges appears in H, a contradiction to Lemma 4.1. Thus the remaining edges are pendent edges attached at u, v or w. Therefore, $H \cong B_3^1(a, b, c)$ for some a, b, c.

If u, v, w have merely one common edge e_1, then there are at least two other edges that each contains two non-pendent vertices. Otherwise H is acyclic or unicyclic. Actually, there can not be three non-pendent edges other than e_1, as in that case, the four non-pendent edges in H will form a 3-cyclic subgraph, which contradicts

Lemma 4.1. Denote by e_2, e_3 the other two non-pendent edges of H. If $|e_2 \cap e_3| = 2$, then $H \cong B_3^2(a,b,c)$. Otherwise $|e_2 \cap e_3| = 1$ and then $H \cong B_3^3(a,b,c)$.

If u, v, w have no common edge, then the connectivity of H indicates that there is a path connecting u, v, w, say ve_1ue_2w. Since H is bicyclic, there are exactly two other non-pendent edges e_3 and e_4, with each of them contains two of u, v, w. Otherwise H is acyclic, unicyclic or has a 3-cyclic subgraph formed by five non-pendent edges. Observe that $|e_3 \cap e_4| < 3$. If $e_3 \cap e_4$ is either $\{u, v\}$ or $\{u, w\}$, then $H \cong B_3^4(a,b,c)$ for some a, b, c. If $e_3 \cap e_4 = \{u\}$, then $H \cong B_3^5(a,b,c)$. Otherwise $e_3 \cap e_4$ is $\{v, w\}$, $\{v\}$ or $\{w\}$, then $H \cong B_3^6(a,b,c)$ for some a, b, c.

Therefore, there are in total six forms $B_3^j(a,b,c)$, $j = 1, \cdots, 6$ of k-graphs in $\mathbb{B}_3^m$, where $a, b, c \in \mathbb{N}$ with $c \geq 1$ for $j = 2, 4$.

Lemma 4.12. *Let H be a k-graph in $\mathbb{B}_3^m \backslash \{B_3^1(m-2,0,0)\}$. If $m \geq 5$, then*

$$\rho(\mathcal{A}(H)) < \rho(\mathcal{A}(B_2(m-4,1))) < \rho(\mathcal{A}(B_3^1(m-2,0,0))).$$

Proof. The first inequality will be proved through the six possible forms of H.

Case 1. $H \cong B_3^1(a,b,c)$. As $H \ncong B_3^1(m-2,0,0)$, at least two of a, b, c are positive, say a, b.

If $c = 0$, then through moving $b-1$ pendent edges from v to u, or moving $a-1$ pendent edges from u to v, the k-graph $B_3^1(m-3,1,0)$ can be obtained. By Lemma 3.3 and Lemma 4.5,

$$\rho(\mathcal{A}(H)) \leq \rho(\mathcal{A}(B_3^1(m-3,1,0))) < \rho(\mathcal{A}(B_2(m-4,1))).$$

If $c \geq 1$, then from Corollary 3.1 and set $u_1 = v, u_2 = w$, it follows

$$\rho(A(H)) < \rho(\mathcal{A}(B_3^1(a,b+c,0))) \leq \rho(\mathcal{A}(B_3^1(m-3,1,0))) < \rho(\mathcal{A}(B_2(m-4,1))).$$

Case 2. $H \cong B_3^3(a,b,c)$.

Set $a \geq c$ throughout this case. If $a = c = 0$, then Lemma 4.5 implies $\rho(\mathcal{A}(H)) = \rho(\mathcal{A}(B_3^3(0, m-3, 0))) < \rho(\mathcal{A}(B_2(m-4, 1)))$. If $a \geq 1$, then Corollary 3.1 indicated that

$$\rho(\mathcal{A}(H)) < \rho(\mathcal{A}(B_3^3(0, a+b, c))) \leq \rho(\mathcal{A}(B_3^3(0, a+b+c, 0))) < \rho(\mathcal{A}(B_2(m-4, 1))).$$

Case 3. $H \cong B_3^2(a, b, c)$ with $c \geq 1$.

Let $a \geq b$ in this case. If $a \geq 1$, then according to Corollary 3.1 and set $u_1 = v, u_2 = w$, for $b + c \geq 1$ we have that

$$\rho(\mathcal{A}(H)) < \rho(\mathcal{A}(B_2(a, b+c))) \leq \rho(\mathcal{A}(B_2(m-4, 1))).$$

If $a = b = 0$ (Figure 4.3), then removing one pendent edge from w to u will generate $B_3^2(1, 0, m-4)$. In addition, through removing a non-pendent edge without w from u to w, we have $B_3^3(0, 0, m-3)$. Then according to Lemma 3.3 and the discussion of Case 2, 3,

$$\rho(\mathcal{A}(H)) < \max\{\rho(\mathcal{A}(B_3^2(1, 0, m-4))), \rho(\mathcal{A}(B_3^3(0, 0, m-3)))\} < \rho(\mathcal{A}(B_2(m-4, 1))).$$

Case 4. $H \cong B_3^j(a, b, c)$ for $j = 4, 5, 6$.

If $H \cong B_3^4(a, b, c)$ and $c \geq 1$, then we can obtain $B_3^2(a+1, b, c)$ through moving c pendent edges of w to an arbitrary pendent vertex in an edge containing both u, v. According to Lemma 3.2 and the discussion in Case 3, it follows for $c \geq 1$ that

$$\rho(\mathcal{A}(H)) < \rho(\mathcal{A}(B_3^2(a+1, b, c))) < \rho(\mathcal{A}(B_2(m-4, 1))).$$

If $H \cong B_3^5(a, b, c)$, then through moving c pendent edges and one edge incident with u, w from w to an arbitrary pendent vertex in an edge containing u, v, the k-graph $B_3^3(b, a+1, c)$ can be obtained. According to Lemma 3.2 and the discussion in Case 2,

$$\rho(\mathcal{A}(H)) < \rho(\mathcal{A}(B_3^3(b, a+1, c))) < \rho(\mathcal{A}(B_2(m-4, 1))).$$

If $H \cong B_3^6(a,b,c)$, then through moving c pendent edges and the edge incident with v, w from w to an arbitrary pendent vertex in an edge containing u, v, $B_3^3(a+1,b,c)$ can be obtained. According to Lemma 3.2 and the discussion in Case 2,

$$\rho(\mathcal{A}(H)) < \rho(\mathcal{A}(B_3^3(a+1,b,c))) < \rho(\mathcal{A}(B_2(m-4,1))).$$

The second inequality of this lemma results from Lemma 4.5 and Lemma4.11. $\square$

Lemma 4.13. *Let H be a k-graph in $\mathbb{B}_i^m$ with $i \geq 4$. Then*

$$\rho(\mathcal{A}(H)) < \max\{\rho(\mathcal{A}(F)) : F \textit{ is a } k\textit{-graph in } \mathbb{B}_{i-1}^m\}.$$

Proof. If all non-pendent vertices of H are contained by an edge f, then there exists a pair of non-pendent vertices v_1, v_2 without other common edge. Otherwise each pair of non-pendent vertices have exactly two common edges, which will result in a 3-cyclic subgraph in H, a contradiction. Let H' be the k-graph obtained from H through moving all edges containing v_2 but f from v_2 to v_1. Observe that H' is in $\mathbb{B}_{i-1}^m$. Then according to Corollary 3.1, $\rho(\mathcal{A}(H)) < \rho(\mathcal{A}(H'))$.

Suppose there is a pair of non-pendent vertices v_1, v_2 in H without any common edge. Denote by $P = v_1 e_1 \cdots e_s v_2$ a shortest path connecting v_1, v_2 and $s \geq 2$. Denote by H_1 the k-graph obtained from H through moving all edges containing v_1 except e_1 from v_1 to v_2. Denote by H_2 the k-graph obtained from H through moving all edges containing v_2 except e_s from v_2 to v_1. Note that both H_1, H_2 are in $\mathbb{B}_{i-1}^m$. According to Lemma 3.3, $\rho(\mathcal{A}(H)) < \max\{\rho(\mathcal{A}(H_1)), \rho(\mathcal{A}(H_2))\}$.

Hence $\rho(\mathcal{A}(H))$ is strictly smaller than the maximum spectral radius over k-graphs in $\mathbb{B}_{i-1}^m$ for $i \geq 4$. $\square$

Finally we consider $H \in \mathbb{B}_4^m$.

Lemma 4.14. *Let H be a k-graph in $\mathbb{B}_4^m$ with $m \geq 5$. Then*

$$\rho(\mathcal{A}(H)) < \rho(\mathcal{A}(B_2(m-4,1))).$$

Proof. **Case 1.** H has exactly two non-pendent edges, say e, f.

Since H is bicyclic, $|e \cap f| = 3$. Then we can obtain H from $B_3^1(a,b,c)$ through attaching $d \geq 1$ pendent edges at any pendent vertex t contained by a non-pendent edge.

Set $a \geq b \geq c$. If $a \geq 1$, then it follows from Corollary 3.1 and Lemma 4.12 that $\rho(\mathcal{A}(H)) < \rho(\mathcal{A}(B_3^1(a, b+d, c))) < \rho(\mathcal{A}(B_2(m-4,1)))$ for $b+d \geq 1$. If $a = b = c = 0$, then $H \cong B_4$. By Lemma 4.5, $\rho(\mathcal{A}(H)) = \rho(\mathcal{A}(B_4)) < \rho(\mathcal{A}(B_2(m-4,1)))$.

Case 2. H contains at least three non-pendent edges.

Subcase 2.1 All non-pendent vertices of H are in one edge, say f. According to the proof of Lemma 4.13, there exists a pair of non-pendent vertices v_1, v_2 without common edge other than f. Then we can obtain a k-graph $H' \in \mathbb{B}_{i-1}^m$ from H through moving all incident edges of v_2 except f to v_1, which has as much non-pendent edges as H does. Then there are more than two non-pendent edges in H' and hence $H' \ncong B_3^1(m-2,0,0)$. It follows from Corollary 3.1 and Lemma 4.12 that $\rho(\mathcal{A}(H)) < \rho(\mathcal{A}(H')) < \rho(\mathcal{A}(B_2(m-4,1)))$.

Subcase 2.2 There is a pair of non-pendent vertices v_1, v_2 in H without any common edges.

Denote by $P = v_1 e_1 \cdots e_s v_2$ a shortest path connecting v_1 and v_2 with $s \geq 2$. Let H_1 be the k-graph obtained from H through moving all incident edges of v_1 except e_1 to v_2. Let H_2 be the k-graph obtained from H through moving all incident edges of v_2 except e_s to v_1. Apparently H_1, H_2 belong to $\mathbb{B}_3^m$. We may prove that neither of them are $B_3^1(m-2,0,0)$.

If there exists a pendent edge containing v_1 or v_2 in H, then in H_1, the shortest path connecting v_1 and any pendent vertex in a pendent edge attached at v_2 has length at least 3. This fact indicates that $H_1 \ncong B_3^1(m-2,0,0)$, since all paths in $B_3^1(m-2,0,0)$ are of length at most 2. With a similar reason, $H_2 \ncong B_3^1(m-2,0,0)$.

Assume that v_1, v_2 are not in any pendent edge. Then each of v_1, v_2 is contained by a non-pendent edge other than e_1 and e_s, say f_1 and f_2 respectively. Hence there are three non-pendent edges $(f_1\backslash\{v_1\}) \cup \{v_2\}$, f_2 and e_s in H_1 implying that $H_1 \ncong B_3^1(m-2,0,0)$. Similarly $H_2 \ncong B_3^1(m-2,0,0)$.

Therefore, it follows from Lemma 3.3 and Lemma 4.12 that

$$\rho(\mathcal{A}(H)) < \max\{\rho(\mathcal{A}(H_1)), \rho(\mathcal{A}(H_2))\} < \rho(\mathcal{A}(B_2(m-4,1))).$$

Now we complete the proof. $\square$

Combining Lemma 4.13 and Lemma 4.14, we have the following result.

Lemma 4.15. *Let H be a k-graph in $\mathbb{B}_i^m$ with $i \geq 4$, $m \geq 5$. Then*

$$\rho(\mathcal{A}(H)) < \rho(\mathcal{A}(B_2(m-4,1))).$$

The above discussion based upon the number of non-pendent vertices directly lead to the second main result of this chapter.

Theorem 4.2. *Let H be a k-graph in $\mathbb{B}^m$ with $m \geq 5$. Then*

(1) $\rho(\mathcal{A}(B_2(m-4,1))) < \rho(\mathcal{A}(B_2(m-3,0))) = \rho(\mathcal{A}(B_3^1(m-2,0,0)))$;

(2) *if $H \notin \{B_2(m-4,1), B_2(m-3,0), B_3^1(m-2,0,0)\}$, then*

$$\rho(\mathcal{A}(H)) < \rho(\mathcal{A}(B_2(m-4,1))).$$

Proof. The relation in (1) results directly by Lemma 4.5 and Lemma 4.11.

The inequality of (2) follows from Lemma 4.11, Lemma 4.12 and Lemma 4.15 by specifying the number of non-pendent vertices in H. $\square$

In Figure 4.5, we present the bicyclic hypergraphs in Theorem 4.2 (1), which are the first three with respect to a descending order of spectral radius over all bicyclic hypergraphs.

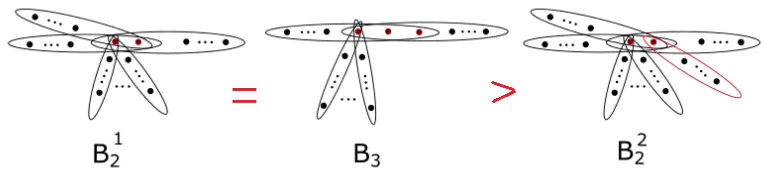


Figure 4.5: The first three bicyclic hypergraphs

4.5 Conclusions

In this chapter, we report our works on ordering uncyclic and bicyclic hypergraphs with respect to the spectra radius of the adjacency tensor, which has made some progress in the developing area of extremal spectral hypergraph theory.

The results of Theorems 4.1 and 4.2 not only provide characterization of hypergraphs with largest spectra radii, but also contains other information. It is easily observed that for supertrees and the unicyclic hypergraphs, the hypergraph maximizing spectral radius among each group are both unique. However, we find two extremal hypergraphs shareing the largest spectral radius in the bicyclic case, which refutes the uniqueness of extremal hypergraphs. Thus it may be reasonable to conjecture that there can be three or even more extremal hypergraphs appeared in cases of larger cyclomatic number.

On the other hand, the methods and the basic framework here are used to deal with unicyclic and bicyclic hypergraphs, they can possibly be applied to study more complicated cases such as hypergraphs with higher cyclomatic numbers.

Chapter 5

Spectral symmetry of odd-colorable hypergrpahs

In this chapter, we investigate another aspect on the spectra of uniform hypergraphs. In details, the property of symmetric spectrum and conditions for equivalent Laplacian spectrum and signless Laplacian spectrum of a k-graph are concerned.

The discussion is based on a newly introduced hypergraph class called odd-colorable hypergraphs.

In 2016, Nikiforov [42] introduced the definition of odd-coloring for tensors (they are called r-matrices therein), here we just restate it for k-graph as follows.

Definition 5.1. *Let $k \geq 2$ and k be even. A k-graph H with $V = [n]$ is called odd-colorable if there exists a map $\varphi : [n] \to [k]$ such that for any edge $\{j_1, j_2, \cdots, j_k\}$ of H, we have*

$$\varphi(j_1) + \cdots + \varphi(j_k) \equiv k/2 (\operatorname{mod} k).$$

The function φ is called an odd-coloring of H.

In the same paper where it was first defined, the odd-colorable hypergraphs were linked with the well-know odd-bipartite hypergraphs. It was proved that an odd-bipartite graph is always odd-colorable (see Proposition 11 in [42]), and furthermore when $k \equiv 2(\text{mod } 4)$, being odd-colorable is equivalent with being odd-bipartite (see

Proposition 12 in [42]).

The spectrum of a tensor $\mathcal{T}$ is called symmetric, if $\mathcal{T}$ and $-\mathcal{T}$ have the same spectrum (i.e., Spec($\mathcal{T}$) is symmetric about the origin) [63]. As the odd-bipartiteness of a connected k-graph are closely relevant to the symmetry of its spectrum, odd-colorable hypergraphs are connected with the existence of a symmetric spectrum as well.

In [42], Nikiforov proved that for a k-graph H, $Spec(\mathcal{A}) = -Spec(\mathcal{A})$ if and only if k is even and H is odd-colorable.

This chapter presents some applications and consequences of spectral symmetry of the odd-colorable k-graphs obtained in [42]. In particular, we obtain some further properties obtained from the symmetric spectra of an odd-colorable k-graph. By employing the Perron-Frobenius theorem on nonnegative weakly irreducible tensors together with the relation between the spectra of a hypergraph and that of its connected components, the paralell results for the disconnected case are proved. Moreover, based on the above outcomes, we discuss the Question 5.1 proposed in [54] about the relations between HSpec($\mathcal{L}$),HSpec($\mathcal{Q}$) and Spec($\mathcal{L}$),Spec($\mathcal{Q}$), and obtain an affirmative answer to this Question 5.1 for the remaining unsolved case $k \equiv 2 \ (mod\ 4)$ in Theorem 5.4.

5.1 Previous results on symmetric spectra of connected k-graphs

Recall that in Theorem 3.2 of [18], Fan et al. proved the following Lemma 5.1 for non-odd-bipartite connected r-graphs. Combining this with the Theorems 2.2 and 2.3 in [54] for the odd-bipartite connected case, we have the following statement.

Lemma 5.1. *Let H be a connected k-graph. Then $Spec(\mathcal{A}) = -Spec(\mathcal{A})$ if and only if $Spec(\mathcal{L}) = Spec(\mathcal{Q})$.*

Theorem 5.1. *[42] Let G be a k-graph. Then $Spec(\mathcal{A}(G)) = -Spec(\mathcal{A}(G))$ if and only if k is even and G is odd-colorable.*

The above result solves a problem in [44] about k-graphs with symmetric spectrum and disproves a conjecture in [69].

Combining Theorem 5.1 and Lemma 5.1, we can see that for a connected k-graph G, its Laplacian spectrum and signless Laplacian spectrum are equal if and only if k is even and G is odd-colorable. To extend this result to the disconnected case, we need the following lemma which is a consequence of Corollary 4.2 in [55].

5.2 Applications of symmetric spectra for odd-colorable hypergraphs

The first result derived from the symmetric spectra is Therorem 5.2 in the following. Before we can prove that, a lemma on the spectra of a hypergraph and that of its connected components is needed, which is a consequence of Corollary 4.2 in [55].

Lemma 5.2. *[55] Let G be a k-graph of order n, $G_1, G_2, \cdots, G_r$ be the connected components of G of orders $n_1, \cdots, n_k$, respectively. Then*

$$Spec(\mathcal{A}(G)) = \bigcup_{i=1}^{r} Spec(\mathcal{A}(G_i))^{(k-1)^{n-n_i}},$$

$$Spec(\mathcal{L}(G)) = \bigcup_{i=1}^{r} Spec(\mathcal{L}(G_i))^{(k-1)^{n-n_i}},$$

$$Spec(\mathcal{Q}(G)) = \bigcup_{i=1}^{r} Spec(\mathcal{Q}(G_i))^{(k-1)^{n-n_i}},$$

where the notation S^t means the repetition of t times of the multi-set S.

Some other preliminaries are also needed to study the disconnected case of spectral equivalence between the Laplacian and signless Laplacian tensors of odd-colorable k-graphs.

Lemma 5.3. *[54] Let G be a connected k-graph. Then $\rho(\mathcal{L}(G)) = \rho(\mathcal{Q}(G))$ if and only if $Spec(\mathcal{L}(G)) = Spec(\mathcal{Q}(G))$.*

Theorem 5.2. *Let G be a k-graph. Then $Spec(\mathcal{L}(G)) = Spec(\mathcal{Q}(G))$ if and only if k is even and G is odd-colorable.*

Proof. Suppose that G is connected. Then by Theorem 5.1 and Lemma 5.1 we have

$$Spec(\mathcal{L}(G)) = Spec(\mathcal{Q}(G)) \Longleftrightarrow Spec(\mathcal{A}(G)) = -Spec(\mathcal{A}(G))$$

$$\Longleftrightarrow G \text{ is odd-colorable and } k \text{ is even.}$$

Now we consider that G is disconnected. Let $G_1, G_2, \cdots, G_t$ be all the connected components of G, and the number of vertices of G_1 be n_1. We prove the sufficiency first.

$$\begin{aligned} G \text{ is odd-colorable} &\Longrightarrow \text{Every } G_i \text{ is odd-colorable } (\forall i = 1, \cdots, t) \\ &\Longrightarrow Spec(\mathcal{L}(G_i)) = Spec(\mathcal{Q}(G_i)) \ (\forall i = 1, \cdots, t) \\ &\qquad\qquad \text{(by the proof of connected case)} \\ &\Longrightarrow Spec(\mathcal{L}(G)) = Spec(\mathcal{Q}(G)) \quad \text{(by Lemma 5.2)} \end{aligned}$$

The necessity of the disconnected case is proved by induction on t, the number of connected components of G. Set $\rho := \rho(\mathcal{Q}(G)) = \rho(\mathcal{L}(G))$. Then ρ is equal to some $\rho(\mathcal{L}(G_j))$, say $\rho = \rho(\mathcal{L}(G_1))$. Since $|\mathcal{L}(G_1)| = \mathcal{Q}(G_1)$ and $\mathcal{Q}(G_1)$ is nonnegative weakly irreducible, by Proposition 2.4 we have $\rho = \rho(\mathcal{Q}(G)) \geq \rho(\mathcal{Q}(G_1)) \geq \rho(\mathcal{L}(G_1)) = \rho$. Thus $\rho = \rho(\mathcal{Q}(G_1))$. Hence for the connected k-graph G_1, $\rho(\mathcal{L}(G_1)) = \rho(\mathcal{Q}(G_1))$. Then by Lemma 5.3, we have $Spec(\mathcal{L}(G_1)) = Spec(\mathcal{Q}(G_1))$, which implies G_1 is odd-colorable by the above arguments for the connected case. Now consider the r-graph $G' = G_2 \cup \cdots \cup G_t$. Since $G = G_1 \cup G'$, by Lemma 5.2 we have

$$Spec(\mathcal{L}(G)) = Spec(\mathcal{L}(G_1))^{(k-1)^{n-n_1}} \bigcup Spec(\mathcal{L}(G'))^{(k-1)^{n_1}},$$

$$Spec(\mathcal{Q}(G)) = Spec(\mathcal{Q}(G_1))^{(k-1)^{n-n_1}} \bigcup Spec(\mathcal{Q}(G'))^{(k-1)^{n_1}},$$

Therefore, $Spec(\mathcal{L}(G)) = Spec(\mathcal{Q}(G))$ together with $Spec(\mathcal{L}(G_1)) = Spec(\mathcal{Q}(G_1))$ imply that $Spec(\mathcal{L}(G')) = Spec(\mathcal{Q}(G'))$. By induction on t we obtain that G' is also odd-colorable. Since $G = G_1 \cup G'$ and both G_1 and G' are odd-colorable, we conclude that G is odd-colorable. $\square$

As the applications of the above theorem, we can further deduce the following two results.

Theorem 5.3. *Let G be a k-graph with k being even, and $G_1, \cdots, G_r$ be all the connected components of G. Then $Spec(\mathcal{L}(G)) = Spec(\mathcal{Q}(G))$ if and only if $Spec(\mathcal{L}(G_i)) = Spec(\mathcal{Q}(G_i))$ for every connected component G_i $(i = 1, \cdots, r)$ of G.*

Proof. The sufficiency follows from Lemma 5.2. It remains to prove the necessity. Since $Spec(\mathcal{L}(G)) = Spec(\mathcal{Q}(G))$, it results from Theorem 5.2 that G is odd-colorable. Thus G_i is also odd-colorable, and then by Theorem 5.2 again we have $Spec(\mathcal{L}(G_i)) = Spec(\mathcal{Q}(G_i))$ $(i = 1, \cdots, r)$. $\square$

Recall that an eigenvalue of a tensor $\mathcal{T}$ is called an H-eigenvalue, if there exists a real eigenvector associated with it. The H-spectrum of a tensor $\mathcal{T}$, denoted by $Hspec(\mathcal{T})$, is the set of H-eigenvalues of $\mathcal{T}$.

In the Theorem 2.2 of [54], it was proved that if k is even and the k-graph G is connected, then

$$Hspec(\mathcal{L}(G)) = Hspec(\mathcal{Q}(G)) \Longrightarrow Spec(\mathcal{L}(G)) = Spec(\mathcal{Q}(G)) \tag{5.1}$$

In the same paper, the following question was proposed:

Question 5.1. *Is the reverse implication of (5.1) true or not?*

In [18], Fan et al. showed that the reverse of (5.1) is false in the case $k \equiv 0 \ (mod\ 4)$ by inducing the generalized power hypergraphs $G^{k,k/2}$ of a non-bipartite ordinary graph G as counterexamples.

Now by using Theorem 5.2, we can show in the following theorem that the reverse implication of (5.1) is true in the case $k \not\equiv 0 \ (mod\ 4)$, even when G is not connected, thus provide an affirmative answer to Question 5.1 for the remaining unsolved case.

Theorem 5.4. *Let G be an k-graph with $k \not\equiv 0 \ (mod\ 4)$, and $Spec(\mathcal{L}(G)) = Spec(\mathcal{Q}(G))$. Then we have $Hspec(\mathcal{L}(G)) = Hspec(\mathcal{Q}(G)))$.*

Proof. Let $G_1, \cdots, G_r$ be all the connected components of G. Since $Spec(\mathcal{L}(G)) = Spec(\mathcal{Q}(G))$, we obtain by Theorem 5.2 that G is odd-colorable. Thus we have $k \equiv 2 \ (mod\ 4)$ by the assumption $k \not\equiv 0 \ (mod\ 4)$. By Proposition 12 of [42] we deduce that G is odd-bipartite since $k \equiv 2 \ (mod\ 4)$. Thus every connected component G_i of G is also odd-bipartite. Now by Theorem 2.2 of [54], we obtain that $\mathcal{L}(G_i)$ and $\mathcal{Q}(G_i)$ have the same H-spectrum for all connected components G_i $(i = 1, \cdots, r)$ of G. Therefore we conclude that $\mathcal{L}(G)$ and $\mathcal{Q}(G)$ have the same H-spectrum. $\square$

Combining Theorem 5.2 together with Lemma 5.3, we obtain the following corollary.

Corollary 5.1. *Let G be a connected k-graph. Then $\rho(\mathcal{L}(G)) = \rho(\mathcal{Q}(G))$ if and only if k is even and G is odd-colorable.*

5.3 Conclusions

We investigate in this chapter the property of symmetric spectra for odd-colorable hypergraphs. Utilizing this property, we deduce new conditions for connected and disconnected uniform hypergraph to have equal Laplacian and signless Laplacian spectra, H-spectra and spectral radius.

Chapter 6

Spectral bounds and properties of general hypergraphs

This chapter reports our recent attempt on general hypergraphs. Unlike the extensive research having been done on spectra of uniform hypergraphs, the spectra theory of general hypergraphs are still in the very initial stage, with less than a handful of references at present. In the late 2016 and the early 2017, two papers appeared in the very first exploration on spectra of general hypergraphs [1, 9]. Later on, Zhang et al [66] investigated the nearly uniform supertrees and adapted many spectral methods of uniform hypergraphs to non-uniform case. Hou, Chang and Zhang [26] studied the clique number of general hypergraphs and provided some spectral bounds.

In this chapter, we obtained some bounds for spectral radius and signless Laplacian spectral radius of general hypergraphs in term of vertex degrees, diameters and an lower bound for (non-uniform) generalized power hypergraphs. Bounds of the signless Laplacian spectral radius are extended to the Lapalcian spectral radius as well. At last, we study the spectral symmetry of general hypergraphs and the condition for a hypergraph to possess equal Laplacian (H-)spectrum (spectral radius) and signless Laplacian (H-)spectrum (spectral radius).

6.1 Upper bounds of spectral radius and signless Laplacian spectral radius

We should first introduce some existing upper bounds on spectral radius of a hypergraph. Let $\Delta = d_1, d_2, \ldots, d_n$ be the nonincreasing sequence of vertex degrees for a hypergraph H.

- According to the distribution of tensor eigenvalues (Proposition 2.2, Theorem 3.1 (2)):

  $$\rho(\mathcal{A}(H)) \leq \Delta, \rho(\mathcal{Q}(H)) \leq 2\Delta$$

  both of the equalities hold if H is regular.

- Yuan, Zhang & Lu [65]:

  $$\rho(\mathcal{A}(H)) \leq \min\left\{\max_{i\sim j}\sqrt{d_i d_j},\ \sqrt[k]{\Delta d_2^{k-1}}\right\}$$

  $$\rho(\mathcal{Q}(H)) \leq \min\left\{\max_{i\sim j}\ d_i + d_j,\ \Delta + \sqrt[k]{\Delta d_2^{k-1}}\right\}$$

- Li, Zhou & Bu [35]: H is not regular ($\exists d_i \neq d_j$), then

  $$\rho(\mathcal{A}(H)) < \Delta - \frac{n\Delta^2 - km\Delta}{2m(k-1)D(n\Delta - km) + n\Delta},$$

  where D is the diameter of H and m is the edge number of H.

In this subsection, we extend those bounds to general hypergraphs.

Theorem 6.1. *Let $H = (V, E)$ be a hypergraph. Then*

$$\rho(\mathcal{A}(H)) \leq \max_{\{i,j\}\subset e\in E}\sqrt{d_i d_j},\ \rho(\mathcal{Q}(H)) \leq \max_{\{i,j\}\subset e\in E}\ d_i + d_j.$$

Proof. Let $\mathbf{x}$ be a nonnegative eigenvector of H corresponding to $\rho = \rho(\mathcal{A}(H))$, H is of rank k. Without lost of generality, we may assume that

$$x_u = \max_{l \in [n]} x_l = 1, \quad x_v = \max_{\{l,u\} \subset e \in E} x_l.$$

Multiplying eigenequations at u and v, we have

$$\begin{aligned} \rho^2 x_v^{k-1} &= (\sum_{e \in E_u} \mathcal{A}_e \mathbf{x}^{e \setminus u})(\sum_{e \in E_v} \mathcal{A}_e \mathbf{x}^{e \setminus v}) \\ &\leq (\sum_{e \in E_u} \mathcal{A}_e \frac{x_v^{k-1}}{\mathcal{A}_e})(\sum_{e \in E_v} \mathcal{A}_e \frac{1}{\mathcal{A}_e}) \\ &= d_u x_v^{k-1} d_v. \end{aligned}$$

Thus $\rho \leq \sqrt{d_u d_v} \leq \max_{\{i,j\} \subset e \in E} \sqrt{d_i d_j}$.

Following the same schedule, similar inequality for $\rho(\mathcal{Q}(H))$ and the corresponding nonnegative eigenvector $\mathbf{y} \in \mathbb{R}^n$ are obtained:

$$(\rho(\mathcal{Q}(H)) - d_i)(\rho(\mathcal{Q}(H)) - d_j) y_j^{k-1} \leq d_i y_j^{k-1} d_j \text{ for some } i, j,$$

and hence $\rho(\mathcal{Q}(H)) \leq \max_{\{i,j\} \subset e \in E} \ d_i + d_j$. $\square$

Theorem 6.2. *Let $H = (V, E)$ be a hypergraph of rank k. Then*

$$\rho(\mathcal{A}(H)) \leq \max_{s\{i_1, \ldots, i_k\} \in E} \sqrt[k]{d_{i_1} \cdots d_{i_k}},$$

$$\rho(\mathcal{Q}(H)) \leq \Delta + \max_{s\{i_1, \ldots, i_k\} \in E} \sqrt[k]{d_{i_1} \cdots d_{i_k}}.$$

Proof. Let $\mathbf{x}$ a nonnegative eigenvector corresponding to $\rho = \rho(\mathcal{A}(G))$ with

$$x_{i_1} \cdots x_{i_k} = \max_{s\{j_1, \ldots, j_k\} \in E} x_{j_1} \cdots x_{j_k}.$$

The eigenequation at i_l is

$$\rho x_{i_l}^{k-1} = \sum_{e \in E_{i_l}} \mathcal{A}_e \mathbf{x}^{e \setminus i_l},\ l \in [k].$$

Note that for each $e \in E$, the number of ordered sets $(j_1, \ldots, j_k)$ generated by e with each vertex appearing at least once is

$$\sum_{\substack{\alpha_1, \ldots, \alpha_{|e|} \geq 1 \\ \alpha_1 + \cdots + \alpha_{|e|} = k}} \frac{k!}{\alpha_1! \cdots \alpha_{|e|}!} = \frac{|e|}{\mathcal{A}_e}.$$

By symmetry, the number of these sets with one vertex j fixed in first order is constant for each $j \in e$, thus it equals $1/\mathcal{A}_e$. Recall that $\mathbf{x}^{e \setminus j} = \sum_{s\{j, j_2, \ldots, j_k\} = e} x_{j_2} \cdots x_{j_k}$, thus there are $1/\mathcal{A}_e$ summing items in $\mathbf{x}^{e \setminus j}$ for each $j \in e$.

Then we have

$$\begin{aligned}
\mathcal{A}_e x_j \mathbf{x}^{e \setminus j} &= \mathcal{A}_e \sum_{s\{j, j_2, \ldots, j_k\} = e} x_j x_{j_2} \cdots x_{j_k} \\
&\leq \mathcal{A}_e \sum_{s\{j, j_2, \ldots, j_k\} = e} x_{i_1} \cdots x_{i_k} \\
&= x_{i_1} \cdots x_{i_k}.
\end{aligned}$$

Hence for $l \in [k]$,

$$\rho x_{i_l}^k = \sum_{e \in E_{i_l}} \mathcal{A}_e x_{i_l} \mathbf{x}^{e \setminus i_l} \leq \sum_{e \in E_{i_l}} x_{i_1} \cdots x_{i_k} = d_{i_l} x_{i_1} \cdots x_{i_k}.$$

Multiplying all k inequalities we obtain

$$\rho^k \prod_{l=1}^{k} x_{i_l}^k \leq d_{i_1} \cdots d_{i_k} x_{i_1}^k \cdots x_{i_k}^k.$$

Thus $\rho \leq \sqrt[k]{d_{i_1} \cdots d_{i_k}}$.

The results for $\rho(\mathcal{Q}(H))$ can be obtained similarly by employing its eigen-equation and $\rho(\mathcal{Q}(H)) - \Delta \leq \rho(\mathcal{Q}(H)) - d_{i_l}$ for each l. $\square$

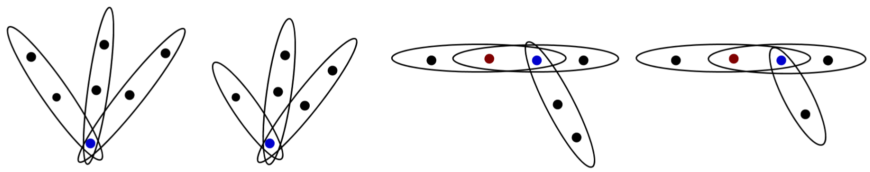


Figure 6.1: Hypergraphs H_1, H_2, H_3, H_4 in Table 6.1

Upper bounds	H_1	H_2	H_3	H_4
$a = \max_{i\sim j} \sqrt{d_i d_j}$	1.7321	1.7321	2.4495	2.4495
$b = \max \sqrt[k]{d_{i_1} \cdots d_{i_k}}$	1.4422	2.0801	1.8171	2.0801
$\sqrt[k]{\Delta^{k-t+1} d_2^{t-1}}$	1.4422	2.0801	2.6207	2.2894
ρ	1.4422	1.5269	1.7100	1.8048
	0	36%	6%	15%

Table 6.1: Comparison of upper bounds

By Definition 1.1, we have the following corollary.

Corollary 6.1. *Let $H = (V, E)$ be a connected hypergraph of rank k and co-rank t, with vertex degrees $d_1 = \Delta \geq d_2 \geq \cdots \geq d_n$. Then*

$$\rho(\mathcal{A}(H)) \leq \sqrt[k]{\Delta^{k-t+1} d_2^{t-1}},$$

$$\rho(\mathcal{Q}(H)) \leq \Delta + \sqrt[k]{\Delta^{k-t+1} d_2^{t-1}}.$$

When $k = t$, H is a uniform hypergraph, Corollary 6.1 coincides with results in [65]. A well known case when equalities hold in the above is that the hypergraph is regular, i.e. all vertices share a common degree.

We test the performance of bounds in Theorems 6.1 and 6.2 in the four hypergraphs depicted in Figure 6.1, respectively. The fourth row are the spectra radii computed from the NQZ algorithm for nonnegative tensors [40]. The last row of the table shows the distance of the bounds to the numerical results.

For simplicity, denote the bounds $\max_{i\sim j} \sqrt{d_i d_j}$ of Theorem 6.1 and $\max \sqrt[k]{d_{i_1} \cdots d_{i_k}}$ in Theorem 6.2 by a and b, respectively. It is found that bound b may be better than

a when the sequence $\Delta, d_2, d_3, \ldots$ has bigger variance (see results for H_3 in Table 6.1). Moreover, $a = \Delta \geq b$ if there exists a pair of adjacent vertices i, j (i, j are contained in one edge) such that $d_i = d_j = \Delta$. However, if any pair of vertices of degree Δ is not adjacent, and one of them is contained in an edge with very few vertices, then b may be much larger than a (see results for H_2 in Table 6.1). More importantly, if there exists $e \in E$, $d_i = \Delta$ for each $i \in e$, then $a = b = \Delta$, which is not sharp when H is connected yet not regular.

Therefore, a bound for non-regular hypergraphs is provided in the next. Before that, we present some lemmas being used.

Lemma 6.1. *[68] Let $a_1, \ldots, a_n$ be nonnegative real numbers. Then*

$$\sum_{i=1}^{n} a_i - n\sqrt[n]{a_1 \cdots a_n} \geq \frac{1}{n-1} \sum_{1 \leq i < j \leq n} (\sqrt{a_i} - \sqrt{a_j})^2.$$

Lemma 6.2. *[68] Let a, b, x_1, x_2 be positive numbers. Then $a(x_1 - x_2)^2 + bx_2^2 \geq \frac{ab}{a+b} x_1^2$.*

Note that $\sum_i d_i = \sum_{e \in E} |e| := p$. Let d be the average degree of H, then $p = nd$. According to the eigenequation of $(\rho(\mathcal{A}), \mathbf{x})$ at vertex i,

$$\rho(\mathcal{A}) x_i^k = x_i \sum_{e \in E} \mathcal{A}_e x^{e \setminus \{i\}} \leq \sum_{e \in E} x_i \mathcal{A}_e \frac{x_{\max}^{k-1}}{\mathcal{A}_e} \leq d_i x_{\max}^k.$$

Summing from $i = 1$ to n we obtain $\rho(\mathcal{A}) \leq \sum_i d_i x_{\max}^k = p x_{\max}^k$, and thus

$$x_{\max} \geq \frac{\rho(\mathcal{A})}{p}.$$

Theorem 6.3. *Let $H = (V, E)$ be a non-regular connected hypergraph of rank k, co-rank t and maximum degree Δ. Then*

$$\Delta - \rho(\mathcal{A}) > \frac{t(n\Delta - p)\Delta}{2pD(k-1)(n\Delta - p) + tn\Delta},$$

$$2\Delta - \rho(\mathcal{Q}) > \frac{4t(n\Delta - p)\Delta}{[4pD(k-1)+t]\,(n\Delta - p) + tn\Delta},$$

where $p = \sum_i d_i = \sum_{e\in E} |e|$.

Proof. Let $\mathbf{x}$ be the Perron vector of $\mathcal{A}$ with $\sum_{i=1}^n x_i^k = 1$. Let $x_u = \max_i x_i$ and $x_v = \min_i x_i$. Since H is connected and non-regular, $x_u > \sqrt[k]{\frac{1}{n}} > x_v > 0$. Note that for $e = \{j_1, \ldots, j_r\}$,

$$x^e = \sum_{c\{i_1,\ldots,i_k\}=e} x_{i_1}\cdots x_{i_k} \le \frac{r}{\mathcal{A}_e} x_{j_1}\cdots x_{j_r}.$$

By Lemma 6.1, together with $\sum_{i=1}^n d_i < k|E|$ we have

$$\begin{aligned}
\Delta - \rho(\mathcal{A}) &= \Delta\sum_{i=1}^n x_i^k - \sum_{e\in E}\mathcal{A}_e x^e \\
&= \sum_{i=1}^n(\Delta - d_i)x_i^k + \sum_{i=1}^n d_i x_i^k - \sum_{e\in E}\mathcal{A}_e x^e \\
&= \sum_{i=1}^n(\Delta - d_i)x_i^k + \sum_{e=\{j_1,\ldots,j_r\}\in E}(x_{j_1}^k + \cdots + x_{j_r}^k - \mathcal{A}_e x^e) \\
&> \sum_{i=1}^n(\Delta - d_i)x_i^k + \sum_{e=\{j_1,\ldots,j_r\}\in E}(x_{j_1}^k + \cdots + x_{j_r}^k - r x_{j_1}\cdots x_{j_r}) \\
&\ge (n\Delta - p)x_v^k + \frac{1}{k-1}\sum_{\substack{\{i,j\}\subset e\in E\\ i<j}}(x_i^{\frac{k}{2}} - x_j^{\frac{k}{2}})^2
\end{aligned}$$

Since H is connected, there is a path $P = (u =)u_0 e_1 u_1 \cdots u_{r-1} e_r u_r (= v)$ connecting u and v, where each e_i contains u_{i-1} and u_i. Now

$$\sum_{\substack{\{i,j\}\subset e\in E(P)\\ i<j}}(x_i^{\frac{k}{2}} - x_j^{\frac{k}{2}})^2 \ge \sum_{i=0}^{r-1}(x_{u_i}^{\frac{k}{2}} - x_{u_{i+1}}^{\frac{k}{2}})^2 + \sum_{v_i\in e_i\setminus\{u_{i-1},u_i\}}\sum_{i=0}^{r-1}[(x_{u_i}^{\frac{k}{2}} - x_{v_{i+1}}^{\frac{k}{2}})^2 + (x_{v_{i+1}}^{\frac{k}{2}} - x_{u_i}^{\frac{k}{2}})^2].$$

It results from the Cauchy-Schwartz inequality that

$$\begin{aligned}
\sum_{\substack{\{i,j\}\subset e\in E(P)\\ i<j}} (x_i^{\frac{k}{2}} - x_j^{\frac{k}{2}})^2 &\geq \frac{1}{r}\left[\sum_{i=0}^{r-1}(x_{u_i}^{\frac{k}{2}} - x_{u_{i+1}}^{\frac{k}{2}})\right]^2 + \sum_{v_i\in e_i\setminus\{u_{i-1},u_i\}} \frac{1}{2r}\left[\sum_{i=0}^{r-1}(x_{u_i}^{\frac{k}{2}} - x_{u_{i+1}}^{\frac{k}{2}})\right]^2 \\
&\geq \frac{1}{r}(x_u^{\frac{k}{2}} - x_v^{\frac{k}{2}})^2 + \frac{t-2}{2r}(x_u^{\frac{k}{2}} - x_v^{\frac{k}{2}})^2 \\
&= \frac{t}{2r}(x_u^{\frac{k}{2}} - x_v^{\frac{k}{2}})^2 \geq \frac{t}{2D}(x_u^{\frac{k}{2}} - x_v^{\frac{k}{2}})^2.
\end{aligned}$$

Thus by Lemma 6.2 and $x_u^k \geq \frac{\rho(\mathcal{A})}{p}$,

$$\begin{aligned}
\Delta - \rho(\mathcal{A}) &> (n\Delta - p)x_v^k + \frac{t}{2D(k-1)}(x_u^{\frac{k}{2}} - x_v^{\frac{k}{2}})^2 \\
&> \frac{t(n\Delta - p)}{2D(k-1)(n\Delta - p) + t}x_u^k \\
&> \frac{t(n\Delta - p)[\Delta - (\Delta - \rho(\mathcal{A}))]}{[2D(k-1)(n\Delta - p) + t]\,p}.
\end{aligned}$$

Finally we obtain

$$\Delta - \rho(\mathcal{A}) > \frac{t(n\Delta - p)\Delta}{2pD(k-1)(n\Delta - p) + tn\Delta}.$$

Following the above schedule, we may deduce a similar bound for $2\Delta - \rho(\mathcal{Q})$ as below:

$$\begin{aligned}
2\Delta - \rho(\mathcal{Q}) &= 2\Delta\sum_{i=1}^{n} x_i^k - \left(\sum_{i=1}^{n} d_i x_i^k + \sum_{e\in E}\mathcal{A}_e x^e\right) \\
&= 2\sum_{i=1}^{n}(\Delta - d_i)x_i^k + \sum_{i=1}^{n} d_i x_i^k - \sum_{e\in E}\mathcal{A}_e x^e \\
&> 2(n\Delta - p)x_v^k + \frac{t}{2D(k-1)}(x_u^{\frac{k}{2}} - x_v^{\frac{k}{2}})^2 \\
&> \frac{2t(n\Delta - p)[2\Delta - (2\Delta - \rho(\mathcal{Q}))]}{[4D(k-1)(n\Delta - p) + t]\,p}.
\end{aligned}$$

Then $2\Delta - \rho(\mathcal{Q}) > \frac{4t(n\Delta-p)\Delta}{[4D(k-1)p+t](n\Delta-p)+tn\Delta}$. $\square$

Note that when $t = k$, $p = k|E|$, then the first bound coincides the bound in [68].

Let $\mathcal{I}$ be the identity tensor. Since $\mathcal{Q} \leq \mathcal{A} + \Delta\mathcal{I}$, by properties of symmetric tensors in [47], $\rho(\mathcal{Q}) \leq \rho(\mathcal{A}) + \Delta$, which implies the first inequality of this theorem.

Remark. Furthermore if H is f-edge connected, then there are f mutually edge-disjoint paths connecting u and v. Thus

$$\Delta - \rho(\mathcal{A}) > (n\Delta - p)x_v^k + \frac{ft}{2D(k-1)}(x_u^{\frac{k}{2}} - x_v^{\frac{k}{2}})^2.$$

Then by Lemma 6.2, we obtain

$$\Delta - \rho(\mathcal{A}) > \frac{ft(n\Delta - p)\Delta}{2pD(k-1)(n\Delta - p) + ftn\Delta}.$$

6.2 Spectral bounds for generalized power hypergraphs

As is mentioned in Section 3.3, relationship has been found between the spectral radius of a generalized power hypergraph and that of its base. This fact leads us to an idea of characterizing the spectral radius of a non-uniform generalized power hypergraph by its bases.

We will apply the minimax theorem for nonnegative tensors in Section 2.2 to deduce the following result. In the following, the adjacency tensor $\mathcal{A}(H)$ of a hypergraph H is also denoted as $\mathcal{A}^H$ for convenience.

Theorem 6.4. *Let $H \in \mathbb{G}^{k,s}$ be a generalized power hypergraph as in Definition 3.5 of an r-graph $G = (V, E)$, $k \geq \max\{r, 3\}$. Then*

$$\rho(\mathcal{A}^H) \geq [\rho(\mathcal{A}^G)]^{\frac{rs}{k}}.$$

equality holds if H is k-uniform.

Proof. We can suppose that $k \geq rs+1$. Otherwise if $k = rs$, then H is uniform and equality holds according to [30]. Let $\mathbf{x}$ be an eigenvector of $\mathcal{A}^G$ corresponding to $\rho = \rho(\mathcal{A}(G))$. Define a vector $\mathbf{y}$ with $|V(H)|$ components as below:

$$y_l = \begin{cases} (x_i)^{\frac{r}{k}} & \text{if } l \in V_i \text{ for } i \in V, \\ (\rho^{-1}\prod_{j\in e} x_j)^{\frac{1}{k}} & \text{if } l \in V_e \text{ for } e \in E. \end{cases}$$

Let $e \in E$ and $e' \in E'$ with $e' = (\cup_{j\in e} V_j) \cup V_e$. For $i \in e$ and $i' \in V_i$, consider an arbitrary summing item $y_{i_2}\cdots y_{i_k}$ of $\mathbf{y}^{e'\backslash\{i'\}}$. Assume that for $j \in e$, vertices in V_j appear β_j times in total in the sequence $i_2, \ldots, i_k$, then

$$\begin{aligned} y_{i_2}\cdots y_{i_k} &= \prod_{j\in e} x_j^{\frac{r}{k}\beta_j}(\rho^{-1}\prod_{j\in e} x_j)^{\frac{k-1-\sum_{l\in e}\beta_l}{k}} \\ &= \left[\frac{\rho^{-k+\sum_{l\in e}(\beta_l - s)+1+rs} x_i^{k-r+(r-1)(\beta_i-s+1)}\prod_{j\in e\backslash\{i\}} x_j^{k+(r-1)(\beta_j-s)}}{x_i^{\sum_{l\in e\backslash\{i\}}(\beta_l-s)}\prod_{j\in e\backslash\{i\}} x_j^{\sum_{l\in e\backslash\{j\}}(\beta_l-s)+1}}\right]^{\frac{1}{k}}. \end{aligned}$$

By using eigenequations of $(\rho, \mathbf{x})$ at $j \in e$, we have

$$\rho x_j^{r-1} = \sum_{f\in E_j}\prod_{l\in f\backslash\{j\}} x_l \geq \prod_{l\in e\backslash\{j\}} x_l.$$

Thus

$$\begin{aligned} &\rho^{[\sum_{l\in e}(\beta_l-s)+1]} x_i^{(r-1)(\beta_i-s+1)} \prod_{j\in e\backslash\{i\}} x_j^{(r-1)(\beta_j-s)} \\ =\ & \rho^{\beta_i-s+1} x_i^{(r-1)(\beta_i-s+1)} \cdot \prod_{j\in e\backslash\{i\}} [\rho^{\beta_j-s} x_j^{(r-1)(\beta_j-s)}] \\ \geq\ & \prod_{l\in e\backslash\{i\}} x_l^{\beta_i-s+1} \cdot \prod_{j\in e\backslash\{i\}}\left(\prod_{l\in e\backslash\{j\}} x_l^{\beta_j-s}\right) \\ =\ & x_i^{\sum_{l\in e\backslash\{i\}}\beta_l - s} \prod_{j\in e\backslash\{i\}} x_j^{\sum_{l\in e\backslash\{j\}}(\beta_l-s)+1}. \end{aligned}$$

Accordingly,

$$y_{i_2} \cdots y_{i_k} \geq \rho^{\frac{ts}{k}-1} x_i^{\frac{k-r}{k}} \prod_{j \in e \setminus \{i\}} x_j.$$

Therefore, for any of $i' \in V_i$,

$$\begin{aligned}
\frac{(\mathcal{A}^H \mathbf{y}^{k-1})_{i'}}{y_{i'}^{k-1}} &= \frac{\sum_{e' \in E_i'} \mathcal{A}_{e'}^H \mathbf{y}^{e' \setminus i'}}{y_{i'}^{k-1}} \\
&\geq \frac{\rho^{\frac{rs}{k}-1} x_i^{\frac{k-t}{k}} \sum_{e \in E_i} \prod_{j \in e \setminus \{i\}} x_j}{x_i^{\frac{r(k-1)}{k}}} \\
&= \frac{\rho^{\frac{rs}{k}-1} x_i^{\frac{k-r}{k}} \cdot \rho x_i^{r-1}}{x_i^{\frac{r(k-1)}{k}}} = \rho^{\frac{rs}{k}}.
\end{aligned}$$

Next we consider $l \in V_e \subset e'$, $y_l = (\rho^{-1} \prod_{j \in e} x_j)^{\frac{1}{k}}$. Similarly by eigenequations at $j \in e$, we can prove that for each item $y_{l_2 \cdots l_k}$ of $\mathbf{y}^{e' \setminus l}$,

$$\begin{aligned}
y_{l_2 \cdots l_k} &= \prod_{j \in e} x_j^{\frac{r}{k}\beta_j} (\rho^{-1} \prod_{j \in e} x_j)^{\frac{k-1-\sum_{l \in e} \beta_l}{k}} \\
&= \rho^{\frac{rs-k+1}{k}} \prod_{j \in e} x_j^{\frac{k-1}{k}} \cdot \left[\frac{\prod_{j \in e} \rho x_j^{(r-1)(\beta_j - s)}}{\prod_{j \in e} x_j^{\sum_{l \in e \setminus \{j\}} (\beta_j - s)}} \right]^{\frac{1}{k}} \\
&\geq \rho^{\frac{rs-k+1}{k}} \prod_{j \in e} x_j^{\frac{k-1}{k}}.
\end{aligned}$$

Hence,

$$\frac{(\mathcal{A}^H \mathbf{y}^{k-1})_l}{y_l^{k-1}} = \frac{\mathcal{A}_{e'}^H \mathbf{y}^{e' \setminus l}}{y_l^{k-1}} \geq \frac{\rho^{\frac{rs-k+1}{k}} \prod_{j \in e} x_j^{\frac{k-1}{k}}}{(\rho^{-1} \prod_{j \in e} x_j)^{\frac{k-1}{k}}} = \rho^{\frac{rs}{k}}.$$

By Theorem 3.4 (2) and the arbitrariness of e and e', we have that

$$\rho(\mathcal{A}^H) \geq \min_{i \in V'} \frac{(\mathcal{A}^H \mathbf{y}^{k-1})_i}{y_i^{k-1}} \geq \rho^{\frac{rs}{k}}.$$

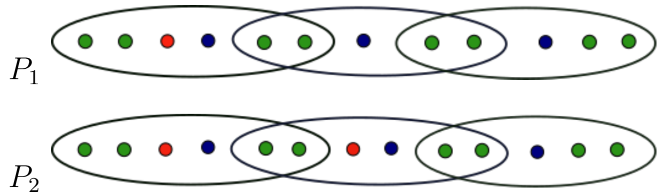


Figure 6.2: Two type of generalized loose paths with rank 6 and three edges

□

To further understand the performance of this bound, we compare the spectral radii (compute by the NQZ algorithm) with the obtained bound for two types of generalized loose paths. The first type, denoted as P_1, is obtained from a generalized power $P^{k-1,s}$ of a path P with $s = 2$ by adding a new vertex to its first edge. The second type, denoted as P_2, is obtained from a generalized power $P^{k,s}$ of a path P with $s = 2$ by deleting a core/pendent vertex from its last edge. Figure 6.2 depicts these two hypergraphs in the case of three edges with $k = 6$.

We first compute P_1, P_2 for $k = 6$ from 2 edges to 8 edges and results are shown in Figure 6.3, where the blue line and green line represent spectral radii of P_1, P_2 respectively while the doted pink line corresponds to the bound in Theorem 6.4. For example, the path P with m edges has spectral radius $2\cos(\frac{\pi}{m+2})$, then the bound is $[2\cos(\frac{\pi}{m+2})]^{\frac{2}{3}}$ (which is actually $\rho(P^{6,2})$). Besides, the dotted blue line corresponds to the spectral radius of the uniform generalized power $P^{5,2}$ of P, which differs from P_1 by one less vertex in the first edge.

It can be observed that as edge number increases, $\rho(P_2)$ approaches the lower bound which is actually the spectral radius of $\rho(P^{6,2})$, while $\rho(P_1)$ is approaching the spectral radius of $\rho(P_{5,2})$ at present. Due to storage limitation of this algorithm, we do not proceed the computation after 8 edges. To further observe the trend, we consider the case of $k = 5$ with edge numbers being $3, 5, 7, \ldots, 15$ and similar shapes are obtained as shown in Figure 6.4.

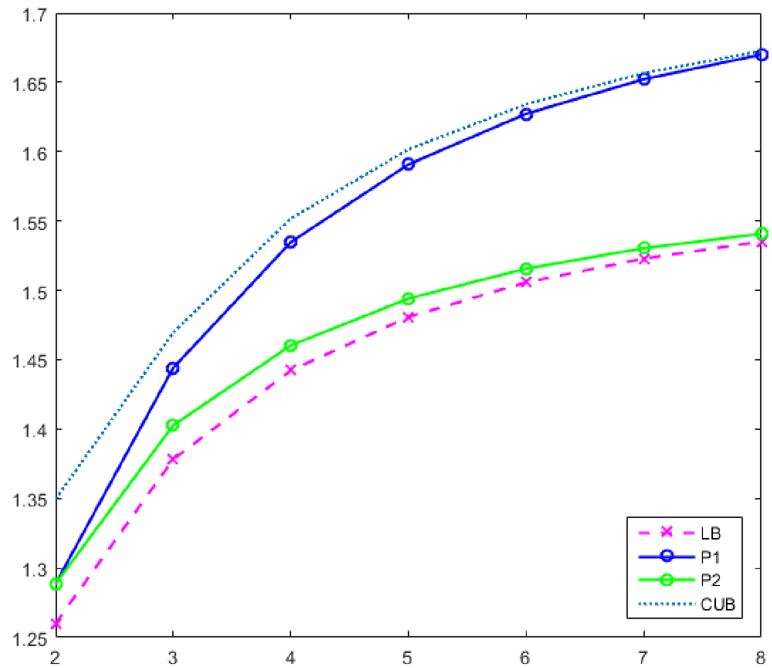


Figure 6.3: Experiment results $k = 6$

We may conjecture that the spectral radius of a generalized power hypergraph in $\mathcal{G}^{k,s}$ can be bound up by $G^{k-1,s}$. Besides, it may be noted from our experiments that the more a non-uniform hypergraph approaches a uniform one in structure, the less difference exists between their spectral radii.

6.3 Symmetric spectra and spectral equivalence of general hypergraphs

In this section, properties relevant to the spectral radius of $\mathcal{L}, \mathcal{Q}$ and the spectral symmetry of $\mathcal{A}$ are discussed.

Recall from Chapter 3 that $\rho(\mathcal{L}) \leq \rho(\mathcal{Q})$ for each hypergraph. Thus we have the following bounds for the Laplacian spectral radius of a general hypergraph deduced from results in Section 6.1.

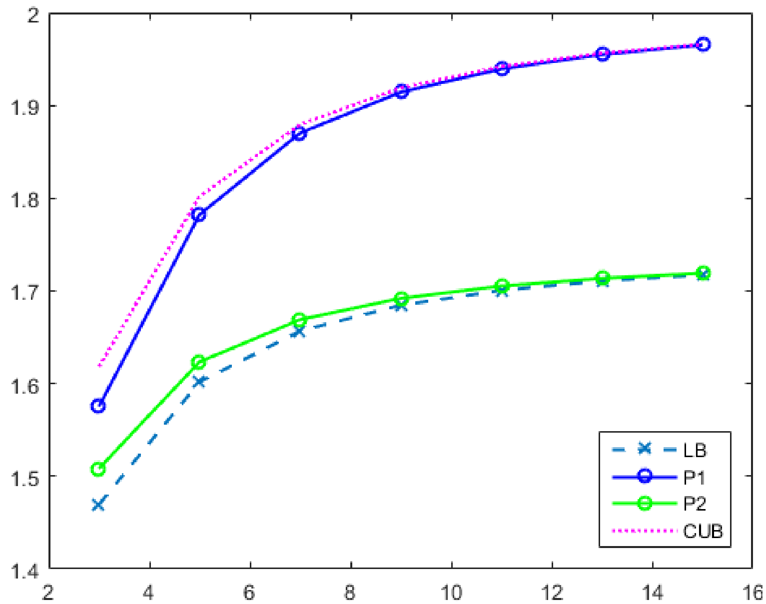


Figure 6.4: Experiment results $k = 5$

Corollary 6.2. *Let H be a hypergraph with rank $k \geq 3$. Then*

$$\rho(\mathcal{L}) \leq \min\{\max_{i\sim j}\{d_i + d_j\},\ \Delta + \max \sqrt[k]{d_{i_1} \cdots d_{i_k}}\}.$$

Note that the equality in Corollary 6.2 can be obtained when H itself is a uniform, regular and odd-bipartite hypergraph (such as the one in Figure 6.5), or the connected component attaining the largest spectral radius among all components of H is such a hypergraph.

In the next, we investigate when will a general hypergraph possess symmetric spectrum or equal Laplacian and signless Lapalacian spectra. Before that, the concept of diagonal similar tensors is to be introduced.

Definition 6.1. *[53, 61] Let $\mathcal{M}, \mathcal{N} \in \mathbb{T}_{k,n}$. If there exists a nonsingular diagonal matrix U of order n such that $\mathcal{N} = U^{-(k-1)}\mathcal{M}U$, where*

$$(U^{-(k-1)}\mathcal{M}U)_{i_1\cdots i_k} = U_{i_1 i_1}^{-(k-1)}\mathcal{M}_{i_1\cdots i_k}U_{i_2 i_2}\cdots U_{i_k i_k},$$

then $\mathcal{M}$ and $\mathcal{N}$ are called diagonal similar.

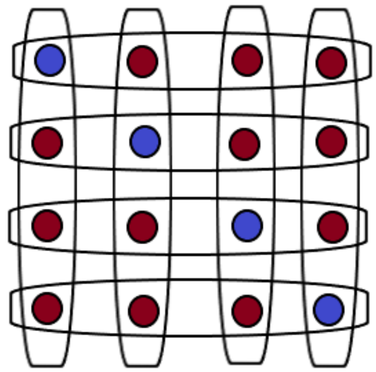


Figure 6.5: A regular and odd-biparitite 4-graph

Lemma 6.3. *[53] If two tensors are diagonal similar, then they have the same spectra.*

In addition, we need the following property for weakly irreducible tensors.

Lemma 6.4. *[61] Let $\mathcal{M}, \mathcal{N} \in \mathbb{T}_{k,n}$ with $|\mathcal{M}| \leq \mathcal{N}$. If $\mathcal{N}$ is weakly irreducible and $\rho(\mathcal{M}) = \rho(\mathcal{N})$, where $\lambda = \rho(\mathcal{N})e^{i\theta}$ and $\mathbf{y} \in \mathbb{C}^n$ is an eigenpair of $\mathcal{M}$, then*
(1) *no component of $\mathbf{y}$ is zero;*
(2) *then $\mathcal{M} = e^{i\theta}U^{-(k-1)}\mathcal{N}U$, where U is defined by $U = diag(y_1/|y_1|, \ldots, y_n/y_n)$.*

To begin with, equivalence between $Spec(\mathcal{A}) = Spec(-\mathcal{A})$ and $Spec(\mathcal{L}) = Spec(\mathcal{Q})$ are verified for connected general hypergraphs by employing Lemma 6.4. The proof is very similar with that for uniform hypergraphs [18, 54].

Proposition 6.1. *Let H be a connected hypergraph with adjacency tensor $\mathcal{A}$, Laplacian tensor $\mathcal{L}$ and signless Laplacian tensor $\mathcal{Q}$. Then the following conditions are equivalent.*

(1) $-\rho(\mathcal{A}) \in Spec(\mathcal{A})$;

(2) $Spec(\mathcal{A}) = Spec(-\mathcal{A})$;

(3) $\rho(\mathcal{L}) = \rho(\mathcal{Q})$;

(4) $Spec(\mathcal{L}) = Spec(\mathcal{Q})$.

Proof. It is apparent that (2) $\Rightarrow$ (1) and (4) $\Rightarrow$ (3). Now we prove (1) $\Rightarrow$ (4) and (3) $\Rightarrow$ (2).

If $-\rho(\mathcal{A}) \in Spec(\mathcal{A})$, then by Lemma 6.4, there is a nonsingular diagonal matrix U such that $\mathcal{A} = e^{i\theta}U^{-(k-1)}\mathcal{A}U$.

Since $-\rho(\mathcal{A}) \in Spec(\mathcal{A})$, we have $e^{i\theta} = -1$, which means $\mathcal{A} = -U^{-(k-1)}\mathcal{A}U$. Note that for the degree diagonal tensor $\mathcal{D}$, $U^{-(k-1)}\mathcal{D}U = \mathcal{D}$.

Hence we have

$$\mathcal{L} = \mathcal{D} - \mathcal{A} = \mathcal{D} + U^{-(k-1)}\mathcal{A}U = U^{-(k-1)}(\mathcal{D} + \mathcal{A})U = U^{-(k-1)}\mathcal{Q}U.$$

Thus $\mathcal{L}$ and $\mathcal{Q}$ are diagonal similar, and by Lemma 6.3, $Spec(\mathcal{L}) = Spec(\mathcal{Q})$.

On the other hand, if $\rho(\mathcal{L}) = \rho(\mathcal{Q})$, then from Lemma 6.1 we have $\mathcal{L} = e^{i\theta}U^{-(k-1)}\mathcal{Q}U$. Comparison of diagonal entries between $\mathcal{L}$ and $\mathcal{Q}$ implies that $e^{i\theta} = 1$. Thus

$$\mathcal{D} - \mathcal{A} = U^{-(k-1)}(\mathcal{D} + \mathcal{A})U = \mathcal{D} + U^{-(k-1)}\mathcal{A}U,$$

and then $-\mathcal{A} = U^{-(k-1)}\mathcal{A}U$. As $\mathcal{A}$ and $-\mathcal{A}$ are diagonal similar, we have $Spec(\mathcal{A}) = Spec(-\mathcal{A})$.

Now (1) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1) means that the four conditions are equivalent. $\square$

It deserves remarking that the above equivalence are restricted in connected hypergraphs. If H has at least two connected components, then it is possible that $\rho(\mathcal{L}) = \rho(\mathcal{Q})$ and $Spec(\mathcal{L}) \neq Spec(\mathcal{Q})$ (for example, the connected component H_i with $\rho(\mathcal{Q}(H_i)) = \rho(\mathcal{Q})$ is an odd-colorable uniform hypergraph while another component is non-uniform).

Next we derive some facts about non-uniform hypergraphs utilizing odd-coloring of tensors.

Note that the definition of odd-colorable uniform hypergraphs mentioned in the last chapter is a specialized version of odd-colorable r-matrices introduced in [42].

The r-matrix therein refers to a square tensor of order r in this thesis. We restate the original definition as following.

Definition 6.2. *Let $\mathcal{T} \in \mathbb{T}_{k,n}$ for an even k. Then $\mathcal{T}$ is called odd-colorable if there is a map $\phi : [n] \to [k]$ satisfying*

$$\phi(i_1) + \cdots \phi(i_k) \equiv \frac{k}{2}(mod\ k),\ \forall\ \mathcal{T}_{i_1 \cdots i_k} \neq 0.$$

Moreover, it was shown in the Theorem 20 of [42] that being odd-colorable is a necessary condition for a symmetric nonnegative tensor to have symmetric spectrum.

Lemma 6.5. *[42] Let $\mathcal{T} \in \mathbb{T}_{k,n}$ be symmetric and nonnegative. If $Spec(\mathcal{T}) = Spec(-\mathcal{T})$, then k is even and $\mathcal{T}$ is odd-colorable.*

Considering the structure of the adjacency tensor for a non-uniform hypergraph, we can deduce the following from Lemma 6.5.

Proposition 6.2. *If H is a non-uniform hypergraph, then the spectrum of its adjacency tensor is not symmetric.*

Proof. Suppose that H is a hypergraph with rank k and co-rank t, $k > t$. If $\mathcal{A}$ is odd-colorable, let ϕ be the map satisfying Definition 6.2. For an edge e of H with cardinality $t < k$, say $e = \{j_1, \ldots, j_t\}$, the entries of $\mathcal{A}$ corresponding to this edge e are nonzero. Let $A_{i_1 \cdots i_k}$ be such an entry. Then

$$\phi(i_1) + \cdots + \phi(i_k) \equiv \frac{k}{2}(\text{mod } k).$$

Note that by replacing i_1 (or other i_l) with any one of $j_1, \ldots, j_t$, the new indices always satisfy the above equation as far as each j_l appears in it at least once. Thus we can conclude that $\phi(j_1) = \cdots = \phi(j_t)$ and the equation implies $k\phi(j) = \frac{k}{2} + ks$ for $j \in e$ and some fixed integer s, which further yields $\phi(j) = \frac{1}{2} + s$, a contradiction with the definition of the map ϕ. Therefore, $\mathcal{A}$ is not odd-colorable for a non-uniform hypergraph. By Lemma 6.5, $\mathcal{A}$ can not have a symmetric spectrum. $\square$

Combining this fact and Proposition 6.1, the following statements can be concluded.

Corollary 6.3. *If H is a connected hypergraph with distinct rank and co-rank. Then*
(1) $\max\{\lambda : \lambda \in HSpec(\mathcal{L})\} \leq \rho(\mathcal{L}) < \rho(\mathcal{Q})$;
(2) $\min\{\lambda : \lambda \in HSpec(\mathcal{A})\} > -\rho(\mathcal{A})$.

The corollary also indicates that if a hypergraph has symmetric H-spectrum or equal Laplacian H-spectrum and signless Laplacian H-spectrum, then it is uniform (and odd-bipartite [54, Theorems 2.2, 2.3]).

6.4 Conclusions

In our study of general hypergraphs, we obtained three types of upper bounds for the spectral radius $\rho(\mathcal{A})$ and the signless Lapalcian spectral radius $\rho(\mathcal{Q})$, each of them has advance under different situations. Additionally, the upper bounds for $\rho(\mathcal{Q})$ also suits the Laplacian spectral radius $\rho(\mathcal{L})$.

Based on the idea of characterizing the spectral radius of general hypergraphs by uniform hypergraphs, we proved a lower bound for (non-uniform) generalized power hypergraph. Experiments show that the actual spectral radius may be approaching to the bound when the non-uniform hypergraph lacks only a few core/pendent vertices from being uniform.

Finally, we extend the equivalent conditions of symmetric spectrum (i.e. $Spec(\mathcal{A}) = Spec(-\mathcal{A})$) for connected uniform hypergraphs to connected general hypergraphs. Moreover, through the odd-coloring of the adjacency tensor, we refute the possibility for a non-uniform hypergraph to have symmetric (H-)spectrum, or some equivalent conditions. Nevertheless, under some conditions, a disconnected non-uniform hypergraph has $-\rho(\mathcal{A})$ as its eigenvalue, or equal Laplacian spectral radius and signless Laplacian spectral radius.

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